A NEW LEARNING TRAJECTORY FOR TRIGONOMETRIC FUNCTIONS

Özcan Demir and André Heck

Korteweg-de Vries Institute for Mathematics, University of Amsterdam, The Netherlands

Educational research has shown that many secondary school students consider the subject of trigonometric functions as difficult and only develop shallow and fragmented understanding. It is not clear which of the two popular approaches to teaching and learning trigonometry, namely the ratio method and the unit circle method, works best. In this study we propose a new framework for trigonometric understanding and a dynamic geometry supported learning trajectory for learning about trigonometric functions. We also report on the results of a classroom study in which the new hypothetical learning trajectory for trigonometric functions has been implemented. We discuss the task-related difficulties that students faced in their concept development and we describe their trigonometric understanding in terms of our framework.

BACKGROUND

Trigonometry is an important subject in secondary mathematics education and beyond, even though its meaning may change from country to country at secondary school level (cf., Delice & Roper, 2006). Understanding of the trigonometric functions is fundamental for studying periodic phenomena and working in many application areas. The curriculum of trigonometry is mostly distributed over several school years. It involves the introduction of sine, cosine and tangent as functions of an angle, either in the geometric context of right-angled triangles or in the context of geometric transformations using the notion of the unit circle, and as functions of a real number. Thus, students encounter trigonometric functions while learning both geometry and calculus.

In the geometry world of triangles, the sine, cosine, and tangent of an acute angle are defined as the ratios of pairs of sides of a right triangle. This is referred to as the ratio method of introducing trigonometric functions. The right triangle is also often embedded in the unit circle, but then the notion of angle actually gets the meaning of rotation angle. The sine and cosine of a particular angle are then defined as the horizontal and vertical coordinate of a point obtained by rotating the point (1,0) about the origin over the angle. This is called the unit circle method. In both approaches, the trigonometric functions are functions of an angle and not of a real number. This is more or less repaired by introducing the notion of radian. It is important to realize and make clear to students that the notion of angle differs in the two approaches: it is in the ratio method an angle of a triangle with values between 0 and 90 degrees, whereas it is in the unit circle method a rotation angle which has both a magnitude and a direction.

However, research findings concerning the question which of the two approaches, ratio method or unit circle method, works better to introduce trigonometry are sparse and not conclusive. For example, Kendal and Stacey (1997) found in a large-scale study that students who were taught trigonometric functions by the ratio method better mastered the skills required for solving tasks with triangles (e.g., correct choice of the trigonometric function, correct formulation and transposition of the trigonometric equation to solve, and correct calculations) and made greater improvements in attitude toward the mathematical subject than students who were introduced to trigonometry via the unit circle method. Some researchers, such as Weber (2005, 2008) and Moore (2012), advocate the unit circle method to teach introductory trigonometry; others favour a combination of both. Teachers and educational researchers who advocate the use of the unit circle method often mention the following advantages:

- it can also deal with obtuse angles, negative angles, and angles of any size;
- it avoids working with ratios, which is a notion that has been proven to be difficult for many a pupil (Hart, 1981);
- sine values and cosine values are not restricted to positive values (like in the ratio method);
- it supports reasoning about various properties of trigonometric functions such as periodicity and trigonometric equalities;
- it links with many circular motions in a real setting (motion of seats in a Ferris wheel or merry-go-round, movement of pedals and wheels of a bicycle, and so on).

Taking these advantages into consideration, Kendall and Stacey (1997) stated that they preferred that basic trigonometry be introduced using the unit circle trigonometric function definitions (and in more than one quadrant!), connecting them to the ratio definitions and then adopting the techniques of the ratio method for the solution of problems about triangles. In the Dutch mathematics curriculum (and textbooks) the opposite road is taken: the ratio method is studied for right triangles at lower level and followed up by the unit circle at the beginning of upper level. Real world contexts such as a ladder resting against a wall, the motion of a robot arm and the motion of a Ferris wheel are used to make the subject more attractive and insightful for students, Typically, the students' learning is supported by dedicated computer programs and dynamic geometry software. The main advantage of ICT is that it enables students to explore relationships between numeric and visual representations of trigonometric ratios in right triangles and relationships between graphical representations of trigonometric functions and positions of points on the unit circle (cf., Kissane & Kemp, 2009; Zengin, Furkan, & Kutluca, 2012).

