SKEW PRODUCTS OF INTERVAL MAPS OVER SUBSHIFTS

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Abstract. We treat step skew products over transitive subshifts of finite type with interval fibers. The fiber maps are diffeomorphisms on the interval; we assume that the end points of the interval are fixed under the fiber maps. Our paper thus extends work by V. Kleptsyn and D. Volk who treated step skew products where the fiber maps map the interval strictly inside itself.

We clarify the dynamics for an open and dense subset of such skew products. In particular we prove existence of a finite collection of disjoint attracting invariant graphs. These graphs are contained in disjoint areas in the phase space called trapping strips. Trapping strips are either disjoint from the end points of the interval (internal trapping strips) or they are bounded by an end point (border trapping strips). The attracting graphs in these different trapping strips have different properties.

1. Introduction

We aim to describe the dynamics of specific step skew products

$$(\omega, x) \mapsto (\sigma\omega, f_{\omega_0}(x))$$

with a shift as dynamics in the base and with interval fiber maps. That is, $\omega = (\omega_i)_{i\in\mathbb{Z}}$ is a sequence using finitely many symbols, and σ is the left shift operator acting on it. We treat such systems in cases where σ is a subshift of finite type and where the f_i's are diffeomorphisms on a compact interval that fix the endpoints of the interval.

Kleptsyn and Volk [5] conducted a study of dynamics of generic step skew products of diffeomorphisms on the line over subshifts of finite type. They looked at diffeomorphisms that are mapping a bounded interval strictly inside itself. They showed that so called bony graphs (after Kudryashov, see [6]) arise as attractors: these attractors are the union of a measurable graph and a zero measure set of intervals inside fibers (the bones).

A different situation occurs for diffeomorphisms on a compact interval that fix the endpoints of the interval. Such systems gained interest with an example by Kan [4] where they gave rise to intermingled basins. This example is over a full shift on two symbols and the end points of the interval are attracting on average. Il'yashenko [2, 3] similarly considered examples of diffeomorphisms over a full shift under an assumption of repulsion on average at the end points. He established attractors with positive standard measure (the standard measure is the product of Markov measure on the shift space and Lebesgue measure on the fiber space). The attractors are the closure of an invariant measurable graph. Note the contrast with bony graphs which have zero standard measure.

We provide a classification of dynamics of generic step skew products of diffeomorphisms on a compact interval (all diffeomorphisms fixing end points of the interval) over subshifts of finite type. Both types of graphs, bony and thick, can arise in a single step skew product.

1.1. **Step skew product systems over subshifts of finite type.** Write Ω for the finite set of symbols $\{1,\dots,N\}$. Let $\mathcal{A} = (a_{ij})_{i,j=1}^N$ be a matrix with $a_{ij} \in \{0,1\}$. Associated to $\mathcal{A}$ is the set $\Sigma_{\mathcal{A}}$ of bilateral sequences $\omega = (\omega_n)_{-\infty}^{\infty}$ composed of symbols in Ω and with transition matrix $\mathcal{A}$:

$$a_{\omega_n \omega_{n+1}} = 1$$

for all $n \in \mathbb{Z}$. Let $(\Sigma_{\mathcal{A}}, \sigma)$ be the subshift of finite type on $\Sigma_{\mathcal{A}}$. The map σ shifts every sequence $\omega \in \Sigma_{\mathcal{A}}$ one step to the left, $(\sigma\omega)_i = \omega_{i+1}$. We can also consider the left shift operator σ acting on the one-sided symbol space $\Sigma_{\mathcal{A}}^+$, i.e. the space of sequences $\omega = (\omega_n)_0^\infty$ composed of symbols in Ω with $a_{\omega_n\omega_{n+1}} = 1$ for all $n \geq 0$. The spaces $\Sigma_{\mathcal{A}}$ and $\Sigma_{\mathcal{A}}^+$ are endowed with the product topology. We assume that $\mathcal{A}$ is primitive, i.e.

$$\exists n_0 \in \mathbb{N}\ \forall i, j \in \Omega\ (\mathcal{A}^{n_0})_{ij} > 0.$$

This implies that the subshift σ is topologically transitive and topologically mixing.

Consider the interval $I = [0,1]$ and $\{f_1, ..., f_N\}$, a finite family of orientation preserving (strictly increasing) C^2-diffeomorphisms defined on I assuming that $f_i(0) = 0$ and $f_i(1) = 1$ for every $i \in \Omega$. Write F^+ for the skew product system

$$F^+(\omega, x) = (\sigma\omega, f_\omega(x))$$

on $\Sigma_{\mathcal{A}}^+ \times I$, where the fiber maps f_ω depend only on ω_0, i.e. $f_\omega = f_{\omega_0}$. We also write $(F^+)^n(\omega, x) = (\sigma^n\omega, f_\omega^n(x))$ for iterates of F^+ in which

$$f_\omega^n(x) = f_{\omega_{n-1}} \circ \cdots \circ f_{\omega_0}(x).$$

Likewise, on $\Sigma_{\mathcal{A}} \times I$ we have

$$F(\omega, x) = (\sigma\omega, f_\omega(x)).$$

In this paper we consider the following set of step skew product systems.

Definition 1.1. *We denote by* $\mathbf{S}$ *the set of step skew product systems* $F : \Sigma_{\mathcal{A}} \times I \to \Sigma_{\mathcal{A}} \times I$ *of the form*

$$F(\omega, x) = (\sigma\omega, f_{\omega_0}(x)),$$

for orientation preserving diffeomorphisms $f_i : I \to I$ *that fix end points of* I.

1.2. **Markov measures.** Let $\Pi = (\pi_{ij})_{i,j=1}^N$ be a right stochastic matrix, i.e. $\pi_{ij} \geq 0$ and $\sum_{j=1}^N \pi_{ij} = 1$, such that $\pi_{ij} = 0$ precisely if $a_{ij} = 0$. By the Perron-Frobenius theorem for stochastic matrices, there exists a unique positive left eigenvector $p = (p_1, ..., p_N)$ for Π that corresponds to the eigenvalue 1; i.e.

$$\sum_{i=1}^N p_i \pi_{ij} = p_j,\ \forall j \in \Omega. \tag{1}$$

We assume that p is normalized so that it is a probability vector, $\sum_{i=1}^N p_i = 1$.

For a finite word $\omega_{k_1}...\omega_{k_n}$, $k_i \in \mathbb{Z}$, the cylinder $C^{k_1,\dots,k_n}_{\omega_{k_1},\dots,\omega_{k_n}}$ (we will also use the notation $C^{k_1,\dots,k_n}_{\omega}$) is the set

$$C^{k_1,\dots,k_n}_{\omega_{k_1},\dots,\omega_{k_n}} = \{\omega' \in \Sigma_{\mathcal{A}}\ ;\ \omega'_{k_i} = \omega_{k_i},\ \forall 1 \leq i \leq n\}.$$

As cylinders form a countable base of the topology on $\Sigma_{\mathcal{A}}$, Borel measures on $\Sigma_{\mathcal{A}}$ are determined by their values on the cylinders. A Borel measure ν on $\Sigma_{\mathcal{A}}$ is called a Markov measure constructed from the distribution p_i and the transition probabilities π_{ij}, if for every $\omega \in \Sigma_{\mathcal{A}}$ and $k \leq l$,

$$\nu(C_\omega^{k,\dots,l}) = p_{\omega_k} \prod_{i=k}^{l-1} \pi_{\omega_i \omega_{i+1}}.$$

One can easily check that with this definition ν is well-defined and is a probability measure. Moreover, ν is invariant under the shift map σ; it is ergodic and $\mathrm{supp}(\nu) = \Sigma_{\mathcal{A}}$. From now on, we consider a fixed ergodic Markov measure ν on $\Sigma_{\mathcal{A}}$. Write π for the natural projection $\Sigma_{\mathcal{A}} \mapsto \Sigma_{\mathcal{A}}^+$. Then, $\nu^+ = \pi\nu$ is the Markov measure on $\Sigma_{\mathcal{A}}^+$.

We do not consider measures on $\Sigma_{\mathcal{A}}$ that are not Markov measures. The reason is the connection of Markov measures to stationary measures for the stochastic process induced by F^+, see Section 3.

Definition 1.2. *The standard measure s on $\Sigma_{\mathcal{A}} \times I$ is the product of ν and the Lebesgue measure on the fiber.*

1.3. **Trapping strips, bony graphs and thick graphs.** Let $F \in \mathbf{S}$. As in [5], F admits forward invariant regions called trapping strips. Let $\varphi_1, \varphi_2 : \Sigma_{\mathcal{A}} \to I$, be continuous functions such that $\varphi_1 < \varphi_2$, i.e. $\varphi_1(\omega) < \varphi_2(\omega)$ for any $\omega \in \Sigma_{\mathcal{A}}$. Given such functions, we define the strip

$$\mathcal{S}_{\varphi_1,\varphi_2} = \{(\omega, x) \,;\, \varphi_1(\omega) \leq x \leq \varphi_2(\omega)\}.$$

We distinguish two types of strips:

(1) An internal strip has $0 < \varphi_1 < \varphi_2 < 1$;
(2) For a border strip, $\varphi_1 = 0$ or $\varphi_2 = 1$, or both.

