ResearchGate

See discussions, stats, and author profiles for this publication at: https://www.researchgate.net/publication/263177255

Periodic attractors, strange attractors and hyperbolic dynamics near homoclinic orbits to saddle-focus equilibria

Article *in* Nonlinearity · January 2002

CITATIONS
24

READS
24

1 author:

Ale Jan Homburg
University of Amsterdam

57 PUBLICATIONS **689** CITATIONS

SEE PROFILE

All content following this page was uploaded by Ale Jan Homburg on 07 December 2015.

The user has requested enhancement of the downloaded file.

Periodic attractors, strange attractors and hyperbolic dynamics near homoclinic orbits to saddle-focus equilibria

Ale Jan Homburg
Korteweg-de Vries Institute for Mathematics
University of Amsterdam
Plantage Muidergracht 24
1018 TV Amsterdam
The Netherlands
alejan@science.uva.nl

Abstract

We discuss dynamics near homoclinic orbits to saddle-focus equilibria in three dimensional vector fields. The existence of periodic and strange attractors is investigated not in unfoldings, but in families for which each member has a homoclinic orbit. We consider how often, in the sense of measure, periodic and strange attractors occur in such families. We also discuss the fate of typical orbits, and establish that despite the possible existence of attractors, a large proportion of points from a small vicinity of the homoclinic orbit, lies outside the basin of an attractor.

1 Introduction

The homoclinic orbit to a saddle-focus equilibrium provides one of the main examples of the occurrence of chaotic dynamics in three dimensional vector fields. Examples of dynamical systems from applications where these homoclinic orbits play a basic role, can be found in [ArnCouTre82, ArgArnRic93, GhiChi87]. The striking complexity of the dynamics near these homoclinic orbits has been discovered and investigated by Shil'nikov, see [Shi65, Shi70, OvsShi87]. This involves hyperbolic horseshoes close to the homoclinic orbit, but possibly also periodic attractors and strange attractors. An important question is what type of dynamics typically occurs; either for single vector fields how do typical points evolve with the flow, or for families of vector fields what sort of dynamics is encountered for typical parameter values.

In this paper, we will study dynamics from this point of view near a homoclinic orbit to a saddle-focus equilibrium at which the vector field has negative divergence, so that the flow near the equilibrium contracts volumes. We are particularly interested in the occurrence of periodic attractors, of strange attractors (see below) and of hyperbolic basic sets. With the methods of this paper it can easily be shown that strange attractors occur in unfoldings, i.e. for perturbations that no longer have a homoclinic orbit. We do not consider unfoldings in this paper; we will however see that all complications known to occur in unfoldings (including Newhouse domains [New74, New79, PalTak93] and strange attractors [MorVia93]) already exist for dynamical systems at the homoclinic bifurcation.

Our main results will be formulated and discussed in this introduction. Before doing so we introduce the notions of renormalization, Hénon like map and strange attractor. A study of the dynamics near a homoclinic orbit proceeds through the introduction of a first return map on a cross section, transverse to the homoclinic orbit. Let Π be such a first return map on a cross section Σ. One technique we use to obtain information on the dynamics, is by studying renormalizations of Π; i.e. considering small subsections $\Sigma_n \subset \Sigma$ and first return maps on these subsections. Rescaling the first return map Π_n on Σ_n to a map defined on a region of unit size, usually leads to a small perturbation of a map with a simple expression. In fact, we will study a number of cases where this procedure leads to renormalizations to Hénon like maps (small perturbations of the Hénon map). Facts known for the dynamics of (strongly dissipative) Hénon like maps, for instance the existence of strange attractors, then hold for Π on a small scale. Renormalizations to Hénon like maps in unfoldings of homoclinic tangencies were computed in [TedYor86, PalTak93], see also [MorVia93, GonShiTur96].

Recall that

$$H_{a,b}(x,y) \quad = \quad (a - x^2 + y, bx) \tag{1}$$

is the Hénon family [Hen76]. Note that the Jacobian $\det DH_{a,b}(x,y)$ equals $-b$. The renormalizations we will consider lead to small perturbations of $H_{a,0}$. A Hénon type strange attractor of a two dimensional dissipative diffeomorphism f is a compact invariant set Λ with the following properties:

- Λ equals the closure of the unstable manifold of a hyperbolic periodic point,
- the basin of attraction of Λ contains an open set,
- there is a dense orbit in Λ with positive Lyapounov exponent, i.e. there exists $c \in \Lambda$ with $\omega(c) = \Lambda$ and $\|Df^n(f(c))\| \geq C\lambda^n$, $n \in \mathbb{N}$, for some $C > 0$, $\lambda > 1$,
- Λ is not hyperbolic. In fact we may assume that with c as above, $Df^n(c)$ exponentially contracts some vector both for positive and negative iterates: $\|Df^n(c)v\| \leq C\nu^n$, $n \in \mathbb{Z}$, for some $v \in T_c\mathbb{R}^2$, $C > 0$, $0 < \nu < 1$.

L. Mora and M. Viana [MorVia93], adapting work of M. Benedicks and L. Carleson [BenCar91] on the Hénon map, proved that Hénon like maps, for fixed small b, possess strange attractors for a positive set of parameter values a. We will say that a vector field possesses a Hénon type strange attractor, if a first return map on a cross section does.

Let $\mathfrak{X}$ be the set of smooth vector fields X on $\mathbb{R}^3$ with the following properties.

- X has a hyperbolic singularity p, so that $DX(p)$ has eigenvalues $\lambda_u > 0$, and $\lambda_s \pm i\omega_s$ with $\lambda_s < 0$.
- X has a homoclinic orbit Γ, i.e. an orbit contained in both the stable and the unstable manifold of p.
- $\lambda_s + \lambda_u > 0$ and $2\lambda_s + \lambda_u < 0$.

Smooth in this paper will always mean C^K for some large enough value of K, which we won't specify. The eigenvalue condition $2\lambda_s + \lambda_u < 0$ is equivalent to the divergence of X at the equilibrium being negative. This implies that the first return map on a small cross section will be dissipative.

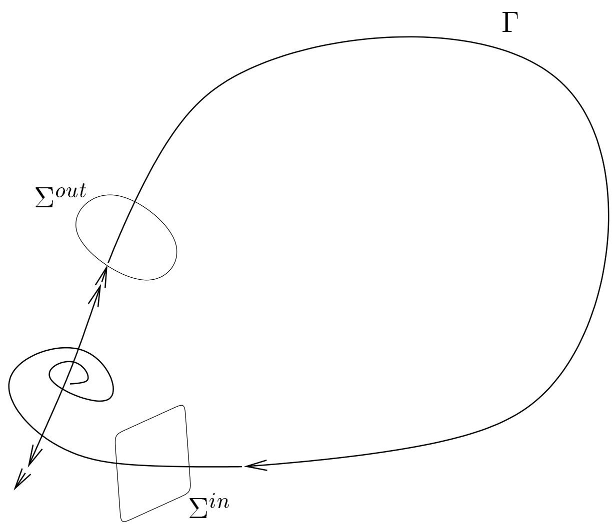

Figure 1: A saddle-focus homoclinic orbit Γ. A first return map Π on Σ^{in} is the composition of a local transition map $\Pi_{\text{loc}} : \Sigma^{in} \to \Sigma^{out}$ and a global transition map $\Pi_{\text{far}} : \Sigma^{out} \to \Sigma^{in}$.

In [Shi65, Shi70], Shil'nikov established the existence of countably many horseshoes in any neighborhood of $\overline{\Gamma}$ (the condition $2\lambda_s + \lambda_u < 0$ is not needed for this result). A geometric explanation can be found in [Tre84]. The dynamics near Γ is in general not hyperbolic, but substantially more complicated:

Definition 1.1 *Consider a vector field $X \in \mathfrak{X}$ in a small tubular neighborhood $\mathcal{U}$ of the homoclinic orbit Γ. A k-periodic orbit of X is a periodic orbit that makes k rounds in $\mathcal{U}$.*

Theorem 1.2 **[OvsShi87]** *For each $\epsilon > 0$, the set of vector fields in $\mathfrak{X}$ that have an attracting 2-periodic orbit in an ϵ-neighborhood of $\overline{\Gamma}$, is open and dense in $\mathfrak{X}$. The set of vector fields that possess infinitely many attracting 2-periodic orbits in an ϵ-neighborhood of $\overline{\Gamma}$, is dense in $\mathfrak{X}$. The set of vector fields that have an orbit of homoclinic tangency to a hyperbolic periodic orbit in an ϵ-neighborhood of $\overline{\Gamma}$, is dense in $\mathfrak{X}$.*

We extend these results by including statements on strange attractors and by a measure theoretic discussion on how often periodic and strange attractors occur. For this discussion, we need a natural parameter so that we can consider parametrized families, see [GonTurGasNic97]. For families $\{X_\gamma\}$ of vector fields in $\mathfrak{X}$, we express parameter dependence of the equilibrium, the homoclinic orbit and the eigenvalues, by writing p_γ, Γ_γ, $\lambda_s(\gamma) \pm i\omega_s(\gamma)$ and $\lambda_u(\gamma)$.

Definition 1.3 *A monotone Shil'nikov family is a one parameter family of vector fields $\{X_\gamma\}$ in $\mathfrak{X}$, with γ from a small interval I containing 0, so that*

$$\frac{\partial}{\partial \gamma}\left(\frac{-\lambda_s(\gamma)}{\lambda_u(\gamma)}\right) \neq 0.$$

Theorem 1.4 *Let $\{X_\gamma\}$, $\gamma \in I$, be a monotone Shil'nikov family. Let Σ be a cross section transversal to Γ_γ. Let $\mathcal{U}_n$ be a decreasing sequence of tubular neighborhoods of $\overline{\Gamma_\gamma}$. Let $P_{2,n}$ be the set of parameter values $\gamma \in I$ for which X_γ possesses an attracting 2-periodic orbit in $\mathcal{U}_n$. The following holds for n large enough:*

1. *$P_{2,n}$ is open and dense in I,*
2. *$\lim_{n\to\infty} |P_{2,n}| = 0$ (here $|\cdot|$ denotes Lebesgue measure).*
3. *The set of parameter values $\gamma \in I$ for which X_γ has infinitely many 2-periodic attractors is dense in I, but has zero measure.*

Let $A_{2,n}$ be the set of parameter values in $\gamma \in I$ for which X_γ possesses a Hénon type strange attractor in $\mathcal{U}_n$, intersecting Σ in 2 connected components. The following holds for n large enough:

4. *$A_{2,n}$ has positive measure and is dense in I,*
5. *$\lim_{n\to\infty} |A_{2,n}| = 0$.*

In the above theorem only periodic attractors and strange attractors that intersect a cross section in two connected components are discussed. In a study of n-periodic attractors with $n > 2$, additional complications occur, compare [GonTurGasNic97]. So, we do not have conclusive statements on the measure of the parameter set for which (periodic) attractors exist. Despite this complexity, the following result shows that for any single vector field $X \in \mathfrak{X}$ and for a large proportion of initial conditions, orbits starting in a cross section close to the homoclinic orbit move away. Though attractors and nonhyperbolic dynamics may exist near the homoclinic orbit, they involve only small portions of a tubular neighborhood of the homoclinic orbit.

Theorem 1.5 *Let $X \in \mathfrak{X}$ and let Σ be a cross section transverse to the homoclinic orbit Γ. Let $\mathcal{U}_n$ be a decreasing sequence of tubular neighborhoods of $\overline{\Gamma}$. Let D_n be the set of points x in $\Sigma \cap \mathcal{U}_n$ for which the positive orbit $\mathcal{O}^+(x)$ leaves $\mathcal{U}_n$. Then*

$$|D_n|/|\Sigma \cap \mathcal{U}_n| \quad \to \quad 1,$$

as $n \to \infty$.

One of our motivations to start this research was the result in [PumRod97], where the existence of strange attractors in a model family of vector fields with a homoclinic orbit to a saddle-focus equilibrium is studied. In [PumRod97], the eigenvalue condition $\lambda_s + \lambda_u = 0$ as well as the existence of locally linearizing coordinates is assumed. In a model of piecewise smooth vector fields, [PumRod97] achieve the existence of infinitely many coexisting strange attractors, for some parameter value.

However, combining results of [OvsShi87, Col98] shows that the coexistence of infinitely many strange attractors is true more generally. We remark that the existence of a dense subset of vector fields in $\mathfrak{X}$ with any finite number of strange attractors can be deduced by combining [OvsShi87] with [GonShiTur93, GonShiTur01], see further [Kal00].