The research literature on students' understanding of trigonometric functions is sparse, but multiple studies are consistent in their conclusions that students develop in the aforementioned approaches a shallow and disconnected understanding of trigonometric functions and its underlying concepts, and have difficulty using sine and cosine functions defined over the domain of real numbers (Brown, 2005; Weber, 2005; Challenger, 2009; Moore, 2010). The sine and cosine functions may have been defined in the ratio method, unit circle method or a combination of both approaches, but the graphs of these real functions remain mysterious or merely diagrams produced by a graphing calculator or mathematics software. Trigonometric function are different from other types of functions in that they cannot be computed by carrying out certain arithmetic calculations revealed by an algebraic formula. Analytic definitions of trigonometric functions, as presented by Van Asch and Van der Blij (1997), are of little help at secondary school level. But, as Chin and Tall (2012) noted, this sophisticated viewpoint on trigonometric function, which is learned in undergraduate mathematics education, might affect the way in which mathematics teachers view the teaching and learning of their students. Anyway, the complex nature of trigonometry makes it challenging for students to understand the topic deeply and conceptually.

In this paper we present a new framework based on a model of trigonometric understanding. This framework played an important role in the design of a new instructional approach to sine and cosine functions and in the analysis of the data regarding students' understanding of sine and cosine functions that were collected in a classroom study. Unlike in the aforementioned traditional approaches, students start in the new learning trajectory with a winding function on a regular polygon instead of a winding function on the unit circle, engage in meaningful mathematical activities using the dynamic mathematics system GeoGebra as a cognitive tool, proceed to the unit circle through approximation with regular polygons, and finally arrive at sine and cosine graphs. The hope and expectation is that the new method helps students understand trigonometric functions in a connected manner based on an integrated understanding model of three trigonometric contexts, namely the triangle trigonometry, the circle trigonometry and trigonometric function graphs.

A MODEL OF TRIGONOMETRIC UNDERSTANDING

Brown (2005) presented a model of students' understanding of trigonometric functions within the geometry world of triangles and angles in degrees, and within the context of the unit circle. In her model, sine and cosine of an angle are ratios, distances, and coordinates. For example, the coordinate view pairs the angle with one of the coordinates of a point on the unit circle. Students with a robust understanding are able to integrate these different views. Brown identified several cognitive obstacles in learning trigonometry. For example, she found that students with a distance-dominant understanding had difficulties with the sign of a trigonometric value such as $\sin 210^{\circ}$. Drawback of Brown's model is that it is mainly based on the connections between triangle trigonometry and unit circle geometry, but hardly informs about the relations between the unit circle and the graphs of the real functions sine and cosine. The latter point is crucial if students are to gain an integrated understanding of the different aspects of trigonometry. For this reason we propose a model with a broader focus, namely a model that also includes graphs of trigonometric functions.

Our model of trigonometric understanding is based on a conceptual analysis of mathematical ideas within and among three contexts of trigonometry, namely triangle trigonometry, unit circle trigonometry, and trigonometric function graphs (Figure 1). The conceptual analysis of trigonometry based on angle measure by Thompson (2008) was a source of inspiration: we also strive for coherence between mathematical meanings at various levels of trigonometry. But our approach differs in two aspects: (1) the fundamental idea in the new model of trigonometric understanding is the functional relationship between arc length and corresponding horizontal and vertical position for the case of cosine and sine in the unit circle prior to angle measure; (2) our model is broader than Thompson's example using angle measure in the sense that it includes trigonometric functions in the domain of real numbers. The model of trigonometric understanding puts emphasis on coherent meanings within the three trigonometric contexts and among them. For example, we presume that it is easier for students to study first covariation of quantities having the same unit for measurement — in our approach a relationship between path length and horizontal or vertical displacement — than to start with a function from angle measure (degrees or radians) to length measure (using the length of the hypotenuse of a right triangle or the radius of a circle as a unit).