If the graph of φ_1 (or φ_2) is disjoint from $\Sigma_{\mathcal{A}} \times \{0\}$ (from $\Sigma_{\mathcal{A}} \times \{1\}$), then this graph is called an internal boundary.

Definition 1.3. *A strip $\mathcal{S}_{\varphi_1,\varphi_2}$ is called a trapping strip if $F(\mathcal{S}_{\varphi_1,\varphi_2}) \subseteq \mathcal{S}_{\varphi_1,\varphi_2}$. The strip $\mathcal{S}_{\varphi_1,\varphi_2}$ is called a strict trapping strip if moreover internal boundaries are mapped inside the interior of $\mathcal{S}_{\varphi_1,\varphi_2}$.*

Likewise one can consider trapping strips for F^+. It is clear that internal and border trapping strips are the only two possible kinds of trapping strips. Consider a trapping strip $\mathcal{S}$ with boundary functions $\varphi_1 < \varphi_2$. Because of monotonicity of the fiber maps, the images $F^n(\mathcal{S})$ are strips. Since for a trapping strip $\mathcal{S}$ also $F^n(\mathcal{S}) \subseteq \mathcal{S}$, we get that for every $n \geq 0$ the image $F^n(\mathcal{S})$ is a trapping strip. Therefore any trapping strip $\mathcal{S}$ has a non-empty maximal attractor

$$A_{\max} = \bigcap_{n=0}^{\infty} F^n(\mathcal{S}).$$

We encounter two different types of maximal attractors.

Definition 1.4. *A measurable graph B in $\Sigma_{\mathcal{A}} \times I$ is called a bony graph if it is contained in a closed set that intersects ν-almost every fiber in a single point and every other fiber in an interval, which is called a bone.*

Note that the standard measure of the closure of a bony graph is zero;

$$s\left(\overline{B}\right) = 0.$$

Following [5] we also call the closed set that is the union of the measurable graph and the bones, a bony graph. A bony graph can have an empty set of bones; a bony graph with an empty set of bones is a continuous graph. It is easy to construct examples where the maximal attractor is in fact a continuous graph.

Definition 1.5. *A measurable graph B in $\Sigma_{\mathcal{A}} \times I$ is called a thick graph if its closure has positive standard measure, i.e.*

$$s\left(\overline{B}\right) > 0.$$

We also call the closure of the thick graph, a thick graph.

2. Classification of dynamics for generic skew products

The Lyapunov exponent of a system $F \in \mathbf{S}$ at a point $(\omega, x) \in \Sigma_{\mathcal{A}} \times I$ is

$$\lim_{n\to\infty} \frac{1}{n} \ln\left(f'_{\omega_{n-1}} \circ \cdots \circ f'_{\omega_0}(x)\right) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln\left(f'_{\sigma^i\omega}(f^i_\omega(x))\right), \tag{2}$$

in case the limit exists. Since for every $i \in \Omega$, $x = 0, 1$ are fixed points of f_i, by the definition of Markov measure and Birkhoff's ergodic theorem, we obtain for $x = 0, 1$ that

$$L(x) = \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \ln\left(f'_{\sigma^i\omega}(x)\right) = \int_{\Sigma^+_{\mathcal{A}}} \ln\left(f'_\omega(x)\right)\, d\nu^+(\omega) = \sum_{i=1}^{N} p_i \ln\left(f'_i(x)\right)$$

for ν^+-almost all $\omega \in \Sigma^+_{\mathcal{A}}$. Note that generically $L(0)$ and $L(1)$ differ from zero.

We have introduced all notions needed to present our description of the dynamics of generic step skew product systems. The following theorem holds for step skew product systems from an open and dense subset of $\mathbf{S}$ which is given explicitly in Section 2.1 below.

Theorem 2.1. *There is an open and dense set $\mathbf{G}$ of $\mathbf{S}$, so that $F \in \mathbf{G}$ satisfies the following.*
F admits a finite collection of disjoint trapping strips $\mathcal{S}^t$, $1 \le t \le T$, of the form

$$\mathcal{S}^t = \cup_{k=1}^{N} C^0_k \times [A^t_k, B^t_k].$$

Furthermore,

1. *(1) $\mathcal{S}^t$ contains a unique attracting invariant graph Γ^t: Γ^t is the graph of a measurable function $X^t : D^t \subset \Sigma_{\mathcal{A}} \to I$ defined on a set D^t with $\nu(D^t) = 1$. Given $x_i \in [A^t_i, B^t_i]$, for $\sigma^{-n}\omega \in D^t$,*

$$|f^n_{\sigma^{-n}\omega}(x_{\omega_{-n}}) - X^t(\omega)| \to 0 \text{ as } n \to \infty.$$

2. *(2) If $L(0) < 0$, then $\Gamma = \Sigma_{\mathcal{A}} \times \{0\}$ is an attracting invariant graph: there is a set $D^t \subset \Sigma_{\mathcal{A}}$ with $\nu(D^t) = 1$, and a positive function $r : D^t \to (0, 1]$ so that for (ω, x) with $\omega \in D^t$, $0 \le x < r(\omega)$,*

$$f^n_\omega(x) \to 0 \text{ as } n \to \infty.$$

A similar statement applies to $\Sigma_{\mathcal{A}} \times \{1\}$ if $L(1) < 0$.

(3) (a) if the strict trapping strip is a border trapping strip, then its maximal attractor is a thick graph.
(b) if the strict trapping strip is an internal trapping strip, then its maximal attractor is a bony graph.

(4) With respect to the standard measure on $\Sigma_{\mathcal{A}} \times I$, the positive orbit of almost every initial point converges to one of the attracting graphs from items (1), (2).

Kleptsyn and Volk [5] show that the bony graphs in internal strict trapping strips are upper-semicontinuous:

$$\forall \omega \in \Sigma_{\mathcal{A}} \, \forall \varepsilon > 0 \, \exists \delta > 0 \, d(\omega, \omega') < \delta \Rightarrow B_{\omega'} \subset U_{\varepsilon}(B_{\omega}),$$

where d metrizes the product topology on $\Sigma_{\mathcal{A}}$, $B_{\omega} = B \cap (\{\omega\} \times I)$ and $U_{\varepsilon}(B_{\omega})$ denotes the ε-neighborhood of B_{ω} in $\{\omega\} \times I$. They refer to these bony graphs as continuous bony graphs.

2.1. **Genericity conditions.** The open and dense set $\mathbf{G}$ of $\mathbf{S}$ in Theorem 2.1 is determined by a number of genericity conditions. Here we list the imposed genericity conditions. They are equal to those appearing in [5], with two additional conditions related to the fixed boundary points of I (items (1) and (5) below). The first condition gives that the end points of I are repelling or attracting, on average.

(1) $L(0), L(1) \neq 0$.

To formulate the further conditions we introduce the notions of simple transition and simple return.

Definition 2.1. *A finite word $\omega_1 \dots \omega_n$ is called admissible if each pair of consecutive symbols $\omega_i \omega_{i+1}$ is admissible; i.e. $\pi_{\omega_i \omega_{i+1}} \neq 0$. A map of the form*

$$f_{\omega_1 \dots \omega_n} := f_{\omega_n} \circ \dots \circ f_{\omega_1} : I \to I,$$

is called an admissible composition if the word $\omega_1 \dots \omega_n$ is admissible.

Definition 2.2. *A simple transition is an admissible composition $f_{\omega_1 \dots \omega_n} : I \to I$ in which all the symbols ω_i, $1 \leq i \leq n$ are different. It is called a simple return if also $\omega_1 = \omega_{n+1}$.*

We can now state the following genericity conditions.

(2) Any fixed point q of any simple return g is hyperbolic: $g'(q) \neq 1$;

and if we consider the restriction of f_i's to the open interval $(0, 1)$ then

(3) No attracting fixed point of a simple return is mapped to a repelling fixed point of a simple return by a simple transition. Also, no repelling fixed point of a simple return is mapped to an attracting fixed point of a simple return by a simple transition;

(4) One can not choose from the interior of each interval I_k, $k \in \Omega$, a single point a_k such that for any admissible couple i, j one could have $f_i(a_i) = a_j$.