Theorem 1.6 [OvsShi87, Col98] *For each $\epsilon > 0$, there is a dense subset $\mathcal{D}$ of $\mathcal{X}$ such that for all $X \in \mathcal{D}$, X possesses infinitely many coexisting strange attractors in an ϵ-neighborhood of $\overline{\Gamma}$.*

Proof. Proposition 3.4 below shows how families of strongly dissipative Hénon like maps appear as renormalizations of a first return map. This shows the occurrence of persistent homoclinic tangencies (alternatively, this follows from [OvsShi87] where it is shown that there are arbitrarily small perturbations of X in $\mathfrak{X}$ with a periodic orbit as close as desired near Γ whose stable and unstable manifolds are tangent). Applying [Col98] proves the result; since the required perturbations can be chosen with support off Γ, the perturbations can be made within $\mathfrak{X}$. ∎

Of course, in this type of result the number of connected components with which the strange attractors intersect a cross section is not specified, nor is the size of their basins of attraction.

In section 2 we derive asymptotic expansions for first return maps on a small cross section, transverse to the homoclinic orbits. In section 3 we prove theorem 1.4. Theorem 1.5 will be proved in section 4.

Acknowledgments: I wish to thank Sergey Gonchenko, Oleg Sten'kin, Dmitry Turaev and Jim Yorke for helpful discussions. I thank the International center for advanced studies and the Scientific research institute for applied mathematics and cybernetics in Nihzny Novgorod for their gracious hospitality. I am further grateful for the hospitality of the Institute for physical science and technology in College Park, Maryland.

2 Exponential expansions

Let $\{X_\gamma\}$ be a curve of vector fields in $\mathfrak{X}$ with γ from a small interval I containing 0. Let Σ^{in} be a small cross section transverse to the homoclinic orbit Γ_γ of X_γ. In this section we will obtain the following asymptotic expansions for the first return map Π_γ on Σ^{in}. The result is similar to [OvsShi87], but treats parameter dependence and contains more precise asymptotics. To make this part of the paper self-contained, we give a complete proof and do not refer to [OvsShi87] for parts of the argument. When bounding higher order derivatives in the following, the formulas apply to derivatives up to some order $\tilde{K}$ close to K, even though we won't explicitly state so.

Proposition 2.1 *One can choose the cross section Σ^{in} and smooth coordinates (θ, z) on it, so that Π_γ has the following asymptotic expansion.*

$$\Pi_\gamma(\theta, z) = \begin{pmatrix} \phi_1(\theta) z^{\frac{-\lambda^s}{\lambda^u}} \sin\left(-\frac{\omega^s}{\lambda^u} \ln z\right) + \phi_2(\theta) z^{\frac{-\lambda^s}{\lambda^u}} \cos\left(-\frac{\omega^s}{\lambda^u} \ln z\right) + R_{1,\gamma}(\theta, z) \\ \phi_3(\theta) z^{\frac{-\lambda^s}{\lambda^u}} \sin\left(-\frac{\omega^s}{\lambda^u} \ln z\right) + \phi_4(\theta) z^{\frac{-\lambda^s}{\lambda^u}} \cos\left(-\frac{\omega^s}{\lambda^u} \ln z\right) + R_{2,\gamma}(\theta, z) \end{pmatrix}. \tag{2}$$

The functions ϕ_i, $i = 1, 2, 3, 4$, are smooth functions of θ and γ (we suppress the dependence on γ from the notation) and satisfy $\det \begin{pmatrix} \phi_1 & \phi_2 \\ \phi_3 & \phi_4 \end{pmatrix} \neq 0$. *Furthermore,*

$$\left| \frac{\partial^{k+l+m}}{\partial\theta^k \partial z^l \partial\gamma^m} R_{i,\gamma}(\theta, z) \right| \leq C_{k+l+m} z^{\frac{-2\lambda_s}{\lambda_u} - l} (\ln z)^m. \tag{3}$$

PROOF. Proposition 2.4 below provides asymptotic expansions for a local transition map $\Pi_{\text{loc},\gamma} : \Sigma^{in} \to \Sigma^{out}$, where Σ^{out} is a cross section transverse to the local unstable manifold of the equilibrium p_γ. The global transition map $\Pi_{\text{far},\gamma} : \Sigma^{out} \to \Sigma^{in}$ is a local diffeomorphism by the flow box theorem, while $\Pi_\gamma = \Pi_{\text{far},\gamma} \circ \Pi_{\text{loc},\gamma}$. ■

2.1 Expansions for local transition maps

Let $\{X_\gamma\}$, $\gamma \in I$, be a curve of vector fields in $\mathfrak{X}$, as above. Take local coordinates (x, y, z) near the equilibrium p_γ of X_γ in which p_γ becomes the origin $\mathbf{0}$ and

$$DX_\gamma(\mathbf{0}) \;=\; \begin{pmatrix} \lambda_s & -\omega_s & 0 \\ \omega_s & \lambda_s & 0 \\ 0 & 0 & \lambda_u \end{pmatrix}. \tag{4}$$

We suppress the dependence of the eigenvalues on the parameter from the notation. For some small δ, take cross sections

$$\begin{aligned} \Sigma^{in} &\subset \{x^2 + y^2 = \delta^2, |z| \le \delta\}, \\ \Sigma^{out} &= \{x^2 + y^2 \le \delta^2, z = \delta\}, \end{aligned}$$

that intersects the homoclinic orbit Γ_γ transversally, compare figure 1. After a linear rescaling we may assume that $\delta = 1$. Note that the quadratic and higher order terms in the expression of X_γ, in the rescaled coordinates, are bounded in norm by $C\delta$, for some $C > 0$.

A workable asymptotic expression for the first return map Π_γ on Σ^{in} can be obtained after a local coordinate change near the equilibrium. As indicated in figure 1, Π_γ is the composition $\Pi_{\text{far},\gamma} \circ \Pi_{\text{loc},\gamma}$ of a local transition map through a neighborhood of the equilibrium, and a global transition map. By the flow box theorem, $\Pi_{\text{far},\gamma}$ is a local diffeomorphism. Proposition 2.4 below discusses asymptotics of $\Pi_{\text{loc},\gamma}$, in coordinates given by the following proposition.

Proposition 2.2 *Let $\{X_\gamma\}$, $\gamma \in I$, be as above. X_γ is locally smoothly equivalent to*

$$(\lambda_s x - \omega_s y + f_\gamma(x, y, z))\frac{\partial}{\partial x} + (\lambda_s y + \omega_s x + g_\gamma(x, y, z))\frac{\partial}{\partial y} + \lambda_u z \frac{\partial}{\partial z}, \tag{5}$$

where f_γ and g_γ are of the order $\mathcal{O}(\|(x, y)\|^3 z)$.

Remark 2.3 *Smoothly equivalent means that (5) is obtained from X_γ by a smooth local coordinate change and a time reparametrization (that is, multiplication by a smooth positive function). The proof uses only the eigenvalue condition $\lambda^s + \lambda^u > 0$; the condition $2\lambda^s + \lambda^u < 0$ plays no role.*

Proof. For notational convenience, we consider a single vector field $X \in \mathfrak{X}$. Start with local coordinates (x, y, z) as in (4), in which we can write

$$X(x, y, z) = \\ (\lambda_s x - \omega_s y + f(x, y, z))\frac{\partial}{\partial x} + (\omega_s x + \lambda_s y + g(x, y, z))\frac{\partial}{\partial y} + (\lambda_u z + h(x, y, z))\frac{\partial}{\partial z},$$

where f, g, h are of quadratic and higher order. A coordinate change that straightens the local stable and unstable manifolds, yields $f, g = \mathcal{O}(\|(x,y)\|)$ and $h = \mathcal{O}(z)$. A time reparametrization makes $h \equiv 0$. We may moreover assume that X restricted to the local stable manifold is linear, so that $f, g = \mathcal{O}(\|(x,y)\|z)$. A polynomial coordinate change removes monomials xz and yz from f and g, so that $f, g = \mathcal{O}(\|(x,y)\|z^2) + \mathcal{O}(\|(x,y)\|^2 z)$, compare [KatHas95].

We will first remove the terms of order $\mathcal{O}(\|(x,y)\|z^2)$ from f, g. For this we follow [OvsShi87] and consider a coordinate change of the form

$$\begin{aligned} \bar{x} &= x + g_{11}(z)x + g_{12}(z)y, \\ \bar{y} &= y + g_{21}(z)x + g_{22}(z)y, \\ \bar{z} &= z. \end{aligned}$$

Write the differential equations in the new coordinates as

$$\begin{aligned} \dot{\bar{x}} &= \lambda_s \bar{x} - \omega_s \bar{y} + F_1(\bar{x},\bar{y},\bar{z})\bar{x} + F_2(\bar{x},\bar{y},\bar{z})\bar{y}, \\ \dot{\bar{y}} &= \omega_s \bar{x} + \lambda_s \bar{y} + G_1(\bar{x},\bar{y},\bar{z})\bar{x} + G_2(\bar{x},\bar{y},\bar{z})\bar{x}, \\ \dot{\bar{z}} &= \lambda_u z. \end{aligned}$$

Treating $g_{11}, \ldots, g_{22}$ as variables, one checks that

$$\begin{aligned} F_1(\bar{x},\bar{y},\bar{z}) &= \dot{g}_{11} + \omega_s g_{12} + \omega_s g_{21} + h.o.t., \\ F_2(\bar{x},\bar{y},\bar{z}) &= \dot{g}_{12} - \omega_s g_{11} + \omega_s g_{22} + h.o.t., \\ G_1(\bar{x},\bar{y},\bar{z}) &= \dot{g}_{21} + \omega_s g_{22} - \omega_s g_{11} + h.o.t., \\ G_2(\bar{x},\bar{y},\bar{z}) &= \dot{g}_{22} - \omega_s g_{21} - \omega_s g_{12} + h.o.t., \end{aligned}$$

where the higher order terms are quadratic and higher order in $\bar{z}, \bar{y}, \bar{z}, g_{11}, \ldots, g_{22}$. The demand that F_1, F_2 and G_1, G_2 vanish at $(\bar{x}, \bar{y}) = (0,0)$, yields differential equations for $\bar{z}$ and $g_{11}, \ldots, g_{22}$;

$$\dot{\bar{z}} = \lambda_u \bar{z}, \tag{6}$$

$$\begin{pmatrix} \dot{g}_{11} \\ \dot{g}_{12} \\ \dot{g}_{21} \\ \dot{g}_{22} \end{pmatrix} = \omega_s \begin{pmatrix} 0 & -1 & -1 & 0 \\ 1 & 0 & 0 & -1 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} g_{11} \\ g_{12} \\ g_{21} \\ g_{22} \end{pmatrix} + h.o.t. \tag{7}$$

The spectrum of the above antisymmetric matrix consists of eigenvalues with zero real part. The one dimensional strong unstable manifold of the set of differential equations (6) and (7), provides the functions $g_{11}, \ldots, g_{22}$.

We can thus assume that $f, g = \mathcal{O}(\|(x,y)\|^2 z)$. By the nonresonance condition $\lambda_s + \lambda_u \neq 0$, there is a polynomial coordinate change that removes the monomials $x^2 z, xyz, y^2 z$ from f and

g, see e.g. [KatHas95]. So $f, g = \mathcal{O}(\|(x,y)\|^3 z) + \mathcal{O}(\|(x,y)\|^2 z^2)$. To remove the terms of order $\mathcal{O}(\|(x,y)\|^2 z^2)$ from f, g, consider a coordinate change of the form

$$\begin{aligned}
\tilde{x} &= x + g_{11}(z)x^2 + g_{12}(z)xy + g_{22}(z)y^2, \\
\tilde{y} &= y + h_{11}(z)x^2 + h_{12}(z)xy + h_{22}(z)y^2, \\
\tilde{z} &= z.
\end{aligned}$$

Write the differential equations in the new coordinates as

$$\begin{aligned}
\dot{\tilde{x}} &= \lambda_s \tilde{x} - \omega_s \tilde{y} + F_1(\tilde{x},\tilde{y},\tilde{z})\tilde{x}^2 + F_2(\tilde{x},\tilde{y},\tilde{z})\tilde{x}\tilde{y} + F_3(\tilde{x},\tilde{y},\tilde{z})\tilde{y}^2, \\
\dot{\tilde{y}} &= \omega_s \tilde{x} + \lambda_s \tilde{y} + G_1(\tilde{x},\tilde{y},\tilde{z})\tilde{x}^2 + G_2(\tilde{x},\tilde{y},\tilde{z})\tilde{x}\tilde{y}, + G_3(\tilde{x},\tilde{y},\tilde{z})\tilde{y}^2, \\
\dot{\tilde{z}} &= \lambda_u z.
\end{aligned}$$

As above, treat $g_{11}, g_{12}, g_{22}, h_{11}, h_{12}, h_{22}$ as variables and calculate the expressions for $F_1, \ldots, G_3$. The demand that $F_1, \ldots, G_3$ vanish at $(\tilde{x}, \tilde{y}) = (0,0)$, yields differential equations for $\tilde{z}$ and $g_{11}, g_{12}, g_{22}, h_{11}, h_{12}, h_{22}$;

$$\dot{\tilde{z}} = \lambda_u \tilde{z}, \tag{8}$$

$$\begin{pmatrix} \dot{g}_{11} \\ \dot{g}_{12} \\ \dot{g}_{22} \\ \dot{h}_{11} \\ \dot{h}_{12} \\ \dot{h}_{22} \end{pmatrix} = \left(-\lambda_s I + \omega_s \begin{pmatrix} 0 & -1 & 0 & -1 & 0 & 0 \\ 2 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 \end{pmatrix} \right) \begin{pmatrix} g_{11} \\ g_{12} \\ g_{22} \\ h_{11} \\ h_{12} \\ h_{22} \end{pmatrix} + h.o.t., \tag{9}$$

where h.o.t. are terms of quadratic and higher order in $\tilde{z}$ and in $g_{11}, g_{12}, g_{22}, h_{11}, h_{12}, h_{22}$. The linear part of (9) is a matrix of the form $-\lambda_s I + \omega_s A$. Its spectrum consists of eigenvalues with real part equal to $-\lambda_s$. This can be checked by considering A^2; the eigenvalues of A^2 can easily be computed and they turn out to be real and negative. So A has its eigenvalues on the imaginary axis.

