A jump forwards with mathematics and physics

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We jump on human body motions such as bouncing on a jumping stick, hopping, and making kangaroo jumps. Students can record the movements with a digital camera and use their video clips to investigate the motions with suitable video analysis and modelling software. We discuss some mathematical models of these motions using basic biomechanical principles and we compare modelling results with experimental data obtained from video measurements. Highlight is the application of the model of a planar inverted spring-mass system: this rather simple model works qualitatively and quantitatively well for the complex motions of hopping, skipping and running at moderate speeds. The examples of video analysis and modelling activities give a good impression of the potential of the subject of human gait for student practical investigations and as a context for applied mathematics and physics at secondary and undergraduate level.

Introduction

Human gait can take many forms such as sauntering, walking, hopping, skipping, jogging, running, sprinting, and so on. In this paper we construct mathematical models of the following bouncing gaits: bouncing on a jumping stick, hopping, and making kangaroo jumps. This seems very ambitious because such vivid motions are at first sight not easily modelled. The pushing-off and landing of a vertically hopping person savour strongly of the motion of an extending and compressing inverted spring-mass system. All sorts of models of this type are used in biomechanical studies. But how simple or complex must such a mathematical model be to describe reality to a reasonable extent? Can students at secondary and undergraduate level with modest knowledge of mathematics and physics actually do such investigations? In an attempt to answer these questions we investigate bouncing gaits and corresponding inverted spring-mass models of increasing complexity.

Vertical bouncing on a jumping stick

In a 2008 nationwide secondary physics examination in the Netherlands, Thomas was put on the scene with his pneumatic jumping stick (See the left part of Figure 1). The main components of this type of jumping stick are: (i) foot and hand supports; (2) an air-filled cylinder; (3) a piston that can slide within the cylinder and forms the bottom of an air chamber; and (4) an elongate shaft coupled to the bottom of the piston and moveable therewith. When Thomas repeatedly jumps in the vertical direction, we can distinguish two phases: (1) the aerial phase, in which the jumping stick is off ground, maximally extended, and assumingly moving as a rigid body under gravity; and (2) the contact phase, in which one end of the jumping stick remains in contact with a fixed point on the ground. At landing, the shaft and piston are forced into the cylinder, the stick length is shortened and consequently the volume of the air chamber decreases and the pressure therein increases. After shortening of the jumping stick, the shaft springs back and the jumping stick elongates again, just like a compressed spring. It turns out that Thomas could comfortably jump when the airborne and contact phase took almost equal time. The motion of Thomas on his jumping stick serves as source of inspiration for describing some natural gaits of humans and animals with an inverted spring-mass model.

This rather clear situation of a periodic motion of a person on a jumping stick can be described well with a model based on simple mathematics and physics. The quality of the chosen model can be evaluated by comparing the model results with data acquired through video analysis of the motion. A schematic drawing of the one-dimensional spring-mass model is shown in the right part of Figure 1. In this model we ignore the mass of the spring and we divide the period of one jump into two phases, viz., the aerial phase (with aerial time t_a) and the contact phase (with contact time t_c). We assume that Thomas is able to vertically jump on his stick without changing his posture and that his body centre is near the hand supports. At landing, the jumping stick has rest length L. We further assume that during aerial phase only the gravitational force $F_g = -mg$, where m is the body mass and g is the acceleration of gravity, plays a role and that during contact phase also the spring force $F_s = C(L - y)$ must be taken into account, i.e., for heights $y \le L$ the linear elastic motion of the spring depends on a spring stiffness C (for heights $y > L$ we may take $C = 0$).

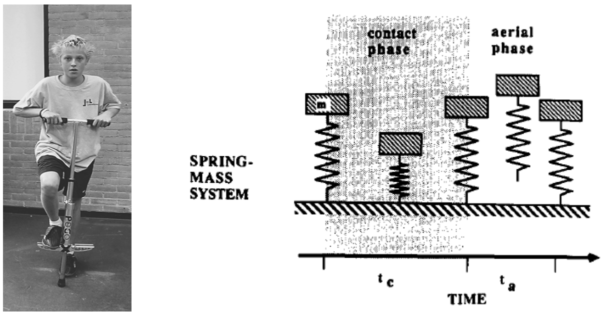

Figure 1. Thomas on his jumping stick and the corresponding spring-mass model.