Condition (4) precludes finite invariant sets, see [5]. The final condition relates to minimal iterated function systems. First we recall the definition of minimality of an iterated function system. Suppose given an iterated function system IFS $\{g_1, \dots, g_k\}$ of continuous maps g_i on a metric space X. Let Y be a subset of X with $g_i(Y) \subset Y$ for all i. We say that

IFS $\{g_1, \ldots, g_k\}$ is minimal on Y if for every points $x, y \in Y$ and every neighborhood V of y, there is a composition $g_{i_n} \circ \cdots \circ g_{i_1}$ that maps x into V.

The proof of [2, Lemma 3] gives the following result.

Proposition 2.1. *Let $f, g : I \to I$ be diffeomorphisms fixing the boundary points of I. Assume that $\lambda = f'(0) < 1$, $\mu = g'(0) > 1$. Assume further that either*

$$\ln(\lambda)/\ln(\mu) \notin \mathbb{Q},$$

or

$$\frac{f''(0)}{\lambda^2 - \lambda} \neq \frac{g''(0)}{\mu^2 - \mu}.$$

Then the iterated function system generated by f, g is minimal on some interval $(0, u)$.

Proof. Il'yashenko [2, Lemma 3] considers, for $x, y \in (0,1)$, compositions $g^l \circ f^k(x)$ that converge to y for suitable $k, l \to \infty$. His analysis uses linearizing coordinates $h \circ f \circ h^{-1}(x) = \lambda x$ with $x \in [0, s]$ for an $s < 1$. Here h is a local diffeomorphism. The two cases where $\ln(\lambda), \ln(\mu)$ are rationally dependent or not, are distinguished. In case $\ln(\lambda), \ln(\mu)$ are rationally dependent, the argument works if the second order derivative of $h \circ g \circ h^{-1}$ at 0 is not zero. An explicit calculation shows that this gives the condition in the proposition. □

(5) The admissible returns f, g introduced in Lemma 4.6 satisfy the conditions formulated in Proposition 2.1.

3. Stationary measures

A key role in our study is played by ergodic invariant measures for the skew product systems. The necessary material is collected in this section.

Write $\mathcal{I} = \Omega \times I$. For every $i, j \in \Omega$, π_{ij} equals the probability of the transition from a point (i, x) in $\mathcal{I}$ to another point $(j, f_i(x))$. For every $i \in \Omega$ we denote $\{i\} \times I \in \mathcal{I}$ by I_i. We can identify I_i with I. Denote by $\mathcal{B}$ the Borel sigma-algebra on I. We consider Borel probability measures $\mathfrak{m}$ on the space $\mathcal{I}$ with $\mathfrak{m}(I_i) = p_i$. For such a measure $\mathfrak{m}$, define the probability measure $\mathfrak{m}_i$ on I_i by

$$\mathfrak{m}_i = \frac{\mathfrak{m}|_{I_i}}{\mathfrak{m}(I_i)}.$$

We denote by $f_i \mathfrak{m}_i$ the push-forward measure of $\mathfrak{m}_i$ by f_i, where $f_i \mathfrak{m}_i(B) = \mathfrak{m}_i(f_i^{-1}(B))$ for $\mathcal{B}$-measurable sets B. Define $\mathcal{T}$ on the space of probability measures on $\mathcal{I}$ by

$$(\mathcal{T}\mathfrak{m})_k = \frac{1}{p_k} \sum_{i=1}^{N} p_i \pi_{ik} f_i \mathfrak{m}_i, \ \forall k \in \Omega,$$

with an understanding that $\mathcal{T}\mathfrak{m}(I_i) = p_i$.

Definition 3.1. *A measure $\mathfrak{m}$ on the space $\mathcal{I}$ is stationary if $\mathcal{T}\mathfrak{m} = \mathfrak{m}$.*

Recall the notation $C_k^0 = \{\omega \in \Sigma_{\mathcal{A}} \mid \omega_0 = k\}$. Write $C_k^{+,0} = \{\omega \in \Sigma_{\mathcal{A}}^+ \mid \omega_0 = k\}$. For $k \in \Omega$, write ν_k^+ for the restriction of the Markov measure ν^+ to the cylinder $C_k^{+,0}$. A direct computation gives the following correspondence between stationary measures and invariant measures for the skew product system with one sided time.

Lemma 3.1. *A probability measure* $\mathfrak{m}$ *is a stationary probability measure if and only if* μ^+ *defined by*

$$\mu^+ = \sum_{k=1}^{N} \nu_k^+ \times \mathfrak{m}_k \tag{3}$$

is an invariant measure of F^+ *with marginal* ν^+ *on* $\Sigma_{\mathcal{A}}^+$.

Let $\mathcal{F}^+$ be the Borel sigma-algebra on $\Sigma_{\mathcal{A}}^+$. It yields a sigma-algebra $\mathcal{F}_0 = \pi^{-1}\mathcal{F}^+$ on $\Sigma_{\mathcal{A}}$, where $\pi : \Sigma_{\mathcal{A}} \to \Sigma_{\mathcal{A}}^+$ is the natural coordinate projection. Write $\mathcal{F}$ for the Borel sigma-algebra on $\Sigma_{\mathcal{A}}$. A measure μ on $\Sigma_{\mathcal{A}} \times I$ with marginal ν has conditional measures μ_ω on the fibers $\{\omega\} \times I$, such that

$$\mu(A) = \int_{\Sigma_{\mathcal{A}}} \mu_\omega(A \cap (\{\omega\} \times I))\, d\nu(\omega)$$

for measurable sets A. A measure μ^+ on $\Sigma_{\mathcal{A}}^+ \times I$ with marginal ν^+ likewise has conditional measures μ_ω^+. It is convenient to consider ν^+ also as a measure on $\Sigma_{\mathcal{A}}$ with sigma-algebra $\mathcal{F}_0$ and μ^+ also as a measure on $\Sigma_{\mathcal{A}} \times I$ with sigma-algebra $\mathcal{F}_0 \otimes \mathcal{B}$. When $\omega \in \Sigma_{\mathcal{A}}$ we will write μ_ω^+ for the conditional measures $\mu_{\pi\omega}^+$. The spaces of measures are equipped with the weak star topology. The following result relates invariant measures for the one-sided and the two-sided skew product systems. It is a special case of [1, Theorem 1.7.2]. We write $\Sigma_{\mathcal{A}} = \Sigma_{\mathcal{A}}^- \times \Sigma_{\mathcal{A}}^+$. and with this $\omega = (\omega^-, \omega^+)$ for $\omega \in \Sigma_{\mathcal{A}}$.

Proposition 3.1. *Let* μ^+ *be an* F^+*-invariant probability measure with marginal* ν^+*. Then there exists an* F*-invariant probability measure* μ *with marginal* ν *and conditional measures*

$$\mu_\omega = \lim_{n\to\infty} f^n_{\sigma^{-n}\omega} \mu^+_{\sigma^{-n}\omega}, \tag{4}$$

ν*-almost surely.*

Let μ *be an* F*-invariant probability measure with marginal* ν *and* $\Pi : \Sigma_{\mathcal{A}}^- \times \Sigma_{\mathcal{A}}^+ \times I \to \Sigma_{\mathcal{A}}^+ \times I$ *be the natural projection where* $\Sigma_{\mathcal{A}} = \Sigma_{\mathcal{A}}^- \times \Sigma_{\mathcal{A}}^+$*. Then*

$$\mu^+ = \Pi\mu \tag{5}$$

is an F^+*-invariant probability measure with marginal* ν^+*.*

The correspondence $\mu \leftrightarrow \mu^+$ *given by* (4), (5) *is one-to-one and* μ *is ergodic if and only if* μ^+ *is ergodic. An invariant measure* μ *for which* μ_ω *depends on the past* $\omega^- \in \Sigma_{\mathcal{A}}^{-1}$ *only, corresponds to a measure* μ^+ *that comes from a stationary measure* $\mathfrak{m}$ *as in* (3)*.*

4. Bony graphs and thick graphs

The proof of Theorem 2.1 is divided into different steps. We will first discuss the case where both $L(0) > 0$ and $L(1) > 0$. The other cases are then easy to treat and will be considered later.

4.1. **Repelling end points.** We assume $L(0) > 0$ and $L(1) > 0$. We briefly outline the different steps in the proof of Theorem 2.1, which will be worked out below.

Step 1: Stationary measures: By a Krylov-Bogolyubov procedure on a suitable class of probability measures we construct stationary measures that do not assign measure to the endpoints 0 or 1 of the interval $[0, 1]$.

Step 2: Trapping strips: The convex hull of the support of an ergodic stationary measure, as constructed in the first step, provides a trapping strip. Trapping strips can be border trapping strips or internal trapping strips.

Step 3: Conditional measures: A stationary measure gives rise to an invariant measure of the skew product system with two sided time. We prove that such an invariant measure has delta measures as conditional measures on fibers. For each trapping strip there is a unique invariant measure with support in the trapping strip.