Since $\lambda_u > -\lambda_s$, the one dimensional strong unstable manifold of the set of differential equations (8), (9) provides the functions $g_{11}, g_{12}, g_{22}, h_{11}, h_{12}, h_{22}$. ■

As the following proposition shows, in coordinates for which (5) holds, the local transition map $\Pi_{\mathrm{loc},\gamma} : \Sigma^{in} \to \Sigma^{out}$ (compare figure 1) is up to some small terms identical to the local transition map for a linear vector field.

Proposition 2.4 *After applying the smooth coordinate change given by proposition 2.2, the local transition map* $\Pi_{\mathrm{loc},\gamma} : \Sigma^{in} \to \Sigma^{out}$ *has the following asymptotic expansion.*

$$\Pi_{\mathrm{loc},\gamma}(x,y,z) = \begin{pmatrix} z^{\frac{-\lambda_s}{\lambda_u}} \left(x \cos\left(-\frac{\omega_s}{\lambda_u} \ln z\right) - y \sin\left(-\frac{\omega_s}{\lambda_u} \ln z\right)\right) + S_{1,\gamma}(x,y,z) \\ z^{\frac{-\lambda_s}{\lambda_u}} \left(x \sin\left(-\frac{\omega_s}{\lambda_u} \ln z\right) + y \cos\left(-\frac{\omega_s}{\lambda_u} \ln z\right)\right) + S_{2,\gamma}(x,y,z) \end{pmatrix}. \tag{10}$$

Here $S_{1,\gamma}$ and $S_{2,\gamma}$ are smooth functions for $z > 0$. Moreover, there exist positive constants $C_{k+l+m+r}$, k, l, m, r nonnegative integers, so that for $i = 1, 2$,

$$\left| \frac{\partial^{k+l+m+r} S_{i,\gamma}}{\partial x^k \partial y^l \partial z^m \partial \gamma^r}(x,y,z) \right| \leq C_{k+l+m+r} z^{\frac{-\lambda_s}{\lambda_u}+1-m} (\ln z)^r. \tag{11}$$

PROOF. We will first consider a single vector field $X \in \mathfrak{X}$. By assumption, X is given by (5). By the variation of constants formula, the solution $(x,y,z)(t)$ with $(x,y,z)(0) = (x_0, y_0, z_0)$ satisfies

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= e^{At} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \int_0^t e^{A(t-s)} \begin{pmatrix} f((x,y,z)(s)) \\ g((x,y,z)(s)) \end{pmatrix} ds, \\ z(t) &= e^{\lambda_u t} z_0, \end{aligned}$$

where $A = \begin{pmatrix} \lambda_s & -\omega_s \\ \omega_s & \lambda_s \end{pmatrix}$. To estimate the difference of $(x(t), y(t))$ with the linear flow $e^{At}(x_0, y_0)$, define

$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} - e^{At} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}. \tag{12}$$

Then

$$\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \int_0^t e^{A(t-s)} \begin{pmatrix} f(x(s), y(s), z(s)) \\ g(x(s), y(s), z(s)) \end{pmatrix} ds, \tag{13}$$

$$z(t) = e^{\lambda_u t} z_0, \tag{14}$$

where $(x(s), y(s))$ should be substituted by $(p(s), q(s)) + e^{As}(x_0, y_0)$. To bound the first two coordinates $p(t)$ and $q(t)$, we define Σ_α to be the set of curves $(\mathbf{u}, \mathbf{v}) : [0,T] \to \mathbb{R}^2$ with $(\mathbf{u}, \mathbf{v})(0) = (0,0)$ and for which $\sup_t \|(\mathbf{u}(t), \mathbf{v}(t))\| e^{-\alpha t}$ is bounded. Equipped with the norm

$$\|(\mathbf{u}, \mathbf{v})\|_\alpha = \sup_{0 \leq t \leq T} \|(\mathbf{u}(t), \mathbf{v}(t))\| e^{-\alpha t},$$

Σ_α is a Banach space. Let $\mathfrak{Y}$ be the map on $C^0([0,T], \mathbb{R}^2)$ given by the right hand side of (13). We claim that there is a ball B_{λ_s} in Σ_{λ_s}, so that $\mathfrak{Y}$ maps B_{λ_s} into itself. It follows from standard theory of differential equations that $\mathfrak{Y}^i(\mathbf{u}, \mathbf{v}) \to (p(t), q(t))$ as $i \to \infty$. Alternatively one can show that $\mathfrak{Y}$ is a contraction on B_{λ_s}, using similar estimates as needed to show that $\mathfrak{Y}$ maps B_{λ_s} into itself. From this and the claim we obtain that $(\mathbf{p}, \mathbf{q}) \in \Sigma_{\lambda_s}$.

To establish the claim, let $(\mathbf{u}_1, \mathbf{v}_1) = \mathfrak{Y}(\mathbf{u}_0, \mathbf{v}_0)$ for $(\mathbf{u}_0, \mathbf{v}_0) \in \Sigma_{\lambda_s}$ with $\|(\mathbf{u}_0, \mathbf{v}_0)\|_{\lambda_s} \leq K$. In the following estimates we will use

$$\|(f,g)(x,y,z)\| = \mathcal{O}(\|(x,y)\|^3 |z|).$$

Further, C will denote a positive constant, which may vary from line to line.

$$\begin{aligned}\|(u_1(t),v_1(t))\| &= \left\| \int_0^t e^{A(t-s)} \begin{pmatrix} f((u_0(s),v_0(s))+e^{As}(x_0,y_0),z(s)) \\ g((u_0(s),v_0(s))+e^{As}(x_0,y_0),z(s)) \end{pmatrix} ds \right\| \\ &\leq \int_0^t \left\|e^{A(t-s)}\right\| \left\| \begin{pmatrix} f((u_0(s),v_0(s))+e^{As}(x_0,y_0),z(s)) \\ g((u_0(s),v_0(s))+e^{As}(x_0,y_0),z(s)) \end{pmatrix} \right\| ds \\ &\leq C\delta \int_0^t e^{\lambda_s(t-s)}(K+C)^3 e^{3\lambda_s s} e^{\lambda_u s}|z_0| ds \\ &\leq C\delta(K+C)^3 \left(e^{3\lambda_s t}e^{\lambda_u t} + e^{\lambda_s t}\right)|z_0| \\ &\leq C\delta e^{\lambda_s t}(K+C)^3(e^{(2\lambda_s+\lambda_u)t}|z_0| + |z_0|). \end{aligned} \tag{15}$$

Hence, for K large enough, $\|(\mathbf{u}_1,\mathbf{v}_1)\|_{\lambda_s}$ will be smaller then $\|(\mathbf{u}_0,\mathbf{v}_0)\|_{\lambda_s}$. Note that these estimates rely on $2\lambda_s+\lambda_u \leq 0$.

It follows that $(\mathbf{p},\mathbf{q})$ lies in Σ_α. In fact, from (15) one gets

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{At} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + e^{\lambda_s t}\mathcal{O}(e^{(2\lambda_s+\lambda_u)t}z_0) + e^{\lambda_s t}\mathcal{O}(z_0). \tag{16}$$

The transition time from $(x_0,y_0,z_0) \in \Sigma^{in}$ to Σ^{out} equals $-\frac{1}{\lambda_u}\ln z_0$. Putting $T = -\frac{1}{\lambda_u}\ln z_0$ yields the identity in the statement of the proposition (for $k,l,m,r=0$).

To obtain estimates on derivatives with respect to (x_0,y_0,z_0), one differentiates (13) and (14) with respect to (x_0,y_0,z_0) and to t and treats the resulting integral equations as above. To illustrate the procedure, let us consider derivatives with respect to x_0. From (13), (14) one obtains

$$\frac{\partial(p,q)}{\partial x_0}(t) = \int_0^t e^{A(t-s)} \frac{\partial}{\partial x_0} \begin{pmatrix} f(x(s),y(s),z(s)) \\ g(x(s),y(s),z(s)) \end{pmatrix} ds, \tag{17}$$

$$\frac{\partial z}{\partial x_0}(t) = 0. \tag{18}$$

Here $(x(s),y(s))$ are functions of $(p(s),q(s))$ through (12). Considering $\frac{\partial(f,g)}{\partial x_0}$ as functions of $(x,y,\frac{\partial x}{\partial x_0},\frac{\partial y}{\partial x_0},z)$, one has

$$\left\| \frac{\partial}{\partial x_0} \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \end{pmatrix} \right\| = \mathcal{O}\left(\|(x,y)\|^2 \left\| \left(\frac{\partial x}{\partial x_0}, \frac{\partial y}{\partial x_0} \right) \right\| z \right).$$

As above, replacing (u_i,v_i), $i=0,1$, with $(u_i,v_i,\frac{\partial u_i}{\partial x_0},\frac{\partial v_i}{\partial x_0})$, one concludes that

$$\left\| \frac{\partial(p,q)}{\partial x_0}(t) \right\| \leq C|z_0|e^{\lambda_s t},$$

for some $C > 0$.

We claim that in general derivatives of $(p(t), q(t))$ satisfy estimates of the form

$$\left\| \frac{\partial^{k+l+m+n}(p,q)}{\partial x_0^k \partial y_0^l \partial z_0^m \partial t^n}(t) \right\| \leq \begin{cases} C_{k+l+n} e^{\lambda_s t} |z_0|, & \text{if } m = 0, \\ C_{k+l+m+n} e^{\lambda_s t} e^{\lambda_u t(m-1)}, & \text{if } m > 0, \end{cases} \tag{19}$$

where k, l, m, n are nonnegative integers. The above reasoning for derivatives with respect to x_0 can be followed. From (13) one gets

$$\begin{aligned} \frac{\partial^{k+l+m+n}(p,q)}{\partial x_0^k \partial y_0^l \partial z_0^m \partial t^n}(t) &= \int_0^t A^n e^{A(t-s)} \frac{\partial^{k+l+m}}{\partial x_0^k \partial y_0^l \partial z_0^m} \begin{pmatrix} f(x(s), y(s), z(s)) \\ g(x(s), y(s), z(s)) \end{pmatrix} ds + \\ & \frac{\partial^{k+l+m}}{\partial x_0^k \partial y_0^l \partial z_0^m} \left[\sum_{i=0}^{n-1} A^{n-1-i} \frac{\partial^i}{\partial t^i} \begin{pmatrix} f(x(t), y(t), z(t)) \\ g(x(t), y(t), z(t)) \end{pmatrix} \right]. \end{aligned} \tag{20}$$

Partial derivatives of $z(t)$ are easily obtained from $z(t) = e^{\lambda_u t} z_0$. Note that

$$\begin{aligned} & \frac{\partial^{k+l+m+i}}{\partial x_0^k \partial y_0^l \partial z_0^m \partial t^i} \begin{pmatrix} f(x(t), y(t), z(t)) \\ g(x(t), y(t), z(t)) \end{pmatrix} = \\ & \quad (\mathcal{F}_i, \mathcal{G}_i) \left(x, y, z, D(x,y,z), \ldots, \frac{\partial^m}{\partial z_0^m} D^{k+l+i-1}(x,y,z), \frac{\partial^{k+l+m+i}(x,y,z)}{\partial x_0^k \partial y_0^l \partial z_0^m \partial t^i} \right)(t), \end{aligned}$$

for smooth functions $\mathcal{F}_i, \mathcal{G}_i$. Here $D^j(x,y,z)$ stands for the j^{th} order derivative of (x,y,z) with respect to (x_0, y_0, t). Moreover, for some $C = C_{k+l+m+i} > 0$,

$$\begin{aligned} & \|(\mathcal{F}_i, \mathcal{G}_i)\| \leq \\ & \begin{cases} C \left\| \left(x, y, D(x,y), \ldots, D^{k+l+i-1}(x,y), \dfrac{\partial^{k+l+i}(x,y)}{\partial x_0^k \partial y_0^l \partial t^i} \right) \right\|^3 |z|, & \text{if } m = 0, \\ C \left\| \left(x, y, D(x,y), \ldots, \dfrac{\partial^m}{\partial z_0^m} D^{k+l+i-1}(x,y), \dfrac{\partial^{k+l+m+i}(x,y)}{\partial x_0^k \partial y_0^l \partial z_0^m \partial t^i} \right) \right\|^3 |z| + & \\ C \displaystyle\sum_{j=1}^{m} \left\| \left(x, y, \ldots, \dfrac{\partial^{m-j}}{\partial z_0^{m-j}} D^{k+l+i-1}(x,y), \dfrac{\partial^{k+l+m-j+i}(x,y)}{\partial x_0^k \partial y_0^l \partial z_0^{m-j} \partial t^i} \right) \right\|^3 \left| \dfrac{\partial z}{\partial z_0} \right|^j, & \text{if } m > 0. \end{cases} \end{aligned}$$

By induction (19) holds for derivatives up to order $k+l+m+n-1$. As before partial derivatives of order $k+l+m+n$ are bounded in $\Sigma_{\lambda_s + (m-1)\lambda_u}$ and thus (19) is derived. Filling in $T = -\frac{1}{\lambda_u} \ln z_0$, one shows (11). Observe that T depends explicitly on z_0, so that the chain rule is needed to compute derivatives with respect to z_0.