The dynamics of the spring-mass system is now determined by a second order differential equation and two initial conditions: $a = y'' = \left(F_g + F_s\right)/m = F/m$, with $y(0) = y_0$ and $y'(0) = v_0$. This can be rewritten as a system of first-order differential equations: $v = y',\quad a = v' = \left(F_g + F_s\right)/m$, $y(0) = y_0$, $v(0) = v_0$. We have done this because the rewritten equations can then be easily implemented in a system-dynamics based modelling tool. We use the modelling tool of the Coach learning environment (Heck et al., 2009). Figure 2 is a screen shot of a graphical model that implements the spring-mass model expressed by the above equations. In the graphical model, each combination of a rectangle and an inflow double arrow represents the integration of a quantity. In the example to the right, the arrow represents the derivative of the variable *y* and this quantity is integrated in time during a simulation run.

The quality of the mathematical model is determined by comparing the model results with data coming from real experiments. The diagram in the middle of Figure 2 shows the graph of the computed height and the point plot of the vertical heights measured in a digital video recorded while Thomas was jumping on his stick. The measured data suggest that a sinusoidal regression curve would describe the data quite well, and indeed it does from mathematical point of view, but the spring-mass model is considered better because it is based on physics laws. Then a sinusoidal displacement during contact phase is followed by a parabolic aerial phase. This judgment of the quality of a model is what students should learn.

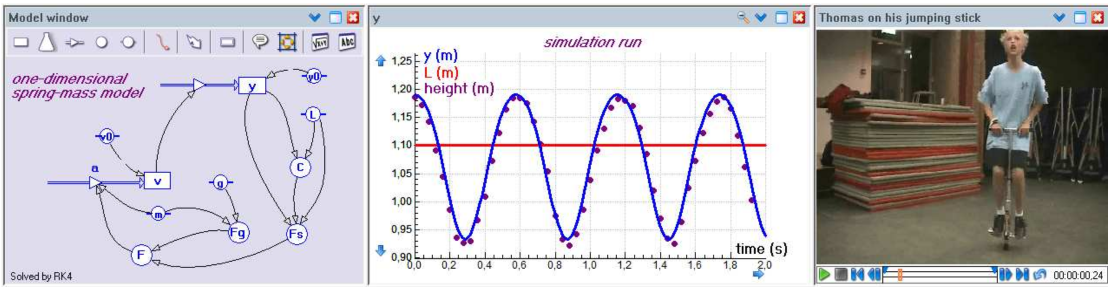

Figure 2. A graphical model implementing the one-dimensional spring-mass model and the results of a simulation run compared with data obtained from a video analysis of a movie clip.

Hopping upward

One may truly wonder whether the previous one-dimensional spring-mass model is a good model for human hopping in the upward direction without the use of a device. The proof of the pudding is in the eating. So, for the purpose of data collection, we went with students to the University Sports Centre of the UvA to let them hop in vertical and forward direction on a motorised treadmill. We recorded motions with a high speed camera at a speed of 300 frames per second so that we could observe as many details as needed and work with a high time resolution. We present the results in this section.