Step 4: Attracting graphs: The points of the delta measures constitute an invariant graph. We discuss its properties in this final step.

For internal trapping strips these results have been obtained by Kleptsyn and Volk [5]. We now elaborate the different steps.

Step 1: Stationary measures. In the construction of stationary measures we iterate the transformation $\mathcal{T}$, whose fixed points are the stationary measures. For $k \in \Omega$ and for any $n \in \mathbb{N}$, the nth iterate of $\mathfrak{m}$ under the transformation $\mathcal{T}$ is calculated on I_k as

$$(\mathcal{T}^n \mathfrak{m})_k = \frac{1}{p_k} \sum_{i_1,\ldots,i_n=1}^{N} p_{i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1}i_n}\pi_{i_n k} f^n_{i_1\ldots i_n}\mathfrak{m}_{i_1}. \tag{6}$$

The above sum is over all N^n possible symbol sequences of length $n+1$ in Ω^n ending with the symbol k, and $p_{i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1}i_n}\pi_{i_n k}$ is the probability of the transition to the symbol k in n steps along the symbol sequence $i_1,\ldots,i_n,k$.

We will need the following arithmetic bound that is connected to formula (6). Recall the assumptions $L(0) > 0$ and $L(1) > 0$. Write $\lambda_i = f_i'(0)$ and $\bar\lambda_i = f_i'(1)$.

Lemma 4.1. *For n large enough and any k, $1 \le k \le N$,*

$$\frac{1}{p_k} \sum_{i_1,\ldots,i_n=1}^{N} p_{i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1}i_n}\pi_{i_n k}\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n}) > 0, \tag{7}$$

$$\frac{1}{p_k} \sum_{i_1,\ldots,i_n=1}^{N} p_{i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1}i_n}\pi_{i_n k}\frac{1}{n}\ln(\bar\lambda_{i_1}\cdots\bar\lambda_{i_n}) > 0. \tag{8}$$

Proof. We consider the end point 0. First note that for ν^+-almost all ω,

$$\begin{aligned} L(0) &= \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\ln(f'_{\sigma^i\omega}(0)) \\ &= \lim_{k\to\infty}\frac{1}{kn}\sum_{i=0}^{k-1}\ln((f^n_{\sigma^{ni}\omega})'(0)) \\ &= \frac{1}{n}\int_{\Sigma^+_{\mathcal{A}}}\ln((f^n_\omega)'(0))\,d\nu^+(\omega) \\ &= \sum_{i_1,\ldots,i_n=1}^{N} p_{i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1}i_n}\frac{1}{n}\ln(f'_{i_1}(0)\cdots f'_{i_n}(0)). \end{aligned}$$

Hence

$$L(0) = \sum_{i_1,\dots,i_n=1}^{N} p_{i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1}i_n}\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n}). \tag{9}$$

A similar equality as (9) holds for $L(1)$, the Lyapunov exponent at $x=1$.

The sum in (7) is an average over all symbol sequences of length $n+1$ ending with a symbol k:

$$\frac{1}{p_k}\sum_{i_1,\dots,i_n=1}^{N} p_{i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1}i_n}\pi_{i_n k}\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n}) = \frac{1}{p_k}\int_{P_{n,k}}\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})\,d\nu^+(i),$$

where $i=(i_1,\dots)$ and $P_{n,k}=\{i\in\Sigma^+_{\mathcal{A}}\ ;\ \sigma^{n+1}i\in C_k\}$. Since ν^+ is invariant we have $\nu^+(C_k)=\nu^+(\sigma^{-(n+1)}(C_k))=\nu^+(P_{n,k})$ for any $n\in\mathbb{N}$. We observe that $\nu^+(P_{n,k})=p_k$ independent of n and we suppress the dependence of $P_{n,k}$ to n.

Write

$$\Gamma(\varepsilon,M)=\left\{\omega\in\Sigma^+_{\mathcal{A}}\ ;\ \left|\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})-L(0)\right|<\varepsilon \text{ for } n\geq M\right\}.$$

By ergodicity (2), $\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})$ converges to $L(0)$ for ν^+-almost all $(i_1,\dots)\in\Sigma^+_{\mathcal{A}}$, as $n\to\infty$. We therefore have that for all $\varepsilon>0$ there exists M so that $\nu^+(\Gamma(\varepsilon,M))>1-\varepsilon$.

Take the positive constant K such that $|\ln\lambda_j-L(0)|\leq K$ for all j. Choose ε small and $M=M(\varepsilon)$ so that $\nu^+(\Gamma(\varepsilon,M))>1-\varepsilon$. Write $\Gamma(\varepsilon,M)^c=\Sigma^+_{\mathcal{A}}\setminus\Gamma(\varepsilon,M)$. For any $n\geq M$ we can compute,

$$\begin{aligned}
d_n &= \left|\frac{1}{p_k}\int\limits_{P_k}\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})\,d\nu^+(i)-L(0)\right|\\
&\leq \frac{1}{p_k}\int\limits_{P_k\cap\Gamma(\varepsilon,M)}\left|\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})-L(0)\right|d\nu^+(i)\\
&\quad+\frac{1}{p_k}\int\limits_{P_k\cap\Gamma(\varepsilon,M)^c}\left|\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})-L(0)\right|d\nu^+(i)\\
&\leq \varepsilon+K\varepsilon.
\end{aligned}$$

Therefore, $d_n\to 0$, as $n\to\infty$. Likewise,

$$\left|\frac{1}{p_k}\int\limits_{P_k}\frac{1}{n}\ln(\bar\lambda_{i_1}\cdots\bar\lambda_{i_n})\,d\nu^+(i)-L(1)\right|\to 0,\ \text{as } n\to\infty.$$

Since $L(0)$ and $L(1)$ are positive, for n large both

$$\frac{1}{p_k}\int_{P_k}\frac{1}{n}\ln(\lambda_{i_1}\cdots\lambda_{i_n})\,d\nu^+(i)>0$$

and

$$\frac{1}{p_k}\int_{P_k}\frac{1}{n}\ln(\bar\lambda_{i_1}\cdots\bar\lambda_{i_n})\,d\nu^+(i)>0.$$

□

Let $\mathcal{M}$ be the space of all Borel probability measures on $\mathcal{I}$ endowed with the weak-star topology. For small $0 < \alpha < 1$, $q > 0$ and $c > 0$ define

$$\mathcal{N}_c = \{\mathfrak{m} \in \mathcal{M};\ \forall\, 0 \leq x \leq q,\ \mathfrak{m}_k\big([0,x)\big) \leq cx^\alpha \text{ and } \mathfrak{m}_k\big((1-x,1]\big) \leq cx^\alpha\ \forall k \in \Omega\}.$$

The condition on the measure of small intervals $[0,x)$ and $(1-x,1]$ excludes measures supported on the end points 0 and 1. Note that $\mathcal{N}_c$ depends on α and q; but we do not include this dependence in the notation. We first show that there exist ergodic stationary measures which belong to $\mathcal{N}_c$.

Proposition 4.1. *Under the assumptions of Theorem 2.1 and in particular* $\sum_{i=1}^N p_i \ln f_i'(x) > 0$ *for* $x = 0, 1$*, there exist positive* α, c, q *and* $n_1 \in \mathbb{N}$ *such that* $\mathcal{T}^{n_1}\mathcal{N}_c \subset \mathcal{N}_c$.

Proof. Note that by (1), for each $k \in \Omega$,

$$\sum_{i_1,\ldots,i_n=1}^{N} p_{i_1}\pi_{i_1 i_2}\cdots\pi_{i_{n-1}i_n}\pi_{i_n k} = p_k. \tag{10}$$

Let n_1 be a number such that for any $n \geq n_1$ the inequality (7) holds in Lemma 4.1. In the following, fix any $n \geq n_1$. Since for each k there are N^n possible transitions in $n+1$ steps ending with k we may rewrite (7) as

$$\sum_{i=1}^{N^n} \rho_i^k \ln \gamma_i > 0$$

in which $\sum_{i=1}^{N^n} \rho_i^k = 1$ by (10). We claim that there is a small $\alpha > 0$ such that our assumption $\sum_{i=1}^{N^n} \rho_i^k \ln \gamma_i > 0$ implies $\sum_{i=1}^{N^n} \rho_i^k \gamma_i^{-\alpha} < 1$. Namely, since $\lim_{\alpha\to 0} \frac{1-\gamma_i^{-\alpha}}{\alpha} = \ln \gamma_i$, $1 \leq i \leq N^n$, $\sum_{i=1}^{N^n} \rho_i^k \ln \gamma_i > 0$ implies that for sufficiently small $\alpha > 0$,

$$\sum_{i=1}^{N^n} \rho_i^k \frac{1-\gamma_i^{-\alpha}}{\alpha} > 0.$$

Multiplying by α we get

$$\sum_{i=1}^{N^n} \rho_i^k - \sum_{i=1}^{N^n} \rho_i^k \gamma_i^{-\alpha} > 0,$$

which implies $\sum_{i=1}^{N^n} \rho_i^k \gamma_i^{-\alpha} < 1$, because $\sum_{i=1}^{N^n} \rho_i^k = 1$.