It remains to consider parameter dependence. We will study $\frac{\partial p}{\partial \gamma}$ and $\frac{\partial q}{\partial \gamma}$. From (13) and (14) one obtains

$$\begin{aligned}\frac{\partial(p,q)}{\partial\gamma} &= \int_0^t (t-s)e^{A(t-s)}\frac{\partial A}{\partial\gamma}\begin{pmatrix} f(x(s),y(s),z(s)) \\ g(x(s),y(s),z(s))\end{pmatrix} ds + \\ & \int_0^t e^{A(t-s)}\frac{\partial}{\partial\gamma}\begin{pmatrix} f(x(s),y(s),z(s)) \\ g(x(s),y(s),z(s))\end{pmatrix} ds, \qquad (21)\\ \frac{\partial z}{\partial\gamma}(t) &= t\frac{\partial\lambda_u}{\partial\gamma}e^{\lambda_u t}z_0. \qquad (22)\end{aligned}$$

As above one show that

$$\left\| \frac{\partial(p,q)}{\partial\gamma}(t)\right\| \leq C|z_0|te^{\lambda_s t},$$

for some $C > 0$. Continuing this reasoning for higher order derivatives, one shows

$$\left\| \frac{\partial^{k+l+m+n+r}(p,q)}{\partial x_0^k \partial y_0^l \partial z_0^m \partial t^n \partial\gamma^r}(t)\right\| \leq \begin{cases} C_{k+l+n+r}t^r e^{\lambda_s t}|z_0|, & \text{if } m=0, \\ C_{k+l+m+n+r}t^r e^{\lambda_s t}e^{\lambda_u t(m-1)}, & \text{if } m>0. \end{cases} \qquad (23)$$

■

3 Attractors

In this section we prove theorem 1.4. We will use the following notation to compare elements of sequences. By $a_n \approx b_n$ we mean that the quotient a_n/b_n is bounded and bounded away from zero, uniformly in n. And $a_n \lesssim b_n$ means that a_n is bounded by some constant times b_n, uniformly in n.

Let $\{X_\gamma\}$ be a monotone Shil'nikov family as defined in definition 1.3. We may take $\mu = -\lambda_s(\gamma)/\lambda_u(\gamma)$ as the parameter to parametrize the family $\{X_\gamma\}$. We will write $\{X_\mu\}$ for the family of vector fields, where we consider parameter values μ from an interval J containing $-\lambda_s(0)/\lambda_u(0)$ and satisfying $\overline{J} \subset (\frac{1}{2}, 1)$.

Write $\Pi_\mu = (F_\mu, G_\mu)$ for the first return map Π_μ on Σ. Denoting $\nu = -\omega_s/\lambda_u$, one has

$$\begin{aligned} G_\mu(\theta,z) &= \phi_3 z^\mu \sin(\nu\ln z) + \phi_4 z^\mu \cos(\nu \ln z) + \mathcal{O}(z^{2\mu}) \\ &= az^\mu \sin(\nu\ln z + \phi) + \mathcal{O}(z^{2\mu}), \qquad (24)\end{aligned}$$

where $a = \sqrt{\phi_3^2+\phi_4^2}$ and $\phi = \arctan \phi_4/\phi_3$. Observe that in the expression for $G_\mu(\theta, z)$, θ appears only in the coefficients and in the higher order terms. Though θ is not a parameter, but varies through F_μ, it seems plausible that the z coordinate largely controls the dynamics of Π_μ. For fixed θ, $z \mapsto G_\mu(\theta, z)$ is an infinite modal map. As a first step, one can ignore the θ dependence and study infinite modal interval maps of the form $z \mapsto az^\mu \sin(\nu \ln z + \phi) + \mathcal{O}(z^{2\mu})$,

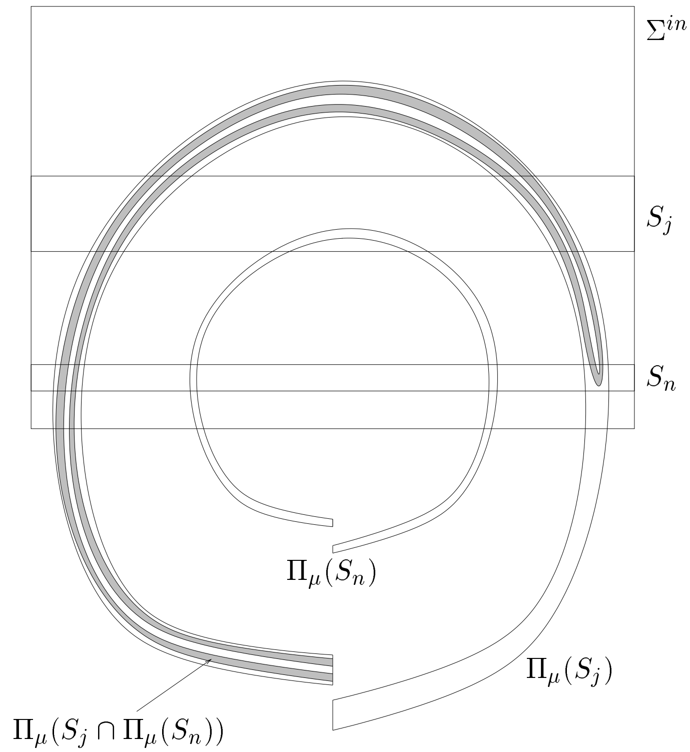

Figure 2: This figure illustrates the possibility of finding Hénon like maps in rescalings of the second iterate of the first return map $\Pi_\mu : \Sigma^{in} \to \Sigma^{in}$, and thus of finding periodic and strange attractors.

see figure 3. An analog of theorem 1.4 can be stated and proved for such infinite modal interval maps [Hom00].

To obtain the critical points of $z \mapsto G_\mu(\theta, z)$, calculate

$$\begin{aligned}\frac{\partial}{\partial z} G_\mu(\theta, z) &= a\mu z^{\mu-1} \sin(\nu \ln z + \phi) + a\nu z^{\mu-1} \cos(\nu \ln z + \phi) + \mathcal{O}(z^{2\mu-1}) \\ &= a z^{\mu-1} \sqrt{\mu^2 + \nu^2} \sin(\nu \ln z + \psi) + \mathcal{O}(z^{2\mu-1}),\end{aligned}$$

with $\psi = \arctan(\nu/\mu)$. To compute the zeros of this expression, let $\tilde{z}_n$ satisfy $\nu \ln \tilde{z}_n + \psi = n2\pi$ that is

$$\tilde{z}_n = e^{\frac{2\pi n - \psi}{\nu}}. \tag{25}$$

The factor 2 is included, since we are only interested in the critical points with positive critical values. Write $z = \tilde{z}_n y$. Then $\frac{\partial}{\partial z} g_\mu(z) = 0$ if $a\sqrt{\mu^2 + \nu^2} y^{\mu-1} \sin(\nu \ln y) + \mathcal{O}(z^\mu) = 0$, which is solved by the implicit function theorem for $y = \tilde{y}_n(\theta)$. Write $z_n = z_n(\theta) = \tilde{z}_n \tilde{y}_n(\theta)$ for the critical points of $G_\mu(\theta, z)$ with positive critical values. We suppress the dependence of z_n on μ

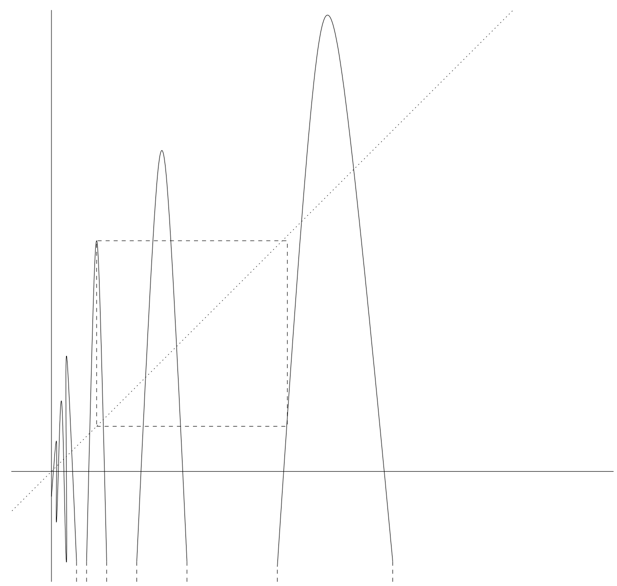

Figure 3: Graph of an infinite modal interval map $z \mapsto az^{\mu} \sin(\nu \ln z + \phi) + \mathcal{O}(z^{2\mu})$. Indicated is an attracting (as it contains a critical point) periodic orbit of period 2.

(and often on θ as well) from the notation. Note that $\tilde{y}_n \to 1$ as $n \to \infty$. The critical points z_n converge exponentially fast to zero:

$$z_n \approx c^n, \tag{26}$$

with $c = e^{2\pi/\nu}$. We remark that $\frac{\partial z_n}{\partial \theta} = \mathcal{O}(z_n)$ and $\frac{\partial z_n}{\partial \mu} = \mathcal{O}(z_n \ln z_n)$.

A similar computation makes clear that the domain of Π_μ consists of strips S_j, $j \in \mathbb{N}$, with $z_j \in S_j$ and on which G_μ takes on positive values. Compare figure 2. Denote

$$\begin{aligned} \Omega_j &= \{\text{the maximal invariant set of } \Pi_\mu \text{ in } S_j\}, \\ \Omega_{j,n} &= \{\text{the maximal invariant set of } \Pi_\mu \text{ in } S_j \cup S_n\}. \end{aligned}$$

Note that Ω_j is a hyperbolic horseshoe [Shi65, Shi70, Tre84]. However, $\Omega_{j,n}$ can be more complicated; figure 2 illustrates the possibility that $\Omega_{j,n}$ contains attractors: the second iterate of Π_μ restricted to (a thin strip inside) S_n maps this strip into a horseshoe shape with the fold inside S_n. Below we will perform computations showing that, due to the negative divergence of the vector fields which results in a strong area contraction of Π_μ, in situations as depicted in figure 2 rescalings to strongly dissipative Hénon maps exist. We will moreover study how frequently such rescalings can be found for different strips S_j, S_n, when varying μ. Finally we establish that all periodic attractors that intersect Σ^{in} in two strips can be obtained from such rescalings. These steps lead to the proof of theorem 1.4.

3.1 Renormalizations

Proposition 3.1 *Let $\{\Pi_\mu\}$, $\mu \in J$, be as above. For positive integers j, n with $\frac{j}{n} \in J$ and j, n large enough, there exist precisely two parameter values $\mu^l_{j,n}, \mu^r_{j,n}$, both within distance $\mathcal{O}(\frac{1}{n})$ of $\frac{j}{n}$, so that at $\mu = \mu^i_{j,n}$, $i = l, r$, Π_μ possesses a periodic point $(\theta^i_{j,n}, z^i_{j,n}) \in S_n$ of period 2, with $\Pi_\mu(\theta^i_{j,n}, z^i_{j,n}) \in S_j$ and so that $z^i_{j,n}$ is a critical point of $z \mapsto G_\mu \circ \Pi_\mu(\theta^i_{j,n}, z)$.*

Remark 3.2 *The periodic orbits from the above proposition are attractors. In particular the proposition shows the occurrence of 2-periodic attractors for a dense set of parameter values.*

PROOF. From $G_\mu(\theta, z_n) = a\tilde{y}_n^\mu \tilde{z}_n^\mu + \mathcal{O}(z_n^{2\mu})$ with $\tilde{z}_n$ given by (25), we get

$$G_\mu(\theta, z_n)/z_j \quad = \quad a\frac{\tilde{y}_n^\mu}{\tilde{y}_j} e^{\frac{2\pi(n\mu - j)}{\nu}} + \mathcal{O}(e^{\frac{2\pi(2n\mu - j)}{\nu}}).$$

With c defined in (26) we get

$$G_\mu(\theta, z_n)/z_j \approx z_n^\mu/z_j \approx c^{\mu n - j}. \tag{27}$$

Writing $\tilde{\mu} = \frac{j}{n}$, this yields $G_{\tilde{\mu}}(\theta, z_n)/z_j \approx 1$. Further,

$$\frac{\partial}{\partial \mu}\left(\frac{G_\mu(\theta, z_n)}{z_j}\right) \quad \approx \quad n, \tag{28}$$

if $\mu - \tilde{\mu} = \mathcal{O}(\frac{1}{n})$. Hence, $G_\mu(\theta, z_n)$ moves with positive speed through I_j. It follows from this and (27) that there are parameter intervals for which $G_\mu \circ \Pi_\mu(\theta, z_n) \lesssim z_n$. For these parameter values, the next lemma provides a critical point for $z \mapsto G_\mu \circ \Pi_\mu(\theta, z)$ close to z_n. Following the proof of the lemma, we continue the proof of the proposition.

