Intermittency and Jakobson's theorem near saddle-node bifurcations

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Abstract

We discuss one parameter families of unimodal maps, with negative Schwarzian derivative, unfolding a saddle-node bifurcation. We show that there is a parameter set of positive but not full Lebesgue density at the bifurcation, for which the maps exhibit absolutely continuous invariant measures which are supported on the largest possible interval. We prove that these measures converge weakly to an atomic measure supported on the orbit of the saddle-node point. Using these measures we analyze the intermittent time series that result from the destruction of the periodic attractor in the saddle-node bifurcation and prove asymptotic formulae for the frequency with which orbits visit the region previously occupied by the periodic attractor.

1 Introduction

By Jakobson's celebrated work [Jak81], the logistic family $x \mapsto \mu x(1-x)$, $x \in [0,1]$, admits absolutely continuous invariant measures (a.c.i.m.'s) for μ from a set of positive measure. In fact, $\mu = 4$ is a (Lebesgue or full) density point of this set. A different argument for this result was given by Benedicks and Carleson [BenCar85], [BenCar91]. Their reasoning was generalized to unfoldings $\{f_\gamma\}$ of unimodal Misiurewicz maps f_0, with eventually periodic critical point c (and possessing negative Schwarzian derivative). It was shown that the bifurcation value $\gamma = 0$ is a density point of the set of parameter values for which f_γ admits an absolutely continuous invariant measure [MelStr93], [ThiTreYou94].

In the first part of this paper, we generalize these results to unfoldings of saddle-node bifurcations in families $\{f_\gamma\}$ of unimodal maps with negative Schwarzian derivative. We establish that a saddle-node bifurcation value occurs as a point of positive density of the parameter set for which there are absolutely continuous invariant measures. Under the assumption that f_0 is not more than once renormalizable, we construct parameters for which f_γ possesses a.c.i.m.'s supported on the maximal interval $[f_\gamma^2(c), f_\gamma(c)]$, see Theorem A below. Our proof is inspired by Luzzatto's approach to the work of Benedicks and Carleson [Luz00]. In contrast to the Misiurewicz bifurcation values, the saddle-node bifurcation value is not a full (one-sided) density point of this parameter set. Both periodic attractors and a.c.i.m's occur with positive density at the bifurcation value. This part of our paper is related to results of M.J. Costa [Cos03] on absolutely continuous measures in certain families of unimodal maps near a saddle-node bifurcation of a fixed point. She focused on the sink-horseshoe bifurcation in which a sink and a horseshoe collapse, see [Zee82, Cos98], and studied families $\{g_\gamma\}$ of unimodal interval maps to describe bifurcations. The class of families of interval maps studied in [Cos03] consists of unfoldings of C^∞ unimodal maps $g_0 : [0,1] \to [0,1]$, that possess a saddle node fixed point at $p \in (0,1)$, so that the critical point $c > p$ satisfies $g_0^2(c) < p$. For such families Costa derived the analogue of Theorem A. Our proof differs from Costa's in significant ways and we will point out those differences in the text.

Following the construction of a.c.i.m.'s, we continue with a detailed discussion of the intermittency that occurs due to the saddle-node bifurcation. That saddle-node bifurcations can give rise to intermittency has been known since [PomMan80], who called intermittency associated with a saddle-node bifurcation type I intermittency. Pomeau and Manneville studied type I intermittency in connection with the Lorenz model. In the model, simplifying (hyperbolicity) assumptions on the dynamics outside a neighborhood of the saddle-node periodic orbit are made. In perhaps the most basic example of type I intermittency, in families of unimodal maps, such simplifications are not justified, due to the presence of a critical point. Our discussion of absolutely continuous invariant measures allows us to give a rigorous treatment of intermittent time series, where we explain and prove quantitative aspects earlier discussed numerically in [HirHubSca82], see Theorems B and C below. Our results continue an investigation of intermittent time series occurring near boundary crisis bifurcations in [HomYou02].

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1.1 Assumptions and statement of main results

Let $\{f_\gamma\}$ be a family of unimodal maps of the interval $[0,1]$, with critical point at c. Suppose that each f_γ is at least C^3 smooth and that $f_\gamma(x)$, $Df_\gamma(x)$, and $D^2 f_\gamma(x)$, are C^1 w.r.t. γ. Suppose that each f_γ has negative Schwarzian derivative (see [MelStr93]) and that $D^2 f_\gamma(c) < 0$. Further, suppose that $f_\gamma(1) = f_\gamma(0) = 0$ and that the fixed point at 0 is hyperbolic repelling. We say that $\{f_\gamma\}$ unfolds a (quadratic) saddle-node if,

- There is a q-periodic point a, with $Df_0^q(a) = 1$, and,
- $D^2 f_0^q(a) \frac{\partial}{\partial \gamma} f_\gamma^q(a) \neq 0$ at $\gamma = 0$.

For the sake of clarity, we will assume without loss of generality that

$$\frac{\partial}{\partial \gamma} f_\gamma^q(a)|_{\gamma=0} > 0 \quad \text{and} \quad D^2 f_0^q(a) > 0.$$

With this convention, for $\gamma > 0$ the saddle-node point disappears and $f_\gamma^q(x) > x$ for x close to a.

A unimodal map f is called renormalizable if there is a proper subinterval $I \subset [f^2(c), f(c)]$ containing the critical point c, so that $f^n(I) \subset I$ for some $n > 0$ and $f^i(I) \cap I = \emptyset$ for $0 < i < n$. A renormalizable map is called once renormalizable if the above property defines n uniquely; n is called the period. Let N denote the maximal such periodic interval. We will assume that a is the point in the saddle node orbit such that $a > c$ and is the closest such point to c. Then N is bounded on the right by a since $f^q(x) > x$ for x slightly to the right of a.

For a set A of parameter values, let $m(A)$ be its Lebesgue measure.

Theorem A *Let $\{f_\gamma\}$ be as above, unfolding a saddle-node bifurcation of a q-periodic orbit at $\gamma = 0$. Assume that f_0 is once renormalizable of period q. Consider the set Γ such that for each $\gamma \in \Gamma$, the map f_γ has an absolutely continuous invariant measure ν_γ whose support is the maximal interval $[f_\gamma^2(c), f_\gamma(c)]$. The set Γ has positive density at $\gamma = 0$:*

$$\liminf_{\gamma \searrow 0} \frac{m(\Gamma \cap (0,\gamma))}{\gamma} \quad > \quad 0.$$

It does not have full density:

$$\limsup_{\gamma \searrow 0} \frac{m(\Gamma \cap (0,\gamma))}{\gamma} \quad < \quad 1.$$

Intermittent dynamics manifests itself by alternating phases with different characteristics. In one phase, referred to as the laminar phase, the dynamics appear to be nearly periodic. While in the other phase, the relaminarization phase, the orbit makes large, seemingly chaotic excursions away from the periodic region. These excursions are called chaotic bursts. Let $\bar{E}$ be a neighborhood of the orbit, $\mathcal{O}(a)$ of a, not containing a critical point of f_0^q. Let $\chi_{\bar{E}}$ be defined as

$$\chi_{\bar{E}}(x,\gamma) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_{\bar{E}}(f_\gamma^i(x)), \tag{1}$$

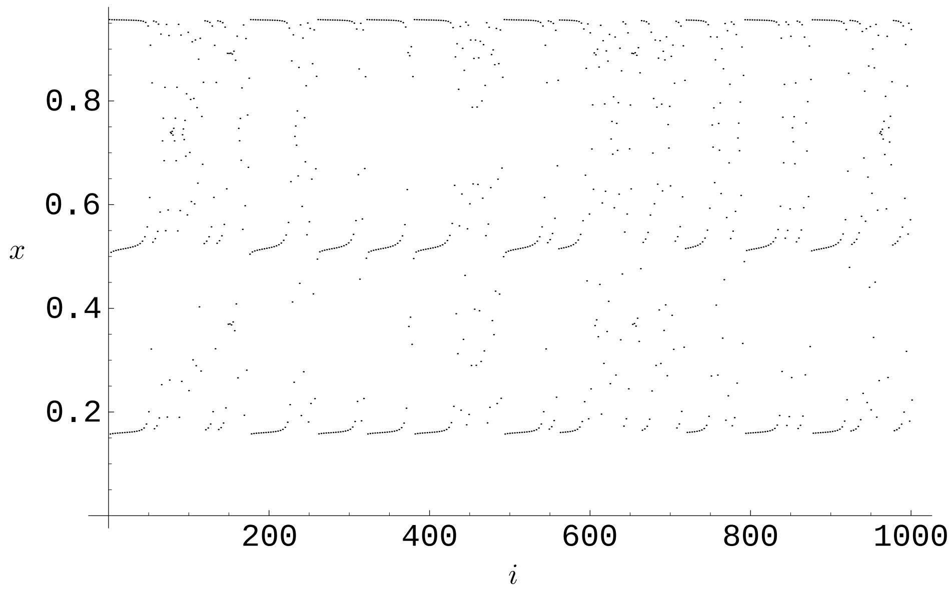

Figure 1: Time series for the quadratic map $x \mapsto \mu x(1-x)$ for $\mu = 3.828$ near the saddle-node bifurcation of the period three orbit. Both the laminar phase (nearly periodic) and chaotic bursts are clearly seen.

whenever the limit exists, where $1_{\bar{E}}$ is the usual indicator function of the set $\bar{E}$. That is, $\chi_{\bar{E}}(x,\gamma)$ is the relative frequency with which the orbit $\mathcal{O}(x)$ visits $\bar{E}$ (for those x for which the limit exists). The following theorem discusses $\chi_{\bar{E}}(x,\gamma)$ for γ near 0.

Denote by ν_0 the atomic measure supported on the orbit of a, given by

$$\nu_0 = \frac{1}{q}\sum_{i=0}^{q-1} \delta_{f_0^i(a)}. \tag{2}$$

We will denote the usual weak convergence of measures by the symbol $\rightharpoonup$.

Theorem B *Let $\{f_\gamma\}$ and Γ be as in Theorem A. There exist sets $\Omega \subset \Gamma$ of parameter values with positive density at $\gamma = 0$, so that*

$$\lim_{\gamma\in\Omega,\gamma\searrow 0} \nu_\gamma \quad \rightharpoonup \quad \nu_0.$$

Restricting to $\gamma \in \Omega$, $\chi_{\bar{E}}(x,\gamma)$ is a constant, $\chi_{\bar{E}}(\gamma)$, for almost every $x \in [0,1]$ and $\chi_{\bar{E}}(\gamma)$ depends continuously on γ at 0. There exist $K_1, K_2 > 0$ so that

$$K_1 \le \lim_{\gamma\in\Omega,\gamma\searrow 0} \frac{1-\chi_{\bar{E}}(\gamma)}{\sqrt{\gamma}} \le K_2. \tag{3}$$

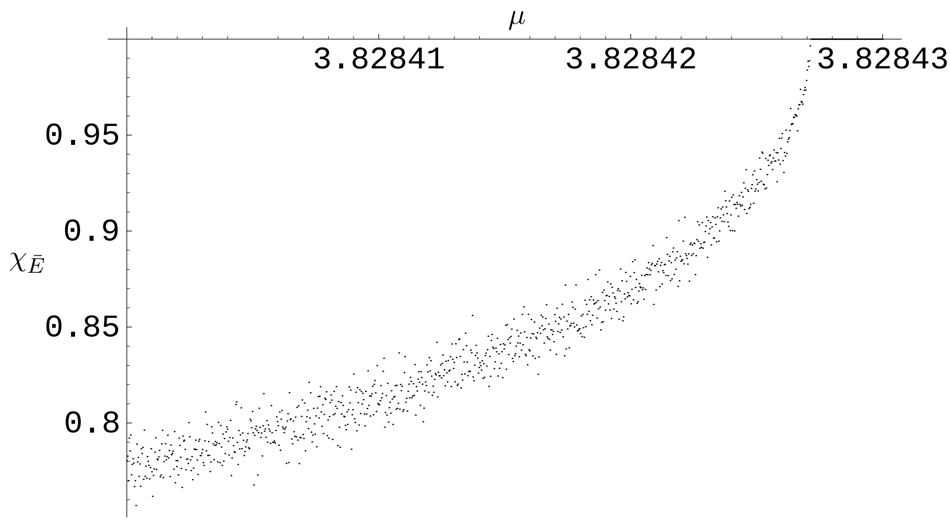

Figure 2: Numerical computation of $\chi_{\bar{E}}(\mu)$ as a function of μ for the logistic family $x \mapsto \mu x(1-x)$ near the saddle-node bifurcation of a period three orbit at $\mu_{sn} = 1 + 2\sqrt{2} \approx 3.828427$.

For comparison we include a result from [HomYou02], which builds on results in [AfrLiuYou96], [DiaRocVia96], and [AfrYou98], showing that also periodic attracting orbits are found for parameters from a set with positive density at the bifurcation point. The frequency with which the dynamics is in the laminar phase behaves in a similar way to the frequency in Theorem B. Note that the fact that the set Γ in Theorem A does not have full density at $\gamma = 0$, follows from the following result.

Theorem C *Let $\{f_\gamma\}$ be as above, unfolding a saddle node bifurcation of a q-periodic orbit at $\gamma = 0$. There exist sets $\mathcal{A}$ of parameter values of positive measure and positive density at 0, i.e.*

$$\lim_{\gamma \searrow 0} \frac{m\,(\mathcal{A} \cap [0,\gamma))}{\gamma} \quad > \quad 0, \tag{4}$$

so that for each $\gamma \in \mathcal{A}$, f_γ has an attracting periodic orbit. Further,

$$\lim_{\gamma \in \mathcal{A}, \gamma \searrow 0} \nu_\gamma \quad \rightharpoonup \quad \nu_0,$$

where ν_γ, $\gamma \in \mathcal{A}$, is the invariant measure supported on the periodic orbit. Restricting to $\gamma \in \mathcal{A}$, $\chi_{\bar{E}}(x,\gamma)$ is a constant, $\chi_{\bar{E}}(\gamma)$, for almost every $x \in [0,1]$ and $\chi_{\bar{E}}(\gamma)$ depends continuously on γ at 0. There exists $K > 0$ so that

$$\lim_{\gamma \in \mathcal{A}, \gamma \searrow 0} \frac{1 - \chi_{\bar{E}}(\gamma)}{\sqrt{\gamma}} \quad = \quad K.$$

Different sets $\mathcal{A}$ lead to different limit values K in Theorem C. In fact, the proof of Theorem C makes clear that arbitrary large numbers occur as the limit values K. This fact, together with Theorems A and C lead us to conjecture that there is a parameter set Λ which has $\gamma = 0$ as a Lebesgue density point, so that

$$\lim_{\gamma \searrow 0, \gamma \in \Lambda} \frac{\ln(1 - \chi_{\bar{E}})}{\ln \gamma} = \frac{1}{2}.$$

It was shown in [HomYou02] that such a limit cannot hold without restricting the parameter set.

2 The saddle node

2.1 Local analysis

Denote by E a small neighborhood of a on which f_0^q is invertible. Let $W_{loc}^s(a)$ and $W_{loc}^u(a)$ denote the usual local stable and local unstable sets for a.

Proposition 2.1 *Let $\{f_\gamma\}$ be a C^1 family of C^r, $r \geq 2$, maps unfolding a saddle-node. Then there exists a family of C^r flows, $\{\phi_\gamma^t\}_{0 \leq \gamma < \bar{\gamma}}$, on E such that $f_\gamma^q \equiv \phi_\gamma^1$ for each $\gamma \geq 0$. Further, $\phi_\gamma^t(\cdot) \to \phi_0^t(\cdot)$ as $\gamma \searrow 0$ in the C^1 topology on E and in the C^r topology on compact intervals away from the fixed point. The flow ϕ_0^t is uniquely determined by f_0.*

Proof. The C^∞ version of this theorem is due to Takens [Tak73]. The C^r result follows from Part 2 of [Yoc95]; the case $\gamma = 0$ follows from Appendix 3 of [Yoc95] and the case $\gamma > 0$ and the convergences as $\gamma \searrow 0$ follow from Theorem IV.2.5 and Lemma IV.2.7 of the same. □

A version of Proposition 2.1 appears in [IlyLi99]. They proved that one may obtain $\phi_\gamma^t(x)$ which depends C^r smoothly on both x and γ, even at the fixed point, if one requires that $(x, \gamma) \mapsto f_\gamma(x)$ be $C^{R(r)}$ smooth, where $R(r)$ may be larger than r. Costa uses the results in [IlyLi99]. Proposition 2.1 allows for our weaker hypotheses and its implications are sufficient for our purposes.

Choose $d \in W_{loc}^s(a)$ (note $d < a$) and let

$$I_\gamma^s = [d, f_\gamma^q(d)].$$

Also, choose a point $e \in W_{loc}^u(a)$ so that

$$I_\gamma^u = [e, f_\gamma^q(e)] \subset E.$$

For the sake of convenience we restrict E to be the interval

$$E = [d, e],$$

so that $W^s_{loc}(a) = [d, a)$ and $W^u_{loc}(a) = (a, e]$. We will use the embedding flow on the interval $E \cup I^u_\gamma = [d, f^q_\gamma(e)]$.

Given d and e as above, let $\{\gamma_l\}_{l=l_0}^\infty$ be the sequence, $\bar{\gamma} > \gamma_{l_0} > \gamma_{l_0+1} > \cdots$, defined by

$$f^{ql}_{\gamma_l}(d) = e,$$

see [MisKaw90]. For each $l \geq l_0$ let $g_l : [0, 1] \to [\gamma_{l+1}, \gamma_l]$, be the reparameterization map defined by

$$\phi^{l+\theta}_{g_l(\theta)}(d) = e.$$

We have that $g_l(0) = \gamma_l$ and $g_l(1) = \gamma_{l+1}$. We may invert $g_l(\cdot)$, for each l, to obtain maps $\theta_l : [\gamma_{l+1}, \gamma_l] \to [0, 1]$.

Proposition 2.2 *The reparameterization maps g_l are smooth monotone decreasing functions with uniformly small distortion: given $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that for every $l \geq N$ and every $\theta \subset [0, 1]$,*

$$(1 - \varepsilon) \leq \frac{Dg_l(\theta)}{|\gamma_l - \gamma_{l+1}|} \leq (1 + \varepsilon).$$

PROOF. Diaz et. al. [DiaRocVia96] proved this result under the hypothesis that $(x, \gamma) \mapsto f_\gamma(x)$ is $C^{R(r)}$, using Il'yashenko and Li's embedding result [IlyLi99]. A proof of this result under the current hypotheses appears in [AfrYou98] based on [Jon90]. □

We remark that $l^2\gamma_l$ converges as $l \to \infty$ (see [MisKaw90]), so that $\gamma_{l+1}/\gamma_l \to 1$ as $l \to \infty$. This fact, together with Proposition 2.2 imply the next proposition [AfrYou98].

Proposition 2.3 *Let Ω be a measurable subset of $[0, \bar{\gamma})$ and denote $\Omega_l = \Omega \cap [\gamma_l, \gamma_{l-1}]$. If the limit*

$$\lim_{l\to\infty} m(\theta_l(\Omega_l))$$

exists and equals Δ, then

$$\lim_{\gamma \searrow 0} \frac{m\left(\Omega \cap [0, \gamma)\right)}{\gamma} = \Delta.$$

We remark that, with Ω_l satisfying the assumptions in the above proposition, $m(\Omega_l)/\gamma_l\sqrt{\gamma_l}$ converges as $l \to \infty$.

Given $\gamma \geq 0$ and $x \in I^u_\gamma$, define $\tau^u_\gamma(x)$ to be the unique number for which

$$\phi_\gamma^{\tau^u_\gamma(x)}(e) = x.$$

For $\gamma \geq 0$ and $x \in I^s_\gamma$, let $\tau^s_\gamma(x)$ be defined by

$$\phi_\gamma^{\tau^s_\gamma(x)}(d) = x.$$

It follows from the smoothness of $\phi_\gamma^t(x)$ that for each $\gamma \geq 0$, the functions $\tau_\gamma^{s,u}$ are C^r diffeomorphisms from $I_\gamma^{s,u}$ to [0,1]. We will use $\tau_\gamma^{s,u}$ as coordinates on $I_\gamma^{s,u}$. In the following, we will associate $[0,1]$ with the unit circle $\mathbf{S}^1$ by making the identification $0 \sim 1$. We also treat $I_\gamma^{s,u}$ as circles through the coordinates $\tau_\gamma^{s,u}$. Let $\mathcal{L}_{l,\theta}$ denote the local (first hit) map from $I^s_{g_l(\theta)}$ to $I^u_{g_l(\theta)}$ induced by $f^q_{g_l(\theta)}$. The convenience of using τ_γ^s and τ_γ^u as coordinates on I_γ^s and I_γ^u is seen in the following proposition.