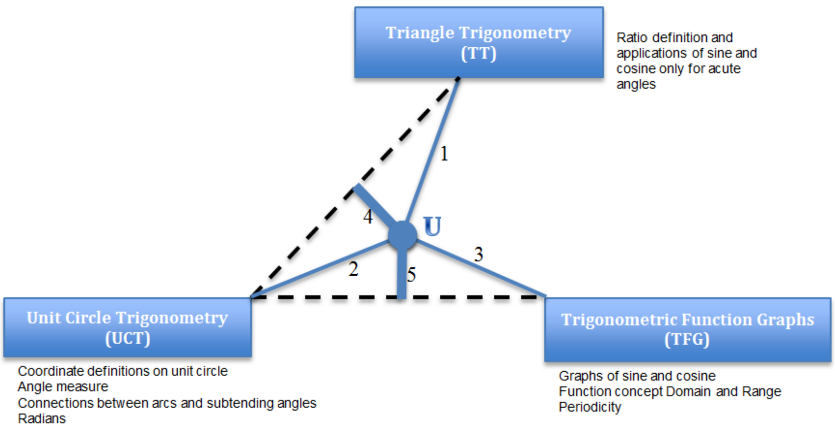

Figure 1: A model of trigonometric understanding

The contexts TT, UCT, and TFG represent three contexts in which trigonometry can be partially understood, while the central point U in the model represents the desired trigonometric understanding of students. The numbered line segments represent that trigonometric understanding should entail aspects in the three contexts and the connections among them. It is important to keep in mind that point U should not be considered static. It may have different places in between the three contexts with respect to the quality of different students' understanding because different tasks may require different aspects of trigonometric understanding.

The line segments labelled 1, 2 and 3 represent understanding of different aspects within three different contexts. These aspects are not only about factual knowledge such as knowing mathematical definitions, but also concern the students' ability to elaborate on them. For example, line segment 2 not only refers to the coordinate definitions of sine and cosine, but also indicates the ability to show trigonometric relationships using the unit circle (e.g., explaining why sine is an odd function). Similarly, explaining relationships between angles and arcs on a circle should go beyond applying the rule of converting degrees and radians. Only when students have developed a process view of these relationships and can apply proportional reasoning in their work with several measures of rotation angle one can start speaking about conceptual understanding of unit circle trigonometry.

A deeper level of understanding is represented by the thicker line segments 4 and 5 in Figure 1, which represent understanding the connections among the contexts represented by the dashed lines. Segment 4 represents understanding aspects of integrating triangle trigonometry and unit circle trigonometry. A task of calculating the sine of an angle greater than 90 degrees, for example, requires integration of the ratio definition of sine with finding the corresponding vertical position of a point on the unit circle. Segment 5 represents understanding the connections between unit circle trigonometry and trigonometric graphs of real functions. This concerns the construction and interpretation of trigonometric graphs as well as explanation of the properties of trigonometric functions. For example, the explanation of why sine is a function of real number requires (1) the coordinate definition of sine; (2) the relations between radians and arcs to transit from angles to real numbers by considering rotation angles in radians or corresponding arc lengths as input of the function; (3) the vertical position as output of the function; and (4) the combination of these into a relationship where each input has a unique output.

Segment	*Context or connection*	*Aspects*
1	Triangle (TT)	Ratio definition of sine and cosine in a right triangle, and applications
2	Unit circle (UCT)	Coordinate definitions of sine and cosine, and applications with directed angles or real numbers
		Application of the coordinate definitions to evaluate sine and cosine for certain inputs
		Trigonometric relationships that can be shown on the unit circle, e.g. sine is an odd function
		Relationship between arcs and subtended angles through the notion of radian
		Converting between degrees and radians via proportionality
3	Graphs (TFG)	Connection of the sine and cosine graph with real functions
		Functional properties of trigonometric graphs, e.g., domain and range of trigonometric functions
		Trigonometric relation revealed by the graphs, e.g., even and odd property of functions
4	Connection TT-UCT	Integration of the ratio method with the unit circle method via reference triangles, affording trigonometric values of angles larger than 90 degress
5	Connection UCT-TFG	Construction and interpretation of trigonometric graphs
		Explanation of properties of trigonometric functions

Table 1: Aspects of the model of trigonometric understanding

Various aspects were hypothesized as important for students' understanding in this model of trigonometric understanding, based on a conceptual analysis of trigonometry. They are listed in Table 1 and formed the basis of the design of a hypothetical learning trajectory of trigonometric functions. Whereas Thompson (2008) exemplified a conceptual analysis of trigonometry based on angle measure and suggested that a coherence of trigonometric ideas can be reached through developing a coherent meaning of angle measure, we emphasize in our approach paths and path lengths in regular polygons and circles before angle measures. We propose the use of arcs of a unit circle to serve as glue between the unit circle trigonometry and trigonometric function graphs and the use of the metaphor of travelling along the rim of a geometric object like a regular polygon or a circle, referring to the paths as travelled distances in order to help students finally develop coherent meanings based on arcs of a unit circle. Angle measure is addressed in our learning trajectory only after almost all connections between the trigonometric function graphs and the unit circle trigonometry have been set. This is a major new aspect of our work. It supplements the approaches implemented by Moore (2010; in press) and by Akkoç and Akbaş Gül (2011) in which the basic idea also was that quantifying angle measure through processes that involve measuring arc length can support coherent angle measure understandings..