A similar reasoning applies to the end point 1 of I, starting with (8) rewritten as $\sum_{i=1}^{N^n} \rho_i^k \ln \bar{\gamma}_i > 0$, to show that for α small, also $\sum_{i=1}^{N^n} \rho_i^k \bar{\gamma}_i^{-\alpha} < 1$.

Thus, there exists a small $\delta > 0$ so that

$$\sum_{i=1}^{N^n} \frac{\rho_i^k}{(\gamma_i - \delta)^\alpha} < 1 \tag{11}$$

and likewise $\sum_{i=1}^{N^n} \frac{\rho_i^k}{(\bar{\gamma}_i - \delta)^\alpha} < 1$. Moreover, for such $\delta > 0$ we are able to choose a sufficiently small $q = q(\delta) > 0$ in such a way that for each symbol sequence $i_1, \ldots, i_n$ in Ω^n,

$$f_{i_1,\ldots,i_n}^{-n}(x) \leq \frac{x}{(\lambda_{i_1}\ldots\lambda_{i_n}) - \delta},\ \forall\, 0 \leq x \leq q. \tag{12}$$

Take c with $cq^\alpha > 1$. Take a measure $\mathfrak{m}$ from the $\mathcal{N}_c$ that corresponds to α and q. We will prove $\mathcal{T}^n\mathfrak{m} \in \mathcal{N}_c$. To do this we must show that if $x \leq q$ then $(\mathcal{T}^n\mathfrak{m})_k\big([0,x)\big) \leq cx^\alpha$

and $(\mathcal{T}^n\mathfrak{m})_k\big((1-x,1]\big) \leq cx^\alpha$ for all $k \in \Omega$. Knowing that $\mathfrak{m}_k\big([0,x)\big) \leq cx^\alpha$ for each $k \in \Omega$ and applying (11), (12) we get:

$$\begin{aligned}(\mathcal{T}^n\mathfrak{m})_k\big([0,x)\big) &= \sum_{i_1,\dots,i_n=1}^{N} \frac{1}{p_k} p_{i_1}\pi_{i_1 i_2}\dots\pi_{i_{n-1}i_n}\pi_{i_n i_k} f^n_{i_1\dots i_n}\mathfrak{m}_{i_1}\big([0,x)\big) \\ &= \sum_{i_1,\dots,i_n=1}^{N} \frac{1}{p_k} p_{i_1}\pi_{i_1 i_2}\dots\pi_{i_{n-1}i_n}\pi_{i_n i_k}\mathfrak{m}_{i_1}\big(f^{-n}_{i_1\dots i_n}[0,x)\big) \\ &\leq \sum_{i_1,\dots,i_n=1}^{N} \frac{1}{p_k} p_{i_1}\pi_{i_1 i_2}\dots\pi_{i_{n-1}i_n}\pi_{i_n i_k}\mathfrak{m}_{i_1}\big([0,\frac{x}{(\lambda_{i_1}\dots\lambda_{i_n})-\delta})\big) \\ &\leq \sum_{i=1}^{N^n} \rho_i^k c\big(\frac{x}{\gamma_i-\delta}\big)^\alpha \\ &= c\left(\sum_{i=1}^{N^n}\frac{\rho_i^k}{(\gamma_i-\delta)^\alpha}\right)x^\alpha \\ &\leq cx^\alpha. \end{aligned} \tag{13}$$

Likewise, $(\mathcal{T}^n\mathfrak{m})_k\big((1-x,1]\big) \leq cx^\alpha$ for $x \leq q$. Thus, for every $\mathfrak{m} \in \mathcal{N}_c$, the image $\mathcal{T}^n\mathfrak{m}$ belongs to $\mathcal{N}_c$. □

Now we know that $\mathcal{T}^{n_1}(\mathcal{N}_c) \subset \mathcal{N}_c$. By the Krylov-Bogolyubov averaging method, for a measure $\mathfrak{m} \in \mathcal{N}_c$ on the compact metric space $\mathcal{I}$ there is a subsequence of $\{\frac{1}{n}\sum_{r=0}^{n-1}\mathcal{T}^{rn_1}\mathfrak{m}\}_{n\in\mathbb{N}}$ which is convergent to a probability measure $\hat{\mathfrak{m}} \in \mathcal{N}_c$ such that $\mathcal{T}^{n_1}\hat{\mathfrak{m}} = \hat{\mathfrak{m}}$. Note that

$$\bar{\mathfrak{m}} = \frac{1}{n_1}\big(\hat{\mathfrak{m}} + \mathcal{T}\hat{\mathfrak{m}} + \ldots + \mathcal{T}^{n_1-1}\hat{\mathfrak{m}}\big)$$

is a probability measure. Since $\mathcal{T}$ is linear and $\mathcal{T}^{n_1}\hat{\mathfrak{m}} = \hat{\mathfrak{m}}$, the measure $\bar{\mathfrak{m}}$ is a fixed point of $\mathcal{T}$:

$$\mathcal{T}\bar{\mathfrak{m}} = \frac{1}{n_1}\big(\mathcal{T}\hat{\mathfrak{m}} + \mathcal{T}^2\hat{\mathfrak{m}} + \ldots + \mathcal{T}^{n_1}\hat{\mathfrak{m}}\big) = \bar{\mathfrak{m}}.$$

We have found a stationary measure $\bar{\mathfrak{m}}$ in $\mathcal{N}_c$ for some c.

The following additional reasoning shows that there is an ergodic stationary measure in $\mathcal{N}_c$. Let $\mathcal{N}$ be the set of stationary measures on $\mathcal{I}$ which is a convex compact subset of $\mathcal{M}$. The ergodic stationary measures are the extreme points of it. Note that $\mathcal{N}_c$ is a convex compact subset of $\mathcal{N}$, which is itself also convex and compact. We claim that the extreme points of $\mathcal{N}_c$ are extreme points of $\mathcal{N}$. Suppose by contradiction that there are $\bar{\mathfrak{m}}_1, \bar{\mathfrak{m}}_2 \in \mathcal{N}\setminus\mathcal{N}_c$ and the convex combination $\bar{\mathfrak{m}} = s\bar{\mathfrak{m}}_1 + (1-s)\bar{\mathfrak{m}}_2 \in \mathcal{N}_c$. In this case, for $0 \leq x \leq q$, $\bar{\mathfrak{m}}_{1,k}([0,x)) \leq (c/s)x^\alpha$ and $\bar{\mathfrak{m}}_{1,k}((1-x,1]) \leq (c/s)x^\alpha$ and similar estimates for $\bar{\mathfrak{m}}_2$. That is, $x \mapsto \bar{\mathfrak{m}}_{i,k}([0,x))/x^\alpha$ and $x \mapsto \bar{\mathfrak{m}}_{i,k}((1-x,1])/x^\alpha$ are bounded. As $\mathcal{T}\bar{\mathfrak{m}} = \bar{\mathfrak{m}}$, we have by (11), (13) that $\bar{\mathfrak{m}} \in \mathcal{N}_{\tilde{c}}$ for some $\tilde{c} < c$. It follows that $t\bar{\mathfrak{m}}_1 + (1-t)\bar{\mathfrak{m}}_2 \in \mathcal{N}_c$ for t close to s. So s is an interior point of the set of values t for which $t\bar{\mathfrak{m}}_1 + (1-t)\bar{\mathfrak{m}}_2 \in \mathcal{N}_c$. Since $\mathcal{N}_c$ is closed it follows that $\bar{\mathfrak{m}}_i \in \mathcal{N}_c$ and the claim is proved. Since the extreme points of $\mathcal{N}$ are ergodic stationary measures, we conclude that the extreme points of $\mathcal{N}_c$ are ergodic stationary measures. Since the set of extreme points of $\mathcal{N}_c$ is nonempty by the Krein-Milman theorem, there are ergodic stationary measures.

Step 2: Trapping strips. Recall from Lemma 3.1 that a stationary measure $\mathfrak{m}$ gives rise to an invariant measure for the one-sided skew product system, with marginal ν^+ on $\Sigma^+_{\mathcal{A}}$. We will see that the supports of such invariant measures are contained in mutually disjoint trapping strips. This step closely follows [5], with adjustments to account for the fixed end points.

Definition 4.1. *A subset $\mathcal{D} = \bigcup_{k=1}^N \mathcal{D}_k \subseteq \mathcal{I}$ is called a domain if for each $k \in \Omega$, $\mathcal{D}_k$ is a closed interval in I_k.*

A boundary point of an interval $\mathcal{D}_k$ different from 0 or 1 is called an internal boundary point.