Proposition 2.4 *For each $l \geq l_0$ and each $\theta \in [0,1]$*

$$\mathcal{L}_{l,\theta} = (\tau^u_{g_l(\theta)})^{-1} \circ R_{-\theta} \circ \tau^s_{g_l(\theta)}, \tag{5}$$

where R_θ denotes a rigid rotation by angle θ.

PROOF. This follows from Proposition 2.1 and the definitions of $\tau^s_{g_l(\theta)}$ and $\tau^u_{g_l(\theta)}$ as the time variables for the embedding flow for $f^q_{g_l(\theta)}$. □

2.2 Global analysis

Let $c_i(\gamma) = f_\gamma^i(c)$, $i \geq 1$. It follows from the assumptions that there is an integer j such that $f_0^j(c) \in I_0^s$. By choosing d so that $f_0^j(c)$ is in the interior of I_0^s we have that $f_\gamma^j(c) \in I_\gamma^s$ for all γ small. Denote this point by $c^s(\gamma)$. It then follows that for any $l \geq l_0$ and $\gamma \in [\gamma_{l+1}, \gamma_l]$, either

$$c_{j+lq}(\gamma) \in I_\gamma^u \qquad \text{or} \qquad c_{j+(l+1)q}(\gamma) \in I_\gamma^u.$$

Denote this intersection of $\{c_i(\gamma)\}$ with I_γ^u by $c^u(\gamma)$, that is, $c^u(\gamma) = \mathcal{L}_{l,\theta}(c^s(\gamma))$. Note that for a fixed l the function $\theta \mapsto c^u(g_l(\theta))$ will have a jump discontinuity at which the value will jump from one endpoint of I_γ^u to the other. Denote by $\theta_l^\sharp$ the point at which the discontinuity takes place.

Lemma 2.5 *There exists a limit*

$$\lim_{l\to\infty} \theta_l^\sharp = \theta_\infty^\sharp.$$

As $l \to \infty$ the sequence of maps $\theta \mapsto c^u(g_l(\theta))$ converges in the C^r topology on compact sets not containing $\theta_\infty^\sharp$.

PROOF. This follows from Proposition 2.1 and the definition of $g_l(\theta)$. □

Specifically, we will make use of the implication that the derivatives of $c^u(g_l(\theta))$ with respect to θ converge uniformly as $l \to \infty$ for θ in compact intervals away from the discontinuity. Later, this will allow us to make estimates of derivatives along $\{c_i(\gamma)\}$ which are uniform in l.

The unstable manifold $W^u(a)$ is the positive orbit of the local unstable manifold $W^u_{loc}(a)$.

Lemma 2.6 $W^u(a) = [f_0^2(c), f_0(c)]$.

PROOF. Recall that f is once renormalizable of period q. From the assumptions it is easy to see that the connected component of the stable manifold $W^s(a)$ containing a is a q periodic interval containing c. From our assumptions $W^u_{loc}(a)$ is an interval of the form $(a, a+\varepsilon)$. Consider

$$M = \overline{\bigcup_{i \geq 0} f_0^{iq}(W^u_{loc}(a))}.$$

Note that M is an invariant interval for f_0^q and that $W^u(a)$ is in fact the orbit of M.

The assumptions give $f_0^q(x) > x$ for $x \in W^u_{loc}(a)$. By this and the assumption of negative Schwarzian derivative, f^q must have a turning point before it has another fixed point to the right of a. This implies that this turning point $\tilde{c}$ is in M. Since the orbit of $\tilde{c}$ must include c, an iterate of f_0^q must map $\tilde{c}$ onto one of the preimages c_i, $-1 \leq i < q$ of c contained in the orbit of $W^s(a)$. Thus M is a periodic interval which contains at least two points in the q periodic saddle-node orbit. This implies that M has period less than q which contradicts the assumption that f_0 is once renormalizable.

□

Note that this argument also shows the connected component of $W^s(a)$ containing a is the maximal interval containing a of period q.

We will use the freedom in the choice of e and take e so that c is not in $\mathcal{O}(e)$ for f_0. By this choice we have that c will be in the interior of $f_0^i(I_0^u)$ for some i.

We will identify parameter values θ for which $f_{g_l(\theta)}$ maps the critical point c onto some repelling hyperbolic periodic point and thus c does not return to a neighborhood of itself, i.e. $f_{g_l(\theta)}$ satisfies the Misiurewicz condition. There are two ways that hyperbolic periodic points can occur: as periodic points whose orbits pass through $\bar{E}$, or as continuations of periodic points for $\gamma = 0$ (outside of $\bar{E}$). In the former case, the periodic orbits exist for parameter values within subintervals of $[\gamma_{l+1}, \gamma_l]$ for each l. In the latter case the periodic orbits exist for all $\gamma > 0$ sufficiently small. We confine ourselves to a discussion of the later case.

Note that the assumptions on f_0 imply that the nonwandering set, L_0, of f_0 restricted to $[0,1]\backslash\bar{E}$ is a hyperbolic set (see Theorem III.5.1 in [MelStr93]). By Lemma 2.6, for any $y^* \in L_0$ there is a point $x^* \in I_0^u$ which is mapped onto y^* by an iterate of f_0. Write y^*_γ for the continuation of y^*. Hyperbolicity of L_0 implies that y^*_γ depends smoothly on γ. Recall that $c^s(\gamma) = c_j(\gamma) \in I^s_\gamma$ is the first point in the orbit of c that hits I^s_γ. Let $\tau_0^c = \tau_0^s(c_j(0))$.

Proposition 2.7 *Given any $y^* \in L_0$, there exists integers $m > 0$ and $l_0 > 0$ and a sequence of parameter values $\gamma_l^* \in [\gamma_{l+1}, \gamma_l]$, $l \geq l_0$ such that*

$$f^{lq+m}_{\gamma_l^*}(c) = y^*_{\gamma_l^*}.$$

(Here y^*_γ is the continuation of y^*). Further, there is a number $\theta^* \in (0,1)$ such that $\theta_l(\gamma_l^*) \to \theta^*$ as $l \to \infty$.

Proof. For each $y^* \in L_0$ there is a point $x^* \in I_0^u$ such that x^* is mapped to y^* by an iterate of f_0. Proposition 2.4 implies the existence of a parameter γ_l^* for each $l \geq l_0$ for which $c^s(\gamma_l^*) \in I^u$ is mapped onto x^* by the local map. Let $\tau^* = \tau_0^u(x^*)$. If we let $\theta^* = 1 - \tau^* + \tau_0^c$ then $R_{-\theta^*}(\tau_0^c) = \tau^*$ and

$$\frac{\partial}{\partial \theta}\Big|_{\theta=\theta^*} (R_{-\theta}(\tau_0^c) - \tau^*) = 1. \tag{6}$$

Since y^*_γ depends smoothly on γ, $\frac{\partial}{\partial\theta} y^*_{g_l(\theta)}$ is close to 0 for l large and $c^u(g_l(\theta^*)) = x^*$, (6) and Lemma 2.5 imply the result. □

We will write $f_{l,\theta}$ for $f_{g_l(\theta)}$. In the following we will work mainly with $f_{l,\theta}$ rather than f_γ. In doing so we avoid problems with the unboundedness of derivatives with respect to γ.

3 Induced maps

As before, a denotes a saddle-node periodic point of f_0, of period q. We may assume that a is nearest to the critical point c of all points in $\mathcal{O}(a)$, so that f_0^q is a homeomorphism on (c,a). We may suppose that

$$\text{there is a periodic point } z_0 \in L_0 \text{ and an integer } j \text{ such that } f_0^j(e) = z_0. \tag{7}$$

Let z_γ be the periodic continuation of z_0 and e_γ the continuation of $e = e_0$ such that $f_\gamma^j(e_\gamma) = z_\gamma$. We will suppress the γ dependence of e in the notation. Let $\bar{E} = \cup_{i=0}^{q-1} f_0^i(E)$, so that $\bar{E}$ is a neighborhood of the orbit of a for $\gamma = 0$. Denote by $e_{-i} = \left(f_\gamma^{-iq}|_E\right)^{-1}(e)$ and

$$E_\gamma^{-i} = [e_{-i-1}, e_{-i}). \tag{8}$$

We call E_γ^{-i}, $-1 \leq i \leq l+1$, a fundamental domain. Denote

$$\tilde{E}_{l,\theta} \;\; = \;\; \cup_{i=0}^{l} E_{l,\theta}^{-i}.$$

An induced map $\tilde{f}_{l,\theta}$ will be defined from $f_{l,\theta}$ by mapping points $\tilde{E}_{l,\theta}$ ahead to the image of $I^u_{l,\theta} = E^0_{l,\theta}$:

$$\tilde{f}_{l,\theta}(x) = \begin{cases} f_{l,\theta}^{iq+1}(x), & \text{if } x \in E_{l,\theta}^{-i}, -l+1 \leq i \leq 0, \\ f_{l,\theta}(x), & \text{otherwise.} \end{cases} \tag{9}$$

For $\gamma \in (\gamma_{l+1}, \gamma_l)$, $\tilde{f}_{l,\theta}$ will have $l+1$ discontinuities, see Figure 3.

We note that Costa [Cos03] defined a similar induced map. Lemma 3.9 in [Cos03] implies that $\left|\frac{\partial}{\partial\theta} D\tilde{f}_{l,\theta}(x)\right| \leq C|D\tilde{f}_{l,\theta}(x)|$.

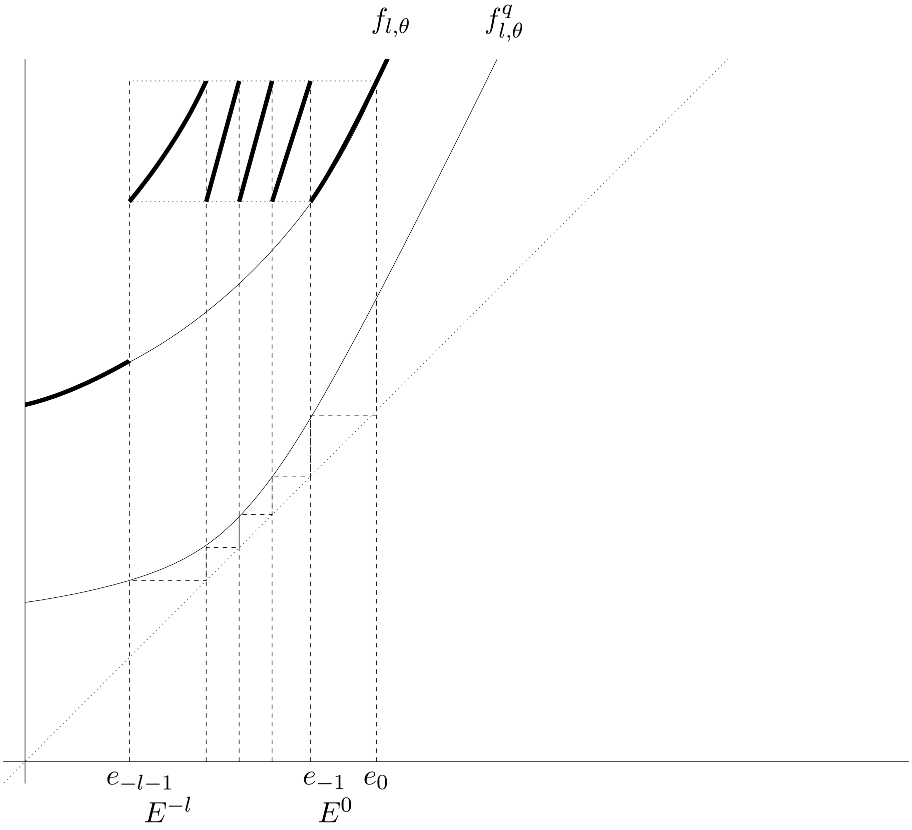

Figure 3: The thick curves form the graph of the induced map $\tilde{f}_{l,\theta}$.

3.1 Iterating intervals

In the following we will consider iterates of intervals. The map $\tilde{f}_{l,\theta}$, $l_0 \leq l < \infty$, is discontinuous along backward iterates of e in E, so that the intervals in the image of an interval under $\tilde{f}_{l,\theta}$ might be arbitrarily small, regardless of the size of the original interval. We therefore slightly adjust the definition of $\tilde{f}_{l,\theta}$ to avoid this problem. Consider an interval $I \subset [0,1]$. For $0 \leq i \leq l$, denote $I^{-i} = I \cap E^{-i}_{l,\theta}$ and let I^{-l-1} be the component of $I \backslash E$ adjacent to $E^{-l}_{l,\theta}$ and I^1 the component of $I \backslash E$ adjacent to $E^0_{l,\theta}$. This yields a partition $\{I^{-i}\}$ of I. If the leftmost or rightmost nonempty intervals of this partition do not contain a fundamental domain $E^{-i}_{l,\theta}$, $-l-1 \leq -i \leq 1$, join them to the adjacent intervals. Note that if I is partitioned into two elements $\{I^{-i}, I^{-i+1}\}$ neither of which is a fundamental domain, this leaves a choice in coding the resulting interval after I^{-i} or I^{-i+1}. This way an interval I that covers one or more fundamental domains is partitioned into subintervals which are at least as large as a fundamental domain. Given $x \in I^{-i} \subset I$, define

$$\breve{f}_{l,\theta}(x; I) \quad = \quad \begin{cases} f_{l,\theta}(x), & \text{if } i = -1 \text{ or } i = l+1, \\ f^{iq+1}_{l,\theta}(x), & \text{if } 0 \leq i \leq l. \end{cases}$$

Note that as long as I does not cover a fundamental domain in E, $\breve{f}_{l,\theta}(\cdot; I)$ equals some fixed iterate of $f_{l,\theta}$. Also note that $\breve{f}_{l,\theta}(I; I)$ consists of at most two components. We define iterations of $\breve{f}_{l,\theta}$ inductively by

$$\breve{f}^j_{l,\theta}(x; I) = \breve{f}_{l,\theta}\left(\breve{f}^{j-1}_{l,\theta}(x; I); \breve{f}^{j-1}_{l,\theta}(I; I)\right).$$

We further remark that, if $\breve{f}^j_{l,\theta}$ maps I onto $f_{l,\theta}(E^0_{l,\theta})$, then there is a fixed number N of iterates after which $\breve{f}^{j+N}_{l,\theta}(I; I)$ contains c in its interior.

Define maps F_l and $\tilde{F}_l$ by

$$\begin{aligned} F_l(x, \theta) &= (f_{l,\theta}(x), \theta), \\ \tilde{F}_l(x, \theta) &= (\tilde{f}_{l,\theta}(x), \theta). \end{aligned}$$

Consider the set $T = (\theta^* - \varepsilon, \theta^* + \varepsilon)$. By a fundamental strip we mean a set $S^{-i}_l = \{(E^{-i}_{l,\theta}, \theta)\}$, $\theta \in T$, with $-1 \leq i \leq l + 1$. Let ω be a parameter interval and consider a curve $C = \{(x(\theta), \theta);\ \ \theta \in \omega\}$ with the following properties. If for a given fundamental strip S^{-i}_l, $0 \leq i \leq l$, C contains points in S^{-i}_l, than either C crosses it (continuous from one boundary component to the other) or ends in it. Define C^{-i}_l to be the intersection of C with the fundamental strip S^{-i}_l and let C^{-l-1}_l be the connected component of C that is adjacent to C^{-l}_l and C^1_l the connected component adjacent to C^0_l. This defines a partition $\{C^{-l-1}_l, \ldots, C^1_l\}$ of C with possibly empty elements. If the leftmost or rightmost nonempty element of this partition does not cross a fundamental strip, join it to the adjacent element. This way a partition of a curve C that crosses at least one fundamental strip is obtained all of whose elements cross a fundamental strip. Define

$$\breve{F}_l(x, \theta; \omega) = \begin{cases} F_l(x, \theta), & \text{if } (x, \theta) \in C^{-l-1}_l \cup C^1_l, \\ F^{iq+1}_l(x, \theta), & \text{if } (x, \theta) \in C^{-i}_l, 0 \leq i \leq l. \end{cases} \tag{10}$$

Iterates of $\breve{F}$ are treated recursively. If the image $\breve{F}^i_l(x, \theta; \omega)$ defines a curve with properties as above, the next iterate $\breve{F}^{i+1}_l(x, \theta; \omega)$ can be defined.

3.2 Expansion for induced maps

The relation between expansion along orbits of $f_{l,\theta}$, $\tilde{f}_{l,\theta}$ and $\breve{f}_{l,\theta}$ is discussed in the next two lemmas. When writing $|D\tilde{f}^n_{l,\theta}(x)| \geq K$ for a point x whose forward orbit goes through a discontinuity of $\tilde{f}_{l,\theta}$, we mean that the estimate holds for the one-sided derivative.

Lemma 3.1 *If there exist $\tilde{C} > 0$, $\tilde{\lambda} > 1$ such that $|D\tilde{f}^n_{l,\theta}(x)| \geq \tilde{C}\tilde{\lambda}^n$ for all $n > 0$, then there are $C_l > 0$ and $\lambda_l > 1$, so that $|Df^n_{l,\theta}(x)| \geq C_l \lambda^n_l$, for all $n > 0$.*

Proof. Write $f^i_{l,\theta}(x) = f^{k(i)}_{l,\theta} \circ \tilde{f}^{m(i)}_{l,\theta}(x)$ with $k(i)$ minimal such nonnegative integer. Compute

$$|Df^i_{l,\theta}(x)| \geq \min |Df^{k(i)}_{l,\theta}| \tilde{C}(\tilde{\lambda}^{m(i)/i})^i.$$

If $k(i) > 0$, the piece of orbit $\tilde{f}_{l,\theta}^{m(i)}(x), \ldots, f_{l,\theta}^{k(i)}(\tilde{f}_{l,\theta}^{m(i)}(x))$ is in $\bar{E}_{l,\theta} = \cup_{i=0}^{q-1} \tilde{E}_{l,\theta}$. Since $c \notin \bar{E}_{l,\theta}$, the term $|Df_{l,\theta}|$ is bounded below in $\bar{E}_{l,\theta}$. Further, $k(i)$ is bounded above by $(l+1)q$ since any point in $\bar{E}_{l,\theta}$ is mapped outside of $\bar{E}_{l,\theta}$ in $(l+1)q$ or fewer iterations. Therefore the quantity $|Df_{l,\theta}^{k(i)}|$ is bounded from below by a constant D_l. Thus, $|Df_{l,\theta}^{i}(x)| \geq \tilde{C} D_l (\tilde{\lambda}^{m(i)/i})^i$. For $k(i) = 0$ this formula holds with $D_l = 1$, so that we can let $C_l = \min\{\tilde{C}, \tilde{C} D_l\}$.

Since there is a minimum number of iterations of $f_{l,\theta}$ needed for an orbit to enter $\bar{E}_{l,\theta}$ after leaving I^u, and the number of consecutive iterations in $\bar{E}_{l,\theta}$ is bounded above by $q(l+1)$, it follows that the fraction $m(i)/i$ is bounded below by a constant $d_l > 0$. Hence, $\tilde{\lambda}^{m(i)/i}$ is strictly larger than some number $\lambda_l > 1$. □

Similarly one derives the following lemma relating expansion of $\breve{f}_{l,\theta}$ to expansion of $\tilde{f}_{l,\theta}$.

Lemma 3.2 *If $|D\breve{f}_{l,\theta}^{n}(x; I)| \geq \breve{C} \breve{\lambda}^n$ for some $\breve{C} > 0$, $\breve{\lambda} > 1$, then there are constants $\tilde{C} > 0$ and $\tilde{\lambda} > 1$ so that $|D\tilde{f}_{l,\theta}^{n}(x)| \geq \tilde{C} \tilde{\lambda}^n$. The converse statement holds as well.*

Given a subinterval $I \subset [0, 1]$, we can write $\breve{f}_{l,\theta}^{j}(x; I) = f_{l,\theta}^{s} \circ \tilde{f}_{l,\theta}^{k(j)}(x)$, where $s \in \{-q, 0\}$. Here $f_{l,\theta}^{-q} : f_{l,\theta}(E_{l,\theta}^0) \to f_{l,\theta}(E_{l,\theta}^{-1})$; the inclusion of a possible composition with this map is to compensate for the fact that $\breve{f}_{l,\theta}^{j}(x; I)$ may end up in $f_{l,\theta}(E_{l,\theta}^{-1})$ instead of $f_{l,\theta}(E_{l,\theta}^0)$.