A NEW LEARNING TRAJECTORY FOR TRIGONOMETRY

The basic idea in the new approach is to avoid an early introduction of radians as angle measure via the unit circle, but instead make the so-called winding function of the real line on the unit circle the central concept and use the concept of arc and arc length to introduce sine and cosine as real functions. We assume here that, in case of introduction of the sine function, it is easier for secondary students to consider a function with distance as input and height as output than to consider a functional relationship between angle measure and height. The nature and unit of the involved quantities is the same in the new approach. Also it is essential in our approach that students can first practice with pencil and paper and develop understanding of the geometric construction of the sine and cosine function via a winding function and coordinate functions. Our reasoning is similar to Weber (2005) and Tall et al. (2000): students better experience first the application of a particular process and reflect on it before they mentally apply the same process. To get hands-on experience with the process of applying a winding function on the unit circle and coordinate functions, students first explore the winding function defined on a regular polygon that circumscribes the unit circle and is oriented in the Cartesian plane such that the point (1,0) is a midpoint of a vertical edge. A point P moves counter-clockwise along the rim of the regular polygon and the covariation of the travelled distance and the vertical position is studied. In case of an n-gon, this leads to a sine-like function s_n as a composition of (1) the winding function, in which every positive real number z is mapped on a point on the rim of the n-gon reached after travelling a distance of z units starting at (1,0); (2) the mapping of a point in the Cartesian plane on the vertical axis; and (3) the mapping from the vertical axis to the real numbers by focussing on the vertical coordinate of a point on the axis.

Students begin with the construction of the function s_4 by pencil and paper. Hereafter they explore the same construction by a dynamic geometry package; see Figure 2 for a screenshot of the GeoGebra activity used in class. We call this activity 'Walking along the unit square.'

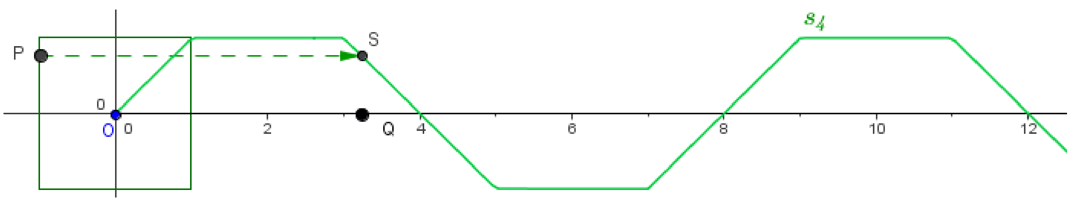

Figure 2: Screenshot of a GeoGebra activity on the graph of the sine-like function s_4.

The purpose of this applet is to promote student construction of knowledge: it is meant to function as a didactic object (Thompson, 2002), in the sense that it is designed as an object to talk about in a way that enables and supports reflective mathematical discourse.

The next step in the trajectory is to extend the graph over the negative horizontal axis and to make a similar construction of a cosine-like function c_4. Various trigonometric-like properties of the functions s_4 and c_4 can be explored such as $s_4(x+8)=s_4(x)$, $s_4(-x)=-s_4(x)$, and $s_4(2-x)=c_4(x)$, for all x. Also note that by construction $s_4(x)=x$ for small values of x.

What can be done for a square can also be done for a pentagon, hexagon, and so on. Students can with some effort explore the functions s_4 and c_4 by pencil and paper, but for other regular n-gons dynamic geometry software is most helpful: see the screenshots of GeoGebra activities in Figure 3 and 4. The graph of s_{10} (Figure 4) already gives the impression of the graph of the sine function.

