YOYO JOY

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If you want your students to investigate a more challenging phenomenon than an object that falls freely under gravity, tie a cord to this object and wrap it around an axle. In other words, make a kind of yoyo out of it. The constrained coupling of translation and rotation and the limited length of the cord make the motion much more interesting. Students can collect experimental data of the yoyo going up and down via video measurement. Mathematics and physics help them to describe and understand the motion. With system dynamics software they can compare their model(s) with reality. We present this work for a self-made yoyo of unusual size.

COLLECTING DATA THROUGH VIDEO MEASUREMENT

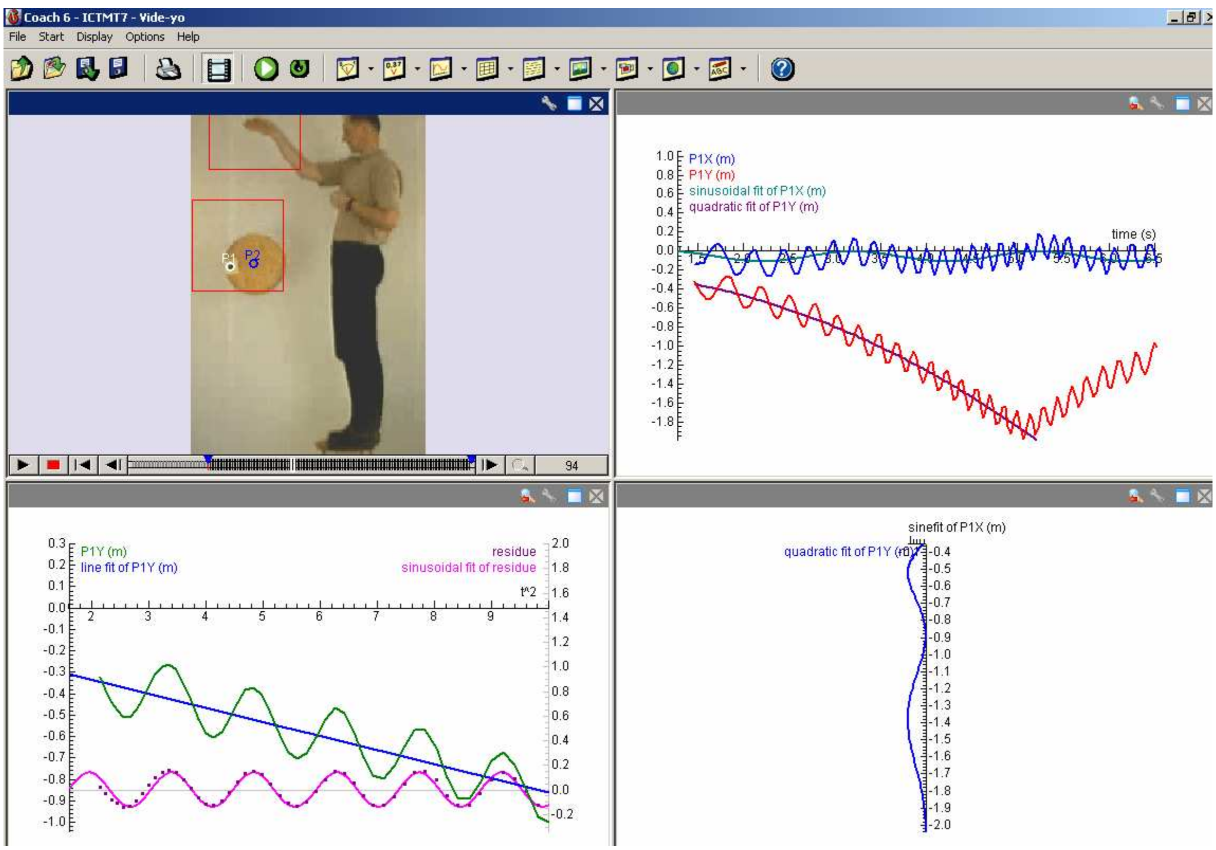


Figure 1. *Screen shot of the video activity*

Figure 1 above is a screen shot of an automatic video measurement with the Coach software (Mioduszewska & Ellermeijer, 2001). In the upper left corner is a video clip in which a teacher plays with a self-made yoyo. It is made of two wooden seats of laboratory stools. Because of the unusual size of this object, students do not believe their eyes when they look at the yoyo winding up and down slowly. You see them thinking “How can this yoyo move so slowly and still go all the way up? What trick is behind this?” No trick at all! Mathematics and physics help students to describe and understand the motion.