Lemma 3.3 *Let ε be a small positive number. By taking the neighborhood E of the saddle node periodic point a small enough, one has*

$$1 - \varepsilon \leq k(j)/j \leq 1 + \varepsilon.$$

PROOF. This holds since for E small, the minimal number of iterates between any two passages through $\tilde{E}_{l,\theta}$ is large. □

Observe that for E small, $\tilde{\lambda}$ in Lemma 3.2 is close to $\breve{\lambda}$. A similar statement can be made for $\breve{F}_l$. Given a parameter interval ω, let

$$\breve{c}_i(l, \theta) \;=\; \Pi \circ \breve{F}_l^{i}(c, \theta; \omega), \tag{11}$$

where Π is the projection $\Pi(x, \theta) = x$, and $\omega_i = \cup_{\theta \in \omega} \breve{c}_i(l, \theta)$. Write

$$\tilde{c}_i(l, \theta) \;=\; \tilde{f}_{l,\theta}^{i}(c).$$

For $\breve{c}_j$ away from $f_{l,\theta}(\tilde{E}_{l,\theta})$, let κ_j be such that

$$\breve{c}_j \;=\; \tilde{c}_{\kappa(j)}. \tag{12}$$

As in the definition of $k(j)$ above, if $\breve{c}_j(l, \theta)$ is in $f_{l,\theta}(\tilde{E}_{l,\theta})$, one can write $\breve{c}_j(l, \theta) = f_{l,\theta}^{s}(\tilde{c}_{\kappa(j)})$ with $s \in \{-q, 0\}$.

Lemma 3.4 *Let ε be a small positive number. By taking the neighborhood E of the saddle node periodic point a small enough, one has*

$$1 - \varepsilon \leq \kappa(j)/j \leq 1 + \varepsilon.$$

4 Parameter values with bounded recurrence

We start the proof of Theorem A. We will construct a set Ω with positive density at $\gamma = 0$, so that f_γ for $\gamma \in \Omega$ has bounded recurrence (see Definition 4.1 below). From this it is deduced that f_γ has an absolutely continuous invariant measure for $\gamma \in \Omega$.

Considering f_γ for $\gamma \in [\gamma_{l+1}, \gamma_l]$, we reparameterize using the parameters l and $\theta \in [0,1]$ defined by $g_l(\theta) = \gamma$, as in Section 2.1, i.e. denote the family of unimodal maps by $f_{l,\theta}$. By Proposition 2.7, there exists a converging sequence of parameter values $\theta_l^* = \theta_l(\gamma_l^*)$ so that $f_{l,\theta_l^*}^{ql+m}(c)$ is a hyperbolic periodic point (it coincides with $y^*_{\gamma_l^*}$, the continuation of $y^* \in L_0$). The map f_{l,θ_l^*} is a Misiurewicz map and by an extension of Jakobson's Theorem, see Theorem V.6.1 in [MelStr93], θ_l^* is a Lebesgue density point of a set of parameters Θ_l for which $f_{l,\theta}$ supports an absolutely continuous invariant measure. Theorem A can be proved by establishing that the measure of Θ_l is bounded away from 0 uniformly in l. Proposition 2.3 will then guarantee that the union $\cup_l g_l(\Theta_l)$ has positive Lebesgue measure and positive density at $\gamma = 0$.

A symmetric neighborhood of the critical point c is a neighborhood whose boundary points have the same image under $f_{l,\theta}$. Write Δ_δ for the symmetric neighborhood of c with points at a maximal distance δ from c. Consider θ near θ_l^* for a fixed value of l. Recall that $\tilde{c}_i(l,\theta) = \tilde{f}^i_{l,\theta}(c)$.

Definition 4.1 *For $\delta > 0, \alpha > 0$, we say that $\tilde{f}_{l,\theta}$ satisfies the bounded recurrence condition* $(\mathrm{BR})_n = (\mathrm{BR})_n(\alpha, \delta)$ *if either $\tilde{c}_i \notin \Delta_\delta$, $1 \leq i \leq n$, or if for all positive integers $k \leq n$,*

$$\prod_{\substack{\tilde{c}_i(l,\theta)\in\Delta_\delta,\\ 1\leq i\leq k}} |\tilde{c}_i(l,\theta) - c| \geq e^{-\alpha k}. \tag{13}$$

We say that $\tilde{f}_{l,\theta}$ satisfies (BR) *if it satisfies* $(\mathrm{BR})_n$ *for all n.*

For $k = \kappa(j)$ as in (12), the bounded recurrence condition is

$$\prod_{\substack{\breve{c}_i(l,\theta)\in\Delta_\delta,\\ 0\leq i\leq j}} |\breve{c}_i(l,\theta) - c| \geq e^{-(1-\varepsilon)\alpha j},$$

for some small $\varepsilon > 0$. The next proposition is the main result of Section 4 and will be shown to imply Theorem A in Section 4.10. Recall that $m(A)$ denotes the Lebesgue measure of a parameter set A.

Proposition 4.2 *Given $\alpha > 0$ there exist $\delta > 0$, $\sigma > 0$ and $\varepsilon_0 > 0$, so that for every l there is a set Θ_l of parameter values for which $\tilde{f}_{l,\theta}$, $\theta \in \Theta_l$, satisfies* (BR)(α, δ) *and*

$$\frac{m(\Theta_l \cap (\theta_l^* - \varepsilon, \theta_l^* + \varepsilon))}{2\varepsilon} \quad > \quad \sigma,$$

for all $0 < \varepsilon < \varepsilon_0$.

To prove Proposition 4.2 we follow the reasoning in [Luz00], where Jakobson's result is proved for the logistic family using a variant of Benedicks-Carleson's proof based on the notion of bounded recurrence. An additional complication is caused by the discontinuities of the induced maps. Moreover, we must do the constructions for countably many families, depending on l, and derive bounds uniformly in l.

To avoid cumbersome notation with many constants, we will in the following write C or K for a constant, not depending on l or θ. In our estimates the constant may change from line to line. In the following sections we will be iterating $\tilde{f}_{l,\theta}$, $\breve{f}_{l,\theta}$ as well as $\breve{F}_l$, depending on whether points or intervals of points, for single maps or families, are considered. In the previous section we related the iterates of these maps. To avoid excessive notation, we will consider the arguments adopting the simplifying assumption that $\tilde{f}_{l,\theta} = \breve{f}_{l,\theta} = \Pi \circ \breve{F}_l(\theta, \cdot)$. Thus we assume $k(j) = j$ and $\kappa(j) = j$ in Section 3.2, see Lemma 3.3 and Lemma 3.4. Although this is not strictly justified, the modifications without the simplifying assumption are small and only the constants are slightly effected (see for instance the lines following Definition 4.1).

We start with three sections on exponential expansion, parameter versus state derivatives and binding, before commencing the inductive constructions. The proof of Proposition 4.2 occupies Sections 4.1 to 4.9. We write $|I|$ for the measure of a subset $I \subset [0, 1]$.

4.1 Exponential expansion and bounded recurrence

The following proposition provides exponential expansion of iterates for $\tilde{f}_{l,\theta}$ that stay away from the critical point c. The estimates are uniform in (l, θ). The proposition is modeled after Theorem III.6.4 in [MelStr93], which treats families of smooth unimodal maps.

Proposition 4.3 *There are constants $\tilde{C} > 0$, $\tilde{\lambda} > 1$ and ε, and a neighborhood W of c, so that for any neighborhood U of c with $U \subset W$ and each large enough integer l, the following holds. For each $|\theta - \theta_l^*| < \varepsilon$, if $\tilde{f}_{l,\theta}^j(x) \notin U$ for $0 \le j \le k - 1$ and $\tilde{f}_{l,\theta}^k(x) \in W$, then*

$$\left| D\tilde{f}_{l,\theta}^k(x) \right| \quad \ge \quad \tilde{C}\tilde{\lambda}^k. \tag{14}$$

If $\tilde{f}_{l,\theta}^j(x) \notin W$ for $0 \le j \le k$, then

$$\left| D\tilde{f}_{l,\theta}^k(x) \right| \quad \ge \quad \tilde{C}\tilde{\lambda}^k. \tag{15}$$

We postpone the proof of this Proposition until later in this section after proving some intermediate results.

We will make use of Koebe's principle, which we quote here. Let $U \subset V$ be two intervals. We say that V contains a δ-scaled neighborhood of U if both components of $V \backslash U$ have at least length $\delta|U|$.

Definition 4.4 *The distortion of a diffeomorphism g on an interval I is defined as*

$$\sup_{x,y \in I} \frac{|Dg(x)|}{|Dg(y)|}.$$

Theorem 4.5 [Koebe principle] *Let f have negative Schwarzian derivative. There is a neighborhood $\mathcal{U}$ of f in C^3, so that the following holds. For each $\delta > 0$, there exists $K < \infty$ so that if $g \in \mathcal{U}$, $I \subset J$ are intervals, $g^n|_J$ is a diffeomorphism, and $g^n(J)$ contains a δ-scaled neighborhood of $g^n(I)$, then the distortion of $g^n|_I$ is bounded by K.*

The neighborhood $\mathcal{U}$ in the statement is chosen so that all maps in it are unimodal maps of negative Schwarzian derivative. See Theorem IV.1.2 in [MelStr93] for the proof of the Koebe principle and additional information.

Another useful property of maps with negative Schwarzian derivative is the following principle, see [MelStr93].

Theorem 4.6 [Minimum principle] *Let f be a map with negative Schwarzian derivative on a closed interval $I = [a, b]$. If Df does not vanish on I, then*

$$|Df(x)| \quad \geq \quad \min\{|Df(a)|, |Df(b)|\}$$

for all $x \in I$.

Note that iterates of maps with negative Schwarzian derivative also have negative Schwarzian derivative. Koebe's principle and the minimum principle therefore apply to branches of the induced maps $\tilde{f}_{l,\theta}$ and $\breve{f}_{l,\theta}$. The following lemma is similar to Theorem III.6.2 in [MelStr93].

Lemma 4.7 *There are constants $K > 0$, $0 < \rho < 1$, so that for all large enough l the following holds. Let I_k be a maximal interval for which $\tilde{f}^k_{l,\theta_l^*}|_{I_k}$ is a homeomorphism. Then*

$$|I_k| \quad \leq \quad K\rho^k.$$

Proof. Denote $\tilde{\mathcal{O}}(c) = \{\tilde{f}^i_{l,\theta_l^*}(c)\}_{i \geq 0}$. The number of elements in $\tilde{\mathcal{O}}(c)$ is fixed. Thus we may let W be a neighborhood of c such that $\tilde{\mathcal{O}}(c) \cap W = \emptyset$ for all l. By assumption, e is preperiodic. Hence, $\tilde{\mathcal{O}}(e) = \{\tilde{f}^i_{l,\theta_l^*}(e)\}_{i \geq 0}$ is a finite set.

Let J_k be a maximal interval on which $\tilde{f}^k_{l,\theta_l^*}$ is a homeomorphism, but $\tilde{f}^{k+1}_{l,\theta_l^*}$ not. Let $\{J^j_{k+1}\}$ be the subintervals of J_k on which $\tilde{f}^{k+1}_{l,\theta_l^*}$ is a homeomorphism. If J_k does not intersect $\tilde{E}_{l,\theta^*}$, then J_k is the union of two intervals J^1_{k+1}, J^2_{k+1}. If J_k does intersect $\tilde{E}_{l,\theta^*}$, there can be up to $l+1$ intervals J^j_{k+1}.

The boundary points of $\tilde{f}^k_{l,\theta_l^*}(J_k)$ are contained in $\tilde{\mathcal{O}}(c) \cup \tilde{\mathcal{O}}(e)$. Since $\tilde{\mathcal{O}}(c)$ and $\tilde{\mathcal{O}}(e)$ are finite, there is a minimum distance between any two points in $\tilde{\mathcal{O}}(c) \cup \tilde{\mathcal{O}}(e)$, uniformly in l. Thus all intervals $\tilde{f}^k_{l,\theta_l^*}(J_k), \tilde{f}^{k+1}_{l,\theta_l^*}(J^i_{k+1})$ have lengths which are bounded from below uniformly in l.

If $c \in \tilde{f}^k_{l,\theta_l^*}(J_k)$, then both components of $\tilde{f}^k_{l,\theta_l^*}(J_k)\backslash\{c\}$ have length bounded away from zero. Applying the Koebe principle one checks that there is a constant $\tau < 1$ with

$$\frac{\left|J^j_{k+1}\right|}{|J_k|} \leq \tau,$$

for both components J^j_{k+1} with $c \in \tilde{f}^k_{l,\theta_l^*}(J^j_{k+1})$. If J_k intersects $\tilde{E}_{l,\theta^*}$, the images $\tilde{f}^{k+1}_{l,\theta_l^*}(J^j_{k+1})$ contain $\tilde{f}_{l,\theta_l^*}(E^0_{l,\theta_l^*})$. Two neighboring fundamental intervals E^{-k}_{l,θ_l^*}, E^{-k+1}_{l,θ_l^*} are of comparable size, since $f^q_{l,\theta_l^*}(E^{-k}_{l,\theta_l^*}) = E^{-k+1}_{l,\theta_l^*}$. It follows again from Koebe's principle that there is a constant $\tau < 1$ with

$$\frac{\left|J^j_{k+1}\right|}{|J_k|} \leq \tau.$$

Further, since $\tilde{\mathcal{O}}(c)$ and $\tilde{\mathcal{O}}(e)$ are each finite it is clear that $\tilde{f}^{k+n}_{l,\theta_l^*}\Big|_{J^i_{k+1}}$ is not a homeomorphism for some uniformly bounded n. The result follows and it is clear that the constants can be chosen uniformly in l. □

The next proposition discusses expansion properties of $\tilde{f}_{l,\theta_l^*}$.

Proposition 4.8 *For any small enough neighborhood W of c, there are constants $\tilde{C} > 0$ and $\tilde{\lambda} > 1$, so that the following holds for all l sufficiently large.*

If $\tilde{f}^j_{l,\theta_l^*}(x) \notin W$ for $0 \leq j \leq k-1$, then

$$\left|D\tilde{f}^k_{l,\theta_l^*}(x)\right| \geq \tilde{C}\tilde{\lambda}^k.$$

If $\tilde{f}^k_{l,\theta_l^*}(x) \in W$, then

$$\left|D\tilde{f}^k_{l,\theta_l^*}(x)\right| \geq \tilde{C}\tilde{\lambda}^k.$$

Proof. Let W be a neighborhood of c, small enough so that $\tilde{f}^i_{l,\theta_l^*}(c) \cap W = \emptyset$ for $i > 0$.

We start with the first estimate. It suffices to show that there exists M with $|D\tilde{f}^M_{l,\theta_l^*}(x)| > 1$ (compare the proof of Theorem III.3.3 in [MelStr93]). Assume on the contrary that there exist points $x_k \in [0,1]$ with $\tilde{f}^j_{l,\theta_l^*}(x_k) \notin W$, $0 \leq j \leq k-1$, and $|D\tilde{f}^k_{l,\theta_l^*}(x_k)| \leq 1$. Since $\tilde{f}_{l,\theta_l^*}$ has negative

Schwarzian derivative, by the minimum principle Theorem 4.6 one finds that on an interval of monotonicity on one side of x_k, $|D\tilde{f}^k_{l,\theta_l^*}(x)| \le 1$.

Let H_k be the maximal interval bounded by x_k so that $\breve{f}^k_{l,\theta_l^*}(\cdot; H_k)$ is a homeomorphism on H_k and $\left|D\breve{f}^k_{l,\theta_l^*}(\cdot; H_k)\right| \le 1$ on H_k. This implies that $|\breve{f}^k_{l,\theta_l^*}(H_k; H_k)| \le |H_k|$. Let y_k be the other boundary point of H_k. Then either $\breve{f}^{m(k)-1}_{l,\theta_l^*}(y_k; H_k) = c$ or $\breve{f}^{m(k)-1}_{l,\theta_l^*}(y_k; H_k) = e$ for some $m(k) \le k$. In both cases, $|\breve{f}^{m(k)}_{l,\theta_l^*}(H_k; H_k)|$ is bounded away from 0. In the first case this is true by the assumption that the orbit of x does not enter W. In the second case, if $\breve{f}^{m(k)-1}_{l,\theta_l^*}(y_k; H_k) = e$, then $\breve{f}^{m(k)-1}_{l,\theta_l^*}(H_k; H_k)$ contains a fundamental interval E^{-i}_{l,θ_l^*} and so $\breve{f}^{m(k)}_{l,\theta_l^*}(H_k; H_k)$ contains $f_{l,\theta_l^*}(E^0_{l,\theta_l^*})$.

By lemma 4.7, $|H_k| \le K\rho^k$ is small as k is big. Since $|\breve{f}^k_{l,\theta_l^*}(H_k; H_k)| \le |H_k|$, also the interval $\breve{f}^k_{l,\theta_l^*}(H_k; H_k)$ is small for large k. The interval $\breve{f}^{m(k)}_{l,\theta_l^*}(H_k; H_k)$ has length bounded away from 0, say $|\breve{f}^{m(k)}_{l,\theta_l^*}(H_k; H_k)| \ge \delta$. It follows that $k - m(k)$ tends to ∞ as $k \to \infty$. In fact, if D denotes the minimum of $|D\breve{f}_{l,\theta_l^*}(\cdot; \cdot)|$ over $[0,1]\backslash W$, then

$$k - m(k) \quad \ge \quad \ln(K\rho^k/\delta)/\ln D. \tag{16}$$

Lemma 4.7 yields that $|\breve{f}^{m(k)}_{l,\theta_l^*}(H_k; H_k)| \le K\rho^{k-m(k)}$, so that

$$k - m(k) \quad \le \quad \ln(\delta/K)/\ln \rho. \tag{17}$$

Now (16) and (17) contradict each other, proving the claim. Observe that M and hence the constants $\tilde{C}$ and $\tilde{\lambda}$ can be chosen uniformly in l. This proves the first estimate.

To prove the second estimate, suppose $\tilde{f}_{l,\theta_l^*}(x) \in W$. Let H_k be the maximal interval containing x such that $\breve{f}^k_{l,\theta}(\cdot; H_k)$ is a homeomorphism on H_k. Because the orbit of c is finite, the interval $\breve{f}^k_{l,\theta}(H_k; H_k)$ extends a positive distance away from $\breve{f}^k_{l,\theta}(x; H_k)$ to both sides. Koebe's principle implies $|D\breve{f}^j_{\eta_l}(x; H_k)| \ge \tilde{C}\tilde{\lambda}^j$ for some $\tilde{C} > 0, \tilde{\lambda} > 1$ which gives the second estimate. □

Proof of proposition 4.3. The Proposition now follows from Lemma 4.7 and Proposition 4.8 as in the proof of Theorem III.6.4 in [MelStr93]. □

The following proposition implies that bounded recurrence implies exponential expansion along the orbit of $\tilde{f}_{l,\theta}(c)$.