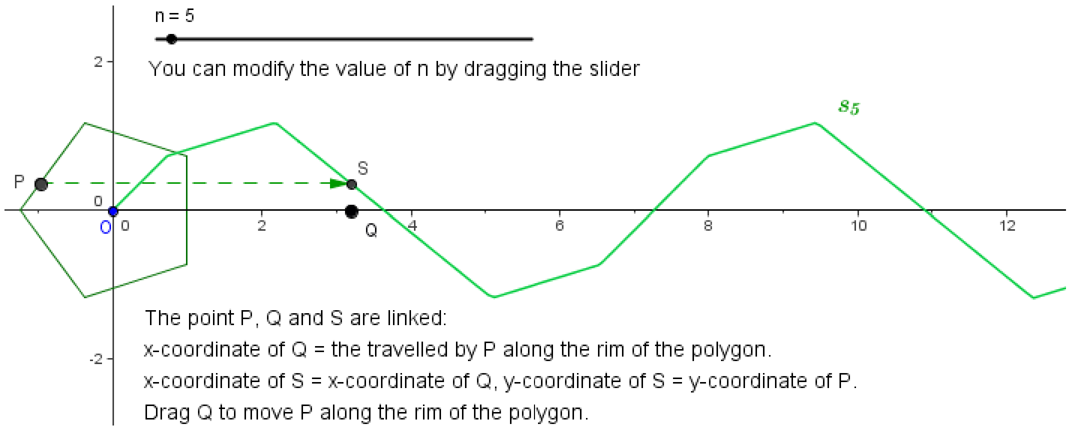

Figure 3: Screenshot of a GeoGebra activity on the graph of the sine-like function s_5.

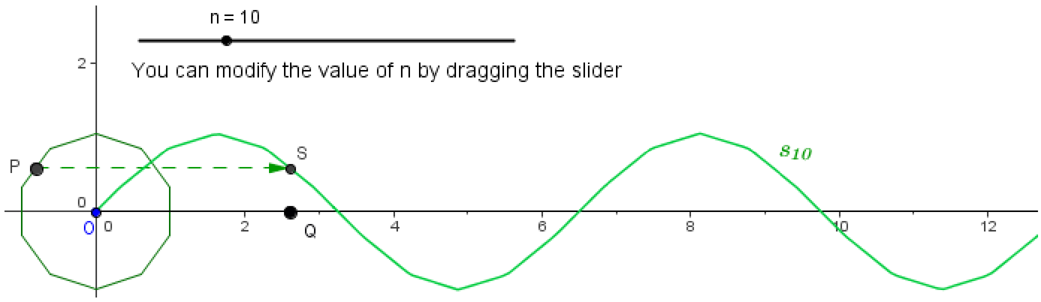

Figure 4: Screenshot of a GeoGebra activity on the graph of the sine-like function s_{10}.

The period of s_n gets closer to 2π with increasing values of n. The graph of s_{30}, associated with the a regular 30-gon, is very smooth and it can hardly be distinguished with the naked eye from the graph of the sine function; see Figure 5. The functions s and c can now be introduced by taking limits: $s=\lim_{n\to\infty} s_n$ and $c=\lim_{n\to\infty} c_n$. Of course, this is not done in a formal way at secondary school level. It is only important that the students realize that the winding function for the unit circle can be defined in a manner similar to the construction using regular n-gons and that the graphs of s_n and c_n look for large n almost the same as the graphs in case the unit circle is used instead of a regular polygon with many edges. We hope and expect that students develop in this way a process view of the sine- and cosine-like mathematical functions that helps them to understand the mental construction of the sine and cosine functions.

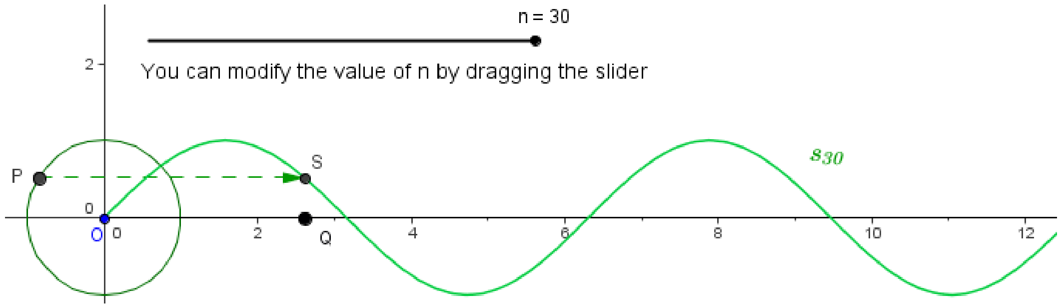

Figure 5: Screenshot of a GeoGebra activity on the graph of the sine-like function s_{30}.

