Intermittency in families of unimodal maps

Ale Jan Homburg
Department of Mathematics
Utrecht University
Budapestlaan 6
3584 CD Utrecht
The Netherlands *
alejan@science.uva.nl

Todd Young
Department of Mathematics
Morton Hall
Ohio University
Athens, OH 45701
U.S.A.
young@math.ohiou.edu

March 22, 2001

Abstract

We consider intermittency in one parameter families of unimodal maps, induced by saddle node and boundary crisis bifurcations. In these bifurcations a periodic orbit or a periodic interval, respectively, disappears to give rise to chaotic bursts. We prove asymptotic formulae for the frequency with which orbits visit the region previously occupied by the attractor. For this, we extend results of Pianigiani on conditionally invariant measures for the logistic family to more general families.

1 Introduction

1.1 Intermittency

Since its first description in [PomMan80], intermittency has received much attention in the mathematical and applied literature. In time series, intermittent dynamics manifests itself by alternating phases with different characteristics. In one phase, referred to as the laminar phase, the dynamics appear to be nearly periodic. While in the other phase, the orbit makes large, seemingly chaotic excursions away the periodic region. These excursions are called chaotic bursts.

In one dimensional dynamics, systems depending on a parameter exhibit parameter intervals (windows) for which the map has a periodic absorbing region. Typically, the absorbing region is created on one side by a saddle node bifurcation. As the parameter continues to change, the stable periodic point within the absorbing region undergoes a period doubling cascade and the dynamics

*present address: KdV-Institute for Mathematics, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands

of the region become chaotic. However the region remains periodic and absorbing. Finally, the region is destroyed by a boundary crisis, that is, the attractor within the region collides with the boundary of the region.

For parameter values outside, but near, a window, the region where the periodic attractor existed continues to act as a pseudo-absorbing region in that orbits which fall into the region typically remain there for a relatively long time, giving the appearance of periodicity. The dynamics outside the pseudo-periodic region exhibit characteristics of hyperbolic dynamics and once an orbit exits the pseudo-periodic region, it typically follows a chaotic-like trajectory until it happens to again fall into the pseudo-absorbing region. Thus, these bifurcations are sources of intermittency. A time series of the logistic family for a parameter value just after a boundary crisis bifurcation is in Figure 1, clearly showing intermittent dynamics. Figure 3 shows an intermittent time series of the logistic family for a parameter just before the saddle node bifurcation of the period three orbit.

We will be interested in the frequency with which iterates are in the laminar phase, as depending on the parameter. Pomeau and Manneville [PomMan80] studied the saddle node bifurcation and based on heuristic arguments, gave an asymptotic estimate of the time spent in each phase. Grebogi, Ott and Yorke [GreOttYor83] made a similar heuristic estimate for the boundary crisis. We will prove the estimates in both of these cases for one parameter families of unimodal maps on an interval, see Theorems A and B below. This is complicated by the complexity of the relaminarization process; as we will see, the frequency of laminar dynamics depends wildly on the parameter. As a consequence, the asymptotics of the frequency of laminar dynamics can only be derived for a subset of parameter values, but this is a relatively large subset. Specifically, the subsets have positive Lebesgue density at the saddle node and full density at the boundary crisis.

1.2 General assumptions and notations

Throughout the paper, $\{f_\gamma\}$ will stand for a one parameter family of unimodal maps on an interval M, of class C^3 jointly in $x \in M$ and $\gamma \in \mathbb{R}$, so that:

- f_γ has a unique maximum at a critical point c,
- $D^2 f_\gamma(c) < 0$,
- $f_\gamma(\partial M) \subset \partial M$,
- f_γ has negative Schwarzian derivative, i.e., $\frac{D^3 f_\gamma}{D f_\gamma} - \frac{3}{2}\left(\frac{D^2 f_\gamma}{D f_\gamma}\right)^2 < 0$.

If a is a periodic point of f_γ, then we denote by $W^s(a)$ the set of all points whose ω-limit sets consist of the orbit, $\mathcal{O}(a)$, of a. We call any periodic orbit $\mathcal{O}(a)$ for which $W^s(a)$ contains open intervals a periodic attractor. The set of components of $W^s(a)$ which contain $\mathcal{O}(a)$, we call the immediate basin of the attractor. The unique component of $W^s(a)$ which contains a we call the immediate basin of a. As a consequence of the last item, any f_γ has at most one periodic attractor and such an attractor contains c in its immediate basin of attraction (see Theorem III.4.1 in [MelStr93]).

We will let m denote Lebesgue measure. Given a set A denote by $K_A : M \times \mathbb{R} \to \mathbb{N}$ the function

$$K_A(x,\gamma) \quad = \quad \min\{j \geq 0 : f_\gamma^j(x) \in A\}. \tag{1}$$

We will denote by Φ_A the function

$$\Phi_A(x,\gamma) \quad = \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} 1_A(f_\gamma^i(x)), \tag{2}$$

whenever the limit exists, where 1_A is the usual indicator function of the set A.

1.3 The boundary crisis bifurcation

We say that $\{f_\gamma\}$ unfolds a boundary crisis if it satisfies:

- At $\gamma = 0$, there is a turning point $c \in M$ and a subinterval N containing c so that $f_0^k(N) = N$ and $f_0^i(N) \cap N = \emptyset$ for $0 < i < k$. The interval N is of the form (a,b), where $f_0^k(a) = f_0^k(b) = a$,
- $D^2 f_0^k(c) \frac{\partial}{\partial\gamma} f_\gamma^k(c) < 0$ at $\gamma = 0$.

The periodic point a is hyperbolic repelling since f_γ has negative Schwarzian derivative and thus its continuation exists for γ close to 0. Hence, for γ close to 0, the continuation of N, which we also denote by N, exists. At $\gamma = 0$, $\bar{N} = \cup_{i=0}^{k-1} f_0^i(N)$ is a union of k disjoint intervals. Its continuation for γ close to 0 will also be denoted by $\bar{N}$. The assumption $f_0^k(N) = N$ implies that $f_0^k(c) = b \in \partial N$. The inequality $D^2 f_0^k(c) \frac{\partial}{\partial\gamma} f_\gamma^k(c) < 0$ implies that for $\gamma > 0$ small, $f_\gamma^k(c) \notin N$ and thus the periodicity of the interval N is destroyed and the attractor in N may 'explode'. For $\gamma > 0$ small, let

$$\bar{E}_\gamma \quad = \quad \{x \in \bar{N};\ \ f_\gamma^k(x) \notin \bar{N}\}$$

be the set of points mapped outside of $\bar{N}$ by f_γ^k; $\bar{E}_\gamma$ is the union of k disjoint small intervals. We will write E_γ for the component of $\bar{E}_\gamma$ inside N.

The following theorem discusses $\Phi_{\bar{N}}(x,\gamma)$ for γ near 0. It will appear that $\Phi_{\bar{N}}$ depends very discontinuously on γ, but is continuous at 0 when restricted to a set of parameter values of large relative measure. The theorem will be proved in section 5 using material in the earlier sections. These sections contain some extensions and additional statements as well.

Theorem A *Let $\{f_\gamma\}$ be as above, unfolding a boundary crisis bifurcation at $\gamma = 0$. For each $\gamma_0 > 0$ and each rational number $\frac{r}{s} \in (0,1)$, there exists $0 < \gamma < \gamma_0$ so that*

$$\Phi_{\bar{N}}(x,\gamma) \quad = \quad \frac{r}{s},$$

for x from a subset of M of full measure.

There exists a set Γ of parameter values of positive measure, with 0 a density point of Γ, i.e.

$$\lim_{\gamma \searrow 0} \frac{m\,(\Gamma \cap [0,\gamma))}{\gamma} \quad = \quad 1,$$

so that for $\gamma \in \Gamma$, f_γ possesses an absolutely continuous invariant measure (a.c.i.m.) ν_γ. Restricting to $\gamma \in \Gamma$, $\Phi_{\bar{N}}(x,\gamma)$ is a constant, $\Phi_{\bar{N}}(\gamma)$, almost everywhere on M and depends continuously on γ at $\gamma = 0$. The following limit holds for $\Phi_{\bar{N}}(\gamma)$:

$$\lim_{\gamma\in\Gamma, \gamma \searrow 0} \frac{1-\Phi_{\bar{N}}(\gamma)}{|\ln\gamma|\sqrt{\gamma}} \quad = \quad \frac{\xi}{2\ln\lambda}, \tag{3}$$

where

$$\begin{aligned} \xi &= 2\frac{d\nu_0}{dm}(c)\sqrt{\left|\frac{\frac{\partial}{\partial\gamma}f_\gamma^k(c)|_{\gamma=0}}{D^2 f_0^k(c)}\right|}, \\ \lambda &= Df_0^k(a), \end{aligned}$$

and where $\frac{d\nu_0}{dm}$ is the Radon-Nikodym derivative of ν_0, i.e. $\frac{d\nu_0}{dm}(c)$ is the density of ν_0 at c.

Remark 1.1 *Discontinuity of invariant measures of f_γ at $\gamma = 0$ was observed by Tsujii [Tsu96] and Thunberg [Thu98]; their observations imply discontinuity of $\Phi_{\bar{N}}$ at $\gamma = 0$. That a set of parameter values of positive measure exists for which f_γ admits an a.c.i.m. follows from the celebrated work by Jakobson [Jak81] and its extensions. The set Γ is fixed in Proposition 5.1 below.*

Formula (3) says that the relative frequency with which orbits are outside of $\bar{N}$, behaves as $\sqrt{\gamma}|\ln\gamma|$ for $\gamma \in \Gamma$ small and positive. The formula makes precise and proves estimates first given in [GreOttYor83]. The measure ν_γ, $\gamma \in \Gamma$, is ergodic, so that by Birkhoff's ergodic theorem

$$\lim_{\gamma\in\Gamma, \gamma \searrow 0} \frac{1-\nu_\gamma(\bar{N})}{|\ln\gamma|\sqrt{\gamma}} \quad = \quad \frac{\xi}{2\ln\lambda}.$$

An example of a plot of $\Phi_{\bar{N}}$ is shown in Figure 2 for a boundary crisis in the quadratic family.