The first step in the investigation is to collect data of the yoyo that is winding up and down. The position of the point near the rim of the disk and marked by a sticker (P1) is measured in a slightly moving coordinate frame whose origin is the hand of the teacher holding the end of the cord of the yoyo. Point tracking is used to collect automatically coordinates of points of interest: the red boxes illustrate the areas that the software searched in the current frame of the video clip for the origin and for the point marked by the sticker. In the upper right diagram, the horizontal and vertical position of P1 are plotted against time. This is combined with a sinusoidal fit of the horizontal displacement of the yoyo, due to an unintentional pendular motion of the yoyo, and a quadratic function fit of the vertical position during the first phase in which the yoyo unwinds. These fits can be used as coordinate functions of a computed point that is displayed in the video clip (P2): it turns out to be the position of the axle during the unwinding phase of the yoyo. The lower right diagram in Figure 1 shows the orbit of point P2 while the yoyo is unwinding.

EXPERIMENTAL MODELLING

Let us focus on the vertical position of the point P1 near the rim of the disk for a yoyo that is unwinding. Its motion is a superposition of the displacement of the centre of mass located at the axle of the yoyo and the vertical projection of an accelerated circular motion. The first part can be described by a quadratic function fit; the second part is an accelerated cycloid. It turns out to be a useful idea to plot the vertical position against the square of time. This is done in the lower left diagram in Figure 1. Then the graph can be described as the sum of a straight-line approximation and a sinusoidal approximation of the residue. The approximate formula for the vertical coordinate of P1 as a function of time *t* is

$$y_{\mathrm{P1}} = -0.0658t^2 + 0.139\sin(4.352t^2 - 0.642) - 0.202$$

A physical interpretation of the above description of the unwinding yoyo is as follows: the centre of mass moves downward with a constant acceleration that is numerically much less than the acceleration due to gravity for a free falling object. The point P1 rotates around the axle with an angular velocity that increases linearly in time. It is an exercise in trigonometry to compute from the vertical displacement of the axle and its radius, how much rotation already has taken place and to determine the coordinates of the point P1. This is applied in the modelling activity shown in Figure 2 below, which is created by the system dynamics module of Coach.

The graphical model is shown in the lower right corner of Figure 2. The meanings of the icons are similar to those of system dynamics software like Stella (Steed, 1992): there are icons for state variables, flows, auxiliary variables, constants, and connectors. What is special for Coach is the possibility of modelling based on discrete events. The 'Events' box on the left in the graphical model contains the hidden subsystem that takes care of the change of direction from downward to upward motion of the yoyo. The 'Ycoord' box on the right contains the subsystem that computes the coordinates of point P1 at any time *t*. In education, the use of subsystems is convenient when students are step by step introduced into the details of a model or when they gradually extend themselves a model while trying to keep a good overview of the system. The diagram in the upper left corner of Figure 2 illustrates that the vertical position of the axle computed in the model for the measured part of the unwinding motion matches well with the quadratic fit of the measured motion. The upper right diagram of Figure 2 shows a phenomenal match between the measured vertical position of the point P1 on the yoyo and the computed *y*-value. The diagram in the lower left corner illustrates that energy loss can be taken into account, so that the yoyo gradually returns to a less high position in each cycle of downward and upward motion.