Until this point at the learning trajectory there has not been paid attention to rotation angles and no link had been laid with the geometric definition of sine and cosine. Assuming that the students already know that the circumference of the unit circle equals 2π and that a point on the rim of the unit circle is mapped by a rotation of 360 degrees about the origin is mapped on itself, they can be led through a few exercises to find out that a counter-clockwise walk along the rim of the unit circle starting from (1,0) over a distance of x units corresponds with a rotation of $x \cdot 180^\circ / \pi$ about the origin. By drawing a right triangle for a point on the unit circle and thinking of the ratio definition of sine, students can figure out that $s(x) = \sin\left(x \cdot 180^\circ / \pi\right)$. This formula allows students to compute function values such as $s(\pi)$ and $s(\pi/3)$. The introduction of radian finally boils down to the understanding that an arc of length 1 corresponds with a rotation angle of $180^\circ / \pi$ and that this leads to $s(x) = \sin\left(x \text{ rad}\right)$. In other words, there is little reason not to call *s* the sine function and to denote it by sin, too. The introduction of the cosine function is similar. This finalizes the linking of the unit circle geometry with the triangle geometry.

A traditional approach to trigonometric function starts with a representation via right triangle, continues with a representation via the unit circle, and finally arrives at a representation via a graph. We start with a representation via right triangle, move straight on to a representation via a graph of a real function, continue with a representation via the unit circle, and finally return to triangle geometry. This new approach has been investigated in two classroom experiments to evaluate its usability in mathematics education. The first experiment was meant as a proof-of-concept investigation in a small class of students who had chosen an optional mathematics course and therefore could be considered as mathematically bright students. It was a success and the cooperating secondary school teacher expressed that in her opinion this approach would also suit 'regular' students. This is why we designed a classroom research study to explore the advantages and weaknesses of our approach for regular mathematics education. We report in the next section about this research study

CLASSROOM CASE STUDY

Based on the model of trigonometric understanding (Figure 1 and Table 1) and the new approach outlined in the previous section, we examined students' concept development and understanding of sine and cosine functions through a single case study.

Subjects and setting

The classroom experiment was conducted at a Dutch secondary school in Amsterdam with a mathematics-B class of 24 pre-university students (17 female and 7 male; age 16-17 yr.), who were in their first year of upper secondary education and classified by their teacher as a high achievement group in mathematics. The main two reasons for selecting this school were: (1) it supplies nice ICT facilities for students and teachers (e.g., every classroom is equipped with an interactive whiteboard and there a several computer labs); and (2) the teacher of this class was the same experienced mathematics teacher who had cooperated in the former pilot study. The first author designed and taught five lessons in class and the mathematics teacher wrote observational notes, translated English mathematical terminology into Dutch when needed, and also helped students during the work in pairs. Two weeks before the classroom experiment started the researcher and teacher met to discuss the lesson plans and course materials. A half-hour diagnostic test, in which students' prior knowledge and skills was evaluated so as to have an idea of their readiness for the trigonometry lessons, was administered a week before the start of the instructional sequence. The evaluation mainly concerned the students' understanding of triangle trigonometry and the function concept, and their graph comprehension skills. It caused a small adaptation of the original instructional materials regarding the single-valuedness of a function.

The students in the research cohort normally had three mathematics 60-minutes lessons per week. However, due to the school schedule, it was not possible finish the trigonometry lessons in two successive weeks and three weeks of teaching were needed. The lessons were implemented in a student-centred approach by creating a learning environment in which the students were encouraged to construct their knowledge through interactions with peers, the researcher (acting as teacher) and the regular classroom teachers. Working in pairs and follow-up whole classroom discussion were important elements of the lessons. GeoGebra applets were used as didactic objects in combination with some tasks presented by way of worksheets. In the first lesson, the grouping of the students in dyads was done on the basis of advice of the cooperating teacher.

Research questions

The research was conducted to find answers to the following two descriptive research questions:

1. What task-related difficulties do students face in their concept development within the designed instructional sequence based on the new hypothetical learning trajectory of trigonometric functions?
2. What characteristics relating to students' understanding of sine and cosine can be found in the data resulting from the intervention based on the new model of trigonometric understanding

Data collection and analysis methods

Classroom observation, and participatory observation in particular, is a standard research method for data collection in case studies. Other classical methods for data collection that were applied in this study were:

- Interviews with the cooperating teacher after each lesson to record her impressions of the activities and of how the instructional materials and the ICT tools had functioned.
- Four semi-structured, audio recorded interviews with students after the instructional sequence, for the purpose of getting an impression of their understanding of the key mathematical points underlying the instructional sequence. Interviewees were selected on the basis of to their performance in class and on advice of the cooperating teacher.
- Audio recordings of students' discussions on worksheet tasks, which gave an impression of the students' concept development. In each lesson, the discussions of four pairs of students were recorded.
- All completed worksheets of pairs of students.
- A 50-minutes trigonometry test after the instructional sequence designed to assess students' understanding and based on our model of trigonometric understanding.