Definition 4.2. *A domain $\mathcal{D} = \bigcup_{k=1}^N \mathcal{D}_k \subseteq \mathcal{I}$ is trapping if any admissible map takes it to itself,*

$$\forall k, l : \ \pi_{kl} > 0, \ f_k(\mathcal{D}_k) \subseteq \mathcal{D}_l.$$

The domain is strict trapping if any internal boundary point of $\mathcal{D}_k$ is mapped inside the interior of $\mathcal{D}_l$.

The following proposition is [5, Proposition 4.5] and holds also here.

Proposition 4.2. *The following conditions are equivalent:*

i) the domain $\mathcal{D} = \bigcup_{k=1}^N \mathcal{D}_k \subseteq \mathcal{I}$ is (strict) trapping;
ii) the strip $\mathcal{S}^+ = \bigcup_{k=1}^N C_k^{+,0} \times \mathcal{D}_k \subseteq \Sigma^+_{\mathcal{A}} \times I$ is (strict) trapping for the skew product F^+;
iii) the strip $\mathcal{S} = \bigcup_{k=1}^N C_k^0 \times \mathcal{D}_k \subseteq \Sigma_{\mathcal{A}} \times I$ is (strict) trapping for F.

Consider an arbitrary ergodic stationary measure $\mathfrak{m} \in \mathcal{N}_c$. Denote the interval that spans the support of $\mathfrak{m}_k$ by $I_{\mathfrak{m},k} = [A_{\mathfrak{m},k}, B_{\mathfrak{m},k}]$:

$$A_{\mathfrak{m},k} = \min \operatorname{supp}(\mathfrak{m}_k),$$
$$B_{\mathfrak{m},k} = \max \operatorname{supp}(\mathfrak{m}_k).$$

For every admissible i, j we have $f_i(\operatorname{supp}(\mathfrak{m}_i)) \subseteq \operatorname{supp}(\mathfrak{m}_j)$. Since the maps f_i are monotone we have that for any admissible transition i, j,

$$f_i(I_{\mathfrak{m},i}) \subseteq I_{\mathfrak{m},j}. \tag{14}$$

Therefore, the collection $\mathcal{I}_{\mathfrak{m}} = \bigcup_{k=1}^N I_{\mathfrak{m},k}$ is a domain, which is trapping by (14).

The imposed genericity conditions imply that for a trapping domain no interval $I_{\mathfrak{m},k}$ can be a single point.

Lemma 4.2. *Consider an arbitrary trapping domain $\mathcal{I}_{\mathfrak{m}}$. Then either $A_{\mathfrak{m},k} = 0$ for all $k \in \Omega$, or $A_{\mathfrak{m},k} \neq 0$ for all $k \in \Omega$. In the latter case, there exist an attracting fixed point A of a simple return and a simple transition f such that $A_{\mathfrak{m},k} = f(A)$. In the former case, i.e. if $A_{\mathfrak{m},k} = 0$ for all $k \in \Omega$, then 0 is an attracting fixed point of a simple return. An analogous statement holds for $B_{\mathfrak{m},k}$.*

Proof. For a chosen trapping domain $\mathcal{I}_{\mathfrak{m}}$ suppose that $A_{\mathfrak{m},k} = 0$ for some $k \in \Omega$. Then, knowing that $x = 0$ is a fixed point of f_k for all k, we have for any $l \in \Omega$ such that k, l is admissible that

$$0 = f_k(0) = f_k(\min \operatorname{supp}(\mathfrak{m}_k)) \in f_k(\operatorname{supp}(\mathfrak{m}_k)) \subseteq \operatorname{supp}(\mathfrak{m}_l).$$

Hence, $A_{\mathfrak{m},l} = \min \operatorname{supp}(\mathfrak{m}_l) = 0$. Since the subshift σ is transitive $A_{\mathfrak{m},k} = 0$ for all $k \in \Omega$.

If $A_{\mathfrak{m},k} \neq 0$ for all k, [5, Lemma 6.3] applies and the result for $A_{\mathfrak{m},k}$ holds by that lemma. If $A_{\mathfrak{m},k} = 0$ for all k, the arguments of [5, Lemma 6.3] apply to yield the same conclusion (the simple transition is redundant since 0 is a fixed point of all maps). □

By Birkhoff's ergodic theorem, a generic sequence of random iterations (k_n, x_n), $x_n \in I_{k_n}$ of a $\mathfrak{m}$-generic initial point is distributed with respect to the measure $\mathfrak{m}$. If we choose such a generic initial point (k_0, x_0) then because the points (k_n, x_n) are distributed with respect to $\mathfrak{m}$, the set $X_k = \{x_n\}|_{k_n = k}$ is dense in $\operatorname{supp}(\mathfrak{m}_k)$ for any k. We apply this observation in the proof of the next lemma, which corresponds to [5, Lemma 6.7].

Lemma 4.3. *For any two trapping domains $\mathcal{I}_{\mathfrak{m}_1}$ and $\mathcal{I}_{\mathfrak{m}_2}$ of two ergodic stationary measures $\mathfrak{m}_1, \mathfrak{m}_2 \in \mathcal{N}_c$ the corresponding intervals $I_{\mathfrak{m}_1,k}$ and $I_{\mathfrak{m}_2,k}$ are either disjoint for any k or coincide for any k.*

Proof. Assume that the intervals $I_{\mathfrak{m}_1,k}$ and $I_{\mathfrak{m}_2,k}$ intersect but do not coincide. Then, there is at least one end point of one of them that does not belong to the other one. Without loss of generality let it be the point $B_{\mathfrak{m}_1,k}$. There is a neighborhood V of $B_{\mathfrak{m}_1,k}$ such that $I_{\mathfrak{m}_2,k} \cap V = \emptyset$.

By genericity condition (4), $A_{\mathfrak{m}_1,k}$ is different from $B_{\mathfrak{m}_2,k}$. So there are generic points of $\mathfrak{m}_1$ in $I_{\mathfrak{m}_1,k} \cap I_{\mathfrak{m}_2,k}$. Choose a generic point p_0 for $\mathfrak{m}_1$ in $I_{\mathfrak{m}_1,k} \cap I_{\mathfrak{m}_2,k}$ which is different from $A_{\mathfrak{m}_1,k}$ and $B_{\mathfrak{m}_2,k}$. There is an admissible return g such that $g(p_0) \in V$ (recall the observation that precedes the lemma), which implies $g(p_0) \notin I_{\mathfrak{m}_2,k}$. On the other hand $p_0 \in I_{\mathfrak{m}_2,k}$ by assumption and $g(I_{\mathfrak{m}_2,k}) \subseteq I_{\mathfrak{m}_2,k}$ by (14). Since the diffeomorphisms f_i's are monotone $g(p_0) \in I_{\mathfrak{m}_2,k}$. This is a contradiction. Therefore, $I_{\mathfrak{m}_1,k}$ and $I_{\mathfrak{m}_2,k}$ have empty intersection or coincide. □

Again consider trapping domains $\mathcal{I}_{\mathfrak{m}}$ corresponding to ergodic stationary measures $\mathfrak{m}$ in $\mathcal{N}_c$. According to Lemma 4.3 these trapping domains are non-intersecting or coincide. By Lemma 4.2 for each trapping domain $\mathcal{I}_{\mathfrak{m}}$ each end point $A_{\mathfrak{m},k}$ and $B_{\mathfrak{m},k}$ which does not coincide with $x = 0$ or $x = 1$ (respectively) is an image of a fixed point of a simple return by a simple transition. On the other hand, since Ω has a finite number of symbols there is only a finite number of simple returns and simple transitions and by condition (2.1) in Section 2.1 any simple return has only finitely many fixed points. Hence, for any $k \in \Omega$ only a finite number of $I_{\mathfrak{m},k}$'s can exist in I. Therefore, we conclude that there are finitely many disjoint trapping domains and corresponding to them finitely many disjoint trapping strips for F by Proposition 4.2. For every stationary measure $\mathfrak{m} \in \mathcal{N}_c$ the corresponding domain $\mathcal{I}_{\mathfrak{m}} = \bigcup_{k=1}^{N} I_{\mathfrak{m},k}$ and strip $\mathcal{S}_{\mathfrak{m}} = \bigcup_{k=1}^{N} C_k^0 \times I_{\mathfrak{m},k}$ are equal to some trapping domain and trapping strip.

We thus obtain a finite number of stationary measures $\mathfrak{m}_t$, $1 \leq t \leq T$, with corresponding trapping domain $\mathcal{I}^t$ and trapping strip $\mathcal{S}^t$.