In order to get more insight in the motion of the body centre during contact phase of upward hopping we determine the exact solution of the one-dimensional spring-mass model. The stance leg (in this case actually both legs) is modelled as a massless, linear spring with stiffness C and rest length L. Here we imitate models of Blickhan (1989), McMahon & Cheng (1990), and many other biomechanical scientists. For ease of computing we assume time $t=0$ when the leg makes first contact with the treadmill (in a video measurement we can easily calibrate time in this way) and we assume that landing speed is equal to $-v$. The vertical position of the body centre has been chosen to be the same as the position of the hip joint of the hopping person (See Figure 3). Under the assumption that only gravitational force and spring force play a role, Newton's second law of motion and Hooke's law of elasticity lead to the following equation of motion for the height y during contact phase: $my''=-mg+C(L-y)$, $y(0)=L,\ y'(0)=-v$. Let $u=y-L$ be the displacement during ground contact. Then the equation of motion can be rewritten as follows: $u''+\omega^2 u=-g,\ u(0)=0,\ u'(0)=-v$, where the natural spring frequency ω is given by $\omega^2=C/m$. This equation can be solved analytically and the solution is the following sinusoid: $u(t)=-\dfrac{v}{\omega}\sin(\omega t)+\dfrac{g}{\omega^2}\cos(\omega t)-\dfrac{g}{\omega^2}$. This leads to the following formula for the speed of the body centre: $v(t)=-v\cos(\omega t)-\dfrac{g}{\omega}\sin(\omega t)$. Halfway contact time the stance leg is maximally bent and the speed of the body centre is equal to zero. This gives the relation: $-v\cos\left(\dfrac{1}{2}\omega t_c\right)-\dfrac{g}{\omega}\sin\left(\dfrac{1}{2}\omega t_c\right)=0$. Thus, contact time is related to speed and frequency as follows: $t_c=\dfrac{2\pi}{\omega}-\dfrac{2}{\omega}\arctan\left(\dfrac{\omega\cdot v}{g}\right)$. What we learn from the last formula is that the motion during contact phase depends on three out of the four factors: (1) the acceleration of gravity g; (2) the natural frequency ω of the spring-mass system; (3) the take-off and landing speed v; and (4) contact time t_c. Because of the definition of the natural frequency one can exchange this factor by the spring constant C (stiffness), provided that the body weight m is known. To conclude, by using exact mathematical methods one can make grounded statements about a bodily motion and investigate the dependencies of determining factors.

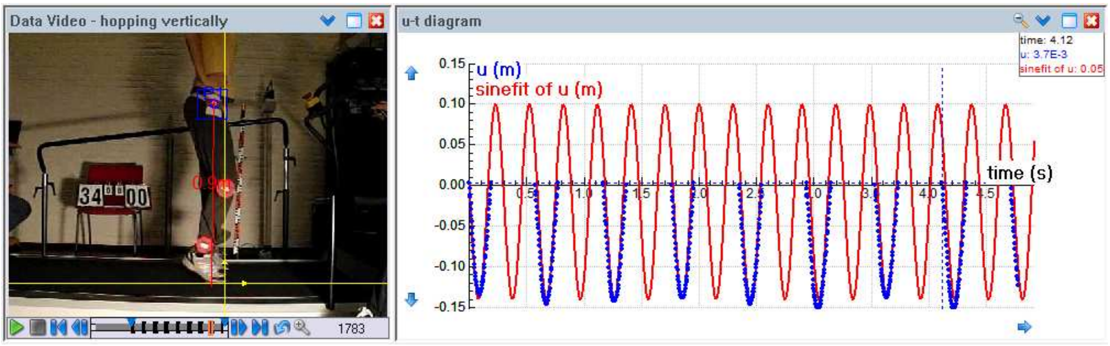

Figure 3. Video analysis of an upward hopping girl on a motorised treadmill that is not turned on.

The exact solution of displacement can also be used to estimate the natural frequency and the landing speed on the basis of measurements. Figure 3 is a screen shot of a video analysis using Coach. The sine curve matching best the data points can be obtained with the data analysis tools. The sinusoidal regression curve $y(t)=0.12\sin(21.25t+2.95)+0.88$ can be rewritten as $u(t)=-0.118\sin(21.25t)-0.023\cos(21.25t)-0.02$. From this formula follow the initial estimates $\omega=21.25\mathrm{Hz},\ v=2.5\,\mathrm{m/s}$. This landing speed is only a little bit greater than the speed of $2.3\,\mathrm{m/s}$ obtained by numerical differentiation of the measured data. The estimated values give a contact time of 0.17s, which is also only a little bit less than the measured contact time of 0.19s in the video clip (For this precision one needs a high speed camera). The take-off speed can be used to compute the duration of the aerial phase, under the assumption that the aerial motion depends only on gravity: $t_a=2v/g$. This gives $t_a\approx 0.47\mathrm{s}$ and the estimated flight time deviates little from the measured flight time of $0.40\mathrm{s}$.