The data were analysed qualitatively to answer the two research questions about students' concept development and understanding related to the sine and cosine functions. The diagnostic test and the trigonometry test before and after the instructional sequence were analysed question by question. However, assuming that the analysis can be reported in a better way with respect to the theoretical underpinnings of the research, it is presented according to the main mathematical points of trigonometry understanding. Audio recordings of the interviews with students were transcribed, analysed interviewee by interviewee, and presented with respect to the trigonometry understanding. Audio recording of group discussions were only transcribed if they related to interesting fragments of students' worksheets ('interesting' in terms of the research question about students' difficulties with the tasks).

Results

The aim of the case study was to examine the students' concept development within and understanding shortly after the implementation of the new learning trajectory for trigonometry. Two research questions were operationalized. In this section we give answers to these questions on the basis of the analysis of the collected data.

Concerning the first research question, the analysis of student's responses to worksheet tasks in combination with the related audio recordings of the group discussions revealed that in general the students were quite successful when working on most tasks. They only faced difficulties in the following four tasks regarding their concept development <u>within</u> the instructional sequence:

1. drawing the graph of the vertical position of a moving point along the rim of the unit square against the travelled distance;
2. deriving the formula $s(x) = \sin\left(x \cdot 180^\circ/\pi\right)$ of the graph of the vertical position of a moving point on the unit circle plotted against arc length.
3. Converting $180^\circ/\pi$ to radians and using it in in the transition to function on real numbers;
4. Calculating the sine and cosine of 210°.

Students' difficulties with the tasks 1 and 2 indicate a more general difficulty for then, namely making mathematical generalizations from the concrete to the abstract. Task 3 and 4, on the other hand, indicate a difficulty in applying known relationships in different contexts.

What made it in task 1 difficult for students to graphs the vertical position of a moving point on the unit square against the travelled distance? It was clear that this task was very uncommon to the students. They were only familiar with drawing the graph of a function for which a formula has been given. However, once they understood the task, they knew what to do and could determine points on the graph. But then it was difficult for them to decide how to connected these points: by line segments or curved segments. Having seen so many smooth graphs in their school career, they tend to do the same in this graph; see Figure 6.

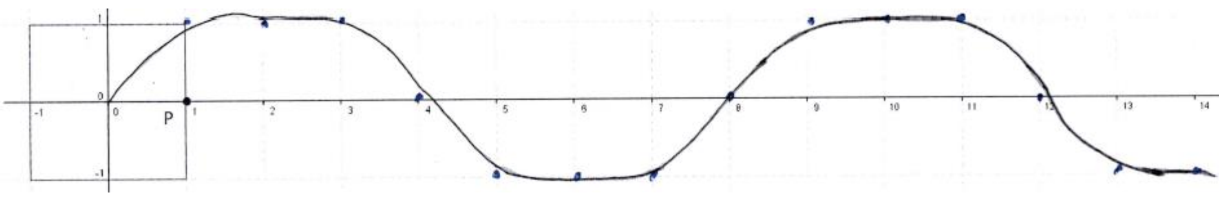

Figure 6: The graph of a student team in the main activity of 'Walking on the unit square.'

The difficulties in deriving the formula $s(x) = \sin\left(x \cdot 180^\circ/\pi\right)$ had to do with proportional reasoning to connect arcs and subtended angles, and with the required movement from specific cases toward a general case expressed by means of a mathematical formula that involves variables.

The group discussions revealed that very few students could link $180^\circ/\pi$ with the notion of radian as angle measure. The radian concept was probably at this stage in the instructional sequence not fully understood yet and the task was therefore too challenging for the students. Later on in the learning sequences, many obstacles with the conversion between degrees and radians had been overcome, but still one could recommend an extra lessons to practice it.

The difficulty of calculating $\sin(210^\circ)$ and $\cos(210^\circ)$ had two sources: (1) some students did not recall the related values of $\sin(30^\circ)$ or the method how to compute them; and (2) many students could not figure out how to link triangle trigonometry with unit circle trigonometry. Luckily whole classroom discussed helped the student overcome the difficulties.