Proposition 4.9 *There exists $\tilde{C} > 0$, $\tilde{\lambda} > 1$ and $\varepsilon_0 > 0$ such that if $|\theta - \theta_l^*| < \varepsilon_0$ and $\tilde{f}_{l,\theta}$ satisfies* (BR)$_k$*, then*

$$\left|D\tilde{f}^k_{l,\theta}(\tilde{c}_1)\right| \quad \ge \quad \tilde{C}\tilde{\lambda}^k.$$

PROOF. Let $0 < \nu_1 < \cdots < \nu_s < k$ be the iterates so that $\tilde{c}_{\nu_j}$ in Δ_δ. Since f_{l,θ_l^*} is a Misiurewicz map, for any given integer N, we get $\nu_1 > N$ by taking θ close enough to θ_l^*, uniformly in l. By the chain rule,

$$\begin{aligned} & D\tilde{f}^k_{l,\theta}(\tilde{c}_1) \\ = \;& D\tilde{f}^{\nu_1-1}_{l,\theta}(\tilde{c}_1)D\tilde{f}_{l,\theta}(\tilde{c}_{\nu_1})\cdots D\tilde{f}^{\nu_s-\nu_{s-1}-1}_{l,\theta}(c_{\nu_{s-1}+1})D\tilde{f}_{l,\theta}(\tilde{c}_{\nu_s})D\tilde{f}^{k-\nu_s}_{l,\theta}(\tilde{c}_{\nu_s+1}). \end{aligned} \tag{18}$$

By Proposition 4.3 and $(\mathrm{BR})_k$ successively,

$$\begin{aligned} |D\tilde{f}^k_{l,\theta}(\tilde{c}_1)| &\geq \tilde{C}^s\tilde{\lambda}^{k-s}\prod_{j=1}^{s}\left|D\tilde{f}_{l,\theta}(\tilde{c}_{\nu_j})\right| \\ &\geq \tilde{C}^s e^{\ln\tilde{\lambda}(k-s)-\alpha k} \end{aligned} \tag{19}$$

Whenever $\tilde{c}_{\nu_j} \in \Delta_\delta$, one has $-\ln|\tilde{c}_{\nu_j} - c| \geq -\ln\delta$. Therefore $(\mathrm{BR})_k$ implies a bound on the number of returns: $-s\ln\delta \leq \sum_{i=1}^{s} -\ln|\tilde{c}_{\nu_i} - c| \leq \alpha k$. Put this into (19). Note that in order to get a positive exponent in (19), α needs to be bounded by a constant depending on $\tilde{\lambda}$. □

4.2 Parameter dependence

Proposition 4.10 *There are constants $D, \varepsilon, K > 0$ so that for all large enough l the following holds. If*

$$|D\tilde{f}^k_{l,\theta}(\tilde{f}_{l,\theta}(c))| \;\geq\; \tilde{C}\tilde{\lambda}^k, \qquad \text{for all } |\theta - \theta_l^*| < \varepsilon \text{ and } k > K,$$

then for all $k > K$,

$$\frac{1}{D} \leq \frac{|\frac{d}{d\theta}\tilde{f}^k_{l,\theta}(c)|}{|D\tilde{f}^{k-1}_{l,\theta}(\tilde{f}_{l,\theta}(c))|} \leq D.$$

PROOF. Let $x(l,\theta)$ be the continuation of $\tilde{f}_{l,\theta_l^*}(c)$ for θ near θ_l^* given by $\tilde{f}^{k-1}_{l,\theta}(x(l,\theta)) = y^*_{l,\theta}$ (where y^*_{l,θ^*} is one of the points in the hyperbolic periodic orbit from the definition of $\theta_l(\gamma_l^*) = \theta_l^*$). Then (6) implies that

$$\frac{1}{C} \leq \left|\frac{d}{d\theta}(x(l,\theta) - \tilde{f}_{l,\theta}(c))\right| \leq C, \tag{20}$$

for some constant $C > 1$. Writing $\tilde{f}^{k-1}_{l,\theta}(\tilde{f}_{l,\theta}(c)) = \tilde{f}^{k-1}_{l,\theta}(x(l,\theta) + \tilde{f}_{l,\theta}(c) - x(l,\theta))$, the chain rule gives

$$\frac{d}{d\theta}\tilde{f}^{k-1}_{l,\theta}(\tilde{f}_{l,\theta}(c))\Big|_{\theta=\theta_l^*} \;=\; \frac{d}{d\theta}\tilde{f}^{k-1}_{l,\theta}(x(l,\theta))\Big|_{\theta=\theta_l^*} + D\tilde{f}^{k-1}_{l,\theta_l^*}(\tilde{f}_{l,\theta_l^*}(c))\frac{d}{d\theta}(\tilde{f}_{l,\theta}(c) - x(l,\theta))\Big|_{\theta=\theta_l^*}.$$

Now $\frac{d}{d\theta}\tilde{f}^{k-1}_{l,\theta}(x_l(l,\theta))\Big|_{\theta=\theta_l^*} = \frac{d}{d\theta}y^*_{l,\theta}$ is arbitrary small for l sufficiently large. By (20) and the exponential growth of $D\tilde{f}^{k-1}_{l,\theta_l^*}(\tilde{f}_{l,\theta_l^*}(c))$, the statement of the proposition holds for $\theta = \theta_l^*$.

By the chain rule,

$$
\begin{aligned}
D\tilde{f}_{l,\theta}^{k-1}(\tilde{f}_{l,\theta}(c)) &= D\tilde{f}_{l,\theta}(\tilde{f}_{l,\theta}^{k-1}(c))D\tilde{f}_{l,\theta}^{k-2}(\tilde{f}_{l,\theta}(c)), \\
\frac{d}{d\theta}\tilde{f}_{l,\theta}^{k}(c) &= D\tilde{f}_{l,\theta}(\tilde{f}_{l,\theta}^{k-1}(c))\frac{d}{d\theta}\tilde{f}_{l,\theta}^{k-1}(c) + \frac{\partial \tilde{f}_{l,\theta}}{\partial\theta}(\tilde{f}_{l,\theta}^{k-1}(c)).
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\left| \frac{\frac{d}{d\theta}\tilde{f}_{l,\theta}^{k}(c)}{D\tilde{f}_{l,\theta}^{k-1}(\tilde{f}_{l,\theta}(c))} - \frac{\frac{d}{d\theta}\tilde{f}_{l,\theta}^{k-1}(c)}{D\tilde{f}_{l,\theta}^{k-2}(\tilde{f}_{l,\theta}(c))} \right| &= \left| \frac{D\tilde{f}_{l,\theta}(\tilde{f}_{l,\theta}^{k-1}(c))\frac{d}{d\theta}\tilde{f}_{l,\theta}^{k-1}(c) + \frac{\partial \tilde{f}_{l,\theta}}{\partial\theta}(\tilde{f}_{l,\theta}^{k-1}(c))}{D\tilde{f}_{l,\theta}(\tilde{f}_{l,\theta}^{k-1}(c))D\tilde{f}_{l,\theta}^{k-2}(\tilde{f}_{l,\theta}(c))} - \frac{\frac{d}{d\theta}\tilde{f}_{l,\theta}^{k-1}(c)}{D\tilde{f}_{l,\theta}^{k-2}(\tilde{f}_{l,\theta}(c))} \right| \\
&= \left| \frac{\frac{\partial \tilde{f}_{l,\theta}}{\partial\theta}(\tilde{f}_{l,\theta}^{k-1}(c))}{D\tilde{f}_{l,\theta}^{k-1}(\tilde{f}_{l,\theta}(c))} \right|. \qquad (21)
\end{aligned}
$$

Note that $|D\tilde{f}_{l,\theta}^{k}(\tilde{f}_{l,\theta}(c))| \geq C\tilde{\lambda}^k$ for $k > K$, by assumption and that $(\partial/\partial\theta)\tilde{f}_{l,\theta}$ is bounded, so the right hand side of (21) is less that $C\tilde{\lambda}^{-k}$ for $k > K$. For each positive integer K, there are constants C, ε so that

$$
\frac{1}{C} \leq \left| \frac{\frac{d}{d\theta}\tilde{f}_{l,\theta}^{K}(c)}{D\tilde{f}_{l,\theta}^{K-1}(\tilde{f}_{l,\theta}(c))} \right| \leq C,
$$

if $|\theta - \theta_l^*| \leq \varepsilon$. Hence, using a telescoping series,

$$
\left| \frac{\frac{d}{d\theta}\tilde{f}_{l,\theta}^{k}(c)}{D\tilde{f}_{l,\theta}^{k-1}(\tilde{f}_{l,\theta}(c))} - \frac{\frac{d}{d\theta}\tilde{f}_{l,\theta}^{K}(c)}{D\tilde{f}_{l,\theta}^{K-1}(\tilde{f}_{l,\theta}(c))} \right| \leq C_K,
$$

for $|\theta - \theta_l^*| \leq \varepsilon$. The constant C_K is small if K is large. □

Proposition 4.11 *Let ω be a parameter interval so that $\theta \mapsto \Pi \circ \breve{F}(c, \theta; \omega)$ is a homeomorphism and there are $\tilde{C} > 0, \tilde{\lambda} > 1$ with $|D\tilde{f}_{l,a}^{k}(\tilde{f}_{l,\theta}(c))| \geq \tilde{C}\tilde{\lambda}^k$ for $\theta \in \omega$ and $0 \leq k \leq n$. Then there is $C > 0$, $\xi \in \omega$ with*

$$
\frac{1}{C}|D\tilde{f}_{l,\xi}^{j-i}(\tilde{c}_i(l,\xi))| \leq \frac{|\omega_j|}{|\omega_i|} \leq C|D\tilde{f}_{l,\xi}^{j-i}(\tilde{c}_i(l,\xi))|,
$$

for all $1 \leq i \leq j \leq n$.

Proof. Consider the map $\varphi(a) = \tilde{c}_j \circ \tilde{c}_i^{-1}(a)$ from ω_i to ω_j. There is $\xi \in \omega$ with $|\omega_j|/|\omega_i| = D\varphi(\xi)$. Proposition 4.10 implies the result. □

Note that the condition of the proposition is satisfied if $\tilde{f}_{l,\theta}$ satisfies $(\mathrm{BR})_n$ for $\theta \in \omega$, by Proposition 4.9.

4.3 Binding

Let Δ^+ be a small symmetric neighborhood of c. In the following section we will specify Δ^+. Given $x \in \Delta^+$, write $\zeta_0 = (c, x)$. Consider iterates ζ_j of the interval ζ_0: $\zeta_j = \breve{f}^j_{l,\theta}(\zeta_0; \zeta_0) = \cup_{y \in \zeta_0} \breve{f}^j_{l,\theta}(y; \zeta_0)$. Define the binding period of x as

$$q_l(x, \theta) \quad = \quad \sup\{m \in \mathbb{N} : |\zeta_j| \leq e^{-2\alpha j} \text{ for all } 0 \leq j \leq m-1\}. \tag{22}$$

Suppose $\tilde{c}_k(l, \theta) \in \Delta^+$ and define the binding period associated with $\tilde{c}_k(l, \theta)$ as

$$p(l, \theta, k) = q_l(\tilde{c}_k(l, \theta), \theta). \tag{23}$$

Assume that $\tilde{f}_{l,\theta}$ satisfies $(\mathrm{BR})_n$. Assume that $\tilde{c}_k(l, \theta) \in \Delta^+$ for some $0 \leq k < n$. Write $\eta_0 = (c, \tilde{c}_k(l, \theta))$ and $\eta_j = \breve{f}^j_{l,\theta}(\eta_0; \eta_0)$. The next lemma implies bounded distortion of iterates of $\breve{f}_{l,\theta}$ on η_0 during the binding period.

Lemma 4.12 *There exists $K > 0$ such that for all $y_1, z_1 \in \eta_1$,*

$$\left| \frac{D\breve{f}^i_{l,\theta}(y_1; \eta_1)}{D\breve{f}^i_{l,\theta}(z_1; \eta_1)} \right| \quad \leq \quad K,$$

for $0 \leq i \leq p(l, \theta, k)$.

Proof. Write $y_j = \breve{f}^{j-1}_{l,\theta}(y_1; \eta_1)$ and $z_j = \breve{f}^{j-1}_{l,\theta}(z_1; \eta_1)$. By the chain rule,

$$\begin{aligned}
\left| \frac{D\breve{f}^i_{l,\theta}(y_1; \eta_1)}{D\breve{f}^i_{l,\theta}(z_1; \eta_1)} \right| &= \left| \prod_{j=1}^{i-1} \frac{D\breve{f}_{l,\theta}(y_j; \eta_1)}{D\breve{f}_{l,\theta}(z_j; \eta_1)} \right| \\
&= \left| \prod_{j=1}^{i-1} \left(1 + \frac{D\breve{f}_{l,\theta}(y_j; \eta_1) - D\breve{f}_{l,\theta}(z_j; \eta_1)}{D\breve{f}_{l,\theta}(z_j; \eta_1)} \right) \right|.
\end{aligned}$$

Split the above product into two parts $j \in J_1 \cup J_2$, where $\eta_j \subset \tilde{E}_{l,\theta}$ for $j \in J_1$ or η_j contains points outside $\tilde{E}_{l,\theta}$ for $j \in J_2$.

If $j \in J_1$, we can write $z_j = \phi^s_{l,\theta}(y_j)$. Then also $\breve{f}_{l,\theta}(z_j; \eta_1) = \phi^s_{l,\theta} \circ \breve{f}_{l,\theta}(y_j; \eta_1)$. Compute

$$D\breve{f}_{l,\theta}(y_j; \eta_1) = D\phi^{-s}_{l,\theta}(\breve{f}_{l,\theta}(y_j; \eta_1)) D\breve{f}_{l,\theta}(z_j; \eta_1) D\phi^s_{l,\theta}(y_j).$$

Now $|D\phi^{-s}_{l,\theta}(\breve{f}_{l,\theta}(y_j; \eta_1)) - 1| \leq Cs \leq C|\eta_{j+1}|$ since $s \leq C|\breve{f}_{l,\theta}(y_j; \eta_1) - \breve{f}_{l,\theta}(z_j; \eta_1)| \leq C|\eta_{j+1}|$. Similarly, $|D\phi^s_{l,\theta}(y_j) - 1| \leq Cs\left(|y_j| + |\eta_j|\right) \leq C|\eta_{j+1}|$. It follows that

$$\begin{aligned}
\left| D\breve{f}_{l,\theta}(y_j; \eta_1) - D\breve{f}_{l,\theta}(z_j; \eta_1) \right| &= \left| D\phi^{-s}_{l,\theta}(\breve{f}_{l,\theta}(y_j; \eta_1)) D\breve{f}_{l,\theta}(z_j; \eta_1) D\phi^s_{l,\theta}(y_j) - D\breve{f}_{l,\theta}(z_j; \eta_1) \right| \\
&\leq C|\eta_{j+1}| \left| D\breve{f}_{l,\theta}(z_j; \eta_1) \right| \qquad (24)
\end{aligned}$$

for some $C > 0$. If $j \in J_2$, then $|D^2 \breve{f}_{l,\theta}(\cdot;\eta_1)|$ is bounded by a constant $C > 0$ and $|D\breve{f}_{l,\theta}(y_j;\eta_1) - D\breve{f}_{l,\theta}(z_j;\eta_1)| \leq C|\eta_j|$. Hence,

$$
\begin{aligned}
\left|\frac{D\breve{f}^i_{l,\theta}(y_1;\eta_1)}{D\breve{f}^i_{l,\theta}(z_1;\eta_1)}\right| &\leq e^{\ln\prod_{j\in J_1}(1+C|\eta_{j+1}|)} e^{\ln\prod_{j\in J_2}\left(1+C\frac{|\eta_j|}{|D\breve{f}_{l,\theta}(z_j;\eta_1)|}\right)} \\
&= e^{\sum_{j\in J_1}\ln(1+C|\eta_{j+1}|)} e^{\sum_{j\in J_2}\ln\left(1+C\frac{|\eta_j|}{|D\breve{f}_{l,\theta}(z_j;\eta_1)|}\right)} \\
&\leq e^{C\sum_{j\in J_1}|\eta_{j+1}|} e^{C\sum_{j\in J_2}\frac{|\eta_j|}{|D\breve{f}_{l,\theta}(z_j;\eta_1)|}}. \qquad (25)
\end{aligned}
$$

We estimate the terms with $j \in J_2$. By the definition of binding period,

$$|\eta_j| \leq e^{-2\alpha j}, \qquad (26)$$

for $0 \leq j \leq p-1$. The bounded recurrence assumption $(\mathrm{BR})_n$ implies

$$|\tilde{c}_k - c| \quad \geq \quad e^{-\alpha k}. \qquad (27)$$

It follows from this and (26) that

$$|z_j - c| \geq e^{-\alpha j} - e^{-2\alpha j} = e^{-\alpha j}(1 - e^{-\alpha j}). \qquad (28)$$

Further $|D\breve{f}_{l,\theta}(z_j;\eta_0)| \geq C|z_j - c|$ for some $C > 0$, so that

$$|D\breve{f}_{l,\theta}(z_j;\eta_1)| \geq Ce^{-\alpha j}, \qquad (29)$$

for some $C > 0$. Combining (26) and (29) shows that $\sum_{j\in J_2}\frac{|\eta_j|}{|D\breve{f}_{l,\theta}(z_j;\eta_1)|}$ is bounded. The terms with $j \in J_1$ are bounded by (26). □

We note here a difference between our proof and Costa's. Costa uses an estimate equivalent to (25) for all points, whereas we treat points that pass close to the saddle-node separately in estimate (24).

Proposition 4.13 *There exist $C, \tilde{C} > 0$ such that if $\tilde{f}_{l,\theta}$ satisfies condition $(\mathrm{BR})_n$ and $\tilde{c}_k \in \Delta^+$ for some $0 \leq k < n$, then $p = p(l,\theta,k)$ satisfies*

$$p \leq -C\ln|\tilde{c}_k - c| \leq C\alpha k$$

and

$$|D\tilde{f}^j_{l,\theta}(\tilde{c}_1)| \quad \geq \quad \tilde{C}\tilde{\lambda}^j,$$

for $0 \leq j < k+p$. Moreover, for α small enough,

$$
\begin{aligned}
|D\tilde{f}^p_{l,\theta}(\tilde{c}_k)| &\geq K|\tilde{c}_k - c|^{C\alpha-1}, \\
|D\tilde{f}^p_{l,\theta}(\tilde{c}_k)| &\geq K\tilde{\lambda}^{p/3},
\end{aligned}
$$

for some $C, K > 0$.

Proof. Write $\hat{p} = \min\{p(l,\theta,k) - 1, n\}$. Since $(\mathrm{BR})_n$ is satisfied, applying Proposition 4.9, $|D\tilde{f}^j_{l,\theta}(\tilde{c}_1)| \geq Ce^{j\ln\tilde{\lambda}}$ for all $j \leq \hat{p}$. By definition of binding, $e^{-2\alpha\hat{p}} \geq |\eta_{\hat{p}}|$. By Lemma 4.12, $|\eta_{\hat{p}}| \geq C|D\check{f}^{\hat{p}-1}_{l,\theta}(\tilde{c}_1;\eta_1)||\eta_1|$. Further, since $|\eta_1| \geq C|\tilde{c}_k - c|^2$ for some $C > 0$, we get

$$e^{-2\alpha\hat{p}} \geq Ce^{(\hat{p}-1)\ln\tilde{\lambda}}|\tilde{c}_k - c|^2, \tag{30}$$

for some $C > 0$. Therefore, $\hat{p} \leq (-2\ln|\tilde{c}_k - c| - \ln C + \ln\tilde{\lambda})/(2\alpha + \ln\tilde{\lambda})$. By $(\mathrm{BR})_k$, we have $\ln|\tilde{c}_k - c| \leq \alpha k$, so that $\hat{p}$ is much smaller than k and therefore $\hat{p} = p - 1$.

By $(\mathrm{BR})_n$, $|D\tilde{f}^j_{l,\theta}(\tilde{c}_1)| \geq Ce^{j\ln\tilde{\lambda}}$ for all $j \leq k$. From the bounded distortion given by Lemma 4.12 it follows that

$$|D\tilde{f}^j_{l,\theta}(\tilde{c}_{k+1})| \geq \tilde{C}e^{j\ln\tilde{\lambda}},$$

for $0 \leq j < p$. From $D\tilde{f}^j_{l,\theta}(\tilde{c}_1) = D\tilde{f}^{j-k}_{l,\theta}(\tilde{c}_{k+1})D\tilde{f}^k_{l,\theta}(\tilde{c}_1)$ if $j \geq k$, the exponential expansion along the orbit of $\tilde{c}_1$, up to time $\max\{n, k+p\}$, follows.

To obtain the expansion during the binding period of the orbit starting at $\tilde{c}_k$, write $|D\tilde{f}^p_{l,\theta}(\tilde{c}_k)| = |D\tilde{f}_{l,\theta}(\tilde{c}_k)||D\tilde{f}^{p-1}_{l,\theta}(\tilde{f}_{l,\theta}(\tilde{c}_k))| \geq C|D\tilde{f}_{l,\theta}(\tilde{c}_k)||\eta_p|/|\eta_1| \geq C|\eta_p|/|\eta_0|$, using the bounded distortion from Lemma 4.12. Since $|\eta_p| \geq e^{-2\alpha p}$ by definition of binding,

$$|D\tilde{f}^p_{l,\theta}(\tilde{c}_k)| \geq C\frac{e^{-2\alpha p}}{|\tilde{c}_k - c|}. \tag{31}$$

By (30), we can write $p \leq -(\ln C + 2\ln|\tilde{c}_k - c|)/(2\alpha + \ln\tilde{\lambda}) \leq -C\ln|\tilde{c}_k - c|$ if $|\tilde{c}_k - c|$ is small, or $|\tilde{c}_k - c|^2 \leq Ce^{-p(\ln\tilde{\lambda}+2\alpha)}$. Putting this in (31), one obtains the two estimates on the growth of $|D\tilde{f}^p_{l,\theta}(\tilde{c}_k)|$ in the statement of the proposition. □

4.4 Induction

We describe the inductive construction of the parameter set Θ_l. For a positive integer r, let $I_r = [c + e^{-r}, c + e^{-r+1})$ and let I_{-r} be the interval on the other side of c with the same image under $\tilde{f}_{l,\theta}$. Let ι be a small positive number. Given $\delta > 0$, write $r_\delta = -\ln\delta$ and $r_{\delta+} = -\iota\ln\delta$. We can suppose that r_δ and $r_{\delta+}$ are integers. Let

$$\Delta = \{c\} \cup \bigcup_{|r|\geq r_\delta+1} I_r,$$
$$\Delta^+ = \{c\} \cup \bigcup_{|r|\geq r_{\delta+}+1} I_r.$$

Note that $\Delta \subset \Delta^+$. Subdividing each interval I_r into r^2 subintervals, $I_{r,m}$, of equal length provides partitions $\mathcal{I}$ of Δ and $\mathcal{I}^+$ of Δ^+.