1.4 The saddle node bifurcation

We include a similar study of intermittency triggered by a saddle node bifurcation, called intermittency of type I in [PomMan80]. We say that $\{f_\gamma\}$ unfolds a saddle node if

- there is a k-periodic point a, with $Df_0^k(a) = 1$, and
- $D^2 f_0^k(a) \frac{\partial}{\partial\gamma} f_\gamma^k(a) > 0$ at $\gamma = 0$.

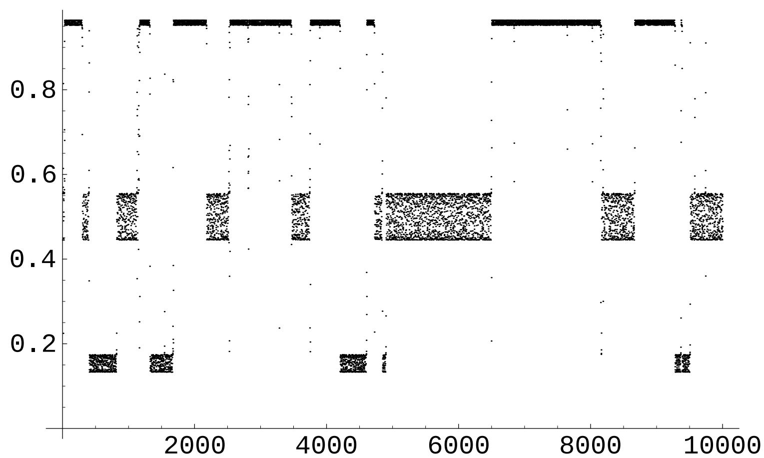

Figure 1: Time series for the third iterate of the quadratic map $x \mapsto \mu x(1-x)$ for $\mu = 3.856804$ near the boundary crisis bifurcation of the period three window.

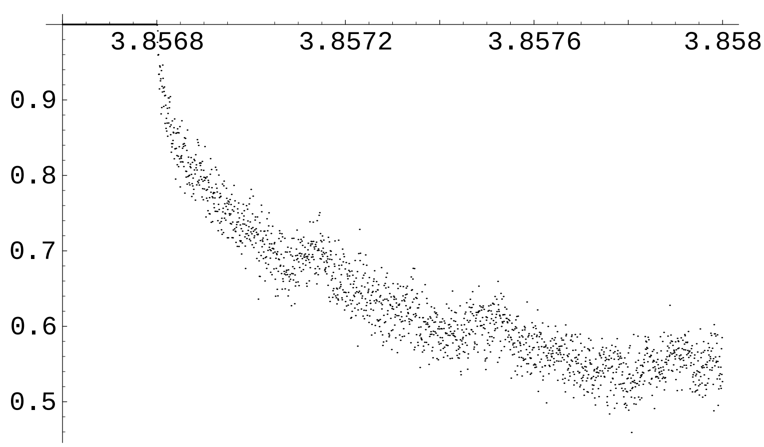

Figure 2: Numerically computed values of $\Phi_{\bar{N}}(\gamma)$ for the quadratic map $x \mapsto \mu x(1-x)$ near the boundary crisis bifurcation of the period three window.

For the sake of clarity we will assume that

$$\frac{\partial}{\partial\gamma} f_\gamma^k(a)|_{\gamma=0} > 0 \quad \text{and} \quad D^2 f_0^k(a) > 0.$$

Note that this implies $f_\gamma^k(x) > x$ for $\gamma > 0$ and all x in a small neighborhood U of a.

If we let a be the point in the saddle node periodic orbit whose immediate basin contains c, then the immediate basin of a is a k-periodic interval $N = [a, b]$, where $f_0^k(b) = a$ and $c \in (a, b)$. Let $\bar{U}$ be a small neighborhood of $\mathcal{O}(a)$, not containing a critical point of f_0^k. Let $\Phi_{\bar{U}}$ be defined as in (2) to be the relative frequency with which the orbit $\mathcal{O}(x)$ visits $\bar{U}$ (for those x for which the limit exists). The following theorem discusses $\Phi_{\bar{U}}(x, \gamma)$ for γ near 0. Its proof can be found in sections 6, 7 and 8.

Theorem B *Let $\{f_\gamma\}$ be as above, unfolding a saddle node bifurcation at $\gamma = 0$. For each $\gamma_0 > 0$ and each rational number $\frac{r}{s} \in (0, 1)$, there exists $0 < \gamma < \gamma_0$ so that*

$$\Phi_{\bar{U}}(x, \gamma) \quad = \quad \frac{r}{s},$$

for x from a subset of M of full measure.

There exist sets Γ of parameter values of positive measure and positive density at 0, i.e.

$$\lim_{\gamma \searrow 0} \frac{m\left(\Gamma \cap [0, \gamma)\right)}{\gamma} \quad > \quad 0, \tag{4}$$

so that restricting to $\gamma \in \Gamma$, $\Phi_{\bar{U}}(x, \gamma)$ is a constant, $\Phi_{\bar{U}}(\gamma)$, almost everywhere on M and depends continuously on γ at $\gamma = 0$. For each such set Γ, there exists $C > 0$ so that

$$\lim_{\gamma \in \Gamma, \gamma \searrow 0} \frac{1 - \Phi_{\bar{U}}(\gamma)}{\sqrt{\gamma}} \quad = \quad C. \tag{5}$$

Remark 1.2 *Each set Γ will consist of a countable union of intervals approaching zero, i.e.*

$$\Gamma \quad = \quad \cup_{n=n_0}^{\infty} \Gamma_n.$$

Each interval Γ_n will itself be a 'periodic window' and the periodic intervals within that set will have period $m + nk'$ for some m and k'. For each set Γ in Theorem B, the number of iterates in the relaminarization phase, that is, between two passages through $\bar{U}$, is constant, but the number of iterates in $\bar{U}$ grows with n. In each Γ_n there are parameter values γ for which f_γ has either a.c.i.m's or periodic attractors. Both occur with positive density at $\gamma = 0$, as in unfoldings of saddle node cycles in diffeomorphisms [DiaRocVia96].

In Figure 4 we show the plot of $\Phi_{\bar{U}}$ in the quadratic family near a saddle-node.

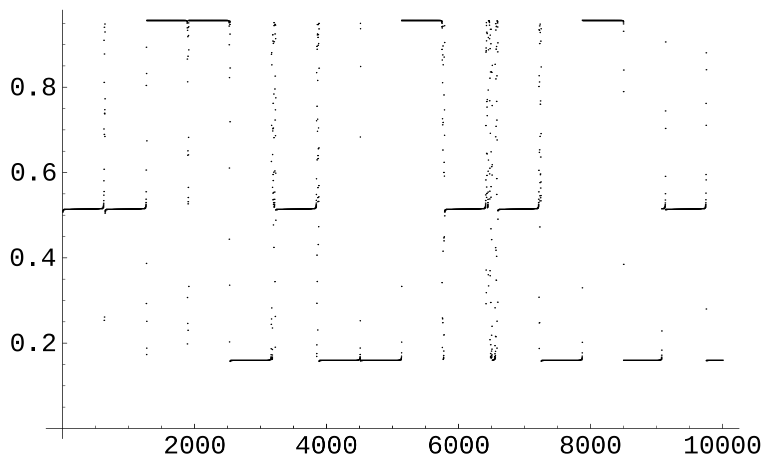

Figure 3: Time series for the third iterate of the quadratic map $x \mapsto \mu x(1-x)$ for $\mu = 3.82842676$ near the saddle node bifurcation of the period three orbit.

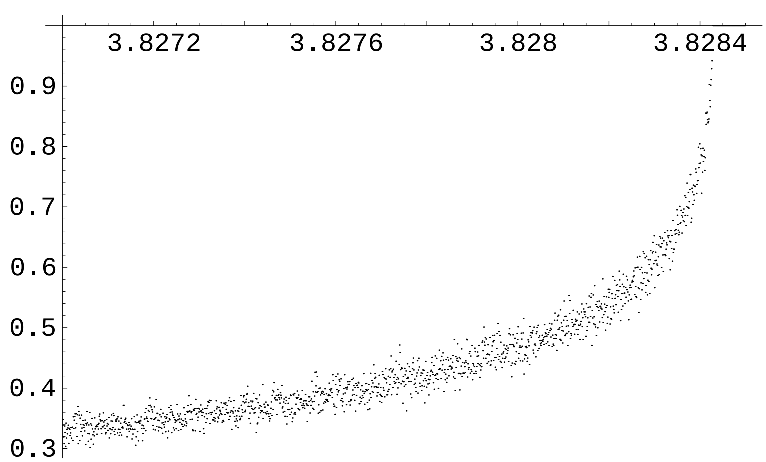

Figure 4: Numerically computed values of $\Phi_{\bar{U}}$ for the quadratic map $x \mapsto \mu x(1-x)$ near the saddle node bifurcation of the period three orbit.