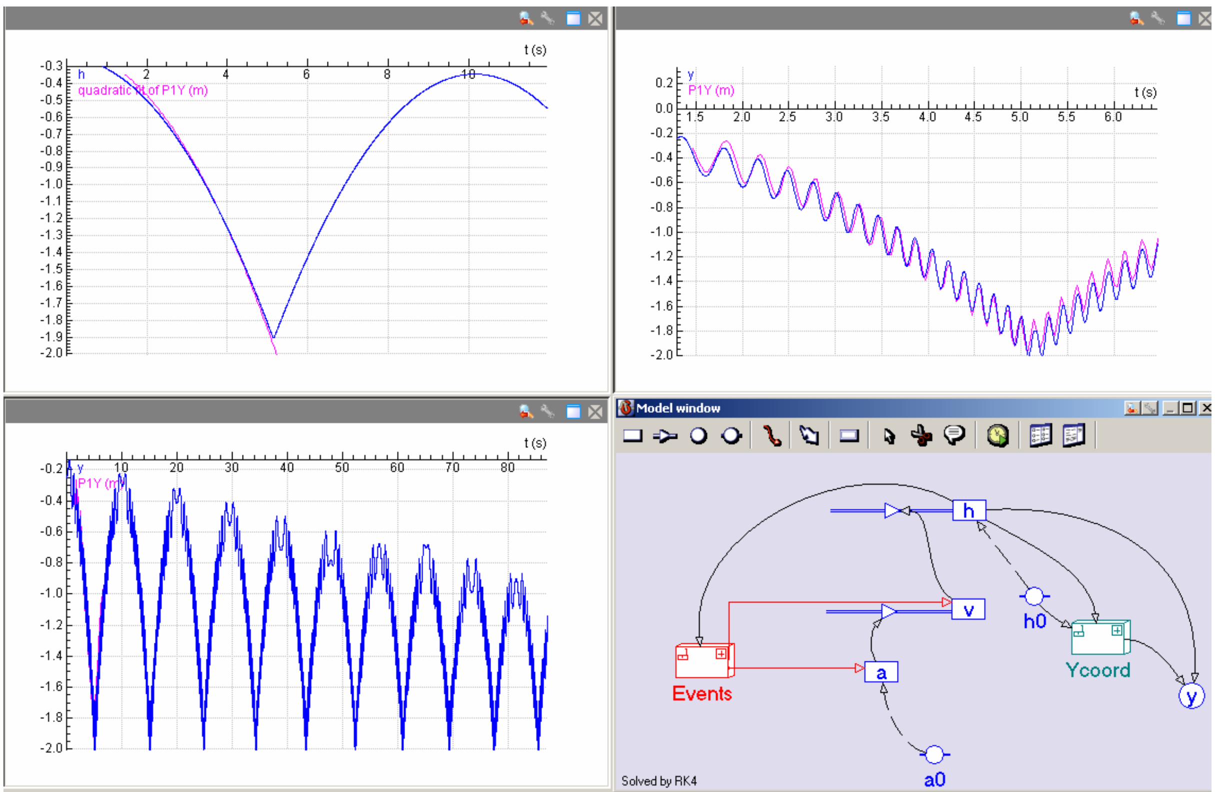
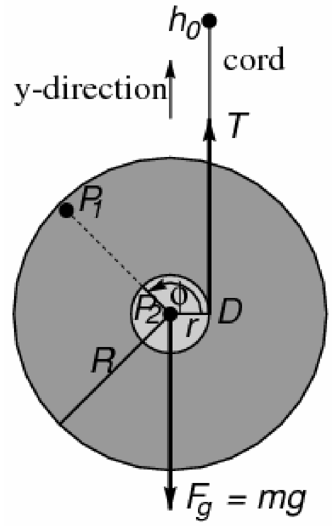


Figure 2. *Screen shot of the modelling activity*

The above experimental approach to study the motion of the yoyo perhaps satisfies an engineer or a student who models just this particular system, but he or she must start all over again for every new kind of yoyo. In other words, what do you learn from it about the motion of a yoyo in general? To this end, mathematics and physics must come into play, possibly in combination with a system dynamics approach.

MATHEMATICS AND PHYSICS OF THE YOYO

The coupling of translation and rotation complicates the motion of a yoyo. How precisely and what this means for the motion can be found by applying the laws of classical mechanics. Here we will only study the phase in which the yoyo is unwinding, but the upward motion can be treated similarly (signs change in the formulas).