Lemma 3.3 *For θ fixed and μ such that $\Pi_\mu(\theta, z_n) \in S_j$ and $G_\mu \circ \Pi_\mu(\theta, z_n) \lesssim z_n$, the map $z \mapsto G_\mu \circ \Pi_\mu(\theta, z)$ has a critical point $z_n^{(2)} = z_n^{(2)}(\theta)$ in S_n within distance $\mathcal{O}(z_n^{\tilde{\mu}+1})$ of z_n. Further, for i, j nonnegative integers there are constants $C_{i,j}$ so that*

$$\left|\frac{\partial^{i+j} z_n^{(2)}}{\partial \theta^i \partial \mu^j}\right| \quad \leq \quad C_{i,j} z_n (\ln z_n)^j, \tag{29}$$

PROOF. Using $\mathcal{O}(z_n^{\tilde{\mu}}) = \mathcal{O}(z_n^\mu)$ if $\tilde{\mu} - \mu = \mathcal{O}(\frac{1}{n})$, compute

$$\begin{aligned} D\Pi_\mu^2(\theta, z_n) &= D\Pi_\mu(\Pi_\mu(\theta, z_n)) D\Pi_\mu(\theta, z_n) \\ &= \begin{pmatrix} \approx z_n^{\tilde{\mu}^2} & \mathcal{O}(z_n^{\tilde{\mu}^2 - \tilde{\mu}}) \\ \mathcal{O}(z_n^{\tilde{\mu}^2}) & \approx z_n^{\tilde{\mu}^2 - \tilde{\mu}} \end{pmatrix} \begin{pmatrix} \mathcal{O}(z_n^{\tilde{\mu}}) & \approx z_n^{\tilde{\mu}-1} \\ \approx z_n^{\tilde{\mu}} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{O}(z_n^{\tilde{\mu}^2}) & \approx z_n^{\tilde{\mu}^2 + \tilde{\mu} - 1} \\ \approx z_n^{\tilde{\mu}^2} & \mathcal{O}(z_n^{\tilde{\mu}^2 + \tilde{\mu} - 1}) \end{pmatrix}. \end{aligned} \tag{30}$$

We claim that there exists $z_n^{(2)}$ within a distance $\mathcal{O}(z_n^{\tilde{\mu}+1})$ of z_n, so that

$$D\Pi_\mu^2(\theta, z_n^{(2)}) \;=\; \begin{pmatrix} \mathcal{O}(z_n^{\tilde{\mu}^2}) & \approx z_n^{\tilde{\mu}^2+\tilde{\mu}-1} \\ \approx z_n^{\tilde{\mu}^2} & 0 \end{pmatrix}. \tag{31}$$

To prove the claim, first note that (30) also holds if, for z near z_n, $\frac{\partial G_\mu}{\partial z}(\theta, z)$ is not 0, but small of order $\mathcal{O}(z_n^{2\tilde{\mu}-1})$. Hence, (30) holds at a point $(\theta, z_n^{(2)})$ so that $z_n^{(2)}$ is from an interval of order $\mathcal{O}(z_n^{\tilde{\mu}+1})$ around z_n. To see this, note that a graph $x \mapsto \mathcal{O}(z_n^{\tilde{\mu}-2})x^2$ has derivative $\mathcal{O}(z_n^{\tilde{\mu}-2})x$, which is of order $\mathcal{O}(z_n^{2\tilde{\mu}-1})$ if $x = \mathcal{O}(z_n^{\tilde{\mu}+1})$. Since $\frac{\partial G_\mu}{\partial z}(\Pi_\mu(\theta, z_n)) \approx z_n^{\tilde{\mu}^2-\tilde{\mu}}$, it follows that there exists a unique point $z_n^{(2)}$ within a distance $\mathcal{O}(z_n^{\tilde{\mu}+1})$ of z_n, so that $\frac{\partial G_\mu \circ \Pi_\mu}{\partial z}(\theta, z_n^{(2)}) = 0$. Since $\mathcal{O}(z_n^{\tilde{\mu}^2}) = \mathcal{O}((z_n^{(2)})^{\tilde{\mu}^2})$ and $\mathcal{O}(z_n^{\tilde{\mu}^2+\tilde{\mu}-1}) = \mathcal{O}((z_n^{(2)})^{\tilde{\mu}^2+\tilde{\mu}-1})$, this concludes the proof of the existence of $z_n^{(2)}$.

To estimate $\frac{\partial z_n^{(2)}}{\partial \theta}$, differentiate the defining equation $\frac{\partial G_\mu \circ \Pi_\mu}{\partial z}(\theta, z)$ for $z_n^{(2)}$ with respect to θ. Writing $P = (\theta, z_n^{(2)})$ and $\Omega = \frac{\partial z_n^{(2)}}{\partial \theta}$, this gives

$$\begin{aligned} 0 = \\ &\frac{\partial^2 G_\mu}{\partial \theta^2}(\Pi_\mu(P)) \left(\frac{\partial F_\mu}{\partial \theta}(P) + \frac{\partial F_\mu}{\partial z}(P)\Omega \right) \frac{\partial F_\mu}{\partial z}(P) + \\ &\frac{\partial^2 G_\mu}{\partial \theta \partial z}(\Pi_\mu(P)) \left(\frac{\partial G_\mu}{\partial \theta}(P) + \frac{\partial G_\mu}{\partial z}(P)\Omega \right) \frac{\partial F_\mu}{\partial z}(P) + \\ &\frac{\partial^2 G_\mu}{\partial \theta \partial z}(\Pi_\mu(P)) \left(\frac{\partial F_\mu}{\partial \theta}(P) + \frac{\partial F_\mu}{\partial z}(P)\Omega \right) \frac{\partial G_\mu}{\partial z}(P) + \\ &\frac{\partial^2 G_\mu}{\partial z^2}(\Pi_\mu(P)) \left(\frac{\partial G_\mu}{\partial \theta}(P) + \frac{\partial G_\mu}{\partial z}(P)\Omega \right) \frac{\partial G_\mu}{\partial z}(P) + \\ &\frac{\partial G_\mu}{\partial \theta}(\Pi_\mu(P)) \left(\frac{\partial^2 F_\mu}{\partial \theta \partial z}(P) + \frac{\partial^2 F_\mu}{\partial z^2}(P)\Omega \right) + \frac{\partial G_\mu}{\partial z}(\Pi_\mu(P)) \left(\frac{\partial^2 F_\mu}{\partial \theta \partial z}(P) + \frac{\partial^2 F_\mu}{\partial z^2}(P)\Omega \right). \end{aligned}$$

Solving for Ω yields

$$\begin{aligned} &\Omega \left[\frac{\partial^2 G_\mu}{\partial \theta^2}(\Pi_\mu(P)) \left(\frac{\partial F_\mu}{\partial z}(P) \right)^2 + 2\frac{\partial^2 G_\mu}{\partial \theta \partial z}(\Pi_\mu(P)) \frac{\partial G_\mu}{\partial z}(P) \frac{\partial F_\mu}{\partial z}(P) \right. \\ &\left. + \frac{\partial^2 G_\mu}{\partial z^2}(\Pi_\mu(P)) \left(\frac{\partial G_\mu}{\partial z}(P) \right)^2 + \frac{\partial G_\mu}{\partial \theta}(\Pi_\mu(P)) \frac{\partial^2 F_\mu}{\partial z^2}(P) + \frac{\partial G_\mu}{\partial z}(\Pi_\mu(P)) \frac{\partial^2 G_\mu}{\partial z^2}(P) \right] \\ &\quad = \quad \frac{\partial^2 G_\mu}{\partial \theta^2}(\Pi_\mu(P)) \frac{\partial F_\mu}{\partial \theta}(P) \frac{\partial F_\mu}{\partial z}(P) + \frac{\partial^2 G_\mu}{\partial \theta \partial z}(\Pi_\mu(P)) \frac{\partial G_\mu}{\partial \theta}(P) \frac{\partial F_\mu}{\partial z}(P) + \\ &\qquad\quad \frac{\partial^2 G_\mu}{\partial \theta \partial z}(\Pi_\mu(P)) \frac{\partial F_\mu}{\partial \theta}(P) \frac{\partial G_\mu}{\partial z}(P) + \frac{\partial^2 G_\mu}{\partial z^2}(\Pi_\mu(P)) \frac{\partial G_\mu}{\partial \theta}(P) \frac{\partial G_\mu}{\partial z}(P) + \\ &\qquad\quad \frac{\partial G_\mu}{\partial \theta}(\Pi_\mu(P)) \frac{\partial^2 F_\mu}{\partial \theta \partial z}(P) + \frac{\partial G_\mu}{\partial z}(\Pi_\mu(P)) \frac{\partial^2 G_\mu}{\partial \theta \partial z}(P). \end{aligned}$$

The term multiplied by Ω is $\approx z_n^{\tilde{\mu}^2-2}$, due to the term $\frac{\partial G_\mu}{\partial z}(\Pi_\mu(P))\frac{\partial^2 G_\mu}{\partial z^2}(P)$. Note that $\frac{\partial G_\mu}{\partial z}(P) \lesssim z_n^{2\tilde{\mu}-1}$. The right hand side is $\mathcal{O}(z_n^{\tilde{\mu}^2-1})$. This proves the estimate for Ω. Similarly one obtains estimates for $\frac{\partial z^{(2)}}{\partial \mu}$ and higher order derivatives with respect to (θ, μ); the straightforward but lengthy computations are left to the reader. ■

We continue with the proof of proposition 3.1. By (27) and (28), there are $\mu_{j,n}^i$, $i = l, r$, within a distance $\mathcal{O}(\frac{1}{n})$ of $\tilde{\mu} = \frac{j}{n}$, so that in addition to (31), one has $G_\mu(\Pi_\mu(\theta, z_n^{(2)})) = z_n^{(2)}$ at $\mu = \mu_{j,n}^i$. Because $\frac{\partial \Pi_\mu^2}{\partial \theta} = \mathcal{O}(z_n^{\tilde{\mu}^2})$, one can vary θ to find a point $(\theta_{j,n}^i, z_{j,n}^i)$ which is a periodic point of period 2 for $\Pi_{\mu_{j,n}^i}$ and for which $\frac{\partial^2 G_{\tilde{\mu}} \circ \Pi_{\tilde{\mu}}}{\partial z^2}(\theta_{j,n}, z_{j,n}) = 0$ at $\mu = \mu_{j,n}^i$. ■

At $\mu = \mu_{j,n}^i$, $i = l, r$, one has

$$D\Pi_\mu^2(\theta_{j,n}^i, z_{j,n}^i) = \begin{pmatrix} \mathcal{O}(z_n^{\tilde{\mu}^2}) & \approx z_n^{\tilde{\mu}^2+\tilde{\mu}-1} \\ \approx z_n^{\tilde{\mu}^2} & 0 \end{pmatrix} \tag{32}$$

and

$$\frac{\partial^2 G_\mu \circ \Pi_\mu}{\partial z^2}(\theta_{j,n}^i, z_{j,n}^i) \approx z_n^{\tilde{\mu}-2} z_n^{\tilde{\mu}^2-\tilde{\mu}} = z_n^{\tilde{\mu}^2-2}, \tag{33}$$

as $n \to \infty$.

Proposition 3.4 *Let $\{\Pi_\mu\}$, $\mu \in J$, be as above and $\mu_{j,n}^l, \mu_{j,n}^r \in J$ be as in proposition 3.1. For positive integers j, n with $\frac{j}{n} \in J$ and j, n large enough, there exists a sequence of intervals $J_{j,n}^i \ni \mu_{j,n}^i$, $i = l, r$, so that Π_γ, $\gamma \in J_{j,n}^i$, has the following property. There are affine rescalings*

$$\Phi_{j,n}^i : [-2, 2] \times [-2, 2] \to Y_{j,n}^i \times Z_{j,n}^i$$

to subdomains $Y_{j,n}^i \times Z_{j,n}^i$ of Σ^{in} and affine rescalings

$$\gamma : [-1, 3] \to J_{j,n}^i$$

of the parameter, so that

$$\left(\Phi_{j,n}^i\right)^{-1} \circ \Pi_{\gamma(a)}^2 \circ \Phi_{j,n}^i \quad \to \quad H_{a,0},$$

as $n \to \infty$, together with derivatives. Moreover, denoting $\tilde{\mu} = \frac{j}{n}$,

$$\begin{aligned} |Y_{j,n}^i| &\leq C z_n^{2-2\tilde{\mu}^2}, \\ |Z_{j,n}^i| &\leq C z_n^{2-\tilde{\mu}^2}, \\ |J_{j,n}^i| &\leq C z_n^{2-2\tilde{\mu}^2}, \end{aligned}$$

for some $C > 0$.