2 Conditionally invariant measures

Let $\{g_\gamma\}$ be a smooth family of unimodal maps on a compact interval N, with a maximum at a critical point c. Assume that g_γ has negative Schwarzian derivative and satisfies

- $g_\gamma(\partial N) \subset \partial N$.
- $g_0(c) \in \partial N$,
- $\frac{\partial}{\partial \gamma} g_\gamma(c) > 0$.
- $Dg^2_\gamma(c) < 0$.

Hence, for $\gamma > 0$, g_γ maps a small interval, E_γ, around the critical point to the outside of N. For g_γ one may read f^k_γ on the interval N as in the definition of boundary crisis bifurcation.

A conditionally invariant measure for g_γ, $\gamma \geq 0$, is a measure μ_γ on N so that for some $\alpha \geq 0$ and any measurable $A \subset N$, $\mu_\gamma(g_\gamma^{-1}(A)) = \alpha \mu_\gamma(A)$. Note that $\alpha = \mu_\gamma(g_\gamma^{-1}(N))$. We will study conditionally invariant measures for g_γ, $\gamma \geq 0$. If $\gamma = 0$, then $\alpha = 1$ and the measure μ_0 would be invariant.

We first study high iterates of g_γ.

Lemma 2.1 *Let $\{g_\gamma\}$, $\gamma \geq 0$, be as above. There exists $K > 0$ such that for each n large enough and each interval I on which g^n_γ is monotone and $g^n_\gamma(I) = N$, the following holds. There is a subinterval J_I of $[0,1]$ and an affine rescaling $R : J_I \to I$, so that $g^n_\gamma \circ R$ is equal to $H \circ G|_{J_I}$, where*

$$G(x) = 3x^2 - 2x^3,$$

and H is a diffeomorphism on $[0,1]$ with

$$\frac{|DH(x)|}{|DH(y)|} \leq K,$$

for all $x, y \in [0,1]$.

Proof. As Theorem III.6.1 in [MelStr93]. Observe that there is some freedom in the choice of I_J; one can take the intervals I_J to have length at least $\frac{1}{2}$. □

Theorem 2.2 *Let $\{g_\gamma\}$ be as above. For $\gamma \geq 0$, there is a conditionally invariant measure μ_γ for g_γ. The measure μ_γ is absolutely continuous with respect to Lebesgue measure.*

Proof. For a fixed value of γ, we write g for g_γ and we seek a conditionally invariant measure μ for g. Define the measure μ_i on N by

$$\mu_i(A) = \frac{m(g^{-i}(A))}{m(g^{-i}(N))},$$

and the measure $\hat{\mu}_n$ by

$$\hat{\mu}_n(A) = \left(\prod_{i=0}^{n-1} \mu_i(A) \right)^{1/n},$$

for Borel sets $A \subset N$.

Since the space of probability measures is compact in the weak topology, there is a weak limit $\hat{\mu}$ of $\hat{\mu}_n$ as $n \to \infty$. We claim that $\hat{\mu}$ is absolutely continuous. We must show that for each $\epsilon > 0$ there exists $\delta > 0$ so that for a Borel set A with $m(A) < \delta$, one has $\mu(A) < \epsilon$. It suffices to show that for each $\varepsilon > 0$ there is $\delta > 0$, so that for any Borel set A with $m(A) < \delta$ one has $m(g^{-i}(A))/m(g^{-i}(N)) < \epsilon$ for any positive integer i. This follows from the following estimate: for I a maximal interval on which g^i is monotone and $g^i(I) = N$, by Lemma 2.1 there exists $D > 0$ depending only on g so that

$$m(g^{-n}(A) \cap I)/m(I) \leq D\sqrt{m(A)}.$$

Let I be a maximal interval on which g^n is monotone and $g^n(I) = N$. By Lemma 2.1 there is $C > 1$ so that uniformly in n,

$$\frac{1}{C} m(E_\gamma) \leq \frac{m\left(g^{-n}(E_\gamma) \cap I\right)}{m(I)} \leq C m(E_\gamma).$$

Taking the union over all maximal intervals I in $g^{-n}(N)$ yields

$$\frac{1}{C} m(E_\gamma) m(g^{-n}(N)) \leq m(g^{-n}(E_\gamma)) \leq C m(E_\gamma) m(g^{-n}(N)),$$

so that

$$(1 - C m(E_\gamma)) m(g^{-n}(N)) \leq m(g^{-n-1}(N)) \leq (1 - \frac{1}{C} m(E_\gamma)) m(g^{-n}(N)).$$

Hence, taking a subsequence of $\hat{\mu}_n$ we may assume that $m(g^{-n}(N))^{1/n}$ converges to a number α. Since $g_* \hat{\mu}_n(A) = \frac{m(g^{-n}(N))^{1/n}}{m(A)^{1/n}} \hat{\mu}_{n+1}(A)$, it follows that μ is conditionally invariant:

$$g_* \mu = \alpha \mu.$$

□

For a map g on N and an absolutely continuous measure μ on N, the Perron-Frobenius operator P_μ on $\mathcal{L}^1(N)$ is determined by

$$\int_A P_\mu u d\mu = \int_{g^{-1}(A)} u d\mu \Big/ \int_{g^{-1}(N)} |u| d\mu \,. \tag{6}$$

Fixed points of P_μ correspond to conditionally invariant measures for g, see [PiaYor79, Pia81]. In particular, if μ is a conditionally invariant measure for g, then $P_\mu 1 = 1$. A density is a nonnegative integrable function σ with integral $\int_N \sigma d\mu = 1$. When we speak of a density of a (probability) measure ν we refer to the Radon-Nikodym derivative $\frac{d\nu}{dm}$ with respect to Lebesgue measure. Observe that P_μ maps densities to densities.

The following proposition discusses convergence of the Perron-Frobenius operator.

Proposition 2.3 *Let $\{g_\gamma\}$ be as above. For fixed $\gamma \geq 0$, write $g = g_\gamma$ and let μ be a conditionally invariant measure for g. For any continuous density σ on N,*

$$P_\mu^n \sigma \quad \to \quad 1,$$

in $\mathcal{L}^1(N)$ as $n \to \infty$. If $\gamma = 0$, this holds for any integrable density σ.

Remark 2.4 *It follows that the conditionally invariant measure μ_γ for g_γ constructed in Theorem 2.2 is unique within the class of probability measures with continuous densities. Moreover, μ_γ is obtained as*

$$\mu_\gamma(A) \quad = \quad \lim_{n\to\infty} \frac{m\left(g_\gamma^{-n}(A)\right)}{m\left(g_\gamma^{-n}(N)\right)},$$

for Borel sets $A \subset N$.

Proof. For $\gamma > 0$, the proposition follows from Theorem 3 in [PiaYor79]. For $\gamma = 0$, use Proposition 4.2.11 in [BoyGor97]. This proposition can be applied since μ_0 is exact, see [MelStr93]. □

Next we show that the density σ_γ of μ_γ depends continuously in $\mathcal{L}^1(N)$ on γ. We need the Koebe principle, providing distortion estimates on branches. See [MelStr93] for its proof.

Lemma 2.5 (Koebe principle) *Let g_γ, $\gamma \geq 0$, be as above. Let I be a maximal interval on which g_γ^n is monotone and $g_\gamma^n(I) = N$. Let $\delta > 0$. Then there is K_δ so that on any interval $J \subset I$ on which $g_\gamma(J)$ is at least δ away from ∂N, and for all small $\gamma \geq 0$,*

$$\frac{1}{K_\delta} \leq \frac{|Dg_\gamma(x)|}{|Dg_\gamma(y)|} \leq K_\delta$$

and

$$\frac{|D^2 g_\gamma(x)|}{|Dg_\gamma(x)|^2} \quad \leq \quad \frac{K_\delta}{m(g_\gamma(J))}.$$

Proposition 2.6 *Let g_γ, $\gamma \geq 0$, and μ_γ be as above. For small enough γ, the density σ_γ of μ_γ depends continuously in $\mathcal{L}^1(N)$ on γ.*

Proof. We first show that for each $\delta > 0$ there is $K_\delta > 0$ so that for all small $\gamma \geq 0$,

$$\sigma_\gamma(x) \quad \leq \quad K_\delta, \tag{7}$$

$$|\sigma_\gamma(x) - \sigma_\gamma(y)| \quad \leq \quad K_\delta |x - y|, \tag{8}$$

for x, y in N at a distance at least δ from ∂N.

We write g for g_γ. Write $P = P_m$ for the Perron-Frobenius operator on $\mathcal{L}^1(N)$, where m is Lebesgue measure (compare (6)). Observe

$$Pu(x) = \sum_{z \in g^{-1}(x)} \frac{u(z)}{|Dg(z)|} \Big/ \int_{g^{-1}(N)} |u(x)| dx.$$

Let

$$S_n = P^n 1,$$

so that

$$S_n(x) = \sum_{z \in g^{-n}(x)} \frac{1}{|Dg^n(z)|} / m(g^{-n}(N)). \tag{9}$$

Note that S_n is the density of the measure $(g^n)_* m / m(g^{-n}(N))$ which appeared in the proof of Theorem 2.2. Let $\mathcal{P}_n$ be the set of maximal intervals on which g^n is monotone and surjective onto N. Observe that $g^{-n}(N) = \cup_{I \in \mathcal{P}_n} I$. Write N_δ for the interval in N consisting of points that are at least distance δ away from ∂N. For $x \in N_\delta$, g^n has bounded distortion by the Koebe principle, so that $|Dg^n(z)| \geq \frac{1}{K} \frac{1}{m(I)}$ for some K depending on δ. Putting this in (9) yields, for $x \in N_\delta$,

$$\begin{aligned} S_n(x) &\leq \sum_{I \in \mathcal{P}_n} \frac{K m(I)}{m(g^{-n}(N))} \\ &= K, \end{aligned} \tag{10}$$

for some K depending on δ.