Let us first fix notations (see the picture to the right). The yoyo has mass *m*, an axle with radius *r*, two circular disks with equal radii *R*, and it is tied to a cord with tensional force *T*. Let *D* be the point of application of this tensional force. This is the point around which the yoyo currently unwinds. The torque τ (moment of force) about the centre of mass, which makes the yoyo rotate, is given by $\tau = T\,r$. The origin of the coordinate system is, like in the video measurement, the position of the hand that keeps the end of the cord at (almost) a fixed position. The positive vertical direction is chosen to be upward and the positive horizontal direction is to the right. As usual, the vertical position, velocity, and acceleration of the centre of mass are denoted by *h*, *v*, and *a*, respectively. Note that these quantities are negative for the yoyo in the unwinding phase for our choice of coordinate system. The angular velocity and angular acceleration of the yoyo are denoted by the symbols ω and α, respectively The symbol *g* is the constant of gravity; the letter *I* is used for the moment of inertia of the yoyo. We release the yoyo at time $t = 0$, at initial height h_0, and with P_1 at angle ϕ_0. Newton's laws give:

(1) sum of forces $= m\,a$, i.e. $-mg + T = m\,a$

(2) $\tau = I\alpha$, i.e. $T\,r = I\alpha$

(3) $v = -r\omega$, and thus $a = -r\alpha$

From (2) and (3) follows: $T = -Ia/r^2$. Substitution in (1) gives:

(4) $$a = -\frac{g}{1 + I/mr^2}\,.$$

For the circular yoyo holds: $I = \frac{1}{2} m R^2$. Thus:

(5) $$a = -\frac{g}{1 + \frac{1}{2}\left(R/r\right)^2}\,.$$

The vertical displacement *y* of the point P_1 near the rim of the disk is a superposition of the vertical displacement *h* of the centre of mass P_2 and the vertical projection y_{rot} of the rotation about the axle. Thus: $y_{\mathrm{rot}} = R_{\mathrm{marker}} \sin\phi$, where

(6) $$\phi = \left(h_0 - h\right)/r + \phi_0 = -\tfrac{1}{2} a t^2/r + \phi_0\,.$$

Therefore:

(7) $$y = h + y_{\text{rot}} = h_0 + \tfrac{1}{2}at^2 + R_{\text{marker}} \sin\left(-\tfrac{1}{2}at^2/r + \phi_0\right).$$

This formula has the same shape as the one found in the experimental modelling: substitution of the yoyo data $R = 0.18$ m, $R_{\text{marker}} = 0.135$ m, and $r = 0.0145$ m gives the following results: $y = -0.1257\ \text{m/s}^2$ and $\phi = 4.334t^2 + \phi_0$. So, using mathematics and physics, we are able to describe the motion of the unwinding yoyo by a formula that is in excellent agreement with the experimental results.

What is important about this algebraic formula work for education is that it is not just a hobby of the teacher, but the formulas really help students to understand the phenomenon better. For example, it follows from formula (5) that the acceleration is independent of the mass of the yoyo; it only depends on the constant of gravity and on the ratio $R : r$. For example, a yoyo of larger disk size would move up and down more slowly, and the same holds for yoyo-ing on the moon. The formulas, and especially Newton's laws, also play a fundamental role in the construction of a computer model, as we will see in the next section

A SYSTEM DYNAMICS APPROACH

Let us discuss briefly how the change of direction of the moving yoyo and the energy loss due to friction are taken into account in the computer model that is shown in full bloom in Figure 3. At this point, all subsystems are fully displayed and we will focus on the 'Events' subsystem as an illustration of the power of event-based modelling.