PROOF. Consider first Π_μ^2 at $\mu = \mu_{j,n}^i$, $i = l, r$. We apply an affine rescaling of Π_μ^2. As before, denote $\tilde{\mu} = \frac{j}{n}$.

Recall from (32), (33) asymptotic estimates for $D\Pi_\mu^2$ and $\frac{\partial^2}{\partial z^2} G_\mu \circ \Pi_\mu$ at the point $(\theta_{j,n}^i, z_{j,n}^i)$ for $\mu = \mu_{j,n}^i$, $i = l, r$. Expand

$$\Pi_\mu^2(\theta_{j,n}^i + \vartheta, z_{j,n}^i + \sigma) = \begin{pmatrix} A\vartheta + B\sigma + \mathcal{O}(z_n^{\tilde{\mu}^2-1}\vartheta\sigma) + \mathcal{O}(z_n^{\tilde{\mu}^2}\vartheta^2) + \mathcal{O}(z_n^{\tilde{\mu}^2-2}\sigma^2) \\ C\vartheta + \lambda\sigma^2 + \mathcal{O}(z_n^{\tilde{\mu}^2-1}\vartheta\sigma) + \mathcal{O}(z_n^{\tilde{\mu}^2}\vartheta^2) + \mathcal{O}(z_n^{\tilde{\mu}^2-3}\sigma^3) \end{pmatrix}.$$

Define rescaled coordinates (y, x) by

$$\begin{aligned} \frac{-1}{C\lambda} y + \theta_{j,n}^i &= \theta, \\ \frac{-1}{\lambda} x + z_{j,n}^i &= z. \end{aligned}$$

Note that

$$\begin{aligned} \frac{1}{\lambda} &\approx z_n^{2-\tilde{\mu}^2}, \\ \frac{1}{C\lambda} &\approx z_n^{2-2\tilde{\mu}^2}. \end{aligned}$$

For the map $\bar{\Pi}_\mu$ at $\mu = \mu_{j,n}^i$ in rescaled coordinates, the following holds:

$$D\bar{\Pi}_{\mu_{j,n}}^2(0,0) = \begin{pmatrix} \mathcal{O}(z_n^{\tilde{\mu}^2}) & \approx z_n^{2\tilde{\mu}^2+\tilde{\mu}-1} \\ 1 & 0 \end{pmatrix}$$

and

$$\frac{\partial^2 \bar{G}_{\mu_{j,n}} \circ \bar{\Pi}_{\mu_{j,n}}}{\partial x^2}(0,0) = -1,$$

as $n \to \infty$. Higher order derivatives are of the order $\mathcal{O}(z_n^\epsilon)$ for some $\epsilon > 0$. Note that $2\lambda_s + \lambda_u < 0$ implies $2\tilde{\mu}^2 + \tilde{\mu} - 1 > 0$. Hence, the renormalization of Π_μ^2, $\mu = \mu_{j,n}^i$, converges to the Hénon map $H_{0,0}$.

For μ near $\mu_{j,n}^i$, let $\hat{\theta}_{j,n}^i$ near $\theta_{j,n}^i$ be so that

$$F_\mu \circ \Pi_\mu(\hat{\theta}_{j,n}^i, z_n^{(2)}) = \hat{\theta}_{j,n}^i. \tag{34}$$

Define (y, z) by

$$\begin{aligned} \frac{1}{C\lambda} y + \hat{\theta}_{j,n}^i &= \theta, \\ \frac{1}{\lambda} x + z_n^{(2)} &= z. \end{aligned}$$

Let $\sigma = \frac{\partial}{\partial \mu}\left(G_\mu \circ \Pi_\mu(\hat{\theta}^i_{j,n}, z^i_{j,n})\right)$ at $\mu = \mu^i_{j,n}$ and rescale the parameter μ by

$$\frac{1}{\lambda\sigma}a + \mu^i_{j,n} \quad = \quad \mu.$$

We claim that

$$\frac{\partial \hat{\theta}^i_{j,n}}{\partial \mu} \quad = \quad \mathcal{O}(z_n^{\tilde{\mu}^2} \ln z_n). \tag{35}$$

One obtains (35) by differentiating the defining equation (34) with respect to μ and solving for $\frac{\partial \hat{\theta}^i_{j,n}}{\partial \mu}$, we leave the computations to the reader. Combining (35) with (29) it follows that

$$\frac{\partial z_2^{(2)}(\hat{\theta}^i_{j,n})}{\partial \mu} \quad = \quad \mathcal{O}(z_n \ln z_n). \tag{36}$$

Using (35) and (36) one checks that, with $z_n^{(2)} = z_n^{(2)}(\hat{\theta}^i_{j,n})$ and $P = (\hat{\theta}^i_{j,n}, z_n^{(2)})$,

$$\begin{aligned}
\sigma &= \frac{\partial G_\mu}{\partial \theta}(\Pi_\mu(P))\frac{\partial F_\mu(P)}{\partial \mu} + \frac{\partial G_\mu}{\partial z}(\Pi_\mu(P))\frac{\partial G_\mu(P)}{\partial \mu} + \frac{\partial G_\mu}{\partial \mu}(\Pi_\mu(P)) \\
&= \frac{\partial G_\mu}{\partial \theta}(\Pi_\mu(P))\left[\frac{\partial F_\mu}{\partial \theta}(P)\frac{\partial \hat{\theta}^i_{j,n}}{\partial \mu} + \frac{\partial F_\mu}{\partial z}(P)\frac{\partial z_n^{(2)}}{\partial \mu} + \frac{\partial F_\mu}{\partial \mu}(P)\right] + \\
&\quad \frac{\partial G_\mu}{\partial z}(\Pi_\mu(P))\left[\frac{\partial G_\mu}{\partial \theta}(P)\frac{\partial \hat{\theta}^i_{j,n}}{\partial \mu} + \frac{\partial G_\mu}{\partial z}(P)\frac{\partial z_n^{(2)}}{\partial \mu} + \frac{\partial G_\mu}{\partial \mu}(P)\right] + \\
&\quad \frac{\partial G_\mu}{\partial \mu}(\Pi_\mu(P)) \\
&\approx \frac{\partial G_\mu}{\partial z}(\Pi_\mu(P))\frac{\partial G_\mu}{\partial \mu}(P) \\
&\approx z_n^{\tilde{\mu}^2} \ln z_n
\end{aligned}$$

and thus

$$\lambda\sigma \quad \approx \quad z_n^{2\tilde{\mu}^2-2} \ln z_n.$$

Expand

$$\Pi_\mu^2(\hat{\theta}^i_{j,n} + \vartheta, z_n^{(2)} + \sigma) \quad = \quad \begin{pmatrix} A\vartheta + B\sigma & + & \mathcal{O}(z_n^{\tilde{\mu}^2-1}\vartheta\sigma) + \mathcal{O}(z_n^{\tilde{\mu}^2}\vartheta^2) + \mathcal{O}(z_n^{\tilde{\mu}^2-2}\sigma^2) \\ & + & \mathcal{O}(z_n^{\tilde{\mu}^2}\ln(z_n)\mu\vartheta) + \mathcal{O}(z_n^{\tilde{\mu}^2-1}\ln(z_n)\mu\sigma) \\ C\vartheta + \lambda\sigma^2 & + & \mathcal{O}(z_n^{\tilde{\mu}^2-1}\vartheta\sigma) + \mathcal{O}(z_n^{\tilde{\mu}^2}\vartheta^2) + \mathcal{O}(z_n^{\tilde{\mu}^2-3}\sigma^3) \\ & + & \mathcal{O}(z_n^{\tilde{\mu}^2}\ln(z_n)\mu\vartheta) + \mathcal{O}(z_n^{\tilde{\mu}^2-2}\ln(z_n)\mu\sigma^2) \end{pmatrix}.$$

As above, it follows that the rescaled return map $\bar{\Pi}_\mu^2$ is a small perturbation of the one parameter family $H_{a,0}$. ■

Proposition 3.5 *Let $\{\Pi_\mu\}$, $\mu \in J$, be as above and $J^l_{j,n}, J^r_{j,n} \subset J$ as in proposition 3.4. For μ outside $J^l_{j,n} \cup J^r_{j,n}$, the maximal invariant set $\Omega_{j,n}$ in $S_j \cup S_n$ is hyperbolic.*

PROOF. The proposition is shown by constructing invariant stable and unstable cone fields (see [Mos73, PalTak87]). We will first discuss the existence of an unstable cone field; an invariant cone field whose vectors get expanded under the action of $D\Pi_\mu$. Considering fixed parameter values, we will from here on suppress the dependence on μ from the notation. Consider points $(\theta, z) \in S_n$ for which $\Pi(\theta, z), \Pi^2(\theta, z) \in S_j \cup S_n$ with $j < n$.

From $(\theta, z) \in S_n$ we get $z \approx z_n$. We have $z_n < z_j \lesssim z_n^\mu$. Suppose first that $z_j \approx z_n^\mu$. From the computations for Lemma 3.3, one obtains the following estimate. If Δ is a positive number, then

$$\left| \frac{\partial}{\partial z} G_\mu \circ \Pi_\mu(\theta, z) \right| \quad > \quad \Delta \tag{37}$$

if $(\theta, z) \in S_n$ lies outside a neighborhood of size $\approx z_n^{\mu+1}$ of $z_n^{(2)}$ and $G_\mu \circ \Pi_\mu(\theta, z) \in S_j \cup S_n$. It follows that for such z we can write

$$D\Pi^2(\theta, z) \quad = \quad \begin{pmatrix} a z_n^{\mu^2} & b z_n^{\mu^2+\mu-1} + \tilde{b}\Delta \\ c z_n^{\mu^2} & \Delta \end{pmatrix},$$

where $a, b, \tilde{b}, c$ are bounded and $|\Delta|$ is bounded from below (and from above by some constant times $z_n^{\mu^2-1}$). Indeed, by (37), $|\Delta| > 2$ for μ outside $J^l_{j,n} \cup J^r_{j,n}$.

Take a vector $\begin{pmatrix} v \\ 1 \end{pmatrix}$ in $T_{(\theta,z)}\Sigma^{in}$. The image $D\Pi^2(\theta, z) \begin{pmatrix} v \\ 1 \end{pmatrix}$ is parallel to $\begin{pmatrix} w \\ 1 \end{pmatrix}$, for

$$w \quad = \quad \frac{a z_n^{\mu^2} v + b z_n^{\mu^2+\mu-1} + \tilde{b}\Delta}{c z_n^{\mu^2} v + \Delta}.$$

In particular, there exists $K > 0$, so that $v \leq K \max\{z_n^{\mu^2+\mu-1}, 1\}$ implies that also $w \leq K \max\{z_n^{\mu^2+\mu-1}, 1\}$, for n large enough. Note that these cones can be wide, if $\mu^2 + \mu - 1 < 0$.

If z_j is not $\approx z_n^\mu$, we can write

$$D\Pi^2(\theta, z) \quad = \quad \begin{pmatrix} A z_j^\mu & B z_j^{\mu-1} \\ C z_j^\mu & D z_j^{\mu-1} \end{pmatrix} \begin{pmatrix} a z_n^\mu & b z_n^{\mu-1} \\ c z_n^\mu & d z_n^{\mu-1} \end{pmatrix},$$

where a, b, c, A, B, C are bounded and $|d|, |D|$ are bounded and bounded away from 0. As above, one shows that there exists $K > 0$ so that $D\Pi^2$ maps cones $\left\{ \begin{pmatrix} v \\ 1 \end{pmatrix}; \; v \leq K \right\}$ into themselves. Moreover, one easily checks that $D\Pi^2$ expands vectors in the unstable cones.