Concerning the second research question about the students' integrated understanding of trigonometry after the instructional sequence, several findings could be derived from the interviews with students and the trigonometry test. They are discussed in the following paragraphs, line segment by line segment according to the model of trigonometric understanding shown in Figure 1. Note that the triangle trigonometry context (line segment 1 in Figure 1) was considered prior knowledge.

In general, the students developed a good level of understanding of aspects in the unit circle context. They were able to evaluate trigonometric functions of real numbers by associating them to an arc on the unit circle and they understood the notion of radian well enough for the construction of knowledge of trigonometric functions based on the relationship that the angle measure in radians is equal to the arc length on the unit circle. Although many students understood trigonometric relationship for specific angles, most students were not able not prove a trigonometric equality when expressed in algebraic format. In Figure 7 is shown how a student fail to give a general proof, but instead illustrates a special case.

7. Show that **sin(-x) = - sin x** on the unit circle or on the graph of sine function.

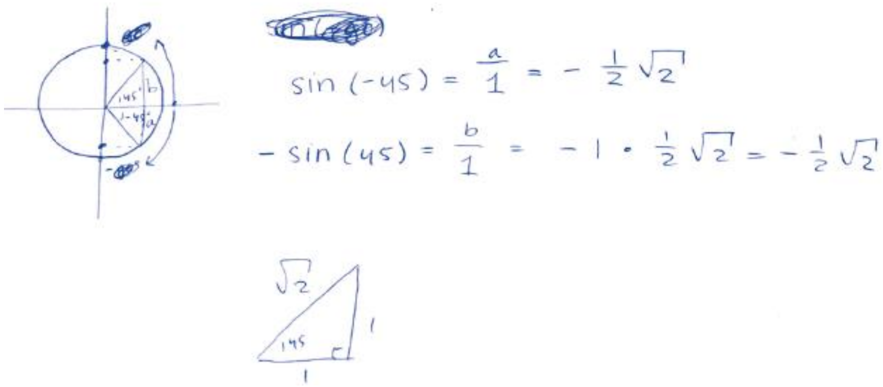

Figure 7: The response of a student to the task to prove sin(-*x*) = - sin(*x*).

For the context of trigonometric function graphs, it was concluded that the students conceptualized sine and cosine as functions of real numbers, and that they grasped how to interpret the graphs in terms of domain, range, periodicity, and symmetry properties.

From the trigonometry test and the interviews with students, we concluded that the students showed good understanding of how to integrate right triangle definitions of sine and cosine with the unit circle method in order to calculate trigonometric values of angles larger than 90°. However, most students could not calculate trigonometric values of well-known angles like 30°, 45°, and 60° because this had not memorized these special values nor the methods to find the values. Despite most students' inability to calculate the trigonometric values of acute angles procedurally, they seemed to understand the connections between triangle and unit circle contexts

Students also developed a deep understanding between the unit circle context and the graph context. The most remarkable finding is that the students' understanding of such connections was based on arcs. This was found on many occasions. Concerning the construction and comprehension of trigonometric graphs, it was found that students conceptualized trigonometric graphs through the arc length on the unit circle. They explained coordinates of point on the graph with a journey metaphor based on arc length as travelled distances and they related the direction of the movement with the sing of an angle or the sign of a real number. Furthermore, students could explain the shape of trigonometric graphs.

Concerning the function concepts and related properties, students could explain the meaning of periodicity and range by referring to the unit circle. The following transcript illustrates how a student (S) explained periodicity when asked about it by the researcher (R):

R: And what is the periodicity of this function, the sine function?

S: Hmm. It is rihght from here and you are starting over again [pointing at 2π on the graph] and on the same thing… [pointing at the second interval]

R: Yeah, the interval, it repeats itself?

S: Yes

R: Let me give you a better version of the graph, maybe it will be easier for you to read it.

S: It is here, it is 2π.

R: Yeah. So you can say the periodicity is 2π. And if you consider the unit circle and the moving point again, what does it mean that the periodicity is 2π?

S: You are going on one circle and then the periodicity..

Although the student could not express herself well in Englsih, she was able to reveal that she knew what periodicity is and understood its meaning both in terms of the sine graph and in the context of the unit circle.

CONCLUSION

The main positive point of the presented study was the effectiveness of the instructional design in promoting students' integrated understanding of trigonometric functions in a way that they did not develop as many difficulties and misconception as reported before in the research literature on trigonometry. Our new approach turned out to be effective in promoting a connected understanding of trigonometric functions as functions defined for angles and as functions defined for the domain of real numbers. Students developed understanding of trigonometric graphs and related properties of the trigonometric functions

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