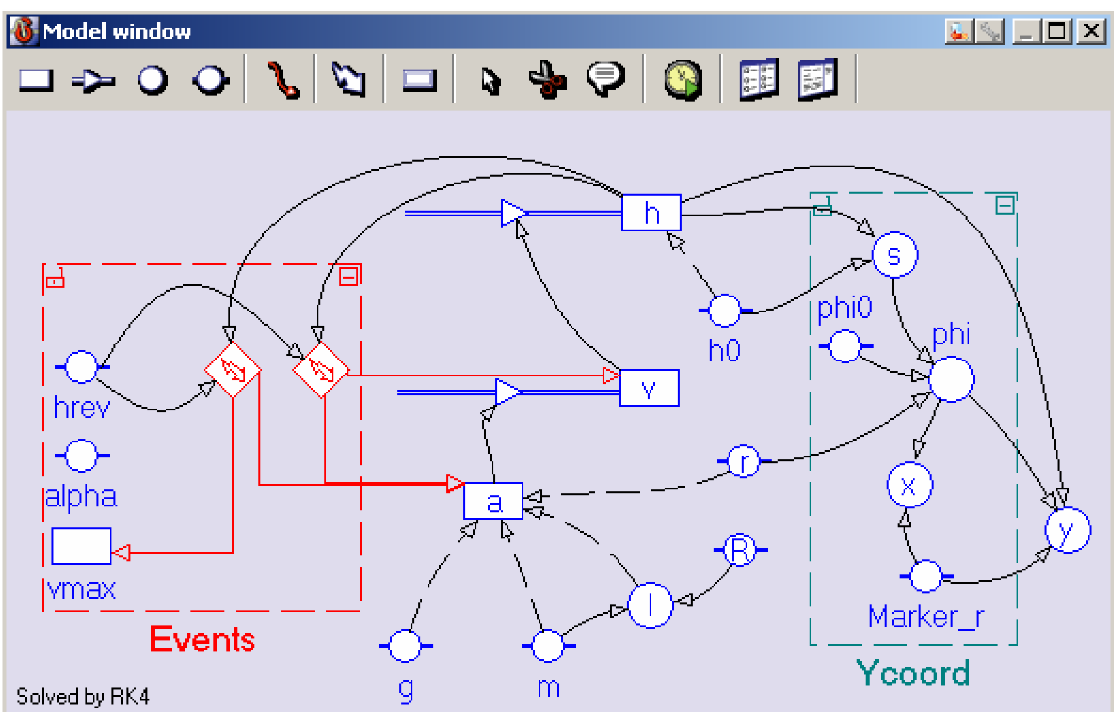


Figure 3. *A graphical model of the yoyo (with events)*

For the change of direction we must update the velocity and acceleration at the following discrete events: firstly, when the end of the cord is reached and the yoyo starts rotating around the turning point *D* (as a result of inertia), and secondly when the turn is completed and the yoyo continues winding upward. Let us look at the details of the first event: it is triggered when the height *h* becomes smaller than the constant h_{rev}. Then we store the absolute value of the current velocity in the variable v_{max}, and we change the acceleration from $a = -\dfrac{g}{1 + I/mr^2}$ into $a = \dfrac{(1+\alpha)^2 v_{\text{max}}^2}{2\pi r}$, where α is a number between 0 and 1. The latter formula is motivated as follows: when the yoyo rotates around *D* until it reaches a position in which the yoyo can wind up again, the orbit of the centre of mass is half a circle with radius *r*. Because of energy loss due to friction – one really feels a pull on end of the cord when the yoyo reaches its lowest position – the turn is not completely elastic and the downward velocity $-v_{\text{max}}$ changes into an upward velocity $\alpha \cdot v_{\text{max}}$, where α (with value between 0 and 1) is a measure for the decrease in velocity; in other words, $\Delta v = (1+\alpha) v_{\text{max}}$. So the mean absolute value of the velocity during the rotating phase at the bottom of the cord is equal to $\frac{1}{2}(1+\alpha) v_{\text{max}}$. The time it take for rotating is therefore estimated as $\Delta t = \dfrac{\pi r}{\frac{1}{2}(1+\alpha) v_{\text{max}}}$, which brings us to $a \approx \dfrac{\Delta v}{\Delta t} = \dfrac{\frac{1}{2}(1+\alpha)^2 v_{\text{max}}}{\pi r}$. When the rotating process is at the end, i.e., when *h* passes through h_{rev}, the second event is triggered and we change the formula for the acceleration back again into $a = -\dfrac{g}{1 + I/mr^2}$ and the velocity is set to $\alpha \cdot v_{\text{max}}$. The action that is triggered by an event is specified in the properties window that pops up when the corresponding events icon (with the thunderbolt symbol) is double-clicked. Using discrete events, the modelling of rather complex system dynamics is not beyond most students anymore. It really widens the scope of computer modelling in education. In our case, it leads to a realistic, physics-based computer model, the simulation of which matches in an excellent way with the experimental data.