Similarly one shows the existence of a stable cone field, a cone field invariant under $D\Pi^{-1}$ consisting of vectors that get expanded by $D\Pi^{-1}$. Here we calculate the fate of vectors under $D\Pi^{-1}$ as follows. Let (θ, z) be as above and suppose $z_j \approx z_n^\mu$. Consider a vector $\begin{pmatrix} 1 \\ v \end{pmatrix}$ in $T_{\Pi^2(\theta,z)}\Sigma^{in}$. Let $\begin{pmatrix} 1 \\ w \end{pmatrix}$ be the vector in $T_{(\theta,z)}\Sigma^{in}$, so that $D\Pi^2(\theta,z)\begin{pmatrix} 1 \\ w \end{pmatrix}$ is parallel to $\begin{pmatrix} 1 \\ v \end{pmatrix}$. By solving for w we get

$$w = \frac{-cz_n^{\mu^2} + az_n^{\mu^2} v}{\Delta - \left(bz_n^{\mu^2+\mu-1} + \tilde{b}\Delta\right) v}.$$

From this formula one checks that, for some $K > 0$, $\left(D\Pi^2(\theta,z)\right)^{-1}$ maps cones consisting of vectors $\begin{pmatrix} 1 \\ v \end{pmatrix}$ with $|v| \le K z_n^{\mu^2}$, strictly into themselves. One treats the easier case where z_j is not $\approx z_n^\mu$ in an analogous manner. It is moreover easily seen that the length of vectors in the stable cones gets expanded under $(D\Pi^2)^{-1}$. ∎

Proof of theorem 1.4. To prove theorem 1.4, we bound the total size in parameter space for which $\{X_\mu\}$ possesses a rescaling to a strongly dissipative Hénon like map as in the above proposition. Restricting to a small tubular neighborhood $\mathcal{U}_n$ translates to restricting to z coordinates bounded by z_N for some high N. Recall that $z_n \approx c^n$ for some $c < 1$. Let $\zeta = \min_{\mu \in J}(2 - 2\mu^2)$. Applying proposition 3.4,

$$\begin{aligned} \sum_{j,n,\frac{j}{n}\in J, n\ge N} \left(|J_{j,n}^l| + |J_{j,n}^r|\right) &\lesssim \sum_{j,n,\frac{j}{n}\in J,n\ge N} z_n^\zeta \\ &\lesssim \sum_{n\ge N} n z_n^\zeta \\ &\lesssim \sum_{n \ge N} n c^{n\zeta} \\ &\lesssim N c^{N\zeta}, \end{aligned} \tag{38}$$

which goes to 0 as $N \to \infty$.

By proposition 3.5, the constructions in proposition 3.4 yield all the existing periodic attractors and strange attractors of the type in the statement of theorem 1.4.

The bounds (38) combined with the Borel-Cantelli lemma also provides an argument to show that infinitely many coexisting 2-periodic attractors occur only for a set of parameter values of measure 0. Indeed, let I_i be an enumeration of the intervals in the parameter interval I, for which an attracting 2-periodic attractor exists. The set of parameter values in I for which there are infinitely many coexisting 2-periodic attractors, is contained in $\cup_{i\geq M} I_i$, for each M. By (38), this set has arbitrarily small measure for large enough M. See also [TedYor86]. ∎

4 Repelling dynamics

In this section we prove theorem 1.5. We start with an investigation of a one dimensional map of the form

$$g(z) \quad = \quad \phi_3 z^\mu \sin(\nu \ln z) + \phi_4 z^\mu \cos(\nu \ln z) + \mathcal{O}(z^{2\mu}). \tag{39}$$

Recall that g has a sequence z_n of critical points with positive critical values, that converge exponentially fast to 0 ($z_n \approx c^n$ with $c = e^{2\pi/\nu}$). Keeping in mind that the one dimensional map is meant to model the first return map Π, it is clear that we are interested only in points x with $g(x) > 0$. Let $\mathcal{V}_n$ be the interval containing z_n on which g is positive. Note that $|\mathcal{V}_n| \approx z_n$. Fix $\sigma > 0$ small. Let $\mathcal{W}_n \subset \mathcal{V}_n$ be the interval around z_n, so that $|g'| \geq \sigma z_n^{\mu-1}$ on $\mathcal{V}_n \backslash \mathcal{W}_n$. Since $|g''(z_n)| \approx z_n^{\mu-2}$ and so $|g'(z - z_n)| \approx z_n^{\mu-2}|z - z_n|$ for z close to z_n, we have

$$|\mathcal{W}_n| \quad \approx \quad \sigma z_n,$$

which is small compared to $\mathcal{V}_n$. So the interval $\mathcal{W}_n$ on which g is not strongly expanding, is a small subinterval of $\mathcal{V}_n$. The following lemma shows that also the set of points that eventually end up in one of the sets $\mathcal{W}_n$, is small.

Lemma 4.1 *Let $C_N = \{x \in [0, z_N];\ \ g^i(x) \in \mathcal{W}_k$ for some $i \in \mathbb{N}, k \geq N\}$. Then $|C_N| \approx \sigma z_N$, for N large. The maximal invariant set of g in $[0, z_N] \backslash \cup_{n\geq N} \mathcal{W}_n$ is hyperbolic and has zero Lebesgue measure.*

Proof. Write $\mathcal{Y}_n = \mathcal{V}_n \backslash \mathcal{W}_n$. For $n_0, n_1 \geq N$, we have

$$\left| g^{-1}(\mathcal{W}_{n_0}) \cap \mathcal{Y}_{n_1} \right| \quad \lesssim \quad \frac{1}{\sigma} z_{n_1}^{1-\mu} |\mathcal{W}_{n_0}|. \tag{40}$$

Because $z_n \approx c^n$, we have $\sum_{n\geq N} z_n \approx z_N$. Summing the left hand side of (40) over $n_1 \geq N$, we obtain

$$\left| g^{-1}(\mathcal{W}_{n_0}) \cap \bigcup_{n_1 \geq N} \mathcal{Y}_{n_1} \right| \quad \lesssim \quad \frac{1}{\sigma} z_N^{1-\mu} |\mathcal{W}_{n_0}|,$$

which is much smaller then $|\mathcal{W}_{n_0}|$ if N is large. Summing over $n_0 \geq N$, we also get

$$\left| g^{-1}\left(\bigcup_{n_0 \geq N} \mathcal{W}_{n_0} \right) \cap \bigcup_{n_1 \geq N} \mathcal{Y}_{n_1} \right| \quad \lesssim \quad \frac{1}{\sigma} z_N^{1-\mu} \left| \bigcup_{n_0 \geq N} \mathcal{W}_{n_0} \right|,$$

This computation holds with $\cup_{n_0\geq N}\mathcal{W}_{n_0}$ replaced by any set in $\cup_{n\geq N}\mathcal{V}_n$, so that for the $m^{\rm th}$ inverse image,

$$\left| g^{-1}\left(\dots\left(g^{-1}\left(g^{-1}(\mathcal{W}_{n_0})\cap\bigcup_{i\geq N}\mathcal{Y}_i\right)\cap\bigcup_{i\geq N}\mathcal{Y}_i\right)\dots\right)\cap\bigcup_{i\geq N}\mathcal{Y}_i\right| \;\leq\; \left(\frac{C}{\sigma}\right)^m z_N^{m(1-\mu)}|\mathcal{W}_{n_0}|,$$

for some constant $C>0$. Summing over $m\in\mathbb{N}$ and $n_0\geq N$, proves $|C_N|\approx\sigma z_N$. That the maximal invariant set of g in $[0,z_N]\setminus\cup_{n\geq N}\mathcal{W}_n$ has zero Lebesgue measure follows similarly, hyperbolicity is obvious. ∎

The above lemma shows that for the one dimensional map g, a set of points of large relative measure in I, either is in a hyperbolic repelling set in I, or leaves I. Generalizing the above computation to the first return map Π, will prove the result. It is precisely the expansion in the second coordinate which makes the generalization work.

Writing $\Pi=(F,G)$, the map $z\mapsto G(\theta,z)$ has a sequence of critical points $z_n(\theta)$ for which $G(\theta,z)>0$. Let $\mathcal{V}_n$ be the connected component of Σ^{in} containing $z_n(\theta)$ on which G is positive; $\mathcal{V}_n$ is a strip of width $\approx z_n$. Let $\mathcal{W}_n\subset\mathcal{V}_n$ be the set of points, for which $\left|\frac{\partial G}{\partial z}(\theta,z)\right|\leq\sigma z_n(\theta)^{\mu-1}$. Recall that

$$\frac{\partial z_n}{\partial\theta}(\theta) \;\leq\; Cz_n(\theta) \tag{41}$$

for some $C>0$. So, $\mathcal{W}_n$ is a strip of width $\approx\sigma z_n$. Denote

$$\mathcal{V} = \bigcup_{n\geq N}\mathcal{V}_n,$$
$$\mathcal{W} = \bigcup_{n\geq N}\mathcal{W}_n.$$

Write $C_a(\theta,z)$ for the cone in $T_{(\theta,z)}\Sigma^{in}$ around the θ-axis with slope bounded by a;

$$C_a(\theta,z) \;=\; \{(u,v);\;\;|v|\leq a|u|\}.$$

Let $D_a(\theta,z)$ be the cone around the z-axis with slope bounded by a;

$$D_a(\theta,z) \;=\; \{(u,v);\;\;|u|\leq a|v|\}.$$

Proposition 4.2 *For N large enough, there is a positive constant $K\approx\frac{1}{\sigma}$ for which the cone field $\{C_{Kz}(\theta,z)\}$ on $\Pi(\mathcal{V}\setminus\mathcal{W})$ is invariant under $D\Pi^{-1}\big|_{\Pi(\mathcal{V}\setminus\mathcal{W})}$. Moreover, $D\Pi^{-1}$ expands vectors lying in the cones. Similarly, there is a constant $K>0$, so that the cone field $\{D_K(\theta,z)\}$ is invariant under $D\Pi\big|_{\mathcal{V}\setminus\mathcal{W}}$ and $D\Pi$ expands vectors lying in these cones.*

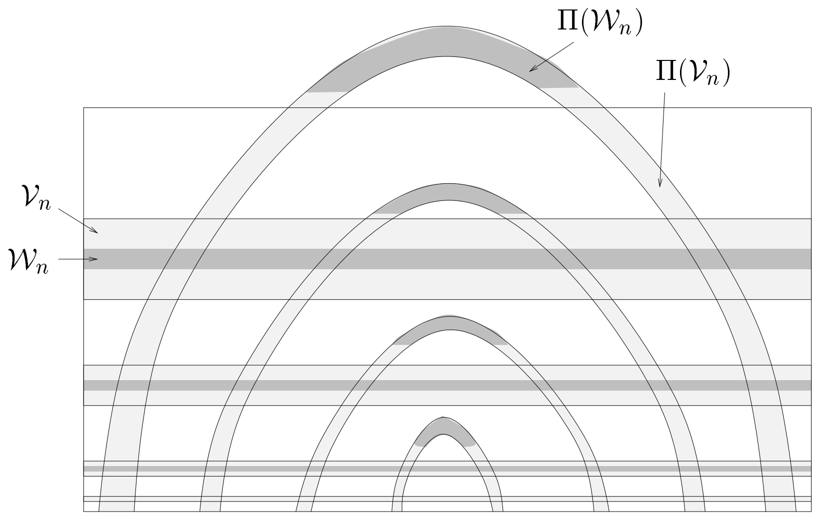

Figure 4: The return map Π maps points in strips $\mathcal{V}_n$ back into $\{(\theta, z) \in \Sigma^{in};\ \ z > 0\}$. Outside smaller strips $\mathcal{W}_n \subset \mathcal{V}_n$, Π has good hyperbolicity properties.

Proof. For $(\theta, z) \in \mathcal{V}_n \backslash \mathcal{W}_n$, we can write

$$D\Pi(\theta, z) \quad = \quad \begin{pmatrix} a z_n^{\mu} & b z_n^{\mu-1} \\ c z_n^{\mu} & d z_n^{\mu-1} \end{pmatrix}. \tag{42}$$

Here $|a|, |b|, |c|$ are uniformly bounded and $|d|$ is bounded from below by σ and from above. From (42) one sees that $D\Pi(\theta, z)$ maps a vector $\begin{pmatrix} 1 \\ w \end{pmatrix}$ in $T_{(\theta,z)}\Sigma$ to a vector parallel to $\begin{pmatrix} 1 \\ v \end{pmatrix}$ in $T_{\Pi(\theta,z)}\Sigma$, where

$$w \quad = \quad \frac{-c z_n^{\mu} + a z_n^{\mu} v}{d z_n^{\mu-1} - b z_n^{\mu-1} v} = \frac{-c z_n + a z_n v}{d - bv}.$$

If $\Pi(\theta, z)$ is contained in $\mathcal{V}_m$ or in the strip between $\mathcal{V}_m$ and $\mathcal{V}_{m-1}$ (so has z-coordinate $\approx z_m$), take v such that $|v| \le K z_m$ with $K = \frac{C}{\sigma}$ for some constant $C > 0$. Then $|w| \lesssim \frac{-c}{d} z_n \lesssim \frac{1}{\sigma} z_n$ if m is large enough. Hence, there exists $C > 0$ so that for m large enough, $D\Pi^{-1}$ maps the cone $C_{K z_m}(\Pi(\theta, z))$ into the cone $C_{K z_n}(\theta, z)$. From (42) one obtains that $D\Pi^{-1}(\Pi(\theta, z))$ expands vectors lying in the cone $\{C_{K z_m}\Pi(\theta, z)\}$.