Further, for $x \in N_\delta$,

$$\begin{aligned} |DS_n(x)| &= \sum_{z \in g^{-n}(x)} \frac{1}{|Dg^n(z)|} \frac{|D^2 g^n(z)|}{|Dg^n(z)|^2} / m(g^{-n}(N)) \\ &\leq C S_n(x). \end{aligned} \tag{11}$$

In view of Theorem 2.2 and Proposition 2.3, the bounds (7) and (8) follow from (10), (11). We remark that for γ from a compact interval not containing 0, K_δ can be bounded uniformly in δ.

To prove continuous dependence of σ_γ on γ in $\mathcal{L}^1(N)$ one employs uniqueness of σ_γ and the above shown equicontinuity properties. Suppose namely that $\gamma \mapsto \sigma_\gamma$ is not continuous. That is, there is a sequence $\gamma_k \to \tilde{\gamma}$ for which $\sigma_{\gamma_k} \not\to \sigma_{\tilde{\gamma}}$. By the above equicontinuity on compact intervals away from ∂N, we may assume that σ_{γ_k} converges in $\mathcal{L}(N)$ to a density τ. Write $P_{m,\gamma}$ for the Perron-Frobenius operator for the map g_γ. Then

$$\left\| P_{m,\tilde{\gamma}} \tau - \tau \right\|_1 \leq \left\| P_{m,\tilde{\gamma}} \tau - P_{m,\gamma_k} \tau \right\|_1 + \left\| P_{m,\gamma_k} \tau - \sigma_{\gamma_k} \right\|_1 + \left\| \sigma_{\gamma_k} - \tau \right\|_1 .$$

The first term on the right hand clearly goes to 0 as $k \to \infty$. The second term does too, since $P_{m,\gamma_k} \sigma_{\gamma_k} = \sigma_{\gamma_k}$. Because also the third term goes to 0 as $k \to \infty$, it follows that τ is a fixed point of $P_{m,\tilde{\gamma}}$, contradicting uniqueness. $\square$

3 Escape rates

Let $\{g_\gamma\}$ be a one parameter family of unimodal maps on an interval N as in the previous section. Let μ_γ be the conditionally invariant measure of g_γ, for $\gamma \geq 0$ small, and let σ_γ be its density. Recall that E_γ is the interval containing c that is mapped outside of N by g_γ, $\gamma > 0$.

Theorem 3.1 *Let*

$$L_{E_\gamma}(x,\gamma) \;=\; \min\{j \geq 0;\;\; g_\gamma^j(x) \in E_\gamma\}$$

and

$$S_\gamma \;=\; \{x;\;\; L_{E_\gamma}(x,\gamma) \leq \frac{z}{\sqrt{\gamma}}\},$$

where $z > 0$. If

$$\xi \;=\; \lim_{\gamma \searrow 0} \frac{1}{\sqrt{\gamma}} \int_{E_\gamma} d\mu_0, \tag{12}$$

then for each absolutely continuous probability measure ν, one has

$$\lim_{\gamma \searrow 0} \int_{S_\gamma} d\nu \;=\; 1 - e^{-\xi z}. \tag{13}$$

If ν is a probability measure satisfying $\nu(A) < D\sqrt{m(A)}$ for some $D > 0$, then

$$\lim_{\gamma \searrow 0} \xi \sqrt{\gamma} \int_N L_{E_\gamma}(x,\gamma) d\nu \;=\; 1. \tag{14}$$

Remark 3.2 *A straightforward computation yields*

$$\xi \;=\; 2\sigma_0(c)\sqrt{\left|\frac{\frac{\partial}{\partial\gamma} g_\gamma(c)|_{\gamma=0}}{D^2 g_0(c)}\right|}.$$

Proof. We closely follow [Pia81], who proves (13) for the quadratic family $x \mapsto \mu x(1-x)$, for measures ν with Lipschitz density.

Let $\bar{P}_\gamma$ on $\mathcal{L}^1(N)$ be determined by

$$\int_A \bar{P}_\gamma u dm \;=\; \int_{g_\gamma^{-1}(A)} u dm.$$

Note that the Perron-Frobenius operator $P_\gamma = P_{m,\gamma}$ and $\bar{P}_\gamma$ are related by $P_\gamma u = \bar{P}_\gamma u \Big/ \int_{g_\gamma^{-1}(N)} |u| dm$.

Let us first formulate a list of properties that the conditionally invariant measures ν_γ and the Perron-Frobenius operator P_γ satisfy. We will demonstrate that these properties allow for the proof of the theorem. We will finish by showing that these properties hold.

I. For each small positive γ, there is a unique (within the class of continuous densities) positive fixed point σ_γ for the Perron-Frobenius operator P_γ.

II. For any density σ and for any positive integer n, the map $\gamma \to P_\gamma^n \sigma$ is continuous in $\mathcal{L}^1(N)$.

III. The fixed point σ_γ depends continuously on γ in $\mathcal{L}^1(N)$.

IV. For any density σ, $\lim_{n\to\infty} \sup_\gamma \|\bar{P}_\gamma^n(\sigma - \sigma_\gamma)\|_1 = 0$.

V. $\lim_{\gamma\searrow 0} \sup_{x\in E_\gamma} |(\sigma_\gamma - \sigma_0)/\sigma_0| = 0$.

Let us demonstrate why this list of properties implies the result. First we establish (13). Take n to be the integer part of $z/\sqrt{\gamma}$. Observe that $n \to \infty$ as $\gamma \to 0$. With $\sigma = \frac{d\nu}{dm}$ we have

$$\begin{aligned} \nu(g_\gamma^{-n}(N)) &= \int_N \bar{P}_\gamma^n \sigma dm \\ &= \int_N \bar{P}_\gamma^n \sigma_\gamma dm + H_n(\gamma), \end{aligned}$$

where $H_n(\gamma) = \int_N \bar{P}_\gamma^n(\sigma - \sigma_\gamma)dm$ goes to 0 as $\gamma \to 0$, by IV. Since σ_γ is a fixed point of P_γ,

$$\begin{aligned} \int_N \bar{P}_\gamma^n \sigma_\gamma dm &= \int_{g_\gamma^{-n}(N)} \sigma_\gamma dm \\ &= \left(\int_{g_\gamma^{-1}(N)} \sigma_\gamma dm\right)^n \\ &= \left(1 - \sqrt{\gamma}\left[\frac{1}{\sqrt{\gamma}}\int_{E_\gamma} \sigma_0 dm + \frac{1}{\sqrt{\gamma}}\int_{E_\gamma} (\sigma_\gamma - \sigma_0)dm\right]\right)^n. \end{aligned} \tag{15}$$

By V and (12) it follows that $\frac{1}{\sqrt{\gamma}}\int_{E_\gamma}(\sigma_\gamma - \sigma_0)dm$ goes to 0 as $\gamma \to 0$. By computing (15) for n equal to the integer part of $z/\sqrt{\gamma}$ and letting $\gamma \to 0$, we obtain

$$\begin{aligned} \lim_{\gamma\searrow 0} \nu(g_\gamma^{-n}(N)) &= \lim_{\gamma\searrow 0}(1 - \sqrt{\gamma}\xi)^n \\ &= e^{-\xi z}. \end{aligned}$$

Since $g_\gamma^{-n}(N) = \{x;\ \ L_{E_\gamma}(x,\gamma) \geq n\}$, (13) follows.

The limit (14) is proved similarly. One has

$$\begin{aligned} \int_{S_\gamma} L_{E_\gamma}(x,\gamma)d\nu &= \sum_{i=0}^{n} i\nu(g_\gamma^{-i}(E_\gamma)) \\ &= \sum_{i=0}^{n} i\left[\nu(g_\gamma^{-i}(N)) - \nu(g_\gamma^{-i-1}(N))\right] \\ &= \sum_{i=1}^{n} \nu(g_\gamma^{-i}(N)) - n\nu(g_\gamma^{-n-1}(N)). \end{aligned}$$

Denoting $\alpha_\gamma = \int_{g^{-1}(N)} d\mu_\gamma$, this gives

$$\begin{aligned}
\sum_{i=0}^{n} i\nu(g^{-i}(E_\gamma)) &= \sum_{i=0}^{n} i(\alpha_\gamma^i - \alpha_\gamma^{i+1}) + \sum_{i=0}^{n} i(H_i(\gamma) - H_{i+1}(\gamma)) \\
&= \sum_{i=1}^{n} \alpha_\gamma^i - n\alpha_\gamma^{n+1} + \sum_{i=1}^{n} H_i(\gamma) - nH_{n+1}(\gamma) \\
&= \frac{\alpha_\gamma(1-\alpha_\gamma^n)}{1-\alpha_\gamma} - n\alpha_\gamma^{n+1} + \sum_{i=1}^{n} H_i(\gamma) - nH_{n+1}(\gamma).
\end{aligned}$$