For each l, let $\mathcal{P}^{(0)}_l$ be the trivial partition $\{(\theta^*_l - \varepsilon, \theta^*_l + \varepsilon)\}$ of the parameter interval $\Theta^{(0)}_l = (\theta^*_l - \varepsilon, \theta^*_l + \varepsilon)$. Inductively we will define parameter sets $\Theta^{(n)}_l$ and partitions $\mathcal{P}^{(n)}_l$ thereof. In order to define $\mathcal{P}^{(n)}_l$ given $\mathcal{P}^{(n-1)}_l$, we first construct a refinement $\hat{\mathcal{P}}^{(n)}_l$ of $\mathcal{P}^{(n-1)}_l$.

Denote

$$\omega_i \quad = \quad \Pi \circ \breve{F}_l^i(c, \omega),$$

where Π is the projection $\Pi(x, \theta) = x$. If ω_k intersects Δ, $0 \leq k \leq n$, then k is called a return time for ω. Recall that the binding period associated to $\tilde{c}_k \in \Delta^+$ is defined in (23). Define the binding period of a parameter interval ω at a time k for which $\omega_k \subset \Delta^+$ as

$$p(\omega, k) = \min_{\substack{\theta \in \omega, \\ \tilde{c}_k(l,\theta) \in \Delta^+}} p(l, \theta, k). \tag{32}$$

We say that a return of ω at time k is a bound return if there is a return time $j < k$ of ω and $k < j + p(\omega, j)$. Let $\omega \in \mathcal{P}_l^{(n-1)}$.

Chopping time. We say that n is a chopping time for ω if

1. ω_n contains at least two elements of $\mathcal{I}^+$, and
2. ω_n is not a bound return for ω.

Non-chopping time. We say that n is a non-chopping time for ω in all other cases, that is if one or more of the following occurs:

1. $\omega_n \cap \Delta^+ = \emptyset$,
2. ω_n is a bound return of ω, or
3. ω_n contains at most one element of the partition $\mathcal{I}^+$ of Δ^+.

In case n is a non-chopping time for ω, we let $\omega \in \hat{\mathcal{P}_l}^{(n)}$. If n is a chopping time for ω we partition ω as follows. Write $\omega_n \cap \Delta^+ = \cup_m \omega_n^m$, so that each ω_n^m fully contains one and at most one element of $\mathcal{I}^+$. If $\omega_n \backslash \Delta^+$ contains an interval of length less then δ^ι, we include this interval in the adjacent interval of $\mathcal{I}^+$. Otherwise an interval of $\omega_n \backslash \Delta^+$ is an element of the partition of ω_n. Write the resulting partition of ω_n as $\omega_n = \cup_m \omega_n^m$. There is a corresponding partition $\cup_m \omega^m$ of ω, given by $\Pi \circ \breve{F}_l^n(c, \omega^m) = \omega_n^m$. Let each element of this partition be an element of $\hat{\mathcal{P}_l}^{(n)}$. As a consequence of the fact that $\tilde{f}_{l,\theta}$ on $\tilde{E}_{l,\theta}$ maps different intervals $E_{l,\theta}^{-i}$ to the same interval, an element of $\hat{\mathcal{P}_l}^{(n)}$ partitioning ω may be a union of several intervals. This happens if $\omega_j \cap \tilde{E}_{l,\theta}$ contains at least two fundamental domains, for some $j < n$.

Let $\omega \in \mathcal{P}_l^{(n-1)}$ and consider $\nu \in \hat{\mathcal{P}_l}^{(n)}|_\omega$. We speak of a bound, essential or inessential return time or an escape time for ν in the following situations.

Bound return time. The interval ν_n intersects Δ and n is a bound return time for ω.

Inessential return time. The interval ν_n intersects Δ and n is a non-chopping time for ω that is not a bound return time.

Essential return time. The interval ν_n intersects Δ and n is a chopping time for ω.

Escape time. The return time n is a chopping time for ω, but ν_n does not intersect Δ. In this case we call ν an escape component of ω.

Any $\nu \in \hat{\mathcal{P}_l}^{(n)}$ belongs to a unique nested sequence of sets

$$\Theta_l^{(0)} = \nu^{(0)} \supset \nu^{(1)} \supset \cdots \supset \nu^{(n-1)} \supset \nu^{(n)} = \nu,$$

where $\nu^{(k)} \in \mathcal{P}_l^{(k)}$ for $0 \leq k < n$. If j is a chopping time for $\nu^{(j-1)}$, then $\nu^{(j)}$ is strictly contained in $\nu^{(j-1)}$. Chopping times are either escape times or essential return times.

The return depth ρ of ν at time k is defined if ν_k intersects Δ, as

$$\rho \quad = \quad \max\{|r|, \;\; \nu_k \cap I_r \neq \emptyset\}.$$

Define the function $\mathcal{R}^{(n)} : \hat{\mathcal{P}}_l^{(n)} \to \mathbb{N}$ which associates to $\nu \in \hat{\mathcal{P}}_l^{(n)}$ the sum of the return depths, over the first n iterates $\Pi \circ \breve{F}_l^i(c,\nu)$. Define the function $\mathcal{E}^{(n)} : \hat{\mathcal{P}}_l^{(n)} \to \mathbb{N}$ which associates to $\nu \in \hat{\mathcal{P}}_l^{(n)}$ the sum of the essential return depths, over the first n iterates $\Pi \circ \breve{F}_l^i(c,\nu)$.

Let

$$\Theta_l^{(n)} = \{\nu \in \hat{\mathcal{P}_l}^{(n)} : \mathcal{E}^{(n)}(\nu) \leq \alpha n/D\}, \tag{33}$$

for some positive constant D which will be fixed later, in the lines following (43). Write

$$\mathcal{P}_l^{(n)} \quad = \quad \hat{\mathcal{P}_l}^{(n)}\Big|_{\Theta_l^{(n)}}. \tag{34}$$

The sets

$$\Theta_l \quad = \quad \bigcap_n \Theta_l^{(n)}$$

will be shown to satisfy the stated properties in Proposition 4.2.

4.5 Bounded recurrence

Proposition 4.14 *Each point in* $\Theta_l^{(n)}$ *satisfies* $(\mathrm{BR})_n$.

PROOF. We start by bounding the sum of the return times for inessential returns following an essential return and before the next chopping time. Then we bound the sum of the return times for bound returns following an essential or inessential return, during its binding period. This will establish that the sum $\mathcal{R}^{(n)}(a)$ of return times of all returns is at most a constant times $\mathcal{E}^{(n)}(a)$, for $a \in \Theta_l^{(n)}$. From this we derive $(\mathrm{BR})_n$.

Let $\omega \in \mathcal{P}^{(\mu_0)}$ for an essential return $\mu_0 < n$. Let $\mu_0 < \mu_1 < \cdots < \mu_u$ be a sequence of inessential returns before the next chopping time. Write $r_0, r_1, \ldots, r_u$ for the corresponding return depths. We will show that

$$\sum_{r=0}^{u} r_i \quad \leq \quad \frac{3}{2} r_0. \tag{35}$$

Let p_i be the length of the binding period for the return at μ_i. By Proposition 4.13, for each fixed $\theta \in \omega$,

$$|D\tilde{f}_{l,\theta}^{p_i}(\tilde{c}_{\mu_i})| \quad \geq \quad K|\tilde{c}_{\mu_i} - c|^{C\alpha - 1}. \tag{36}$$

The right hand side is larger than $Ke^{(1-C\alpha)r_i}$. By Proposition 4.3, $\exists K > 0$ so that

$$|D\tilde{f}_{l,\theta}^{\mu_{i+1}-(\mu_i+p_i)}(\tilde{c}_{\mu_i+p_i})| \quad \geq \quad K. \tag{37}$$

Combining the estimates for $i = 0, \ldots, u$ yields

$$|D\tilde{f}_{l,\theta}^{\mu_u+p_u-\mu_0}(\tilde{c}_{\mu_0})| \quad \geq \quad K^{2u} e^{\sum_{i=0}^{u} r_i(1-C\alpha)}. \tag{38}$$

Proposition 4.11 shows that

$$|\omega_{\mu_u+p_u}| \quad \geq \quad K^{2u}|\omega_{\mu_0}|e^{\sum_{i=0}^{u} r_i(1-C\alpha)}. \tag{39}$$

Since μ_0 is an essential return, $|\omega_{\mu_0}| \geq e^{-r_0}/r_0^2 \geq e^{-r_0(1+\varepsilon)}$ for some ε close to 0. Since $|\omega_{\mu_u+p_u}|$ has bounded length, this shows $C \geq e^{-r_0(1+\varepsilon)+\sum_{i=0}^{u} r_i}$, implying Equation (35).

Next we bound the sum of the return times of bound returns following an essential or inessential return, during its binding period. Suppose $\mu < n$ is a return time, essential or inessential, with return depth ρ. Let $\zeta_1 < \cdots < \zeta_v$ be bound return times between μ and $\mu + p$, where p is the binding period of μ. Write ρ_i for the return depth of ζ_i. We will establish

$$\sum_{i=1}^{v} \rho_i \quad \leq \quad \rho. \tag{40}$$

Take $a \in \omega$ with the deepest sequence of bound returns. Because of binding,

$$|\tilde{c}_{\zeta_i} - \tilde{c}_{\zeta_i-\mu}| \quad \leq \quad e^{-2\alpha(\zeta_i-\mu)}. \tag{41}$$

By $(\mathrm{BR})_{\zeta_i-\mu}$,

$$|\tilde{c}_{\zeta_i-\mu} - c| \quad \geq \quad e^{-\alpha(\zeta_i-\mu)}. \tag{42}$$

It follows that the return depth ρ_i of the bound return at ζ_i is at most one more than the return depth of the return at $\zeta_i - \mu$ (if it is a return). A possible exception is when $\tilde{c}_{\zeta_i}$ falls inside Δ and has return depth r_δ, but $\tilde{c}_{\zeta_i-\mu}$ falls outside Δ and has no return depth defined.

The length p of the binding period of the return at μ is at most $C\rho$ for some $C > 0$, by Proposition 4.13 and $(\mathrm{BR})_\mu$. Therefore, by $(\mathrm{BR})_p$, the sum of those return depths ρ_i with $\rho_i \geq r_\delta + 1$, is at most $C\alpha p \leq C\alpha\rho$, for some $C > 0$. In case $\rho_i = r_\delta$, then for $\theta = \theta_l^*$ it takes at least another Cr_δ iterates before the next return time, since c is mapped to a hyperbolic repelling periodic point. It therefore also takes at least Cr_δ iterates until the next return time, if $|\theta - \theta_l^*|$ is smaller than some constant independent of l. Hence, the sum of return depths ρ_i equal to r_δ is at most $C\rho$, for some $C > 0$.

Combining (35) and (40) shows that

$$\mathcal{R}^{(n)} \leq C\mathcal{E}^{(n)} \tag{43}$$

for some positive constant C. Hence, by (33) with the choice $D = 2C$, $\mathcal{R}^{(n)} \leq \alpha n/2$. Now, if $\nu \leq n$ is a return for ω with return depth r, then $|\tilde{c}_\nu| \geq Ce^{-r} \geq e^{-2r}$ for all points in ω. Therefore

$$\sum_{\nu_j \leq n} -\ln|\tilde{c}_{\nu_j} - c| \leq \sum_{\nu_j \leq n} 2r_j \leq 2\mathcal{R}^{(n)} \leq \alpha n,$$

concluding the proof of Proposition 4.14. □

4.6 Bounded distortion

Proposition 4.15 *Restricted to a connected component of $\omega \in \mathcal{P}_l^{(n)}$, the map $\tilde{c}_k$ is a diffeomorphism with uniformly bounded distortion for all $k \leq \nu + p$ where $\nu < n$ is the last essential or inessential return time of ω and p is the associated binding period. If $n > \nu + p$, then the same statement holds for all $k \leq n$ restricted to any subinterval $\bar{\omega}$ such that $\bar{\omega}_k \subset \Delta^+$.*

Proof. By Proposition 4.10 it suffices to prove a distortion estimate

$$\frac{|D\tilde{f}_{l,a}^k(\tilde{f}_{l,a}(c))|}{|D\tilde{f}_{l,b}^k(\tilde{f}_{l,b}(c))|} \leq D, \tag{44}$$

for parameters $a, b \in \omega$.

Let $0 < \nu_1 < \cdots < \nu_q < k$ be the essential and inessential returns of ω up to time k. By construction a unique element I_{ρ_i,m_i} is associated to ν_i. Write p_i for the length of the binding period following the return ν_i. For notational convenience define ν_0 and p_0 so that $\nu_0 + p_0 + 1 = 0$. Suppose first that $k \leq \nu_q + p_q + 1$. Write

$$\frac{|D\tilde{f}_{l,a}^k(\tilde{c}_1)|}{|D\tilde{f}_{l,b}^k(\tilde{c}_1)|} = \prod_{j=1}^{k-1} \frac{|D\tilde{f}_{l,a}(\tilde{c}_j)|}{|D\tilde{f}_{l,b}(\tilde{c}_j)|} \leq \prod_{j=1}^{k-1} \left(1 + \frac{|D\tilde{f}_{l,a}(\tilde{c}_j) - D\tilde{f}_{l,b}(\tilde{c}_j)|}{|D\tilde{f}_{l,b}(\tilde{c}_j)|}\right).$$

With $D_j = \ln(1 + |D\tilde{f}_{l,a}(\tilde{c}_j) - D\tilde{f}_{l,b}(\tilde{c}_j)|/|D\tilde{f}_{l,b}(\tilde{c}_j)|)$, this yields

$$\frac{|D\tilde{f}_{l,a}^k(\tilde{c}_1)|}{|D\tilde{f}_{l,b}^k(\tilde{c}_1)|} \leq e^{\sum_{i=0}^{q-1} \sum_{j=\nu_i+p_i}^{\nu_{i+1}+p_{i+1}} D_j}. \tag{45}$$

We can further subdivide

$$\sum_{j=\nu_i+p_i}^{\nu_{i+1}+p_{i+1}} D_j = \sum_{j=\nu_i+p_i}^{\nu_{i+1}-1} D_j + D_{\nu_{i+1}} + \sum_{j=\nu_{i+1}+1}^{\nu_{i+1}+p_{i+1}} D_j. \tag{46}$$

Distinguish iterates $j \in J_2$ for which $\tilde{c}_j \notin \tilde{E}_{l,\theta}$ for $\theta \in (a,b)$ and $j \in J_1$ otherwise. If $j \in J_2$, then $|D\tilde{f}_{l,b}(\tilde{c}_j)| \geq C|\tilde{c}_j - c|$ and $|D\tilde{f}_{l,a}(\tilde{c}_j) - D\tilde{f}_{l,b}(\tilde{c}_j)| \leq C|\omega_j|$. Thus

$$D_j \leq C \sup_{\theta\in\omega} |\omega_j|/|\tilde{c}_j(l,\theta) - c| \tag{47}$$

for $j \in J_2$. Suppose $j \in J_1$ and consider

$$|D\tilde{f}_{l,a}(\tilde{c}_j(a)) - D\tilde{f}_{l,b}(\tilde{c}_j(b))| \leq |D\tilde{f}_{l,a}(\tilde{c}_j(a)) - D\tilde{f}_{l,a}(\tilde{c}_j(b))| + |D\tilde{f}_{l,a}(\tilde{c}_j(b)) - D\tilde{f}_{l,b}(\tilde{c}_j(b))|$$

The first term on the right hand side is bounded by $C|\omega_{j+1}||D\tilde{f}_{l,a}(\tilde{c}_j(b))|$, as in the proof of Lemma 4.12. By Lemma 3.9 in [Cos03], $\left|\frac{\partial}{\partial\theta} D\tilde{f}_{l,\theta}(x)\right| \leq C|D\tilde{f}_{l,\theta}(x)|$. Therefore, $|D\tilde{f}_{l,a}(\tilde{c}_j(b)) - D\tilde{f}_{l,b}(\tilde{c}_j(b))| \leq C|D\tilde{f}_{l,b}(\tilde{c}_j(b))|m(\omega)$. Proposition 4.10 implies that $m(\omega) \ll |\omega_{j+1}|$. The above shows that,

$$D_j \leq C|\omega_{j+1}|, \qquad \text{for} \quad j \in J_1. \tag{48}$$

We shall estimate each of the three terms on the right hand side of (46). To estimate the first sum on the right hand side of (46), note that Proposition 4.3 implies $|D\tilde{f}_{l,\theta}^{\nu_{i+1}-j}(\tilde{c}_j(l,\theta))| \geq C\tilde{\lambda}^{(\nu_{i+1}-j)}$, $\theta \in \omega$ and $\nu_i + p_i \leq j < \nu_{i+1}$. By Proposition 4.11, $|\omega_j| \leq C\tilde{\lambda}^{-(\nu_{i+1}-j)}|\omega_{\nu_{i+1}}|$. Moreover, $|\tilde{c}_{\nu_j} - c| \geq C\delta^{\iota}$. Therefore,

$$\sum_{j=\nu_i+p_i}^{\nu_{i+1}-1} D_j \leq C \sum_{j=\nu_i+p_i}^{\nu_{i+1}-1} \tilde{\lambda}^{-(\nu_{i+1}-j)}|\omega_{\nu_{i+1}}|\delta^{-\iota} \leq C|\omega_{\nu_{i+1}}|e^{\rho_{i+1}}. \tag{49}$$

For the second term it is clear from (47) that

$$D_{\nu_{i+1}} \leq C|\omega_{\nu_{i+1}}|e^{\rho_{i+1}}. \tag{50}$$

Finally we estimate the last sum on the right hand side of (46). By Proposition 4.13 and Proposition 4.11,

$$|\omega_j| \leq C|\omega_{\nu_{i+1}+1}| \sup_{a\in\omega} |D\tilde{f}_{l,a}^{j-\nu_{i+1}-1}(\tilde{c}_{\nu_{i+1}+1}(a))|,$$

for $\nu_{i+1} + 1 \leq j \leq \nu_{i+1} + p_{i+1}$. Write $\eta_0 = (c, \tilde{c}_{\nu_{i+1}})$ and $\eta_j = \breve{f}_{l,a}^j(\eta_0; \eta_0)$. Since $|\eta_0| \geq Ce^{-\rho_{i+1}}$, one has $|\eta_1| \geq Ce^{-2\rho_{i+1}}$. By definition of binding, $|\eta_{j-\nu_{i+1}}| \leq e^{-2\alpha(j-\nu_{i+1})}$ for $\nu_{i+1}+1 \leq j < n_{i+1}+p_{i+1}$. Hence,

$$|D\tilde{f}_{l,a}^{j-\nu_{i+1}-1}(\tilde{c}_{\nu_{i+1}+1}(a))| \leq C\frac{|\eta_{j-\nu_{i+1}}|}{|\eta_1|} \leq Ce^{-2\alpha(j-\nu_{i+1})}e^{2\rho_{i+1}}.$$

Since $|D\tilde{f}_{l,a}(\tilde{c}_{\nu_{i+1}})| \leq Ce^{-\rho_{i+1}}$, this shows $\sup_{a\in\omega} |D\tilde{f}_{l,a}^{j-\nu_{i+1}}(\tilde{c}_{\nu_{i+1}}(a))| \leq Ce^{-2\alpha(j-\nu_{i+1})}e^{\rho_{i+1}}$, and thus

$$|\omega_j| \leq C|\omega_{\nu_{i+1}+1}|e^{-2\alpha(j-\nu_{i+1})}e^{\rho_{i+1}}.$$

By $(\mathrm{BR})_n$, $|\tilde{c}_j - c| \geq Ce^{-\alpha j}$ (compare (28) in the proof of Lemma 4.12). This gives

$$\sum_{j=\nu_{i+1}+1}^{\nu_{i+1}+p_{i+1}} D_j \leq C|\omega_{\nu_{i+1}}|e^{\rho_{i+1}}. \tag{51}$$

Combining (49), (50), (51) shows that

$$\sum_{j=\nu_i+p_i+1}^{\nu_{i+1}+p_{i+1}} D_j \leq C|\omega_{\nu_{i+1}}|e^{\rho_{i+1}}. \tag{52}$$

Therefore,

$$\sum_{j=1}^{\nu_q+p_q} D_j \leq C\sum_{i=0}^{q-1} |\omega_{\nu_{i+1}}|e^{\rho_{i+1}}. \tag{53}$$

The proposition (for $k \leq \nu_q + p_q$) follows once we establish

$$\sum_{\{i;\rho_i=r\}} |\omega_{\nu_i}| \quad \leq \quad Ce^{-r}/r^2, \tag{54}$$

for $r \geq r_\delta^\iota$. To show (54), let μ_i, $j = 1, \ldots, m$, be the returns among $\nu_1, \ldots, \nu_q$ with return depths equal to r. By construction, $|\omega_{\mu_m}| \leq Ce^{-r}/r^2$. For any $\mu_i \leq \nu_j < \nu_{j+1} \leq \mu_{i+1}$, one has by Proposition 4.13 and Proposition 4.3 that

$$|D\tilde{f}_{l,a}^{\nu_{j+1}-\nu_j}(\tilde{c}_{\nu_i})| \geq Ce^{(1-C\alpha)\rho_i} \geq e^{(1-C\alpha)\rho_i}. \tag{55}$$

Therefore,

$$|D\tilde{f}_{l,a}^{\mu_{i+1}-\mu_i-1}(\tilde{c}_{\mu_i+1}(a))| \geq C|D\tilde{f}_{l,a}^{p_i}(\tilde{c}_{\mu_i+1}(a))| \geq Ce^{(1-C\alpha)r}.$$

By Proposition 4.11, $|\omega_{\mu_i}| \leq Ce^{-(1-C\alpha)r}|\omega_{\mu_{i+1}}| \leq C\delta^{\iota(1-C\alpha)}|\omega_{\mu_{i+1}}|$ and

$$\sum_{j=1}^{m} |\omega_{\mu_i}| \leq C|\omega_{\mu_m}|.$$

This implies (54) and therefore Proposition 4.15 in case $k \leq \nu_q + p_q$.