In short, the one-dimensional spring-mass model applied to a person hopping upward using no special device leads to model results that are in good agreement with results obtained from video analysis measurement on recorded movie clips. The agreement between model results and measured data get even better when we do not find out a sinusoidal regression curve for the measured data, but instead try to find the best values for parameters in the spring-mass model by the method of trial and improvement. For the hopping girl in Figure 3 we found a very good match between model and video analysis, which holds for nine consecutive hops, using $C = 19\,\mathrm{kN/m}$, $v = 1.95\,\mathrm{m/s}$. This value of the stiffness C is in good agreement with values found in the literature (Farley et al., 1991). From these values we obtain the following results: $\omega = 18.9\,\mathrm{Hz}$, $t_c = 0.19\,\mathrm{s}$, $t_a = 0.40\,\mathrm{s}$. Whereas aerial and contact time were almost equal for hopping on a jumping stick, they differ for human hopping without a device.

Hopping forward like a kangaroo

What should set the seal on our work is the application of a planar inverted spring-mass model to human double legged forward hopping, i.e., mimicking kangaroo jumping. The model is in this case two-dimensional. So to start with, the one-dimensional spring-mass model of upward hopping is extended. The new model contains besides the kinematical variables (y, v_y, a_y, F_y) also the variables (x, v_x, a_x, F_x), as it were 'doubled'. Quantities like speed and force are decomposed in the *x*- and *y*-direction. The planar inverted spring-mass model for bouncing gaits such as hopping and running is schematised in Figure 4.

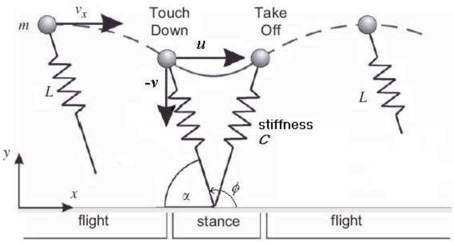

Figure 4. Planar inverted spring-mass model for forward hopping and running.

In comparison with the one-dimensional spring-mass model of upward hopping, we have now two new conditions: the leg angle of attack α, when the leg makes ground contact, and the angle of take-off velocity β, when the leg looses ground contact. These angles are most easily defined when we select the stance point as the origin of the coordinate system during contact phase, with the positive *x*-axis in the direction of motion and the positive *y*-axis in the upward direction, and when we assume that the stance leg lands at time $t = 0$: $\tan\alpha = -y(0)/x(0)$ and $\tan\beta = v/u$, where u is the horizontal landing speed (equal to the speed of the motorised treadmill when the gait is on such device) and $-v$ is the landing and take-off speed. The leg angle of attack and the angle of take-off velocity are not necessarily equal. It is not difficult to determine these angles in a recorded video clip because Coach, like any professional video analysis tool, provides its user a digital ruler, a digital protractor, and graphs of position and velocity. Notice that one cannot freely change the parameters α and β in a computer model for given values of leg length $L = \sqrt{x(0)^2 + y(0)^2}$ and landing velocity $v(0) = \sqrt{u^2 + v^2}$ if the model must be periodic. After all the leg angle of attack and the leg angle of take-off must be equal for a periodic motion. Under the given circumstances the following condition can be used to distinguish between aerial and contact phase: when $y \le L\sin\alpha$. there is ground contact and the leg can be considered as a linear spring with stiffness C.