Putting n equal to the integer part of $z/\sqrt{\gamma}$, one treats the terms $\frac{\alpha_\gamma(1-\alpha_\gamma^n)}{1-\alpha_\gamma} - n\alpha_\gamma^{n+1}$ as in (15). When multiplied with $\xi\sqrt{\gamma}$, these terms converge to $1 - e^{-\xi z} - z\xi e^{-\xi z}$ as $\gamma \to 0$. To estimate the remainder terms, write $\sum_{i=1}^n H_i = \sum_{i=1}^{\epsilon/\sqrt{\gamma}} H_i + \sum_{i=\epsilon/\sqrt{\gamma}}^n H_i$ with $\epsilon = \gamma^{1/4}$. The first sum is then bounded by a constant times $\gamma^{-1/4}$. Because $\epsilon/\sqrt{\gamma} \to \infty$ as $\gamma \to 0$, each term in the second sum is small, by IV. Hence, in the limit $\gamma \to 0$, $\xi\sqrt{\gamma}\,(\sum_{i=1}^n H_i(\gamma) - nH_{n+1}(\gamma))$ goes to 0. This proves

$$\lim_{\gamma \searrow 0} \xi\sqrt{\gamma} \int_{S_\gamma} L_{E_\gamma}(x,\gamma) d\nu = 1 - e^{-\xi z} - \xi z e^{-\xi z}. \tag{16}$$

Now assume that for some $D > 0$ and any Borel set A, $\nu(A) < D\sqrt{m(A)}$. We claim that

VI. The function $\sqrt{\gamma} L_{E_\gamma}(x,\gamma)$ is integrable with respect to ν, uniformly in γ.

Indeed, noting that $\frac{1}{C} m(A) \leq \mu_\gamma(A)$ for some $C > 1$,

$$\begin{aligned}
\int_N L_{E_\gamma}(x,\gamma) d\nu &= \sum_{i\geq 1} \nu(g_\gamma^{-i}(N)) \\
&\leq D \sum_{i \geq 1} \sqrt{m\left(g_\gamma^{-i}(N)\right)} \\
&\leq D\sqrt{C} \sum_{i\geq 1} \sqrt{\mu_\gamma(g_\gamma^{-i}(N))} \\
&\leq D\sqrt{C} \sum_{i \geq 1} \sqrt{\alpha_\gamma}^i \\
&= \frac{D\sqrt{C}\alpha_\gamma}{1-\sqrt{\alpha_\gamma}} \\
&\leq \frac{D\sqrt{C}}{1 - \sqrt{1 - \sqrt{\gamma}\left[\frac{1}{\sqrt{\gamma}}\int_{E_\gamma} \sigma_0 dm + \frac{1}{\sqrt{\gamma}} \int_{E_\gamma} (\sigma_\gamma - \sigma_0) dm\right]}} \\
&\leq \frac{4D\sqrt{C}}{\xi\sqrt{\gamma}},
\end{aligned}$$

for $\gamma > 0$ small enough. Similarly, with n the integer part of $z/\sqrt{\gamma}$,

$$\begin{aligned} \int_{N\backslash S_\gamma} L_{E_\gamma}(x,\gamma) d\nu &= \sum_{i\geq n} \nu(g_\gamma^{-i}(N)) \\ &\leq \frac{D}{\sqrt{\gamma}} e^{-\frac{1}{4}\xi z}, \end{aligned} \tag{17}$$

for some $D > 0$ and $\gamma > 0$ small. By (16) and (17), noting that $\int_N L_{E_\gamma}(x,\gamma)d\nu$ does not depend on z, one obtains (14).

It remains to check properties I to V. Properties I and III have been shown in the previous section. The second property is clear. To prove IV, define $h_n(\gamma) = \int_N \left|\bar{P}_\gamma^n(\sigma - \sigma_\gamma)\right| dm$. Note that h_n is a continuous function of γ. For fixed $\gamma > 0$,

$$\begin{aligned} h_n(\gamma) &\leq \int_N \left|\bar{P}_\gamma^n(\sigma-\sigma_\gamma)^+\right| + \left|\bar{P}_\gamma^n(\sigma-\sigma_\gamma)^-\right| dm \\ &= \int_N \bar{P}_\gamma^n \left|\sigma - \sigma_\gamma\right| dm \\ &= \int_{g_\gamma^{-n}(N)} \left|\sigma-\sigma_\gamma\right| dm \end{aligned}$$

goes to 0 as $n \to \infty$, since $m\left(g_\gamma^{-n}(N)\right) \to 0$ (in the above, $f = f^+ - f^-$ is obtained when writing f as the difference of two nonnegative functions). For $\gamma = 0$, $h_n(0)$ converges to 0 as $n \to \infty$ because the invariant measure μ_0 for g_0 is exact [MelStr93] and therefore $\bar{P}_0^n\sigma \to \sigma_0$ in $\mathcal{L}^1(N)$, see Proposition 2.3. Hence, h_n converges pointwise to 0 for γ in a small interval $[0,\gamma_0]$. In addition, h_n satisfies

$$\begin{aligned} h_{n+1}(\gamma) &\leq \int_{g^{-1}(N)} \left|\bar{P}_\gamma^n(\sigma-\sigma_\gamma)\right| dm \\ &\leq \int_N \left|\bar{P}_\gamma^n(\sigma-\sigma_\gamma)\right| dm, \end{aligned}$$

so that $h_{n+1}(\gamma) \leq h_n(\gamma)$. By Dini's lemma, see [LasYor81], it follows that h_n converges to 0 uniformly, that is, IV holds. Finally, V follows from the proof of Proposition 2.6. □

4 Relaminarization

Let $\{f_\gamma\}$ be as in Theorem A or Theorem B. A point in $\bar{E}_\gamma$ leaves $\bar{N}$ under iteration by f_γ. Its orbit then spends a number of iterates outside of $\bar{N}$. In this section we study iterates in $M\backslash\bar{N}$ in preparation of the proof of Theorem A in the following section.

Write O for the union of $\bar{N}$ and those intervals in $f_\gamma^{-k}(\bar{N})$ that contain a critical point for f_γ^k, compare Figure 5. Observe that O is a finite union of subintervals of M. Let $h_\gamma : M\backslash O \to M$ denote the restriction of f_γ^k to $M\backslash O$. Observe that $h_\gamma(M\backslash O) = f_\gamma(M)\backslash f_\gamma^k(N)$ Also, $|Dh_\gamma|$ is bounded away from 0 since all critical points of f_γ^k lie in O. For some $\lambda > 1$, one has $\left|Dh_\gamma^n\right| \geq \lambda$ for all large enough integers n (see Theorem III.2.1 and Theorem III.3.3 in [MelStr93]).

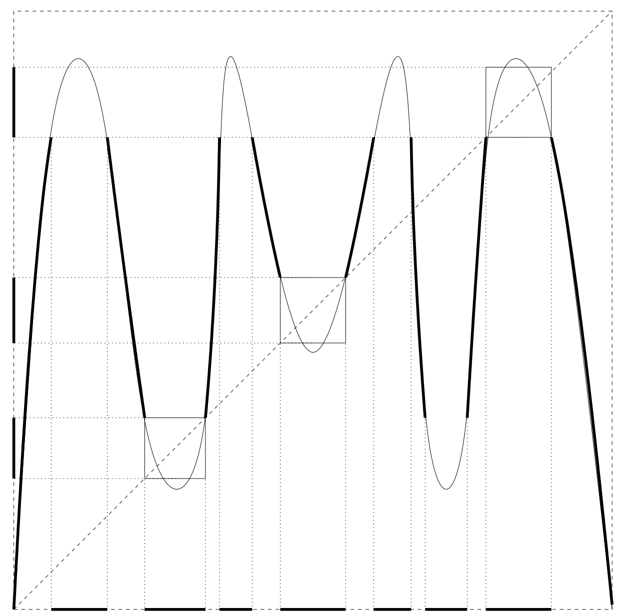

Figure 5: The figure shows the graph of the third iterate of a unimodal map just past a boundary crisis bifurcation. Projected on the left side of the box are the intervals that make up $\bar{N}$. Projected on the bottom one finds O. The branches on the graph of f_γ^k that define h_γ are drawn thicker.

Theorem 4.1 *Let $\{h_\gamma\}$ be as above. For $x \in M$, let $L_O(x,\gamma)$ be the number of iterations under h_γ required for x to enter O. If ν is a probability measure on $M \setminus O$ satisfying $\nu(A) < D\sqrt{m(A)}$ for some $D > 0$ and Borel sets A, then there is $L > 0$ so that for all small $\gamma \geq 0$,*

$$\int_M L_O(x,\gamma) d\nu \quad \leq \quad L.$$

Proof. By [PiaYor79], h_γ possesses a conditionally invariant measure ζ_γ which is absolutely continuous with respect to Lebesgue measure and has density bounded and bounded away from 0 on its support, uniformly in γ. Say

$$\frac{1}{C} m(A) \quad \leq \quad \zeta_\gamma(A)$$

for Borel sets A. Write $(h_\gamma)_* \zeta_\gamma = \beta_\gamma \zeta_\gamma$ for some $0 < \beta_\gamma < 1$. Since $m(O)$ is bounded from below, β_γ is bounded away from 1. Now

$$\begin{aligned} \nu(h_\gamma^{-n}(M)) &\leq D\sqrt{m\left(h_\gamma^{-n}(M)\right)} \\ &\leq D\sqrt{C}\sqrt{\zeta_\gamma(h_\gamma^{-n}(M))} \\ &= D\sqrt{C}\sqrt{\beta_\gamma^{\,n}}. \end{aligned}$$

It follows that

$$\int_M L_O(x,\gamma)d\nu \quad = \quad \sum_{i\geq 1} \nu(h_\gamma^{-i}(M))$$

is bounded uniformly in γ. □

5 Intermittency near the boundary crisis bifurcation

In this section we collect results proved in previous sections to establish Theorem A. The following proposition quotes results by Jakobson and L.-S. Young among others and fixes the parameter set γ.