If $k > \nu_q + p_q$, we consider the additional terms D_j restricting to $\bar{\omega} \subset \omega$ with $\bar{\omega}_k \subset \Delta^+$. The preceding estimates are not affected. By Proposition 4.3 and Proposition 4.11, $|\bar{\omega}_j| \leq Ce^{-\ln\tilde{\lambda}(k-j)}|\bar{\omega}_k| \leq Ce^{-\ln\tilde{\lambda}(k-j)}\delta^\iota$. Further $|\tilde{c}_j - c| \geq C\delta^\iota$ since $\omega_j \cap \Delta^+ = \emptyset$. Therefore

$$\sum_{j=\nu_q+p_q+1}^{k-1} D_j \leq C.$$

□

4.7 Combinatorial properties

To each $\omega \in \hat{\mathcal{P}}_l^{(n)}$ is associated a sequence $0 = \eta_0 < \eta_1 < \cdots < \eta_s \leq n$, $s = s(\omega)$ of escape times and a corresponding sequence of escape components $\omega \subset \omega^{(\eta_s)} \subset \cdots \subset \omega^{(\eta_0)}$ with $\omega^{(\eta_i)} \in \mathcal{P}_l^{(\eta_i)}$. Let $\omega_i^* = \omega^{(\eta_i)}$ for $1 \leq i \leq s$ and $\omega_i^* = \omega$ for $s+1 \leq i \leq n$. This defines ω_i^* for each $0 \leq i \leq n$. Observe that for $\omega, \nu \in \mathcal{P}_l^{(n)}$ and $0 \leq i \leq n$, the sets ω_i^* and ν_i^* are either disjoint or coincide. Define

$$Q_l^{(i)} = \bigcup_{\omega \in \hat{\mathcal{P}}_l^{(n)}} \omega_i^*$$

and let

$$\mathcal{Q}_l^{(i)} = \{\omega_i^*\}$$

be the natural partition of $Q_l^{(i)}$ into sets of the form ω_i^*. Observe that $Q_l^{(n)} \subset \cdots \subset Q_l^{(0)} = \Theta_l^{(0)}$ and $\mathcal{Q}_l^{(n)} = \hat{\mathcal{P}}_l^{(n)}$. For $\omega \in \mathcal{Q}_l^{(i)}$, $0 \leq i \leq n-1$, let

$$Q_l^{(i+1)}(\omega) = \left\{\omega' \in \mathcal{Q}_l^{(i+1)};\ \omega' \subset \omega\right\}.$$

Denote by $\mathcal{Q}_l^{(i+1)}(\omega)$ the partition $\mathcal{Q}_l^{(i+1)}$ restricted to ω. Define $\mathcal{D}^{(i+1)} : Q_l^{(i+1)}(\omega) \to \mathbb{N}$ by

$$\mathcal{D}^{(i+1)}(\theta) = \mathcal{E}^{(\eta_{i+1})}(\theta) - \mathcal{E}^{(\eta_i)}(\theta) \quad \text{for } 0 \leq i < s,$$

and $\mathcal{D}^{(i+1)} = 0$ for $i \geq s$. For an integer $R > 0$ let

$$Q_l^{(i+1)}(\omega, R) = \left\{\omega' \in Q_l^{(i+1)}(\omega) :\ \mathcal{D}^{(i+1)}|_{\omega'} = R\right\}.$$

Lemma 4.16 *For all $0 \leq i \leq n-1$, $\omega \in \mathcal{Q}_l^{(i)}$ and $R \geq 0$,*

$$\textit{card } \mathcal{Q}_l^{(i+1)}(\omega, R) \leq e^{C\alpha R}.$$

Proof. Let S_R denote the set of pairs of integers $(r_1, m_1), \ldots, (r_t, m_t)$ with $t \geq 1$, $|r_j| \geq r_\delta$, $1 \leq m_j \leq r_j^2$ and $|r_1| + \cdots + |r_t| = R$. For δ small enough,

$$\text{card } S_R \leq e^{C\alpha R}, \tag{56}$$

see [Luz00].

Fix a sequence $(r_1, m_1), \ldots, (r_t, m_t)$ in S_R. By construction, there is a unique set $\bar{\omega} \subset \omega$, $\bar{\omega} \in \mathcal{P}_l^{(\eta_i + \nu_t)}$, so that there is a sequence of essential returns $\eta_i < \nu_1 < \cdots < \nu_t \leq \eta_{i+1}$ with $\bar{\omega}_{\nu_j} \cap I_{r_j, m_j} \neq \emptyset$, $1 \leq j \leq t$. At the escape time η_{i+1}, $\bar{\omega}_{\eta_{i+1}}$ intersects partition elements in $\mathcal{I}^+$ of $\Delta^+ \backslash \Delta$. There are less than $2(r_{\delta^+} - r_\delta) r_\delta^2$ such elements. Therefore there are less than r_δ^3 elements of $\mathcal{Q}^{(i+1)}(\omega, R)$ inside $\bar{\omega}$. Combining this estimate with (56) proves the lemma. □

4.8 Metric estimates

Lemma 4.17 *There exists $C > 0$ with the following property. Let ω be a connected component in $\mathcal{P}_l^{(k)}$, $0 \leq k < n$, with k an essential return time and return depth r. Let $p = p(\omega, k)$ be the binding period of ω at time k. Then*

$$|\omega_{k+p}| \quad \geq \quad |\omega_k|^{C\alpha}.$$

PROOF. If $p(l, \theta, k)$ is constant on ω, then by Proposition 4.13, $|D\tilde{f}^p_{l,\theta}(c_k)| \geq C|c_k - c|^{C\alpha - 1} \geq Ce^{-(C\alpha-1)r}$. Thus by Proposition 4.11, $|\omega_{k+p}| \geq e^{-(C\alpha-1)r}|\omega_k|$. Since k is an essential return, $Ce^{-r}/r^2 \leq |\omega_k|$, implying the claim. If p is not constant on ω, then for all $\theta_1, \theta_2 \in \omega$,

$$|D\tilde{f}^{p-1}_{l,\theta_1}(\tilde{c}_{k+1}(l, \theta_1))| \geq C|D\tilde{f}^{p-1}_{l,\theta_1}(\tilde{c}_1(l, \theta_1))| \geq C|D\tilde{f}^{p-1}_{l,\theta_2}(\tilde{c}_1(l, \theta_1))| \geq C|D\tilde{f}^{p-1}_{l,\theta_2}(\tilde{c}_{k+1}(l, \theta_2))|.$$

Here we used Proposition 4.13, Proposition 4.15 and again Proposition 4.13. □

Lemma 4.18 *Let ω be a subset of $\mathcal{P}_l^{(k)}$ so that ω_k is mapped several to one onto ω_{k+1}. Then $\exists C > 0$ so that $|\omega_k| \leq C|\omega_{k+1}|$.*

PROOF. Let S^{-i} be a fundamental strip as introduced in Section 3.1. Consider the map $\breve{F}_l^k(c, \omega; \omega) \cap S^{-i} \mapsto \breve{F}_l^k(c, \omega; \omega) \cap S^{-j}$ that maps a point to the point with the same image in $\breve{F}_l^{k+1}(c, \omega; \omega)$. By $(\mathrm{BR})_k$, Proposition 4.9 and Proposition 4.10, this map has bounded distortion. The lemma is a consequence, since it is true for each connected component of ω_k. □

Proposition 4.19 *Let $\omega \in \mathcal{Q}^{(i)}$, $0 \leq i \leq n - 1$. For $R \geq 0$, let $\tilde{\omega} \in \mathcal{Q}^{(i+1)}(\omega, R)$. Then*

$$m(\tilde{\omega}) \leq e^{(C\alpha-1)R} m(\omega).$$

PROOF. From the construction of $\tilde{\omega}$ there is a nested sequence of sets

$$\tilde{\omega} \subset \omega^{(\nu_s)} \subset \cdots \subset \omega^{(\nu_1)} \subset \omega = \omega^{(\nu_0)}.$$

Each $\omega^{(\nu_j)}$, $1 \leq j \leq s$, has an essential return at time ν_j with return depth r_j. Write

$$\frac{m(\tilde{\omega})}{m(\omega)} \quad = \quad \frac{m(\omega^{(\nu_1)})}{m(\omega)} \frac{m(\omega^{(\nu_2)})}{m(\omega^{(\nu_1)})} \cdots \frac{m(\omega^{(\nu_s)})}{m(\omega^{(\nu_{s-1})})} \frac{m(\tilde{\omega})}{m(\omega^{(\nu_s)})}. \tag{57}$$

We estimate the terms in the product on the right hand side, where the first and the last term are handled separately. We first show that

$$\frac{m(\omega^{(\nu_{j+1})})}{m(\omega^{(\nu_j)})} \quad \leq \quad Ce^{-r_{j+1} + C\alpha r_j}, \tag{58}$$

for $1 \leq j \leq s-1$. We start to prove the estimate for connected components $\bar{\omega}^{(\nu_{j+1})} \subset \bar{\omega}^{(\nu_j)}$. By Proposition 4.15,

$$\frac{m(\bar{\omega}^{(\nu_{j+1})})}{m(\bar{\omega}^{(\nu_j)})} \leq C\frac{|\bar{\omega}^{(\nu_{j+1})}_{\nu_j+p_j}|}{|\bar{\omega}^{(\nu_j)}_{\nu_j+p_j}|}.$$

To estimate the numerator of the right hand side, note that $\bar{\omega}^{(\nu_{j+1})}_{\nu_{j+1}} \subset \Delta^+$. Write $|D\tilde{f}^{\nu_{j+1}-\nu_j}_{l,\theta}(\tilde{c}_{\nu_j})| = |D\tilde{f}^{p_j}_{l,\theta}(\tilde{c}_{\nu_j})||D\tilde{f}^{\nu_{j+1}-\nu_j-p_j}_{l,\theta}(\tilde{c}_{\nu_j+p_j})|$. The last term is bounded from below by the same reasoning as for (55). Therefore, by Proposition 4.13, $|D\tilde{f}^{\nu_{j+1}-\nu_j}_{l,\theta}(\tilde{c}_{\nu_j})| \geq e^{(1-C\alpha)r_j}$. By Proposition 4.11, $|\bar{\omega}^{(\nu_{j+1})}_{\nu_{j+1}}| \geq C|\bar{\omega}^{(\nu_{j+1})}_{\nu_j+p_j}|$ and thus

$$|\bar{\omega}^{(\nu_{j+1})}_{\nu_j+p_j}| \leq C|\bar{\omega}^{(\nu_{j+1})}_{\nu_{j+1}}| \leq Ce^{-r_{j+1}}. \tag{59}$$

The denominator is estimated using Lemma 4.17:

$$|\bar{\omega}^{(\nu_j)}_{\nu_j+p_j}| \geq |\bar{\omega}^{(\nu_j)}_{\nu_j}|^{C\alpha} \geq (e^{-r_j}/r_j^2)^{C\alpha}.$$

The bound (58) follows for ω replaced by $\bar{\omega}$. If there is more than one connected component of $\omega^{(\nu_{j+1})}$ contained in $\bar{\omega}^{(\nu_j)}$, then there is an iterate $\nu_j < s \leq \nu_{j+1}$ so that $\breve{F}^s_l(c, \bar{\omega}^{(\nu_j)}; \bar{\omega}^{(\nu_j)})$ crosses two fundamental strips. Note that $\nu_{j+1}-s$ is uniformly bounded since $\bar{\omega}^{(\nu_j)}_{s+1}$ has size bounded away from 0 and will hit c in a bounded number of iterates. Reasoning as for (59), applying Lemma 4.18 for the second inequality,

$$|\omega^{(\nu_{j+1})}_{\nu_j+p_j} \cap \bar{\omega}^{(\nu_j)}_{\nu_j+p_j}| \leq C|\omega^{(\nu_{j+1})}_{s} \cap \bar{\omega}^{(\nu_j)}_{s}| \leq C|\omega^{(\nu_{j+1})}_{s+1} \cap \bar{\omega}^{(\nu_j)}_{s+1}| \leq C|\omega^{(\nu_{j+1})}_{\nu_{j+1}} \cap \bar{\omega}^{(\nu_j)}_{\nu_{j+1}}| \leq Ce^{-r_{j+1}}.$$

Thus (58) holds with $\omega^{(\nu_{j+1})} \cap \bar{\omega}^{(\nu_j)}$ and $\bar{\omega}^{(\nu_j)}$ replacing $\omega^{(\nu_{j+1})}$ and $\omega^{(\nu_j)}$, respectively. Putting together the connected components of $\omega^{(\nu_j)}$ proves (58).

Next we estimate the first term in the product on the right hand side in (57). Again we first restrict to connected components $\bar{\omega}^{(\nu_1)} \subset \bar{\omega}$. Suppose first $\bar{\omega}_{\nu_1} \subset \Delta^+$. Applying Proposition 4.15,

$$\frac{m(\bar{\omega}^{(\nu_1)})}{m(\bar{\omega})} \leq C\frac{|\bar{\omega}^{(\nu_1)}_{\nu_1}|}{|\bar{\omega}_{\nu_1}|} \leq Ce^{-r_1}/|\bar{\omega}_{\nu_1}|.$$

We will estimate $|\bar{\omega}_{\nu_1}|$. To do this, first note that $\bar{\omega}_{\nu_0}$ either does or does not intersect Δ^+. Suppose $\bar{\omega}_{\nu_0} \cap \Delta^+ \neq \emptyset$. Then $I_{r,m} \subset \bar{\omega}_{\nu_0} \subset \hat{I}_{r,m}$ for some $r_{\delta^+} \leq |r| \leq r_\delta$. Here $\hat{I}_{r,m}$ is the union of $I_{r,m}$ with the two adjacent elements. Reasoning as above shows that $|\bar{\omega}_{\nu_1}| \geq C|\bar{\omega}_{\nu_0+p_0}|$. Applying Lemma 4.17,

$$|\bar{\omega}_{\nu_1}| \geq C|\bar{\omega}_{\nu_0+p_0}| \geq C|\bar{\omega}_{\nu_0}|^{C\alpha} \geq C(e^{-r}/r^2)^{C\alpha} \geq Ce^{-C\alpha r_\delta}.$$

Suppose $\bar{\omega}_{\nu_0} \cap \Delta^+ = \emptyset$. By construction, $|\bar{\omega}_{\nu_0}| \geq C\delta^\iota$. By Proposition 4.3,

$$|\bar{\omega}_{\nu_1}| \geq C|\bar{\omega}_{\nu_0+1}| \geq C\delta^{2\iota} \geq \delta^{C\alpha} \geq e^{-C\alpha r_\delta},$$

as long as ι is small compared to α. Suppose next that $\bar{\omega}_{\nu_1}$ is not completely contained in Δ^+. Restrict to an interval $\hat{\omega} \subset \bar{\omega}$ so that $\hat{\omega}_{\nu_1} \subset \Delta^+$. Now

$$\frac{m(\bar{\omega}^{(\nu_1)})}{m(\bar{\omega})} \leq \frac{m(\bar{\omega}^{(\nu_1)})}{m(\hat{\omega})} \leq C\frac{|\bar{\omega}_{\nu_1}^{(\nu_1)}|}{|\hat{\omega}_{\nu_1}|} \leq Ce^{-r_1}/|\hat{\omega}_{\nu_1}| \leq Ce^{-r_1}e^{C\alpha r_\delta},$$

where the last step holds since $|\hat{\omega}_{\nu_1}| \geq C\delta^\iota \geq Ce^{-C\alpha r_\delta}$. This is true because $\hat{\omega}_{\nu_1}$ contains a connected component of $\Delta^+\backslash\Delta$ by the assumption that $\bar{\omega}_{\nu_1}$ is not completely contained in Δ^+. Concluding, the first term on the right hand side of (57) satisfies $\frac{m(\bar{\omega}^{(\nu_1)})}{m(\bar{\omega})} \leq Ce^{-r_1+C\alpha r_\delta}$. As before one concludes that

$$\frac{m(\omega^{(\nu_1)})}{m(\omega)} \quad \leq \quad Ce^{-r_1+C\alpha r_\delta}. \tag{60}$$

Finally, for the last term in the product in (57),

$$\frac{m(\tilde{\omega})}{m(\omega^{(\nu_s)})} \quad \leq \quad 1, \tag{61}$$

since $\tilde{\omega} \subset \omega^{(\nu_s)}$.