Invariance of the cone field $\{D_K(\theta, z)\}$ under $D\Pi$ is shown analogously. $D\Pi(\theta, z)$ maps a vector $\begin{pmatrix} v \\ 1 \end{pmatrix}$ to a vector parallel to $\begin{pmatrix} w \\ 1 \end{pmatrix}$, where

$$w \quad = \quad \frac{a z_n^{\mu} v + b z_n^{\mu-1}}{c z_n^{\mu} v + d z_n^{\mu-1}} = \frac{a z_n v + b}{c z_n v + d}. \tag{43}$$

For v such that $|v| \leq K$, we find that $|w| \lesssim \frac{b}{d}$. Hence, for a suitable choice of K, $D\Pi(\theta, z)$ maps $D_K(\theta, z)$ into $D_K\Pi(\theta, z)$. From (42) one obtains that $D\Pi(\theta, z)$ expands vectors in $D_K(\theta, z)$. ∎

Take a foliation $\mathfrak{F}$ of $\mathcal{V}\backslash\mathcal{W}$, so that the tangent spaces $T_{(\theta,z)}\mathfrak{F}_{(\theta,z)}$ are contained in the stable cones $C_{Kz}(\theta, z)$. By (41), one can take such a foliation so that the boundary $\partial\mathcal{W}$ consists of leaves of $\mathfrak{F}$.

Proposition 4.3 *Let S_k be a connected component of $\{(\theta, z);\ \Pi^i(\theta, z) \in \mathcal{V}\backslash\mathcal{W}$ for $0 \leq i \leq k\}$. For large enough N, S_k is contained in a thin strip and carries the foliation $\Pi^{-k}(\mathfrak{F})$ which is Lipschitz continuous with Lipschitz constant bounded by $\frac{C}{\sigma^2}$ for some $C > 0$, uniformly in k.*

PROOF. Denote by $(\Pi^{-1})^{(1)}$ the map on $(\mathcal{V}\backslash\mathcal{W}) \times \mathcal{L}(\mathbb{R}, \mathbb{R})$ induced by Π^{-1}:

$$(\Pi^{-1})^{(1)}(\Pi(\theta, z), v) \;=\; \left(\theta, z, \frac{-\frac{\partial G}{\partial\theta}(\theta, z) + \frac{\partial F}{\partial\theta}(\theta, z)v}{\frac{\partial G}{\partial z}(\theta, z) - \frac{\partial F}{\partial z}(\theta, z)v}\right). \tag{44}$$

For $(\theta, z) \in \mathcal{V}_n$, writing

$$D\Pi(\theta, z) \;=\; \begin{pmatrix} az_n^{\mu} & bz_n^{\mu-1} \\ cz_n^{\mu} & dz_n^{\mu-1} \end{pmatrix}$$

as in proposition 4.2, one has

$$(\Pi^{-1})^{(1)}(\Pi(\theta, z), v) \;=\; \left(\theta, z, \frac{-cz_n + az_n v}{d - bv}\right).$$

Define the graph transform $\mathcal{G}$ on $C^0(\mathcal{V}\backslash\mathcal{W}, \mathcal{L}(\mathbb{R}, \mathbb{R}))$ by

$$\text{graph } \mathcal{G}(\eta)(\theta, z) \;=\; D\Pi^{-1}(\Pi(\theta, z))\text{graph } \eta(\Pi(\theta, z)).$$

Thus

$$\mathcal{G}(\eta)(\theta, z) \;=\; \frac{-cz_n + az_n\eta(\Pi(\theta, z))}{d - b\eta(\Pi(\theta, z))} \tag{45}$$

Let $\mathcal{S}(\mathcal{V}\backslash\mathcal{W}, \mathcal{L}(\mathbb{R}, \mathbb{R}))$ denote the space of continuous sections $\eta \in C^0(\mathcal{V}\backslash\mathcal{W}, \mathcal{L}(\mathbb{R}, \mathbb{R}))$ so that $\|\eta(\theta, z)\| \leq Kz_n$ if $z \approx z_n$. Here K is as in proposition 4.2. As $\sigma z_n^{\mu-1} \leq \|D\Pi(\theta, z)\| \leq Cz_n^{\mu-1}$ for $z \in \mathcal{V}_n\backslash\mathcal{W}_n$, it is fairly clear that for N large enough, $\mathcal{G}$ is a contraction on $\mathcal{S}(\mathcal{V}\backslash\mathcal{W}, \mathcal{L}(\mathbb{R}, \mathbb{R}))$. We claim that for some positive constant C, the graph transform $\mathcal{G}$ maps the set of Lipschitz continuous sections in $\mathcal{S}(\mathcal{V}\backslash\mathcal{W}, \mathcal{L}(\mathbb{R}, \mathbb{R}))$ with Lipschitz constant bounded by $\frac{C}{\sigma^2}$, into itself. For this it suffices to show that the cone field $\{C_L(\theta, z, v)\}$, defined by

$$C_L(\theta, z, v) \;=\; \{(u_1, u_2, v_1) \in T_{(\theta,z,v)}(\Sigma \times \mathcal{L}(\mathbb{R}, \mathbb{R}))\,,\;\; v_1 \leq L\|(u_1, u_2)\|\},$$

is an invariant unstable cone field for $(\Pi^{-1})^{(1)}$. To show this, write

$$(\Pi^{-1})^{(1)}(\theta,z,v) \;=\; (\Pi^{-1}(\theta,z), L(\Pi^{-1}(\theta,z),v))$$

and compute

$$D(\Pi^{-1})^{(1)}(\theta,z,v) \;=\; \begin{pmatrix} \multicolumn{2}{c}{D\Pi^{-1}(\theta,z)} & \begin{matrix}0\\0\end{matrix} \\ \frac{\partial}{\partial\theta}L(\Pi^{-1}(\theta,z),v) & \frac{\partial}{\partial z}L(\Pi^{-1}(\theta,z),v) & \frac{\partial}{\partial v}L(\Pi^{-1}(\theta,z),v) \end{pmatrix}.$$

Using

$$D\Pi^{-1}(\theta,z) \;=\; \begin{pmatrix} \mathcal{O}(z_n^{-\mu}) & \mathcal{O}(z_n^{-\mu}) \\ \mathcal{O}(z_n^{1-\mu}) & \mathcal{O}(z_n^{1-\mu}) \end{pmatrix},$$

one shows that for $(\theta,z)\in\mathcal{V}_n\backslash\mathcal{W}_n$ and $\|v\|\leq Kz_n$,

$$\begin{aligned} \frac{\partial}{\partial\theta}L(\Pi^{-1}(\theta,z),v) &\leq C\frac{1}{\sigma^2}z_n^{1-\mu}, \\ \frac{\partial}{\partial z}L(\Pi^{-1}(\theta,z),v) &\leq C\frac{1}{\sigma^2}z_n^{1-\mu}, \\ \frac{\partial}{\partial v}L(\Pi^{-1}(\theta,z),v) &\leq C\frac{1}{\sigma^2}z_n, \end{aligned}$$

for some $C>0$. Let $(u_1,u_2,v_1)\in T_{(\theta,z,v)}(\Sigma\times\mathcal{L}(\mathbb{R},\mathbb{R}))$ with $\|(u_1,u_2)\|=1$ and $(\bar u_1,\bar u_2,\bar v_1) = D(\Pi^{-1})^{(1)}(\Pi(\theta,z),v)(u_1,u_2,v_1)$. Then $(\bar u_1,\bar u_2,\bar v_1)$ is parallel to a vector $(\tilde u_1,\tilde u_2,\tilde v_1)$ with $\|(\tilde u_1,\tilde u_2)\| = 1$ and $\|\tilde v_1\|\leq C\|v_1\|\frac{1}{\sigma^2}$. It is now readily seen that $\{C_L(\theta,z,v)\}$ is an unstable cone field, so that the claim is proved. ∎

Proof of theorem 1.4. Proposition 4.3 allows a reduction to the one dimensional case: the projection to a vertical line along leaves of $\Pi^{-k}(\mathfrak{F})$ does not distort distances by more then $\frac{C}{\sigma^2}$. The reasoning of lemma 4.1 can therefore be followed. This concludes the proof of theorem 1.5. ∎

References

[ArgArnRic93] A. Arneodo, F. Argoul, J. Elezgaray, P. Richetti, Homoclinic chaos in chemical systems, *Physica D* **62** (1993), 134-169.

[ArnCouTre82] A. Arneodo, P. Coullet, C. Tresser, Oscillators with chaotic behavior: an illustration of a theorem by Shil'nikov, *J. Statist. Phys.* **27** (1982), 171-182.

[BenCar91] M. Benedicks, L. Carleson, The dynamics of the Hénon map, *Annals of Math.* **133** (1991), 73-169.

[BirShi92] V.S. Biragov, L.P. Shil'nikov, On the bifurcation of a saddle-focus separatrix loop in a three-dimensional conservative dynamical system, *Selecta Math. Soviet* **11** (1992), 333-340.

[Col98] E. Colli, Infinitely many coexisting strange attractors, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* **15** (1998), 539-579.

[GasKapNic84] P. Gaspard, R. Kapral, G. Nicolis, Bifurcation phenomena near homoclinic systems: a two parameter analysis, *J. Statist. Phys.* **35** (1984), 697-727.

[GhiChi87] M. Ghil, S. Childress, *Topics in geophysical fluid dynamics: atmospheric dynamics, dynamo theory, and climate dynamics*, Springer Verlag, 1987.

[GleSpa84] P. Glendinning, C. Sparrow, Local and global behaviour near homoclinic orbits, *J. Statist. Phys.* **35** (1984), 645-696.

[GonShiTur93] S.V. Gonchenko, L.P. Shil'nikov, D.V. Turaev, On models with non-rough Poincare homoclinic curves, *Physica D* **62** (1993), 1-14.

[GonShiTur96] S.V. Gonchenko, L.P. Shil'nikov, D.V. Turaev, Dynamical phenomena in systems with structurally unstable Poincaré homoclinic orbits, *Chaos* **6** (1996), 15-31.

[GonShiTur01] S.V. Gonchenko, L.P. Shil'nikov, D.V. Turaev, Homoclinic tangencies of an arbitrary order in Newhouse domains. *J. Math. Sci., New York* **105** (2001), 1738-1778.

[GonTurGasNic97] S.V. Gonchenko, D.V. Turaev, P. Gaspard, G. Nicolis, Complexity in the bifurcation structure of homoclinic loops to a saddle-focus, *Nonlinearity* **10** (1997), 409-423.

[Hen76] M. Hénon, A two-dimensional mapping with a strange attractor, *Commun. Math. Phys.* **50** (1976), 69-77.

[Hom00] A.J. Homburg, Infinite modal maps and homoclinic bifurcations, in: *Equadiff 99, International Conference on Differential Equations Berlin 1999*, eds. B. Fiedler, K. Gröger, J. Sprekels, World Scientific (2000).

[Kal00] V. Kaloshin, Generic diffeomorphisms with superexponential growth of number of periodic orbits, *Commun. Math. Phys.* **211** (2000), 253-271.

[KatHas95] A. Katok, B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.

[MorVia93] L. Mora, M. Viana, Abundance of strange attractors, *Acta Math.* **171** (1993), 1-71.

[Mos73] J. Moser, *Stable and random motions in dynamical systems*, Princeton Univ. Press, 1973.

[New74] S.E. Newhouse, Diffeomorphisms with infinitely many sinks, *Topology* **13** (1974), 9-18.

[New79] S.E. Newhouse, The abundance of wild hyperbolic sets and nonsmooth stable sets for diffeomorphisms, *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 101-151.

[OvsShi87] I.M. Ovsyannikov, L.P. Shil'nikov, On systems with saddle-focus homoclinic curve, *Math. USSR Sbornik*, **58**, 1987, 557-574.

[PalTak87] J. Palis, F. Takens, Hyperbolicity and the creation of homoclinic orbits, *Annals of Math.* **125** (1987), 337-374.

[PalTak93] J. Palis, F. Takens, *Hyperbolicity and sensitive chaotic dynamics at homoclinic bifurcations. Fractal dimensions and infinitely many attractors,* Cambridge University Press, 1993.

[PasRovVia98] M.J. Pacifico, A. Rovella, M. Viana, Infinite-modal maps with global chaotic behavior, *Annals of Math.* **148** (1998), 1-44.

[PumRod97] A. Pumariño, J.A. Rodríguez, *Coexistence and persistence of strange attractors*, LNM **1658**, Springer Verlag, 1997.

[Shi65] L.P. Shil'nikov, A case of the existence of a countable number of periodic motions, *Soviet Math. Dokl.* **6** (1965), 163-166.

[Shi70] L.P. Shil'nikov, A contribution to the problem of the structure of an extended neighborhood of a rough state of saddle-focus type, *Math. USSR Sbornik* **10** (1970), 91-102.

[TedYor86] L. Tedeschini-Lalli, J.A. Yorke, How often do simple dynamical processes have infinitely many coexisting sinks?, *Commun. Math. Phys.* **106** (1986), 635-657.

[Tre84] C. Tresser, About some theorems by L. P. Šil'nikov, *Ann. Inst. Henri Poincaré Phys. Théor.* **40** (1984), 441-461.

View publication stats