Step 3: Conditional measures. We will see that inside each trapping strip $\mathcal{S}^t$, $1 \leq t \leq T$, there exists a unique invariant measurable graph Γ^t to which almost every point of the trapping strip is attracted. First we show that for each $1 \leq t \leq T$, $\mu^t = \mu_{\mathfrak{m}^t}$ has δ-measures as conditional measures along fibers inside the trapping strip $\mathcal{S}^t$, ν-almost surely. To prove the following lemma we follow [1, Theorem 1.8.4].

Lemma 4.4. *For every ergodic stationary probability measure $\mathfrak{m}$, the conditional measure $\mu_{\mathfrak{m},\omega}$ of $\mu_{\mathfrak{m}}$ is a δ-measure for ν-almost every $\omega \in \Sigma_{\mathcal{A}}$.*

Proof. Consider a $\mu_{\mathfrak{m}}$ and its conditional measures $\mu_{\mathfrak{m},\omega}$. Let $X_{\mathfrak{m}}(\omega)$ be the smallest median of $\mu_{\mathfrak{m},\omega}$, i.e. the infimum of all points x for which

$$\mu_{\mathfrak{m},\omega}([0,x]) \geq \frac{1}{2} \text{ and } \mu_{\mathfrak{m},\omega}([x,1]) \geq \frac{1}{2}.$$

The set of medians of $\mu_{\mathfrak{m},\omega}$ is a compact interval and $X_{\mathfrak{m}} : \Sigma_{\mathcal{A}} \to I$ is measurable. Define $C^-_{\mathfrak{m}}(\omega) := [0, X_{\mathfrak{m}}(\omega)]$ for which by definition $\mu_{\mathfrak{m},\omega}(C^-_{\mathfrak{m}}(\omega)) \geq \frac{1}{2}$. The set $C^-_{\mathfrak{m}}(\omega)$ is invariant: Since for every $i \in \Omega$, f_i is increasing for every $x_1 < x_2$ and ω we have $f_\omega(x_1) < f_\omega(x_2)$. This implies that x is a median of $\mu_{\mathfrak{m},\omega}$ if and only if $f_\omega(x)$ is a median of $f_\omega \mu_{\mathfrak{m},\omega}$. By invariance of $\mu_{\mathfrak{m}}$ we have $f_\omega \mu_{\mathfrak{m},\omega} = \mu_{\mathfrak{m},\sigma\omega}$. Hence, $X_{\mathfrak{m}}(\sigma\omega) = f_\omega(X_{\mathfrak{m}}(\omega))$ which implies $C^-_{\mathfrak{m}}(\sigma\omega) = f_\omega(C^-_{\mathfrak{m}}(\omega))$.

Because $\mu_{\mathfrak{m}}$ is ergodic and $C^-_{\mathfrak{m}}(\omega)$ is invariant $\mu_{\mathfrak{m},\omega}(C^-_{\mathfrak{m}}(\omega)) = 1$, ν-almost surely. By the same argument for $C^+_{\mathfrak{m}}(\omega) := [X_{\mathfrak{m}}(\omega), 1]$ for $\{X_{\mathfrak{m}}(\omega)\} = C^-_{\mathfrak{m}}(\omega) \cap C^+_{\mathfrak{m}}(\omega)$ we obtain $\mu_{\mathfrak{m},\omega}(\{X_{\mathfrak{m}}(\omega)\}) = 1$. Thus $\mu_{\mathfrak{m},\omega} = \delta_{X_{\mathfrak{m}}(\omega)}$ for ν-almost every $\omega \in \Sigma_{\mathcal{A}}$. □

Lemma 4.5. *Every trapping strip contains a unique stationary measure with support contained in the trapping strip.*

Proof. Suppose there are two invariant ergodic measures $\mu_{\mathfrak{m}_1} \neq \mu_{\mathfrak{m}_2}$ for which $\mathcal{S}_{\mathfrak{m}_1} = \mathcal{S}_{\mathfrak{m}_2}$. By Lemma 4.4 there are measurable functions $X_{\mathfrak{m}_i} : \Sigma_{\mathcal{A}} \to I$ and $D_i \subset \Sigma_{\mathcal{A}}$ with $\nu(D_i) = 1$, for $i = 1, 2$, such that $\lim_{n\to\infty} f^n_{\sigma^{-n}\omega} \mu^+_{\mathfrak{m}_i,\sigma^{-n}\omega} = \delta_{X_{\mathfrak{m}_i}(\omega)}$ for every $\omega \in D_i$ respectively. From $\nu(D_i) = 1$ we have $D_1 \cap D_2 \neq \emptyset$. Therefore, there is $\bar\omega \in D_1 \cap D_2$ so that $X_{\mathfrak{m}_1}(\bar\omega) \neq X_{\mathfrak{m}_2}(\bar\omega)$. Without loss of generality suppose that $X_{\mathfrak{m}_1}(\bar\omega) < X_{\mathfrak{m}_2}(\bar\omega)$.

Since $\mathcal{S}_{\mathfrak{m}_1} = \mathcal{S}_{\mathfrak{m}_2}$ we have that for every $k \in \Omega$, $I_{\mathfrak{m}_1,k} = I_{\mathfrak{m}_2,k}$. So we can find generic points $(k, x_{1,k})$ and $(k, x_{2,k})$ for $\mathfrak{m}_1$ and $\mathfrak{m}_2$ such that $x_{1,k} > x_{2,k}$. Because $f^n_{\sigma^{-n}\bar\omega}(x_{i,\bar\omega_{-n}})$ converges to $X_{\mathfrak{m}_i}(\bar\omega)$ as $n \to \infty$, and for each $j \in \Omega$, f_j is strictly increasing, we conclude that $X_{\mathfrak{m}_2}(\bar\omega) < X_{\mathfrak{m}_1}(\bar\omega)$, contradicting our assumption. Thus, $\mu_{\mathfrak{m}_1} = \mu_{\mathfrak{m}_2}$ is unique in $\mathcal{S}_{\mathfrak{m}}$. □

Step 4. Attracting graphs. By Lemma 4.4 and 4.5, for every $1 \leq t \leq T$ there exists a unique measurable function $X^t : \omega \mapsto X^t(\omega)$ for each $\mathcal{S}^t$ with the domain $D^t \subset \Sigma_{\mathcal{A}}$, $\nu(D^t) = 1$, such that $\lim_{n\to\infty} f^n_{\sigma^{-n}\omega} \mu^{t,+}_{\sigma^{-n}\omega} = \delta_{X^t(\omega)}$ for each $\omega \in D^t$. So there are graphs Γ^t of X^t with $\Gamma^t \subset \mathcal{S}^t$ which are invariant because $X^t(\sigma\omega) = f_\omega(X^t(\omega))$. Therefore, for every generic point (k, x_k) for $\mathfrak{m}^t$ we have $\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}(x_{\omega_{-n}}) = X^t(\omega)$. Since the fiber maps are strictly increasing for every choice of (k, x_k) with $x_k \in I_{\mathfrak{m}^t,k}$ (different from $0, 1$) and $\omega \in D^t$,

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}(x_{\omega_{-n}}) \to X^t(\omega). \tag{15}$$

For a trapping strip $\mathcal{S}^t$ denote by A^t_ω the intersection of its maximal attractor, $A^t_{\max}$, with the fiber $I_\omega = \{\omega\} \times I$. For an $\omega \in \Sigma_{\mathcal{A}}$ define

$$A^t_{\omega,n} = f_{\omega_{-1}} \circ \cdots \circ f_{\omega_{-n}}(\mathcal{S} \cap I_{\sigma^{-n}\omega}).$$

Since the strip $\mathcal{S}^t$ is trapping for every $n \in \mathbb{N}$, $A^t_{\omega,n+1} \subseteq A^t_{\omega,n}$. Hence, for each $\omega \in \Sigma_{\mathcal{A}}$, $A^t_\omega = \bigcap_{n\geq 0} A^t_{\omega,n}$ is either an interval or a single point. So for every $\omega \in D^t$, $X(\omega) \in A^t_\omega$ and $\Gamma^t \subset A^t_{\max}$.

We state two theorems on the structure of the maximal attractor in internal and in border trapping strips, respectively. The first result is contained in [5].