Putting (58), (60) and (61) into (57) shows

$$\frac{m(\tilde{\omega})}{m(\omega)} \leq C^s e^{-\sum_{j=1}^s r_j + C\alpha\sum_{j=1}^{s-1} r_j} e^{C\alpha r_\delta} \leq e^{(C\alpha-1)R} e^{-C\alpha r_s + C\alpha r_\delta}.$$

The result follows since $r_s \geq r_\delta$. □

4.9 Measure bounds

Proposition 4.20 *There exists $C > 0$ such that*

$$\sum_{\omega' \in \mathcal{Q}_l^{(i+1)}(\omega,R)} m(\omega') \quad \leq \quad e^{(C\alpha-1)R} m(\omega).$$

Proof. This follows by combining Proposition 4.19 and Lemma 4.16. □

Proposition 4.21

$$\int_{\Theta_l^{(n-1)}} e^{\mathcal{E}^{(n)}/2} = \sum_{\omega \in \mathcal{Q}_l^{(n)}} e^{\mathcal{E}^{(n)}/2} m(\omega) \leq e^{3n/r_\delta} m(\Theta_l^{(0)}).$$

Proof. The equality follows immediately from the definitions. For the inequality, let $0 \leq i < n$, $\omega \in \mathcal{Q}_l^{(i)}$ and write

$$\sum_{\omega' \in \mathcal{Q}_l^{(i+1)}(\omega)} e^{\mathcal{D}_l^{(i)}(\omega')/2} m(\omega') \quad = \quad \sum_{\omega' \in \mathcal{Q}_l^{(i+1)}(\omega,0)} m(\omega') + \sum_{R \geq r_\delta} e^{R/2} \sum_{\omega' \in \mathcal{Q}_l^{(i+1)}(\omega,R)} m(\omega').$$

Proposition 4.20 implies

$$\sum_{\omega'\in\mathcal{Q}_l^{(i+1)}(\omega)} e^{\mathcal{D}_l^{(i)}(\omega')/2}m(\omega') \le \sum_{R\ge r_\delta} e^{R/2} \sum_{\omega'\in\mathcal{Q}_l^{(i+1)}(\omega,R)} m(\omega') \le \left(1+\sum_{R\ge r_\delta} e^{(C\alpha-\frac{1}{2})R}\right) m(\omega),$$

so that

$$\begin{aligned}\sum_{\omega'\in\mathcal{Q}_l^{(i+1)}(\omega)} e^{\mathcal{D}_l^{(i)}(\omega')/2}m(\omega') &\le (1+e^{-r_\delta/3})m(\omega)\\ &\le e^{3/r_\delta}m(\omega),\end{aligned} \tag{62}$$

assuming that α has been chosen small enough and r_δ large enough. Since $\mathcal{E}^{(n)} = \mathcal{D}_l^{(0)}+\cdots+\mathcal{D}_l^{(n-1)}$ and $\mathcal{D}_l^{(i)}$ is constant on elements of $\mathcal{Q}_l^{(i)}$, we have

$$\sum_{\omega\in\mathcal{Q}_l^{(n)}} e^{\mathcal{E}^{(n)}(\omega)/2}m(\omega) =$$

$$\sum_{\omega_1^*\in\mathcal{Q}_l^{(1)}(\omega_2^*)} e^{\mathcal{D}_l^{(0)}(\omega_1^*)/2}\cdots \sum_{\omega_{n-1}^*\in\mathcal{Q}_l^{(n-1)}(\omega_n^*)} e^{\mathcal{D}_l^{(n-2)}(\omega_{n-1}^*)/2} \sum_{\omega=\omega_n^*\in\mathcal{Q}_l^{(n)}} e^{\mathcal{D}_l^{(n-1)}(\omega_n^*)/2}m(\omega).$$

Applying (62) repeatedly gives

$$\sum_{\omega\in\mathcal{Q}_l^{(n)}} e^{\mathcal{E}^{(n)}(\omega)/2}m(\omega) \le e^{3n/r_\delta}m(\Theta_l^{(0)}).$$

□

Observe

$$m(\Theta_l^{(n-1)}\backslash\Theta_l^{(n)}) = m(\{\omega\in\mathcal{Q}_l^{(n)};\ \ e^{\mathcal{E}^{(n)}/2}\ge e^{\alpha n/20}\}).$$

Chebyshev's inequality and Proposition 4.21 yield

$$\begin{aligned}m(\Theta_l^{(n-1)}\backslash\Theta_l^{(n)}) &\le e^{-\alpha n/20}\int_{\Theta_l^{(n-1)}} e^{\mathcal{E}^{(n)}/2}\\ &\le e^{\left(\frac{3}{r_\delta}-\frac{\alpha}{20}\right)n}m(\Theta_l^{(0)})\\ &\le e^{-\alpha n/30}m(\Theta_l^{(0)}),\end{aligned}$$

if r_δ is large enough. This implies

$$m(\Theta_l^{(n)}) \ge m(\Theta_l^{(n-1)}) - e^{-\alpha n/30}m(\Theta_l^{(0)}).$$

Write $\Theta_l^{(0)} = (\theta_l^*-\varepsilon,\theta_l^*+\varepsilon)$. For ε small, there exists N so that $\Theta_l^{(j)} = \Theta_l^{(j+1)}$ for all $j\le N$. Hence

$$m(\Theta_l^{(n)}) \ge \left(1-\sum_{i=N}^{n} e^{-\alpha i/30}\right) m(\Theta_l^{(0)}).$$

Since N goes to ∞ as $\sigma\to 0$, it follows that Θ_l has positive measure, moreover, that a uniform lower bound for $m(\Theta_l)$ exists. This concludes the proof of Proposition 4.2.

4.10 Proof of Theorem A

Combining Proposition 4.2 with Proposition 2.3, implies that $\Omega = \cup_l g_l(\Theta_l)$ has positive measure and has positive density at $\gamma = 0$. Proposition 4.9 shows that bounded recurrence implies exponential expansion along the orbit of $\tilde{c}_1$; $|D\tilde{f}^k_{l,\theta}(\tilde{c}_1)| \geq \tilde{C}\tilde{\lambda}^k$ for some $\tilde{C} > 0, \tilde{\lambda} > 1$, for $\theta \in \Theta_l$. Together with Lemma 3.1 this gives

Proposition 4.22 *For each $\gamma \in \Omega$, there are $C > 0$, $\lambda > 1$, so that*

$$\left| Df^k_\gamma(f_\gamma(c)) \right| \geq C\lambda^k.$$

Thus f_γ is a Collet-Eckmann map if $\gamma \in \Omega$. Collet-Eckmann maps are known to admit absolutely continuous invariant measures, see Theorem V.4.6 in [MelStr93]. This concludes the proof of Theorem A, except for the conclusion that $\operatorname{supp}(\nu_\gamma) = [f^2_\gamma(c), f_\gamma(c)]$.

Lemma 4.23 $\operatorname{supp}(\nu_\gamma) = [f^2_\gamma(c), f_\gamma(c)]$.

Proof. It suffices to show that for any interval $I \subset [0,1]$, there is an $N \in \mathbb{N}$ with $[f^2_\gamma(c), f_\gamma(c)] \subset f^N_\gamma(I)$. Note that $\breve{f}^j_\gamma(I;I)$ will intersect Δ for some j, by Proposition 4.3. The interval Δ is divided into intervals J_k on which the binding period $q_l(x,\theta)$ is constant. If $\breve{f}^j_\gamma(I;I)$ does not contain an interval J_k, keep iterating until the next intersection with Δ. By Proposition 4.13 and Proposition 4.3, the next intersection following the binding period is a larger interval. Therefore, there is an iterate j_1 so that $\breve{f}^{j_1}_\gamma(I;I) \supset I_{k_{j_1}}$. Continuing to iterate, this gives a sequence of iterates j_i with $\breve{f}^{j_i}_\gamma(I;I) \supset I_{k_{j_i}}$. Since each of these intersections is larger than the previous, $\breve{f}^m_\gamma(I;I) \supset I_1$ for some m. By Lemma 2.6, we may assume that the repelling fixed point is contained in an iterate $\breve{f}^k_0(I_1;I_1)$. Since this is an open condition, the repelling fixed point is in $\breve{f}^{m+k}_\gamma(I;I)$ for all $\gamma \in \Omega$ near 0. Clearly, $[f^2_\gamma(c), f_\gamma(c)] \subset \breve{f}^n_\gamma(I;I)$ for some $n \geq m+k$. □

Note that we do not need (as in [Cos03]) to exclude parameter values for which the orbit of the critical point comes too near to the saddle-node point. This is due to our use of different arguments in the distortion estimates in Lemma 4.12 and Proposition 4.15. We treat orbits close to the saddle-node point separately, whereas Costa treats all orbits together. This leads to her need to exclude some parameter values in her condition $(CP3)_n$.

5 Intermittency

In this section we study intermittent time series of f_γ at parameter values γ for which f_γ admits an absolutely continuous invariant measure. We conclude the proof of Theorem B in Section 5.3.

We start with a closer look at invariant measures for $\tilde{f}_\gamma$ and f_γ. In Section 4 we constructed a set Ω of parameter values with positive density at $\gamma = 0$, so that f_γ has bounded recurrence (see Definition 4.1) for $\gamma \in \Omega$. Because we need bounds on the density of the invariant measures in our discussion of intermittency, we give an alternative way to produce invariant measures following [You92].

5.1 Construction of invariant measures

Proposition 5.1 *For $\gamma \in \Omega$, $\tilde{f}_\gamma$ possesses an absolutely continuous invariant measure $\tilde{\nu}_\gamma$. There is a constant $K > 0$ not depending on γ so that for any Borel set $A \subset [0,1]$,*

$$\tilde{\nu}_\gamma(A) \quad \leq \quad K\sqrt{|A|}.$$

Proof. By Proposition 4.13, there is a function $\tilde{p}(x)$ defined near c with

(i) $\tilde{p}(x) \leq -\tilde{C} \ln |x - c|$ for some $\tilde{C} > 0$ not depending on γ,

(ii) $|D\tilde{f}_\gamma^j(f_\gamma(x))| \geq \tilde{C}\tilde{\lambda}^j$ for some $\tilde{C} > 0$, $\tilde{\lambda} > 1$ and for all $0 \leq j < \tilde{p}(x)$,

(iii) $|D\tilde{f}_\gamma^{\tilde{p}(x)}(x)| \geq \sigma^{\tilde{p}(x)}$ for some $\sigma > 1$.

Outside the domain of definition Δ^+ of $\tilde{p}$, let $\tilde{p} = 1$. Define the induced map R_γ on $[0,1]$ by

$$R_\gamma(x) \quad = \quad \tilde{f}_\gamma^{\tilde{p}(x)}.$$

For any $K > 0$ and all $\gamma \in \Omega$ we may assume that $\tilde{p}|_\Delta$ is bounded from below by K, by taking Δ small enough. By Proposition 4.3, if $x, \ldots, \tilde{f}_\gamma^{m-1}(x) \notin \Delta$ and $\tilde{f}_\gamma^m(x) \in \Delta$, then $|D\tilde{f}_\gamma^m(x)| \geq \tilde{C}\tilde{\lambda}^m$ for some $\tilde{C} > 0, \tilde{\lambda} > 1$. As stated in Proposition 4.3, the constant $\tilde{C}$ does not depend on Δ. By taking Δ small, $\tilde{C}\tilde{\lambda}^K > 1$. It follows that some power of R_γ is expanding.

Following [Ryc83], the measure $\tilde{\nu}_\gamma$ is constructed by finding its density as a fixed point of a Perron-Frobenius operator P_γ, defined by

$$P_\gamma \phi(x) = \sum_{y \in R_\gamma^{-1}(x)} \frac{1}{|DR_\gamma(x)|} \phi(y).$$

Note that

$$P_\gamma^n \phi(x) = \sum_{y \in R_\gamma^{-n}(x)} \frac{1}{|DR_\gamma^n(x)|} \phi(y).$$

Let $g_n : [0,1] \to \mathbb{R}$ be the function $g_n(x) = 1/|DR_\gamma^n(x)|$ where R_γ^n is continuous and $g_n(x) = 0$ elsewhere. We claim that the variation $V(g_1)$ of g_1 is bounded. The variation $V\left(g_1|_{\Delta^+}\right)$ is bounded as in [You92]. To bound $V\left(g_1\Big|_{\tilde{E}_{l,\theta}}\right)$, note that by Proposition 2.1 the map $\tilde{f}_{l,\theta} : E_{l,\theta}^{-i} \to f_{l,\theta}^q(E_{l,\theta}^0)$

has bounded distortion. For $x_i \in E^{-i}_{l,\theta}$, $|D\tilde{f}_{l,\theta}(x_i)| \geq C|E^{-i}_{l,\theta}|/|f^q_{l,\theta}(E^0_{l,\theta})|$ for some $C > 0$. Therefore, for any $x_i \in E^{-i}_{l,\theta}$,

$$\sum_{i=0}^{l-1} \frac{1}{|D\tilde{f}_{l,\theta}(x_i)|} \leq C,$$

for some $C > 0$. As a consequence, $V\left(g_1\Big|_{\tilde{E}_{l,\theta}}\right)$ is bounded. Since the variation of g_1 outside $\Delta^+ \cup \tilde{E}_{l,\theta}$ is also bounded, the claim follows. By induction the variation of g_n is bounded, see [Ryc83]. Let N be so that $\sup|g_N| < 1/2$. Next, choose a finite partition $\mathcal{Q}$ of $[0,1]$ consisting of closed intervals, so that $A_N = \sup|g_N| + \max V\left(g_N|_{K\in\mathcal{Q}}\right) < 1$. This is possible since for any $\varepsilon > 0$ and for any $x \in [0,1]$, one can take a neighborhood U_x of x so that $V\left(g_N|_{U_x}\right) < 2\sup|g_N| + \varepsilon$. From the set of such neighborhoods, one takes a finite subcover of $[0,1]$. Then

$$V\left(P^N_\gamma \phi\right) \leq A_N V(\phi) + C,$$

for some $C > 0$, see [Ryc83]. It follows from this construction that P_γ has a fixed point with uniformly bounded variation.

If ζ_γ denotes the measure whose density is the fixed point of P_γ, then $\tilde{\nu}_\gamma$ is obtained by pushing forward ζ_γ,

$$\tilde{\nu}_\gamma(A) = \sum_{k=1}^{\infty}\sum_{j=0}^{k-1} \zeta_\gamma(\tilde{f}^{-j}_\gamma(A) \cap B_k),$$

where B_k is the set on which $R_\gamma = \tilde{f}^k_\gamma$. By property (i) above, B_k is exponentially small in k ($|B_k| \leq Ce^{-k/C}$ for some $C > 0$). This shows that $\tilde{\nu}_\gamma$ is a finite measure. The uniform bound for $\tilde{\nu}_\gamma$ follows from the properties of R_γ as in [You92]. Indeed, $\sum_{k=2}^{\infty}\zeta_\gamma(\tilde{f}^{-1}(A)\cap B_k) = \zeta_\gamma(\tilde{f}^{-1}(A) \cap \cup_{k\geq 2}B_k) \leq C\sqrt{|A|}$ for some $C > 0$. Property (ii) above gives for $l > 2$, $\sum_{k=l}^{\infty}\zeta_\gamma(\tilde{f}^{-(l-1)}(A)\cap B_k) = \zeta_\gamma(\tilde{f}^{-(l-1)}(A) \cap \cup_{k\geq l}B_k) \leq C\tilde{\lambda}^{-(l-1)}\sqrt{|A|}$ for some $C > 0$. Taking a union over $l \geq 2$, this implies the given uniform bound for $\tilde{\nu}_\gamma$. □

The invariant measure ν_γ for f_γ is constructed by pushing forward $\tilde{\nu}_\gamma$. Recall that $\tilde{E}_\gamma = \cup_{i=0}^{l-1}E^{-i}$ with $E^{-i} = (f_\gamma|_E)^{-iq}(I^u_\gamma)$. Let

$$\bar{\nu}_\gamma(A) \quad = \quad \tilde{\nu}_\gamma(A \cap ([0,1]\backslash\tilde{E}_\gamma)) + \sum_{k=0}^{l-1}\sum_{i=0}^{kq-1} \tilde{\nu}_\gamma(f^{-i}_\gamma(A) \cap E^{-k}).$$

This measure is obviously finite and can thus be rescaled to a probability measure ν_γ.

Lemma 5.2 *The measure ν_γ, $\gamma \in \Omega$, is an absolutely continuous invariant probability measure for f_γ.*

Proof. To prove that ν_γ is invariant for f_γ, recall that $\tilde{f}_\gamma|_{E^{-j}} = f_\gamma^{jq}$ and $\tilde{f}_\gamma$ equals f_γ outside $\tilde{E}_\gamma$. Compute

$$\begin{aligned}
\bar{\nu}_\gamma(f_\gamma^{-1}(A)) &= \tilde{\nu}_\gamma(f_\gamma^{-1}(A) \cap ([0,1]\backslash \tilde{E}_\gamma)) + \sum_{k=0}^{l-1} \sum_{i=1}^{kq-1} \tilde{\nu}_\gamma(f_\gamma^{-i}(A) \cap E^{-k}) + \\
&\quad \sum_{k=0}^{l-1} \tilde{\nu}_\gamma(f_\gamma^{-kq}(A) \cap E^{-k}) \\
&= \tilde{\nu}_\gamma(\tilde{f}_\gamma^{-1}(A) \cap ([0,1]\backslash \tilde{E}_\gamma)) + \sum_{k=0}^{l-1} \sum_{i=1}^{kq-1} \tilde{\nu}_\gamma(f_\gamma^{-i}(A) \cap E^{-k}) + \\
&\quad \sum_{k=0}^{l-1} \tilde{\nu}_\gamma(\tilde{f}_\gamma^{-1}(A) \cap E^{-k}). \qquad (63)
\end{aligned}$$

By $\tilde{f}_\gamma$ invariance of $\tilde{\nu}_\gamma$, that is $\tilde{\nu}_\gamma(\tilde{f}_\gamma^{-1}(A)) = \tilde{\nu}_\gamma(A)$, the union of $\tilde{\nu}_\gamma(\tilde{f}_\gamma^{-1}(A) \cap ([0,1]\backslash \tilde{E}_\gamma))$ and $\sum_{k=0}^{l-1} \tilde{\nu}_\gamma(\tilde{f}_\gamma^{-1}(A) \cap E^{-k})$ equals the union of $\tilde{\nu}_\gamma(A \cap ([0,1]\backslash \tilde{E}_\gamma))$ and $\sum_{k=0}^{l-1} \tilde{\nu}_\gamma(A \cap E^{-k})$. The right hand side of (63) therefore is $\bar{\nu}_\gamma(A)$, showing f_γ-invariance of $\bar{\nu}_\gamma$ and thus of ν_γ. □

5.2 Laminar phase and relaminarization

Paraphrasing, the following proposition shows that a typical (with respect to the invariant measure ν_γ) point $x \in [0,1]$ needs a bounded number of iterates to enter $\tilde{E}_\gamma$. With Proposition 5.4 below it is used to provide estimates for the average length of laminar and relaminarization phases. This is the main ingredient for the proof of Theorem B in Section 5.3.

For a set $A \subset [0,1]$ and a point $x \in [0,1]$, let

$$L_A(x) = \min\{k \in \mathbb{N};\ \ f_\gamma^k(x) \in A\}.$$

Proposition 5.3 *There is $L > 0$ so that for all $\gamma \in \Omega$,*

$$\int_{[0,1]\backslash \tilde{E}_\gamma} L_{\tilde{E}_\gamma} d\nu_\gamma / \nu_\gamma([0,1]\backslash \tilde{E}_\gamma) \ \ \leq \ \ L.$$

Proof. We claim that the denominator $\nu_\gamma([0,1]\backslash \tilde{E}_\gamma)$ in the estimate is bounded away from 0, uniformly in γ. Recall that ν_γ equals $\tilde{\nu}_\gamma$ outside $\tilde{E}_\gamma$. To establish the claim, note that $\tilde{f}_\gamma^{-1}([0,1]\backslash \tilde{E}_\gamma)$ and $[0,1]\backslash \tilde{E}_\gamma$ together cover $[0,1]$. By $\tilde{f}_\gamma$-invariance of $\tilde{\nu}_\gamma$, the measures of both sets are the same, and, since they add up to at least 1, are bounded from below by $\frac{1}{2}$.

Since ν_γ equals $\tilde{\nu}_\gamma$ outside $\tilde{E}_\gamma$, applying Proposition 5.1, we get a uniform bound of the form

$$\nu_\gamma(A) \ \ \leq \ \ K\sqrt{|A|} \qquad (64)$$

for Borel sets $A \subset [0,1]\backslash \tilde{E}_\gamma$.