Proposition 5.1 *Let $\{f_\gamma\}$ be as in Theorem A. There is a set Γ of parameter values containing 0 and satisfying*

$$\lim_{\gamma_0 \searrow 0} \frac{m\,(\Gamma \cap [0,\gamma_0))}{\gamma_0} \quad = \quad 1,$$

so that f_γ has an a.c.i.m. ν_γ for $\gamma \in \Gamma$. Moreover, ν_γ is ergodic and there exists $D > 0$ so that for all $\gamma \in \Gamma$ and Borel sets A,

$$\nu_\gamma(A) \quad \leq \quad D\sqrt{m(A)}. \tag{18}$$

Proof. The existence of Γ follows from work of Jakobson [Jak81] and generalizations, see Theorem V.6.1 in [MelStr93]. The bound (18) follows from Theorem 1 in [You92] (L.-S. Young provides a bound for the density of ν_γ; by integration the desired bound for ν_γ is derived). See also [Now93] and Theorem V.4.1 in [MelStr93]. □

Proof of Theorem A. We start by establishing the statement on the discontinuity of $\Phi_{\bar{N}}$ at $\gamma = 0$. Let q be a hyperbolic periodic point for f_0 and denote its continuation for $\gamma > 0$ by q. Within any interval $(0, \gamma_0)$ there are parameter values γ for which $f_\gamma^{2R}(c) = q$. Converging to q there are points x_L for any large enough L, so that $f_\gamma^{kL}(x_L) = a$. Hence there are γ_L converging to γ as $L \to \infty$ with $f_{\gamma_L}^{kR+kL}(c) = a$. There are $\gamma_{L,P}$ converging to γ_L as $P \to \infty$ with $f_{\gamma_{L,P}}^{kR+kL}(c) \in N$ and $f_{\gamma_{L,P}}^{kR+kL+kP}(c) = c$. Moreover, we can get $f_{\gamma_{L,P}}^{kR+kL}(c)$ to be the first positive iterate $f_\gamma^i(c)$ that is in $\bar{N}$. Thus,

$$\begin{aligned} \Phi_{\bar{N}}(x, \gamma_{L,P}) &= \frac{kP}{kR + kL + kP} \\ &= \frac{P}{R + L + P}. \end{aligned}$$

Any fraction $\frac{r}{s}$ can be obtained this way.

Let γ be as in Proposition 5.1. The measure ν_γ is ergodic, so that by Birkhoff's ergodic theorem,

$$\lim_{m\to\infty} \frac{1}{m}\sum_{i=0}^{m-1} 1_A(f_\gamma^i(x)) \quad = \quad \nu_\gamma(A),$$

for any Borel set $A \subset M$ and almost all $x \in M$. Let m be a large number and write, for $0 \leq i \leq m-1$, $x_i = f_\gamma^i(x)$. Consider maximal orbit pieces $\{x_{j_1}, \dots, x_{j_2}\}$, $0 \leq j_1 < j_2 \leq m-1$ that are contained in $\bar{N}$ (maximal meaning that x_{j_1-1} and x_{j_2+1} are outside of $\bar{N}$) and have precisely i intersections with N. Let S_i^m be the total number of points in $\{x_0, \dots, x_{m-1}\} \cap N$ belonging to such orbit pieces. Assuming that $x_{m-1} \notin \bar{N}$,

$$S_i^m \quad = \quad i\left\{ \sum_{j=0}^{m-1} 1_{g_\gamma^{-i+1}(E_\gamma)}(f_\gamma^j(x)) - 1_{g_\gamma^{-i}(E_\gamma)}(f_\gamma^j(x)) \right\}.$$

Let $S^m = \sum_{i=1}^m S_i^m$ be the total number of iterates x_j, $0 \leq j \leq m-1$, in N. By ergodicity of ν_γ, for almost all $x \in M$ and any $i \in \mathbb{N}$,

$$\lim_{m\to\infty} \frac{S_i^m}{S^m} \quad = \quad \frac{i}{\nu_\gamma(N)}\left\{ \nu_\gamma(g_\gamma^{-i+1}(E_\gamma)) - \nu_\gamma(g_\gamma^{-i}(E_\gamma)) \right\}.$$

The average $\sum_{i=1}^n i\frac{S_i^m}{S^m}$ is therefore close to $\frac{1}{\nu_\gamma(N)}\sum_{i=1}^n i^2\left\{ \nu_\gamma(g_\gamma^{-i+1}(E_\gamma)) - \nu_\gamma(g_\gamma^{-i}(E_\gamma)) \right\}$, for m large. Compute

$$\begin{aligned} &\frac{1}{\nu_\gamma(N)}\sum_{i=1}^n i^2\left\{ \nu_\gamma(g_\gamma^{-i+1}(E_\gamma)) - \nu_\gamma(g_\gamma^{-i}(E_\gamma)) \right\} = \\ &\quad \frac{1}{\nu_\gamma(N)}\left\{ \sum_{i=1}^n (2i-1)\nu_\gamma(g_\gamma^{-i+1}(E_\gamma)) - n^2\nu_\gamma(g_\gamma^{-n}(E_\gamma)) \right\}. \end{aligned}$$

We claim that

$$\lim_{n\to\infty} \sqrt{\gamma}\frac{1}{\nu_\gamma(N)} n^2 \nu_\gamma(g_\gamma^{-n}(E_\gamma)) \quad = \quad 0 \tag{19}$$

and

$$\lim_{\gamma\searrow 0} \xi\sqrt{\gamma}\frac{1}{\nu_\gamma(N)}\sum_{i=1}^\infty (2i-1)\nu_\gamma(g_\gamma^{-i+1}(E_\gamma)) \quad = \quad 2. \tag{20}$$

By Proposition 5.1, $\nu_\gamma(A) \leq D\sqrt{m(A)}$. Also, $m(A) \leq C\mu_\gamma(A)$ for some $C > 0$. Compute

$$\begin{aligned} n^2\nu_\gamma(g_\gamma^{-n}(E_\gamma)) &\leq Dn^2\sqrt{m(g_\gamma^{-n}(E_\gamma))} \\ &\leq CDn^2\sqrt{\mu_\gamma(g_\gamma^{-n}(E_\gamma))} \\ &= CDn^2\sqrt{\alpha_\gamma^n \mu_\gamma(E_\gamma)}, \end{aligned}$$

with $\alpha_\gamma = \mu_\gamma(g_\gamma^{-1}(N)) < 1$. The first part of the claim, (19), follows easily from this.

With L_{E_γ} defined as in Theorem 3.1, we can write

$$\lim_{\gamma \searrow 0} \xi\sqrt{\gamma}\int_N L_{E_\gamma}(x,\gamma)d\nu_\gamma \quad = \quad \lim_{\gamma \searrow 0} \xi\sqrt{\gamma}\frac{1}{\nu_\gamma(N)}\sum_{i=0}^{\infty} i\nu_\gamma(g_\gamma^{-i}(E_\gamma)). \tag{21}$$

Here, with $E_\gamma = \bar{E}_\gamma \cap N$,

$$\xi = \lim_{\gamma \searrow 0}\frac{1}{\sqrt{\gamma}}\int_{\bar{E}_\gamma} d\nu_0/\nu_0(\bar{N}) = \lim_{\gamma \searrow 0}\frac{1}{\sqrt{\gamma}}\int_{E_\gamma} d\nu_0/\nu_0(N).$$

To prove (20), it suffices to show that the limit in (21) equals 1. In (14) in Theorem 3.1, $\int_N L_{E_\gamma}(x,\gamma)d\nu$ was approximated for a single measure ν. However, the same limit holds for a family of probability measures ζ_γ on N if this family satisfies

(a) There is $D > 0$ so that for all small $\gamma \geq 0$, $\zeta_\gamma(A) \leq D\sqrt{m(A)}$ for Borel sets A.

It suffices to check the list of properties I to VI in the proof of Theorem 3.1 with ζ_γ replacing ν. The only property that requires some attention is IV. But it is easily seen that

$$\lim_{n\to\infty}\sup_\gamma \left\|\bar{P}_\gamma^n(\zeta_\gamma - \sigma_\gamma)\right\|_1 \quad = \quad 0$$

by copying the reasoning in the proof of Theorem 3.1.

The estimate (18) in Proposition 5.1 tells that ν_γ satisfies property (a). Hence, applying (14) to the measure $\nu_\gamma/\nu_\gamma(N)$, proves (20). An orbit piece in $\bar{N}$ with i intersections with N has length between ki and $(k+1)i-1$. Formula (20) implies that the average length of the laminar phases in an orbit starting at $x \in M$, multiplied by $\xi\sqrt{\gamma}$, goes to $2k$ as $\gamma \searrow 0$, for almost all x.

Orbits that leave $\bar{N}$ when mapped by f_γ arrive close to the periodic orbit $\mathcal{O}(a)$ and thus spend many iterates near $\mathcal{O}(a)$, before they can possibly re-enter $\bar{N}$. To estimate how many iterates orbits on average spend in $M\backslash\bar{N}$ before re-entering $\bar{N}$, consider $K_{\bar{N}}(x,\gamma)$ on $M\backslash\bar{N}$ (recall that $K_{\bar{N}}(x,\gamma)$ is the number of iterates of f_γ that maps x into $\bar{N}$). Since an orbit must exit $\bar{N}$ through E_γ, the average time an orbit spends in $M\backslash\bar{N}$, once it leaves $\bar{N}$ is given by the integral $\int_{E_\gamma} K_{\bar{N}}(x,\gamma)d\nu_\gamma/\nu_\gamma(E_\gamma)$. We will establish that

$$\lim_{\gamma \searrow 0}\frac{\ln\lambda}{|\ln\gamma|}\int_{E_\gamma} K_{\bar{N}}(x,\gamma)d\nu_\gamma/\nu_\gamma(E_\gamma) \quad = \quad k. \tag{22}$$

In words, the average length of the relaminarization phase, multiplied by $\ln\lambda/|\ln\gamma|$, goes to k as $\gamma \searrow 0$. Combined with the above derived asymptotics on the average length of the laminar phase, this proves (3) in Theorem A.