Let us now derive the equations of motion for bouncing gaits. The spring force F_s during contact phase is according to Hooke's law of elasticity given by $F_s = C(L - r)$, where $r = \sqrt{x^2 + y^2}$ is the length of the spring and C is the stiffness of the spring. This spring force must be decomposed into horizontal and vertical components in order to derive the equations of motion: $F_{s,x} = F_s \cos\phi,\ F_{s,y} = F_s \sin\phi$. Thus: $F_{s,x} = F_s\, x/r$, $F_{s,y} = F_s\, y/r$. After moderate algebraic manipulation we obtain the following initial value problem from Newton's second law of motion: $x'' = -\omega_s^2 x + \frac{g\lambda x}{\sqrt{x^2+y^2}},\ y'' = -g - \omega_s^2 y + \frac{g\lambda x}{\sqrt{x^2+y^2}},\ x(0) = -L\cos\alpha,\ x'(0) = u,\ y(0) = L\sin\alpha,$ $y'(0) = -v.$ Here we have introduced the parameters $\omega_s^2 = C/m$, $\omega_p^2 = g/L$, and $\lambda = \omega_s^2/\omega_p^2$. The main difference between vertical and forward hopping is that the leg length, parameterised by λ comes seriously into play in the modelling of the body motion during ground contact. The main application of the analytical methods like the one discussed in this section is that it allows investigating the influence of gait parameters on the body motion and researching dependencies between the various parameters in the mathematical model.

We are almost ready for constructing a computer model of periodic forward hopping like a kangaroo. But first we must realise that the body centre follows under the given assumptions a parabolic curve during aerial phase that starts in $(L\cos\alpha, L\sin\alpha)$ with a horizontal speed u and a vertical take-off speed v. After all, the initial value problem for the motion during aerial phase is: $x'' = 0,\ x'(t_c) = u,\ x(t_c) = L\cos\alpha,\ y'' = -g,\ y'(t_c) = v,\ y(t_c) = L\sin\alpha.$. These equations of motion of the spring-mass model of forward hopping can easily be implemented in the graphical modelling tool of Coach. Only a solution for moving the coordinate system from one stance point to the other must be found. The fact that Coach is designed as a hybrid system that combines a classical system dynamics approach with event-based modelling for processes that change abruptly helps solve the implementation problem of a moving frame. As triggering condition for the landing of a hopping person we can use the Boolean expression $y \le y_{start}$ and $y' \le 0$ and $x + x_{start} > 0$, where the initial conditions x_{start} and y_{start} are given by $x_{start} = -L\cos\alpha$ and $y_{start} = L\sin\alpha$. The event is defined behind the upper-left icon in the below graphical model. The idea behind the event handling is that one starts with a coordinate system of which the origin coincides with the first stance point. In the variable x_0 we store the current value of the *x*-coordinate of the origin of the moving coordinate frame. Each time when the event of touch down occurs, the variable x_0 is refreshed with the *x*-coordinate of the new stance point via the assignment $x_0 = x + L\cos\alpha$.