At $\gamma = 0$, f_0 is renormalizable: there is an interval N containing c in its interior and a in its boundary, so that $f_0^q(N) \subset N$ and $f_0^q(\partial N) \subset \partial N$. By slightly extending N, we obtain an interval V containing $\tilde{E}_\gamma$ for small values of γ. We may take V symmetric in the sense that both boundary points of V have the same image under f_γ. Let $\bar{V}$ be the union of the intervals in $f_\gamma^{-i}(V)$, $0 \leq i < q$, that are near $f_\gamma^{q-i}(N)$. Consider the convex hull of $\bar{V}$ with $\bar{V}$ excluded. This is a finite union of intervals. Denote by I the interior of these intervals. Let $h_\gamma : I \to [0, 1]$ denote the restriction of

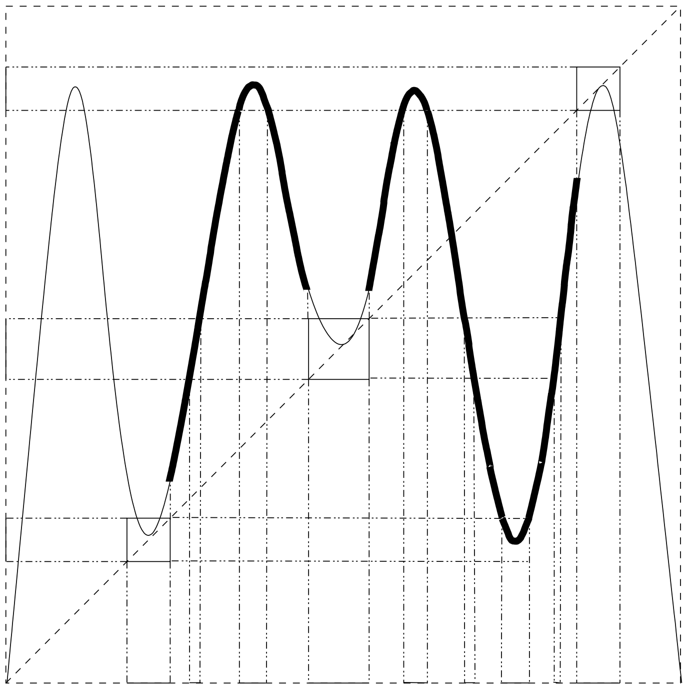

Figure 4: This figure illustrates the proof of Proposition 5.3. It shows the third iterate of a unimodal map at or near a saddle-node bifurcation. Projected on the left side of the box are the intervals that make up $\bar{V}$, I is the union of the intervals in between. The thick branches form the graph of h_γ defined on I. Projected on the bottom one finds $\bar{V}$ with the escape set O.

f_γ^q to I, see Figure 4. Replacing, if necessary, f_γ^q by a power f_γ^{rq} we may assume $|h_\gamma'| \geq \nu > 1$ on $I \backslash h_\gamma^{-1}(I)$. This follows from Proposition 4.3. It follows from Lemma 2.6 that by adjusting the endpoints of V and possibly also further increasing the power r above, we can also have that h_γ maps ∂V into $\bar{V}$. We may thus assume the following two properties.

(i) $|h_\gamma'| \geq \nu > 1$ on $I \backslash h_\gamma^{-1}(I)$,

(ii) $h_\gamma(\partial I) \notin I$.

Write $O = I \cap h_\gamma^{-1}(I)$ for the set of points that escape from I under iteration by h_γ.

Since (i) and (ii) hold, Theorem 1 in [PiaYor79] is applicable. This shows that h_γ possesses a conditionally invariant measure ζ_γ on I; ζ_γ is characterized by

$$\zeta_\gamma(A) = \zeta_\gamma(h_\gamma^{-1}(A))/\zeta_\gamma(h_\gamma^{-1}(I)),$$

for Borel sets A. Moreover, ζ_γ is absolutely continuous with respect to Lebesgue measure and has density that is bounded and bounded away from 0 on its support. The support of ζ_γ is all of I. This follows from Proposition 3 in [PiaYor79]; the required transitivity condition is easily seen to hold. Thus there is a constant $C > 0$ such that

$$\frac{1}{C}|A| \leq \zeta_\gamma(A),$$

for Borel sets $A \subset I$.

Write $(h_\gamma)_* \zeta_\gamma = \beta_\gamma \zeta_\gamma$ for some $0 < \beta_\gamma < 1$. Since $|O|$ is bounded from below, β_γ is bounded away from 1. Applying (64),

$$\begin{aligned} \nu_\gamma(h_\gamma^{-n}(I)) &\leq K\sqrt{|h_\gamma^{-n}(I)|} \\ &\leq K\sqrt{C}\sqrt{\zeta_\gamma(h_\gamma^{-n}(I))} \\ &= K\sqrt{C}\sqrt{\beta_\gamma}^n. \end{aligned}$$

The number of iterates needed for a point to reach O under iteration by h_γ, is identical to n on $h_\gamma^{-n}(I) \backslash h_\gamma^{-(n+1)}(I)$. The sum $\sum_{i \geq 1} \nu_\gamma(h_\gamma^{-i}(I))$ is therefore the expectation of the number of iterates under iteration by h_γ for a point $x \in I$ to reach O, with respect to the measure ν_γ. Observe that $\sum_{i \geq 1} \nu_\gamma(h_\gamma^{-i}(I))$ is bounded uniformly in γ. It follows that

$$\int_{[0,1]} L_O d\nu_\gamma \leq C \sum_{i \geq 1} \nu_\gamma(h_\gamma^{-i}(I)),$$

where a constant $C > 0$ appears because h_γ comes from an iterate of f_γ (one may take $C = rq$), is bounded uniformly in γ. □

The following result is similar, but concerns returns for $\tilde{f}_\gamma$ to a small neighborhood V of c, disjoint from $\tilde{E}_\gamma$. Since V does not contain $\tilde{E}_\gamma$, a similar statement where $\tilde{f}_\gamma$ is replaced by f_γ is untrue. For a set $A \subset [0,1]$ and a point $x \in [0,1]$, let

$$\tilde{L}_A(x) = \min\{k \in \mathbb{N};\ \ \tilde{f}_\gamma^k(x) \in A\}.$$

Proposition 5.4 *Let V be a small neighborhood of c, disjoint from $\tilde{E}_\gamma$. There is $L > 0$ so that for all $\gamma \in \Omega$,*

$$\int_{[0,1]\backslash V} \tilde{L}_V d\tilde{\nu}_\gamma / \tilde{\nu}_\gamma([0,1]\backslash V) \ \ \leq \ \ L.$$

Proof. As above, the denominator $\tilde{\nu}_\gamma([0,1]\backslash V)$ in the estimate is bounded away from zero uniformly in γ. Let N be the interval containing c so that $f_0^q(N) \subset N$ and $f_0^q(\partial N) \subset \partial N$. Let V be the maximal symmetric interval (symmetric meaning that the boundary points of V have the same image under f_γ) with $V \subset N$ and $V \cap \tilde{E}_\gamma = \emptyset$. It suffices to prove the proposition for this interval V. Given V, let $\bar{V} = \cup_{i=0}^{q-1} f_\gamma^{-i}(V) \cap f_\gamma^{q-i}(N)$ be the union of the intervals in $\bar{N} = \cup_{i=0}^{q-1} f_\gamma^i(N)$ that are mapped onto V by iterates of f_γ.

As in the proof of Proposition 5.3, let I be the interior of the convex hull of $\bar{V}$ excluding $\bar{V}$. Recall that the orbit $\mathcal{O}(f_\gamma(\partial\tilde{E}_\gamma))$ is a finite set of points. Include $\mathcal{O}(f_\gamma(\partial\tilde{E}_\gamma))$ into I. Write $\tilde{h}_\gamma$ for the restriction of $\tilde{f}_\gamma$ to I. By Proposition 4.3 and Lemma 2.6 we may adjust the boundary of V and replace $\tilde{h}_\gamma$ by a power $\tilde{h}_\gamma^N$ to achieve as before

(i) $|\tilde{h}_\gamma'| \geq \nu > 1$ on $I \backslash \tilde{h}_\gamma^{-1}(I)$,

(ii) $\tilde{h}_\gamma(\partial I) \notin I$.

The number of branches of $\tilde{h}_\gamma$ is constant for $\gamma \in [\gamma_{l+1}, \gamma_l]$, but increases with l since the branches of $\tilde{f}_\gamma\Big|_{\tilde{E}_\gamma}$ are included.

Consider the construction of a conditionally invariant measure for $\tilde{h}_\gamma$ for fixed l. A conditionally invariant measure $\tilde{\zeta}_\gamma$ for $\tilde{h}_\gamma$ is characterized by

$$\tilde{\zeta}_\gamma(A) = \tilde{\zeta}_\gamma(\tilde{h}_\gamma^{-1}(A))/\tilde{\zeta}_\gamma(\tilde{h}_\gamma^{-1}(I)).$$

Following [PiaYor79], the density of $\tilde{\zeta}_\gamma$ is constructed as the fixed point of a Perron-Frobenius operator $\tilde{P}_\gamma$. The Perron-Frobenius operator $\tilde{P}_\gamma$ defined for nonnegative functions g on I, has an explicit formula

$$\tilde{P}_\gamma g(x) = \sum_i |\tilde{\psi}_i'(x)| g \circ \tilde{\psi}_i(x) / \int_{\tilde{h}_\gamma^{-1}([0,1])} g dm.$$

The functions $\tilde{\psi}_i$ are the inverse branches of $\tilde{h}_\gamma$. By condition (ii), $\tilde{P}_\gamma$ is well defined. Since the f_γ invariant set $\mathcal{O}(f_\gamma(\partial\tilde{E}_\gamma))$ is excluded from I, $\tilde{P}_\gamma g$ is continuous if g is continuous. By Theorem 1 in [PiaYor79], there is a conditionally invariant measure $\tilde{\zeta}_\gamma$ for $\tilde{h}_\gamma$. To obtain uniform bounds in l, we must investigate properties of conditionally invariant measures for $\tilde{h}_\gamma$. The proof of Proposition 5.4 is identical to the above proof of Proposition 5.3, once Lemma 5.5 below is proved. □

Lemma 5.5 *Let $\tilde{h}_\gamma$ be as in the above proof of Proposition 5.4. The density of the conditionally invariant measure $\tilde{\zeta}_\gamma$ for $\tilde{h}_\gamma$ is bounded away from 0 uniformly in γ.*

Proof. Recall that the density of the measure $\tilde{\zeta}_\gamma$ is a fixed point $\tilde{\xi}_\gamma$ of the Perron-Frobenius operator $\tilde{P}_\gamma$. By [PiaYor79], $\tilde{\xi}_\gamma$ is a unique fixed point and for each $g \in \mathcal{C}(I)$,

$$\lim_{n\to\infty} \tilde{P}_\gamma^n g = \tilde{\xi}_\gamma. \tag{65}$$

For a Lipschitz density g, let the regularity of g be given by

$$\text{Reg } g = \sup\{|g'(x)|/g(x);\;\; I, g'(x) \text{ is defined}, g(x) > 0\}.$$

We will show that

$$\limsup_{n\to\infty} \text{Reg } \tilde{P}_\gamma^n g \leq \rho, \tag{66}$$

for some ρ independent of g. We remark that from [PiaYor79] one concludes that such a bound holds when γ is restricted to an interval $[\gamma_{l+1}, \gamma_l]$ (see also [LasYor81]). Their arguments do not imply (66) for all $\gamma \in \Omega$, the difficulty being the growth of the number of branches of $\tilde{h}_\gamma$ as $\gamma \to 0$.

To evaluate Reg $\tilde{P}_\gamma g$, compute

$$\begin{aligned}
\frac{|(\tilde{P}_\gamma g)'(x)|}{\tilde{P}_\gamma g(x)} &= \frac{\left|\sum_i [\tilde{\psi}_i'(x) g \circ \tilde{\psi}_i(x)]'\right|}{\sum_i |\tilde{\psi}_i'(x)| g \circ \tilde{\psi}_i(x)} \\
&\leq \frac{\left|\sum_i \tilde{\psi}_i''(x) g \circ \tilde{\psi}_i(x)\right|}{\sum_i |\tilde{\psi}_i'(x)| g \circ \tilde{\psi}_i(x)} + \frac{\left|\sum_i \tilde{\psi}_i'(x) g' \circ \tilde{\psi}_i(x) \tilde{\psi}_i'(x)\right|}{\sum_i |\tilde{\psi}_i'(x)| g \circ \tilde{\psi}_i(x)} \\
&\leq \max_i \frac{|\tilde{\psi}_i''(x)|}{|\tilde{\psi}_i'(x)|} + \max_i |\tilde{\psi}_i'(x)| \frac{g' \circ \tilde{\psi}_i(x)}{g \circ \tilde{\psi}_i(x)}.
\end{aligned}$$

In the last step, we used that $|\sum a_i / \sum b_i| \leq \max_i |a_i/b_i|$ for numbers a_i and positive numbers b_i [PiaYor79]. We claim that $\max_i \frac{|\tilde{\psi}_i''(x)|}{|\tilde{\psi}_i'(x)|}$ is bounded, uniformly in γ. This is clear for inverse branches $\tilde{\psi}_i$ of f_γ away from $\tilde{E}_\gamma$, for which $\tilde{\psi}_i'$ is bounded away from 0 and $\tilde{\psi}_i''$ is bounded. Consider an inverse branch $\tilde{\psi}_i$ provided by the flow ϕ_γ^t of the adapted vector field ϕ_γ (see Proposition 2.1). Thus $\frac{d}{dt}\phi_\gamma^t(x_0) = \phi_\gamma(\phi_\gamma^t(x_0))$. To bound $|\tilde{\psi}_i''|/|\tilde{\psi}_i'|$ we consider the flow ϕ_γ^t for negative t. Let $J_\gamma^t = (\phi_\gamma^t)''/(\phi_\gamma^t)'$. Then J_γ^t is a solution of

$$\frac{d}{dt} J_\gamma^t(x_0) = \phi_\gamma''\Big|_{\phi_\gamma^t(x_0)} (\phi_\gamma^t)'(x_0). \tag{67}$$

For γ small, ϕ_γ'' is close to ϕ_0'', which is nonzero by assumption. Further, $(\phi_\gamma^t)'(x_0)$ is a solution of $\frac{d}{dt}(\phi_\gamma^t)'(x_0) = \phi_\gamma'\Big|_{\phi_\gamma^t(x_0)} (\phi_\gamma^t)'(x_0)$. One may assume that $\phi_\gamma(x) = x^2 + \gamma + \mathcal{O}(x^2\gamma) + \mathcal{O}(x^3)$. Let $\phi_{\gamma,a}(x) = ax^2 + \gamma$ and denote the corresponding flow by $\phi_{\gamma,a}^t$. For $t < 0$, $\phi_{\gamma,a^+}^t(x_0) \leq \phi_\gamma^t(x_0) \leq \phi_{\gamma,a^-}^t(x_0)$ for some $a_- < a_+$ close to 1. Note that $\frac{d}{dt}(\phi_{\gamma,a}^t)'(x_0) = 2a\phi_{\gamma,a}^t(x_0)(\phi_{\gamma,a}^t)'(x_0)$. One obtains from this

$$(\phi_{\gamma,a^-}^t)'(x_0) \leq (\phi_\gamma^t)'(x_0) \leq (\phi_{\gamma,a^+}^t)'(x_0)$$

for some $a^- < a^+$ close to 1. Solving $\frac{d}{dt}\phi_{\gamma,a}^t = a\left(\phi_{\gamma,a}^t\right)^2 + \gamma$ yields

$$\frac{1}{\sqrt{a}} \arctan\left(\sqrt{\frac{a}{\gamma}}\phi_{\gamma,a}^t(x_0)\right) = \sqrt{\gamma}t + \frac{1}{\sqrt{a}} \arctan\left(\sqrt{\frac{a}{\gamma}}x_0\right). \tag{68}$$

Differentiating (68) yields

$$(\phi_{\gamma,a}^t)'(x_0) = \frac{a(\phi_{\gamma,a}^t(x_0))^2 + \gamma}{ax_0^2 + \gamma}.$$

These bounds and (67) prove that J_γ^t is bounded, uniformly in γ.

The above reasoning shows that

$$\text{Reg } \tilde{P}_\gamma^N g \le M + \frac{1}{\lambda}\text{Reg } g$$

for constants $M > 0$ and $\lambda > 1$. Now (66) follows by iterating the bound on the regularity of $\tilde{P}_\gamma^N g$.

We can now basically follow the proof of Proposition 7 in [PiaYor79] to conclude the lemma. Take $g \in \mathcal{C}(I)$ with Reg $g \le \rho$. Then, for $x \in [0,1]$,

$$\tilde{P}_\gamma^n g(x) = \sum_i |\tilde{\psi}_{n,i}'(x)| g \circ \tilde{\psi}_{n,i}(x) / \int_{\tilde{h}_\gamma^{-n}([0,1])} gdm,$$

where $\tilde{\psi}_{n,i}$ are the inverse branches of $\tilde{h}_\gamma^n$ near x. For g with $\int_I gdm = 1$, there is a component A_j of I on which $\sup_{A_j} g \ge 1$. We may also assume that $\sup_{A_j} g \ge \frac{1}{2}$ on a component A_j with $|A_j| \ge \beta$ for some constant $\beta > 0$. Then $\inf_{A_j} g \ge \frac{1}{2}e^{-\beta\rho}$.

If n is large enough, $I \subset \tilde{h}_\gamma^n(A_j)$ for all small γ. Hence, for each $x \in I$, there exists i_0 with $\tilde{\psi}_{n,i_0}(x) \in A_j$. This gives

$$\begin{aligned}\tilde{P}_\gamma^n g(x) &\ge |\tilde{\psi}_{n,i_0}'(x)| g \circ \tilde{\psi}_{n,i_0}(x), \\ &\ge |\tilde{\psi}_{n,i_0}'(x)| e^{-\rho\beta}.\end{aligned}$$

Since $|A_j| \ge \beta$, we may take i_0 so that $|\tilde{\psi}_{n,i_0}'(x)|$ is bounded from below. Therefore, $\tilde{P}_\gamma^n g(x) \ge d$ for some $d > 0$ which is independent of g and γ. By (65), this proves the lemma. □

5.3 Proof of Theorem B

We will show how Theorem B is proved by combining Propositions 5.3 and 5.4. Let Ω be as constructed in Section 4 and take $\gamma \in \Omega$. Let ν_γ be the absolutely continuous invariant measure for f_γ, obtained in Lemma 5.2. The measure ν_γ is ergodic (see Chapter V in [MelStr93]), so that by Birkhoff's Ergodic Theorem,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} 1_A(f_\gamma^i(x)) = \nu_\gamma(A), \tag{69}$$

for any Borel set $A \subset [0,1]$ and almost all $x \in [0,1]$ (with respect to either ν_γ or Lebesgue measure). With ν_γ also $\tilde{\nu}_\gamma$ is ergodic. It follows that

$$\lim_{m\to\infty} \frac{1}{m} \sum_{i=0}^{m-1} 1_A(\tilde{f}_\gamma^i(x)) = \tilde{\nu}_\gamma(A). \tag{70}$$

Hence, for almost all $x \in [0,1]$, the distribution with which points in the orbit $\{\tilde{f}_\gamma^i(x)\}$ are in $\tilde{E}_\gamma$ is given by $\tilde{\nu}_\gamma$. Let I be a compact interval in $[d,a)$. If V is a neighborhood of c, then by (70) and applying Proposition 5.4, the measure $\tilde{\nu}_\gamma(V)$ of V is bounded from below by a positive constant, uniformly in γ. Observe that by invariance of $\tilde{\nu}_\gamma$, $\tilde{\nu}_\gamma(\tilde{f}_\gamma^k(V)) \geq \tilde{\nu}_\gamma(V)$, $k \geq 0$. Therefore, the measure $\tilde{\nu}_\gamma(I)$ of I is bounded from below by a positive constant, uniformly in γ. An easy computation shows that for any compact interval I inside $[d,a)$, the number of iterates needed for a point $x \in I$ to leave $\tilde{E}_\gamma$ is bounded from below by $K/\sqrt{\gamma}$ for some $K > 0$. It follows that the average duration of orbit pieces of f_γ in $\tilde{E}_\gamma$ is bounded from below by $K/\sqrt{\gamma}$ for some $K > 0$. Combining this with Proposition 5.3 proves the upper bound on $\chi_{\bar{E}}(\gamma)$ in Theorem B. The lower bound is a trivial consequence of the fact that there is always a positive number of iterates between two laminar phases and that the maximum number of consecutive iterations in $\tilde{E}_\gamma$ is bounded above by $K'/\sqrt{\gamma}$ for some constant K'.

The argument to show that ν_γ converges weakly to ν_0 is similar. By definition of $\bar{\nu}_\gamma$,

$$\bar{\nu}_\gamma(E^{-j}) = \sum_{k=j}^{l-1} \tilde{\nu}_\gamma(E^{-k}).$$

Reasoning as above one shows that $\tilde{\nu}_\gamma(E^{-j})$, $0 \leq j \leq l$, is bounded from below, uniformly in γ. It follows that $\bar{\nu}_\gamma(\tilde{E}_\gamma)$ gets arbitrarily large as $\gamma \to 0$ (because then $l \to \infty$). Since $\bar{\nu}_\gamma([0,1]\backslash\tilde{E}_\gamma) = \tilde{\nu}_\gamma([0,1]\backslash\tilde{E}_\gamma)$ is obviously bounded, $\nu_\gamma(\tilde{E}_\gamma) \to 1$ as $\gamma \to 0$. Because this holds for any neighborhood $\tilde{E}_\gamma$, this shows that $\nu_\gamma \rightharpoonup \nu_0$ as $\gamma \to 0$.

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