Let I be an interval in $M\backslash N$ bounded by a on which $Df_\gamma^k > 0$. By a smooth coordinate change, we may assume that f_γ^k is affine on I:

$$f_\gamma^k(x) \quad = \quad a + \lambda(x-a), \tag{23}$$

with $\lambda = Df_\gamma^k(a)$. Let r be so that $f_\gamma^{kr}(f_\gamma^{2k}(c)) \in f_\gamma^k(I)\backslash I$. Because $\frac{\partial}{\partial\gamma} f_\gamma^k(c) \neq 0$ for small γ, there exists $C > 0$ so that

$$\frac{1}{C} \leq \lambda^r \gamma \leq C, \tag{24}$$

uniformly in γ. This implies

$$\lim_{\gamma \searrow 0} \frac{r}{|\ln \gamma|} \quad = \quad \frac{1}{\ln \lambda}. \tag{25}$$

Because f_γ is unimodal, $\left(f_\gamma\big|_{E_\gamma}\right)_* \nu_\gamma = \nu_\gamma\big|_{f_\gamma(E_\gamma)}$. The measure $\nu_\gamma\big|_{f_\gamma(E_\gamma)}$ satisfies the estimate (18). Consider the measure $\mu = \left(f_\gamma^{2k-1}\big|_{f_\gamma(E_\gamma)}\right)_* \nu_\gamma\big|_{f_\gamma(E_\gamma)}$ on $f_\gamma^{2k}(E_\gamma) = (a, f_\gamma^{2k}(c))$. Since f_γ^{2k-1} is strictly monotone on $f_\gamma(E_\gamma)$, there exists $D > 0$ so that for all Borel sets A,

$$\mu(A) \quad \leq \quad D\sqrt{m(A)}. \tag{26}$$

Also,

$$\mu(f_\gamma^{2k}(E_\gamma)) \quad \geq \quad D'\sqrt{\gamma} \tag{27}$$

for some $D' > 0$, since the density of ν_γ on E_γ is bounded away from 0 and $\frac{\partial}{\partial\gamma} f_\gamma^k(c) \neq 0$. Let $\mu_r = \left(f_\gamma^{rk}\Big|_{f_\gamma^{2k}(E_\gamma)}\right)_* \mu$, which is a measure on $f_\gamma^k(I)$. Note that

$$\int_{E_\gamma} K_{\bar{N}}(x,\gamma) d\nu_\gamma / \nu_\gamma(E_\gamma) \quad = \quad (r+2)k + \int_{f_\gamma^{(r+2)k}(E_\gamma)} K_{\bar{N}}(x,\gamma) d\mu_r / \mu(f_\gamma^{(r+2)k}(E_\gamma)). \tag{28}$$

By (23), (24) and (26) it follows that

$$\begin{aligned} \mu_r(A) \quad &\leq \quad D\frac{\sqrt{m(A)}}{\sqrt{\lambda^r}} \\ &\leq \quad D\sqrt{C}\sqrt{\gamma}\sqrt{m(A)}. \end{aligned}$$

By (27), the normalized measure $\mu_r/\mu(f_\gamma^{2k}(E_\gamma))$ satisfies

$$\mu_r(A)/\mu(f_\gamma^{2k}(E_\gamma)) \quad \leq \quad \frac{D\sqrt{C}}{D'}\sqrt{m(A)}.$$

Let $h_\gamma : M\backslash O \to M$ be as in Theorem 4.1. Recall that, for $x \in M$,

$$K_O(x,\gamma) \quad = \quad \min\{j \geq 0;\ \ h_\gamma^j(x) \in O\}.$$

If an orbit enters O, it takes at most k iterates to enter $\bar{N}$. Therefore, an application of Theorem 4.1 to μ_r in equation (28) shows that

$$\int_{f_\gamma^{(r+2)k}(E_\gamma)} K_{\bar{N}}(x,\gamma) d\mu_r / \mu(f_\gamma^{2k}(E_\gamma)) \quad \leq \quad L,$$

for some $L > 0$, uniformly in γ. This, (25) and (28) proves (22). $\square$

6 The saddle node: local embedding flows and fundamental intervals

Denote by U a small neighborhood of a on which f_0^k is invertible. Let $W^s_{loc}(a)$ and $W^u_{loc}(a)$ denote the usual local stable and local unstable sets for a.

Proposition 6.1 *Let $\{f_\gamma\}$ be a family of C^r, $r \geq 2$, maps unfolding a saddle-node. Then there exists a family of C^r flows, $\{\phi^t_\gamma\}_{0 \leq \gamma < \bar{\gamma}}$, on U such that $f^k_\gamma \equiv \phi^1_\gamma$ for each $\gamma \geq 0$. Further, $\phi^t_\gamma(\cdot) \to \phi^t_0(\cdot)$ in the C^1 topology on U and in the C^r topology on compact intervals away from the fixed point. The flow ϕ^t_0 is uniquely determined by f_0.*

Proof. The C^∞ version of this theorem is due to Takens [Tak73]. The C^r result follows from Part 2 of [Yoc95]. The case $\gamma = 0$ follows from Appendix 3 of that reference. The case $\gamma > 0$ and the convergences as $\gamma \searrow 0$ follow from Theorem IV.2.5 and Lemma IV.2.7 of the same. □

Remark 6.2 *This result is known as the Takens Embedding Theorem. A version of it appears in [IlyLi99]. They proved that one may obtain $\phi^t_\gamma(x)$ which depends C^r smoothly on both x and γ, even at the fixed point, if one requires that $(x, \gamma) \mapsto f_\gamma(x)$ be $C^{R(r)}$ smooth, where $R(r)$ may be larger than r. Proposition 6.1 allows for our weaker hypotheses and its implications are sufficient for our purposes.*

Choose a point $e \in W^u_{loc}(a)$, such that

$$I^u_\gamma \equiv [e, f^k_\gamma(e)] \subset U,$$

for all $0 \leq \gamma < \bar{\gamma}$. Similarly, choose $d \in W^s_{loc}(a)$ and let

$$I^s_\gamma \equiv [d, f^k_\gamma(d)] \subset U.$$

Given $\gamma \geq 0$ and $x \in I^u_\gamma$, define $\tau^u_\gamma(x)$ to be the unique number for which

$$\phi_\gamma^{\tau^u_\gamma(x)}(e) \quad = \quad x.$$

For $\gamma \geq 0$ and $x \in I^s_\gamma$, let $\tau^s_\gamma(x)$ be defined by $\phi_\gamma^{\tau^s_\gamma(x)}(d) = x$. It follows from the smoothness of $\phi^t_\gamma(x)$ that for each $\gamma \geq 0$, the functions $\tau^{s,u}_\gamma$ are C^r diffeomorphisms from $I^{s,u}_\gamma$ to [0,1]. Hereout we will identify the interval [0,1] with the unit circle $\mathbb{S}^1$. We will use $\tau^{s,u}_\gamma$ as coordinates on $I^{s,u}_\gamma$.

Given d and e as above, let $\{\gamma_n\}_{n=n_0}^\infty$ be the sequence, $\bar{\gamma} > \gamma_{n_0} > \gamma_{n_1} > \ldots$, defined by

$$f^{kn}_{\gamma_n}(d) \quad = \quad e.$$

For each $n \geq n_0$ let $g_n : [0,1] \to [\gamma_{n+1}, \gamma_n]$, be the reparameterization map defined by

$$\phi^{n+\theta}_{g_n(\theta)}(d) \quad = \quad e.$$

We have that $g_n(0) = \gamma_n$ and $g_n(1) = \gamma_{n+1}$. We may invert $g_n(\cdot)$, for each n, to obtain maps $\theta_n : [\gamma_{n+1}, \gamma_n] \to [0,1]$.

Proposition 6.3 *The reparameterization maps g_n are smooth monotone decreasing functions with uniformly small distortion: given $\varepsilon > 0$ there is $N \in \mathbb{N}$ so that for every $n \geq N$ and every $\theta \subset [0,1]$,*

$$(1-\varepsilon) \leq \frac{Dg_n(\theta)}{|\gamma_n - \gamma_{n+1}|} \leq (1+\varepsilon).$$

PROOF. Diaz et. al. proved this result under the hypothesis that $(x,\gamma) \mapsto f_\gamma(x)$ is $C^{R(r)}$, using Il'yashenko and Li's embedding result (Remark 6.2). A proof of this result under the current hypotheses appears in [AfrYou98] based on [Jon90]. $\square$

We remark that $n^2\gamma_n$ converges as $n \to \infty$ (see [MisKaw90]), so that $\gamma_{n+1}/\gamma_n \to 1$ as $n \to \infty$. This fact, together with Proposition 6.3 imply the next lemma [AfrYou98].

Lemma 6.4 *Let Γ be a measurable subset of $[0,\bar{\gamma})$ and denote $\Gamma_n = \Gamma \cap [\gamma_n, \gamma_{n-1}]$. If the limit*

$$\lim_{n\to+\infty} m(\theta_n(\Gamma_n))$$

exists and equals Δ, then

$$\lim_{\gamma\searrow 0} \frac{m\left(\Gamma \cap [0,\gamma)\right)}{\gamma} \quad = \quad \Delta.$$

Let $\mathcal{L}_{n,\theta}$ denote the local (first hit) map from I^s_γ to I^u_γ induced by $f_{g_n(\theta)}$. The convenience of using τ^s_γ and τ^u_γ as coordinates on I^s_γ and I^u_γ is seen in the following proposition.