Theorem 4.1. *Let $\mathcal{S}^t$ be an internal trapping strip of a skew product F under the assumptions of Theorem 2.1. Then the maximal attractor A^t_{max} is a bony graph. The attracting graph Γ^t forms the graph part of A^t_{max}.*

Theorem 4.2. *Let $\mathcal{S}^t$ be a border trapping strip of a skew product F under the assumptions of Theorem 2.1. Then the maximal attractor A^t_{max} is a thick graph. The attracting graph Γ^t in $\mathcal{S}^t$ is dense in A^t_{max}:*

$$\overline{\Gamma^t} = A^t_{max}. \tag{16}$$

Proof. We suppress the index t from the notation. Consider, without loss of generality, a border trapping strip $\mathcal{S}$ that contains $\Sigma_{\mathcal{A}} \times \{0\}$. For a trapping strip $\Sigma_{\mathcal{A}} \times [0,1]$, the maximal attractor obviously has positive standard measure. For a trapping strip with one internal boundary, since $x = 0$ is a fixed point for each f_ω and $\Gamma \subset \mathcal{S}$, we have $A_\omega = [0, X(\omega)]$, where $X(\omega) > 0$ for $\omega \in D$. So, ν-almost surely A_ω has positive Lebesgue measure and $s(A_{\max}) > 0$.

Now we prove the density of the graph Γ for border trapping strips. We restrict to the case of a trapping strip with one internal boundary. Let $\mathfrak{m}$ be the ergodic stationary measure supported in $\mathcal{S} = \mathcal{S}_{\mathfrak{m}} = \bigcup_{k=1}^N C^0_k \times I_{\mathfrak{m},k}$. By Lemma 4.2, there is a simple return map f with $f'(0) < 1$. Hence, for some $1 \leq k_0 \leq N$ there are admissible return maps $f, g : I_{\mathfrak{m},k_0} \to I_{\mathfrak{m},k_0}$ and J, a small neighborhood of $x = 0$, such that for every $k \in \Omega$, $J \subset I_{\mathfrak{m},k}$, $f'(x) < 1$ and $g'(x) > 1$ for every $x \in J$.

Lemma 4.6. *All orbits of the iterated function system generated by f, g restricted to J are dense in it, i.e., for any point $x \in J$ and an open interval $J' \subset J$ there is a return map $h_{k_0} \in \mathrm{IFS}\,\{f, g\}$ such that $h_{k_0}(x) \in J'$.*

Proof. This follows [2, Lemma 3], see Proposition 2.1. □

Lemma 4.7. *Let $\mathfrak{m}$ be an ergodic stationary measure in $\mathcal{N}_c$, k and k' be two arbitrary symbols and $I_{\mathfrak{m},k}, I_{\mathfrak{m},k'}$ with $A_{\mathfrak{m},k}, A_{\mathfrak{m},k'} = 0$ and $B_{\mathfrak{m},k}, B_{\mathfrak{m},k'} \neq 1$. Then, for any $\varepsilon > 0$ there exists an admissible composition $G : I_{\mathfrak{m},k} \to I_{\mathfrak{m},k'}$ such that $G(I_{\mathfrak{m},k}) \subset U_\varepsilon(A_{\mathfrak{m},k'})$.*

Proof. See [5, Lemma 6.9]. □

To establish (16), we need to show that for every point $P \in A_{\max}$ and every neighborhood $\mathcal{U}$ of P there exists a point Q in Γ such that $Q \in \mathcal{U}$. We may assume $\mathcal{U} = C^{-m,\dots,m}_{\hat{\omega}} \times U$, $m \in \mathbb{N}$, where $\hat{\omega} \in D$ and U is a small interval in $[0, X(\hat{\omega})]$. We need to find $\tilde{\omega} \in C^{-m,\dots,m}_{\hat{\omega}} \cap D$ such that $X(\tilde{\omega}) \in U$. Take a sequence $\omega' \in D$ with past part $\dots\omega'_{-3}\omega'_{-2}\omega'_{-1} = \omega'_{-}$ and denote

$x' = X(\omega')$. Note that X depends only on the past part of the sequence, so x' depends on ω'_-.

By Lemma 4.7 there is an admissible composition $G = f_{\alpha_n} \circ ... \circ f_{\alpha_1} : I_{\mathfrak{m},\omega'_0} \to I_{\mathfrak{m},k_0}$ so that $G(I_{\mathfrak{m},\omega'_0}) \subset J$ and $\alpha_1 = \omega'_0$. Hence, $G(x') \in J$. Let $U_m \subset J \subset I_{\mathfrak{m},k_0}$ be defined by an admissible composition such that

$$f_{\hat{\omega}_{-1}} \circ ... \circ f_{\hat{\omega}_{-m}} \circ f_{\beta_\kappa} \circ \cdots \circ f_{\beta_1}(U_m) = U,$$

where $\beta_1 = k_0$. By Lemma 4.6 there is a return map $h_{k_0} = f_{\eta_r} \circ \cdots \circ f_{\eta_1} : I_{\mathfrak{m},k_0} \to I_{\mathfrak{m},k_0}$ in IFS $\{f, g\}$ that takes $G(x')$ to U_m. Take

$$\tilde{\omega} = \omega'_- \alpha_1 \ldots \alpha_n \eta_1 \ldots \eta_r \beta_1 \ldots \beta_\kappa \hat{\omega}_{-m} ... \hat{\omega}_0 ... \hat{\omega}_m \omega''_+,$$

where $\tilde{\omega}_0 = \hat{\omega}_0$ and ω''_+ is any admissible sequence. Indeed, with such a construction the sequence $\tilde{\omega}$ belongs to $C^{-m,\ldots,m}_{\hat{\omega}} \cap D$, $f^{n+r+\kappa}_{\sigma^{-(n+r+\kappa+m)}\tilde{\omega}}(x') \in U_m$ and $f^m_{\sigma^{-m}\tilde{\omega}}(U_m) \in U$. Therefore, $X(\tilde{\omega}) = f^{(n+r+\kappa+m)}_{\sigma^{-(n+r+\kappa+m)}\tilde{\omega}}(x') \in U$ and $Q = (\tilde{\omega}, X(\tilde{\omega})) \in \Gamma \cap \mathcal{U}$. □

4.2. **An attracting end point.** It remains to consider cases with negative Lyapunov exponents at end points, i.e. where $L(0) < 0$ or $L(1) < 0$ or both. Note that in an internal trapping strip or border trapping strip bounded by an end point with positive Lyapunov exponent, stationary measures are constructed as before and the analysis proceeds as in the previous sections.

The following subcases remain:

(1) $L(0)$ and $L(1)$ have different signs and F has no internal trapping strip,
(2) $L(0)$ and $L(1)$ are both negative and F has no internal trapping strip,
(3) at least one of $L(0)$ or $L(1)$ is negative and F admits an internal trapping strip.

$L(0)$ and $L(1)$ have different signs and F has no internal trapping strip. Here $L(0)$ and $L(1)$ have different signs. To be definite, say $L(0) > 0$ and $L(1) < 0$. We claim that the only ergodic stationary measures are point measures $\mathfrak{d}^0$, $\mathfrak{d}^1$ on 0 and 1; $\mathfrak{d}^0_k = p_k \delta_0$ and $\mathfrak{d}^1_k = p_k \delta_1$ on the intervals I_k. Indeed, let $\mathfrak{m}$ be a stationary measure that assigns mass outside the points $\{0, 1\}$. Suppose that the convex hull of the support of $\mathfrak{m}$ is a union of intervals $[A_k, 1] \subset I_k$. This defines a trapping strip that we denote by $\mathcal{S}$. As in the previous section, the stationary measure $\mathfrak{m}$ gives rise to an attracting invariant graph Γ of a map $X : D \to [0, 1]$ with $\nu(D) = 1$, so that Γ lies inside $\mathcal{S}$ and for every $(\sigma^{-n}\omega, x_{\omega_{-n}}) \in \mathcal{S}$, $\omega \in D$ (and $x_i \neq 0, 1$),

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}(x_{\omega_{-n}}) \to X(\omega) \tag{17}$$

(compare (15)). Since the skew product system has negative Lyapunov exponent at the endpoint 1, $\Sigma_{\mathcal{A}} \times \{1\}$ has a basin of attraction with positive standard measure. This contradicts (17). Hence, there is no stationary measure $\mathfrak{m}$ that assigns mass outside the points $\{0, 1\}$. It follows that for almost all $\omega \in \Sigma$,

$$\lim_{n\to\infty} f^n_{\sigma^{-n}\omega}(x) = 1$$

for any $x \in (0, 1]$.

$L(0)$ and $L(1)$ are both negative and F has no internal trapping strip. By the reasoning in the previous section, the inverse skew product map admits an attracting invariant graph. The skew product system hence has a repelling invariant graph, and attracting invariant graphs $\Sigma_{\mathcal{A}} \times \{0\}$, $\Sigma_{\mathcal{A}} \times \{1\}$.

At least one of $L(0)$ or $L(1)$ is negative and F admits an internal trapping strip. Suppose $L(0) < 0$. Again by following the reasoning in the previous section, the inverse skew product map then admits a border trapping strip, bounded by $\Sigma_{\mathcal{A}} \times \{0\}$, that contains an attracting invariant graph. The skew product system hence has a repelling invariant graph, and an attracting invariant graph $\Sigma_{\mathcal{A}} \times \{0\}$.

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