WHAT ELSE TO DISCOVER ABOUT A YOYO?

What other questions can students ask themselves about a yoyo, research questions to which they can try to find answers via experimentation, modelling, and serious thinking? An interesting and meaningful thing students can do is to look at the various forms of energy that are important in the description of the motion of a yoyo. The forms of energy for which mathematical formulas can actually be derived are the gravitational energy of the yoyo and the kinetic energy of rotation and translation. With these formulas students can check in which phases energy is conserved and how the energies are related with the vertical position of the centre of mass of the yoyo.

Students may wonder whether the time intervals between turning points of the yoyo have constant size or gradually decrease in time. They can study the maximum height of the yoyo after it has gone down and up n times and try to conjecture a formula for the maximum height as function of n.

We have not said anything about the way the cord is tied to the axle, but it has in fact a large effect on the motion of the yoyo (see e.g. the website How Yo-Yo's Work). Also the shape of the yoyo is important: students can raise the question whether one can play with a yoyo of rectangular shape instead of circular shape and, if so, how this motion then relates to that of a regular type of yoyo? Can algebraic formulas also be derived for a rectangular shaped yoyo?

What happens when the path of the yoyo is restricted, e.g., when the yoyo rolls down an incline? But even a yoyo placed on a rough level surface behaves mysteriously at first sight. When a gentle horizontal pull is exerted on the cord so that the yoyo rotates without slipping, then the yoyo will roll towards the pull (why?). When a gentle, almost vertical pull is exerted on the cord it will roll away from the pull. Somewhere in between there exist a critical angle at which the yoyo does not move. We leave it as an exercise to prove that this critical angle is equal to $\arccos(r/R)$. So, it is independent of the mass of the yoyo, the tension in the cord, and the coefficient of friction.

The questions raised so far about a yoyo illustrate that this simple toy provides a rich context in which students can do a lot of experiments, apply physics concepts in order to understand observed phenomena, and have ample chance to experience the power of mathematical language.

WHAT IS THE ISSUE REALLY?

In our point of view, the real educational issue in the investigative work that we described is the ICT-supported interaction between experimental work and modelling, in which the interpretation of results is based on methods from mathematics and physics. The role of ICT is here to allow students to collect high-quality, real-time data, to construct and use computer models of dynamic systems in much the same way as professionals do, and to compare results from experiments, models, and theory. Furthermore, students can develop and practise through their activities research skills, and the fact that they must apply their knowledge of mathematics and physics in a meaningful way in a concrete context leads at the same time to deepening and consolidation of this knowledge. Important research competencies that students practise through this kind of practical investigations, which are by the way part of the upper level curriculum in the Dutch secondary school system, are being able to

- formulate good research questions that guide the work;
- observe and discuss with others about the behaviour of an object in motion;
- design and implement an experiment for collection of relevant data;

- apply mathematical knowledge and physical concepts in new situations;
- construct, test, evaluate, and improve computer models, and have insight in their role in an investigation;
- interpret and theoretically underpin results;
- reflect on own work;
- collaborate with others in an investigation task.

ICT plays an important role in enabling students to carry out investigations at a high level of quality: automatic video measurement turns out to be an effective means to study rotation motion quantitatively, instead of using only a qualitative approach. It also brings the real world into mathematics and physics education in an attractive way. Computer models can be constructed by students themselves on the basis of general physical concepts in situations where algebraic formulas are out of reach or even impossible. In short, ICT provides the students with tools to be actively involved in the process of finding solutions to challenging problems that they come up with themselves. The fact that these tools are bundled and integrated in a single computer learning environment comes in handy: this makes it for students easier to apply these tools in their attempts to solve problems or to get a better understanding of given situations. Actually, we consider the modelling process, the underlying thinking processes, and the discussions with fellow students during the research as more important in the students' work than the obtained results. All the same it is joyful when experiment, model, and theory are in full agreement, as is the case in our study of the motion of a yoyo.

Acknowledgements

The authors would like to thank their colleagues Leendert van Gastel, Mary Beth Key, and Frank Schweickert for their support and for the fruitful discussions about the motion of a yoyo. Leendert van Gastel also helped to implement the events-based computer model.

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