Proposition 6.5 *For each $n \geq n_0$ and each $\theta \in [0,1]$*

$$\mathcal{L}_{n,\theta} \quad = \quad (\tau^u_{\gamma_n(\theta)})^{-1} \circ R_{-\theta} \circ \tau^s_{\gamma_n(\theta)}. \tag{29}$$

PROOF. This follows from Proposition 6.1 and the definitions of τ^s_γ and τ^u_γ as the time variables for the embedding flow for f^k_γ. $\square$

7 The Mather invariant and return maps

Let $\bar{\mathcal{G}}$ be the first hit map from I^u_0 to $W^s_{loc}(a)$. With $\tau^s_\gamma : I^s_\gamma \to [0,1]$ defined in the previous section we also define, for $\gamma = 0$, $\bar{\tau}^s_0 : W^s_{loc}(a) \to [0,\infty)$ by $\phi_0^{\bar{\tau}^s_0(x)}(d) = x$. Note that $\tau^s_0 = \bar{\tau}^s_0|_{I^s_0}$. Define $\bar{\mathbf{M}} : [0,1] \to \mathbb{R}$ by

$$\bar{\mathbf{M}} \quad = \quad \bar{\tau}^s_0 \circ \bar{\mathcal{G}} \circ (\tau^u_0)^{-1},$$

and define $\mathbf{M} : [0,1] \to [0,1]$ by $\mathbf{M} = \bar{\mathbf{M}} \mod 1$. By identifying the endpoints of $[0,1]$ we may consider $\bar{\mathbf{M}}$ as a map from a subset of the circle $\mathbb{S}^1$ into $\mathbb{R}$ and $\mathbf{M}$ as a map from a subset of $\mathbb{S}^1$ into $\mathbb{S}^1$. Following [Yoc95], we call $\bar{\mathbf{M}}$ the Mather invariant for f_0. One may show that $\mathbf{M}$ is a modulus of smooth conjugation, in other words, it is invariant under differentiable changes of variables.

Given J, denote by $V_\gamma(J)$ the subset of I^u_γ defined by

$$V_\gamma(J) \quad = \quad \{x \in I^u_\gamma : \bar{\mathbf{M}}(\tau(x)) < J, f_0^j(x) \in [d,a) \text{ for some } j < J, f_0^j(x) \notin [a, f_0^k(e)] \text{ for any } j < J\}.$$

The points in $V_\gamma(J)$ are those whose forward orbits enter $W^s_{loc}(a)$ in a bounded number of iterations and which do not come too close to a in the process, either by re-entering $W^u_{loc}(a)$ or by landing in $W^s_{loc}(a)$ too close to a. Since the forward orbit of almost every $x \in W^u_{loc}(a)$ has $\mathcal{O}(a)$ as its omega limit set, the relative measure $m(V_\gamma(J))/m(I^u_\gamma)$ may be made close to 1 by taking J to be large and choosing d and e close to a.

For $\gamma > 0$, consider the first return map, κ_γ, of the interval I^u_γ and let $\tilde{\kappa}_{n,\theta}$ be the normalized map given by

$$\tilde{\kappa}_{n,\theta} \equiv \tau^u_{g_n(\theta)} \circ \kappa_{g_n(\theta)} \circ (\tau^u_{g_n(\theta)})^{-1}.$$

Identifying the endpoints of $[0,1]$, we may consider $\tilde{\kappa}_{n,\theta}$ as a map on the circle.

Proposition 7.1 *Given any J,*

$$\lim_{n\to\infty} \left| \tilde{\kappa}_{n,\theta}\big|_{\tau^u(V_\gamma(J))} - R_{-\theta} \circ \mathbf{M}\big|_{\tau^u(V_\gamma(J))} \right|_{C^r} \quad = \quad 0$$

for each $0 \le \theta < 1$.

Proof. Let $\mathcal{G}_\gamma$ denote the global (first hit) map from I^u_γ to I^s_γ induced by f_γ. Note that κ_γ is not equal to $\mathcal{L}_\gamma \circ \mathcal{G}_\gamma$ since some points in I^u_γ will return to I^u_γ before hitting I^s_γ. However, the two maps do agree when restricted to $V_\gamma(J)$ since points in $V_\gamma(J)$ will in fact hit I^s_γ before returning to I^u_γ. Since we are only considering a finite number of iterations, it follows from the construction of $\mathbf{M}$ that $\tau^s_\gamma \circ \mathcal{G}_\gamma \circ (\tau^u_\gamma)^{-1}$ converges to $\mathbf{M}$ on the restricted set $\tau^u_\gamma(V_\gamma(J))$. Proposition 6.5 then implies that

$$\tau^u_\gamma \circ \mathcal{L}_{n,\theta} \circ \mathcal{G}_\gamma \circ (\tau^u_\gamma)^{-1} \tag{30}$$

converges to $R_{-\theta} \circ \mathbf{M}$ on $\tau^u_\gamma(V_\gamma(J))$. □

8 Intermittency near the saddle node bifurcation

Definition 8.1 *Given a piecewise smooth map $T : X \to X$ on an interval X, a periodic interval is a closed interval $M \subset X$ such that $T^n(M) \subset M$ for some $n > 0$, and the orbit of M is bounded away from the discontinuities of T. We will say that a periodic interval is hyperbolic if one of its endpoints is a repelling hyperbolic point, the other endpoint is mapped onto the first endpoint by T^n and the attractor in M is contained in the interior of M.*

Note that hyperbolic periodic intervals are stable under C^1 perturbations of the map T.

Proposition 8.2 *Suppose that for each $\theta \in (\theta^-, \theta^+) \subset \mathbb{S}^1$ the map $R_{-\theta} \circ \mathbf{M}$ has a hyperbolic periodic interval M_θ of period k_1. Then there exists integers n_0 and m, and a sequence of intervals $\{(\beta_n^-, \beta_n^+)\}_{n=0}^\infty$ converging to 0 such that for each $\gamma \in (\beta_n^-, \beta_n^+)$ and $n \geq n_0$, f_γ has a hyperbolic periodic interval N_γ of period $k_1 nk + m$. The following limit exists,*

$$\lim_{n\to\infty} \frac{|\beta_n^+ - \beta_n^-|}{|\gamma_n - \gamma_{n+1}|} = |\theta^+ - \theta^-|.$$

Proof. By the definition of $\mathbf{M}$, if we choose $J > k_1$, then for γ small enough the interval $(\tau^u)^{-1}(M_\theta) \subset V_\gamma(J)$. The existence of the intervals (β_n^-, β_n^+) is then immediate from Proposition 7.1. Further, Proposition 7.1 also implies that $\theta_n(\beta^\pm) \to \theta^\pm$, as $n \to \infty$. Equation (31) then follows from Proposition 6.4. □

The intervals (β_n^-, β_n^+) are periodic windows in the parameter space. The orbits outside of $\bar{U}$ are equivalent for each n, but the number of iterates inside $\bar{U}$ is a multiple of n.

Proof of Theorem B. We first treat the discontinuity of the averages $\Phi_{\bar{U}}$ as $\gamma \searrow 0$. Let q be a hyperbolic periodic point of f_0. Observe that there is a sequence of parameter values γ_l converging to 0 for which $f_{\gamma_l}^{kl}(c)$ equals the periodic point q. Since c is in the unstable manifold of q, there are $\gamma_{l,m}$ for large enough m, converging to γ_l and so that $f_{\gamma_{l,m}}^{kl+km}(c) = c$. This periodic orbit makes one passage through a small neighborhood of $\mathcal{O}(a)$, which takes $kl - kn$ iterates for some fixed value n. Therefore, $\Phi_{\bar{U}}(x, \gamma_{l,m}) = \frac{kl-kn}{kl+km} = \frac{l-n}{l+m}$ for any x in the basin of attraction of $\mathcal{O}(c)$ (which means for almost all x). Since l, m take on all large enough integer values, any fraction can be obtained this way. This proves the first statement of Theorem B.

Let $(\theta^-, \theta^+) \subset \mathbb{S}^1$ be such that $R_{-\theta} \circ \mathbf{M}$ possesses a hyperbolic periodic interval. Now let N_γ and $\Gamma = \cup_{n=n_0}^\infty (\beta_n^-, \beta_n^+)$ be as in the conclusion of Proposition 8.2. Positive density of Γ follows from Proposition 8.2. We claim that there is a constant C for which

$$\lim_{\gamma\searrow 0, \gamma\in\Gamma} \frac{|1 - \Phi_{\bar{U}}|}{\sqrt{\gamma}} = C. \tag{31}$$

For $\gamma \in \Gamma$, the orbit of almost every point eventually enters the periodic interval N_γ. It therefore suffices to consider points in N_γ. We may assume that the orbit of N_γ does not intersect the boundary of $\bar{U}$. The orbit $\mathcal{O}(N_\gamma)$ spends a fixed number of iterations outside $\bar{U}$. As $n \to \infty$, $\mathcal{O}(N_\gamma)$ spends an arbitrary number of iterations in $\bar{U}$. The numbers of iterations inside and outside of $\bar{U}$ are the same for all points in N_γ, so the limiting constant C is the same for every point in N_γ. The specific formula in (31) follows from a standard calculation of the number of iterations an orbit spends inside $\bar{U}$, see [PomMan80]. □

Note that Theorem B and its proof do not take into consideration any aspects of the dynamics inside the periodic interval.

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