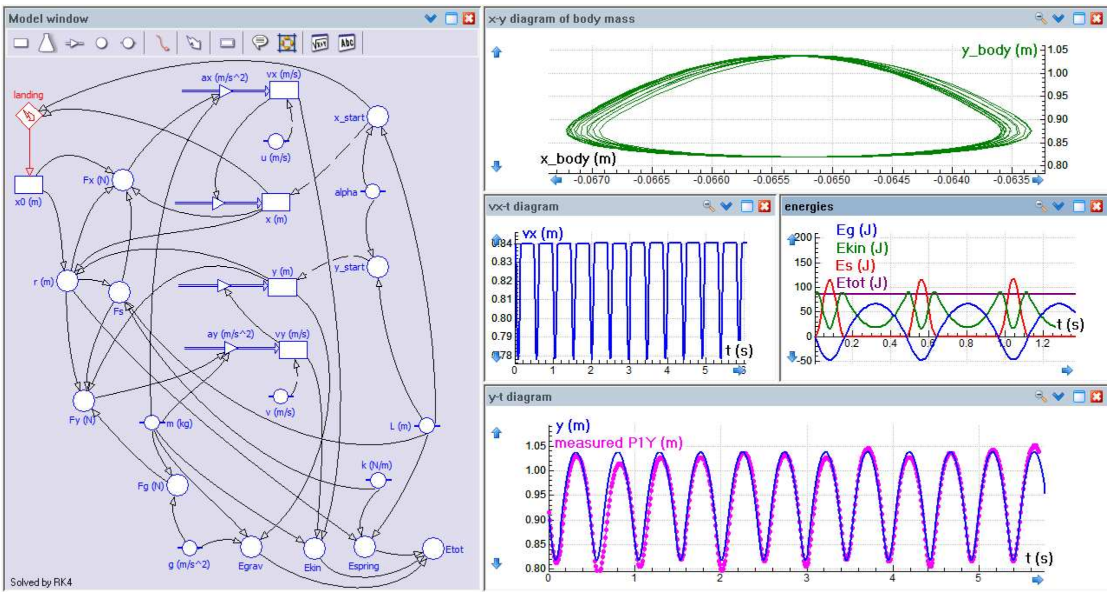

Figure 5. Coach simulation of the planar inverted spring-model of forward hopping like a kangaroo.

In order to evaluate the suitability of the spring-mass model for human hopping like a kangaroo we compare model results with experimental results obtained by motion analysis. To this end we recorded the motion of a hopping girl on a motorised treadmill going at a speed of 3 km/h on video and constructed the height-time graph via automated point tracking of a hip joint marker. We use this measurement as a background graph in the modelling activity to find by trail and improvement appropriate parameter values. It is quite tricky to set the values such that the spring-mass model runs periodically for a long time: the leg angle of attack must match with other parameters in order that the take-off velocity equals the landing velocity. But the reward is great: Figure 5 illustrates a perfect match. The upper-right *y*-*x* diagram illustrates the periodicity of the simulated motion. The computer model can be used to study common forms of energy such as the kinetic energy $E_{kin} = \frac{1}{2}mv^2$, the gravitational energy $E_g = mgy$ and the spring potential energy $E_s = \frac{1}{2}C\left(L-r\right)^2$. The sum E_{tot} of these three energies is constant, as shown in Figure 5. The parameter values found for the hopping girl weighing 53 kg were: $C = 28\,\mathrm{kN/m}$, $u = 0.84\,\mathrm{m/s}$, $v = 1.95\,\mathrm{m/s}$, $L = 0.91\mathrm{m}$, $\alpha = 86.0003$. Other quantities can then be computed: $\omega_s = 23.0\,\mathrm{Hz}$, $\omega_p = 10.8\,\mathrm{Hz}$, $t_a = 0.40\,\mathrm{s}$, $t_c = 0.2\,\mathrm{s}$. What the results indicate is that leg stiffness and spring frequency ω_s are greater in hopping forward like a kangaroo than in hopping upward is greater in this case. Yet this had little or no effect on aerial and contact time. All this is not so very strange considering the low speed of the treadmill. Apparently, only leg stiffness must be increased to maintain a short contact phase.

Conclusion

Finally, let us return to the two questions raised in the introduction. How simple or complex must such a mathematical model be to describe reality to a reasonable extent? It is striking that a relative simple spring-mass model describes bouncing gait patterns so well. Bullimore & Burn (2007) confirmed that the model allows prediction of several gait characteristics such as contact time, vertical momentum, and stride length. But they also noticed that it often overestimates the horizontal ground reaction force, the flight time and the change of mechanical energy of the body centre. Geyer (2005) successfully adapted the spring-mass model to walking and running gait patterns. The actual power of the mathematical models is that they help researcher investigate various aspects of bouncing gaits, such as step frequencies, forces, stability of gait patterns, costs of energy, etc., and compare gait patterns.

The second question was whether students at secondary and undergraduate level with modest knowledge of mathematics and physics can actually do such investigations. Our experience is that students with a keen interest and good ability in mathematics and physics can master the modelling process. Less gifted students are still expected to be able to grasp the one-dimensional inverted spring-mass model, which was after all used in a nationwide examination. More importantly, all students can do with modest means biomechanical research in much the same way as ‘real’ scientists and they can practise herein mathematical knowledge and skills such as graph comprehension, numerical differentiation and integration, data processing and analysis, regression, etc. They can develop the critical attitude that is necessary for successful modelling of natural phenomena. For this it is very important that the students can compare the results of computer models with real data, preferably collected in an earlier measurement activity. Confrontation of a model with reality turns graphical modelling not only into a fun way of learning, but it also makes it exciting, challenging, and concrete work. It is joyful when experiment, model, and theory are in good agreement, as in this paper.

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