Differences between the use of mathematical entities in mathematics and physics and the consequences for an integrated learning environment.

Ton Ellermeijer and André Heck
AMSTEL Institute, University of Amsterdam
Kruislaan 404, 1098 SM Amsterdam, The Netherlands
ellermei@science.uva.nl, heck@science.uva.nl

Abstract

Mathematical concepts are not always used the same in mathematics and science. This is a complicating factor in the implementation of an integrated learning environment for math and science. Differences between the use of variables, functions, and graphs in mathematics and physics are discussed and consequences for the design of a general purpose learning environment are addressed.

Background

In The Netherlands, like in a few other countries, two fields of application of computers in science education have been dominant: for data-acquisition and data processing, and for modelling. Experience with data-acquisition started around 1980, as well by enthusiastic teachers, as well by the Physics Education group of the University of Amsterdam (now part of AMSTEL Institute). The interest in modelling has been triggered by the work of Jon Ogborn (DMS).

Soon the demand for an integration of these tools arose. Even in 1985 at an initiative of the University of Amsterdam group, a small group with Philip Harris, London University, and Dutch Telecom defined a concept for an integrated tool for Science and Technology education (Ellermeijer et. al, 1988). Unfortunately this initiative was not granted by EC, but it did have a major influence on further developments in The Netherlands. Due to the fact for the Dutch school system IBM-PC's (or compatibles) became standard, we could start to develop for more powerful computers. In 1988 the first integrated software with support for an interface for data logging and powerful analysis tools including modelling was realized (IP-Coach 2).

Over the years other valuable tools have been added to the environment, like for Control activities and for analysing digital video. The Coach 4 environment, still a DOS-software package, became de-facto standard in Dutch schools, together with the hardware for data logging. Because of these situation publishers, curriculum committees, teacher-training institutes could focus on this environment, and schools and teachers received a consistent and strong support. Because of its versatile character it is applied not only in Physics, but also in Chemistry, Biology and Technology.

The present environment, Coach 5, internationally achieved a breakthrough. This environment is based on a new concept, made possible by the Windows environment. In addition to all the tools for Science and Technology education teachers/developers have powerful authoring tools to prepare activities for their students. They can select and prepare texts, graphs, videos and measurement settings and choose the right level according to age and skills of their students. In this way, Coach 5 can be adapted for pupils/students ranging from age 10 up to undergraduate students.

Up to date information on the Coach environment can be obtained on the website: http://www.cma.science.uva.nl/english. A demonstration version can be downloaded from this site. A more detailed description of Coach 5 can be found in (Mioduszewska & Ellermeijer, 2001).

Demands for a Learning Environment for Science and Technology

The development of the Coach environment has been driven by a few key starting-points.

The choice for powerful tools, like data-acquisition, processing, modelling, was motivated because we like that what we teach at schools should *reflect the way scientist work* at present. And since the 60's computers have become more and more dominant in all areas of scientific research and also in industry.

Since the 70's many science educators are in favour of more realistic science education. With this we mean the treatment of science concepts in a realistic, everyday life context. The powerful tools have proven themselves as very helpful to *bridge the gap between physics concepts and real life context.*

From the *implementation* point of view an environment that teachers and students can use frequently and throughout the curriculum has many benefits. It means for them that it makes sense to become familiar with the environment, and a learning path from simple use up to the use in advanced projects can be developed. The

integration of tools enables them to move data to other parts of the environment and compare for instance in the same graph data obtained from measurements with data from modelling.

On the other hand, such a versatile tool might become complicated. It has to be possible to adapt the complexity and the features needed to the level of the student. In the Coach 5 environment an author has the facility to create *tailor-made activities*.

The *development* of the Coach environment is an ongoing research project since 1986 driven by classroom experience, educational research and technology. A *multi-disciplinary team* of science educators, curriculum developers, teachers, software and hardware specialists is responsible for the development and for the new implementations. Many smaller improvements are *based on experience* from teachers, from inservice course, and from authors of activities (publishers, teachers, curriculum developers).

More dramatic changes are most of times supported and guided *by intensive scientific research*, for instance in the framework of PhD studies. The extensions planned for Mathematics are an example of this and in the next parts of this contribution we report about it. Another important future development will address the connectivity with Internet. Internet will allow for collaborative work of students, for remote laboratories and remote control and for easy exchange of data and lessons.

Also the hardware tools for students will become more flexible and powerful. Think of dataloggers that can communicate with powerful computers applying wireless techniques. The challenge will be to use these new technologies for the benefit of science education and to keep our teachers informed and prepared.

ICT-rich mathematics activities

The first step in the redesign of Coach towards an integrated environment is to investigate what mathematics can be dealt with already, what kind of problems one encounters in doing so, and what needs can be identified. After all, the present software environment offers many mathematical tools: the use of tables, diagrams, data analysis, computer models, amongst other things. Here, we shall briefly report about recent try-outs of mathematics activities and present some findings about Coach for mathematics. More details can be found elsewhere (Heck, 2000; Heck & Holleman 2001a, 2001b).

Modelling human growth

The recent Dutch human growth study (Wit, 1998) has been used to create learning material for pupils in their first year of the second stage of pre-university education (age 15-16 yr.), who have no experience with practical investigation tasks, and who have not worked with the software environment before. Main objectives are to let pupils

- work with real data and with diagrams that are actually used in health care;
- experience how much useful information can actually be obtained from diagrams;
- see that the change of a quantity is often as important and interesting as the quantity itself;
- practice ICT-skills;
- carry out practical work in which they can apply much of their mathematical knowledge.

Figure 1. Manual function fit of childhood component.

A mathematical highlight is the ICP-model that models mean height for age within millimetres. It is used in medical literature and yet consists of mathematical submodels that are studied at school, viz., exponential growth, quadratic growth, and logistic growth.

Let us discuss some findings out of classroom trials about the Coach tools to graph and analyse growth data. For this, it is important to realise that regression is not a subject in the Dutch mathematics curriculum. How does it work in this activity? During childhood (from 3 years of age to the onset of puberty), a simple quadratic function fits growth during this period very well: $H_2 = a_2 t^2 + b_2 t + c_2$. Pupils are expected to search for a parabola that on the one hand fits well the height between 3 and 10 years of age, and that on the other hand reaches its maximum at the age of 20 years, when height growth usually stops. This curve fitting process is supported in the software by a manual function fit, as shown in the screen dump. The formula of a parabola, $y = ax^2 + bx + c$, has been selected as function type; the selected column corresponds with the mean height of Dutch boys. The icon of the pin on the screen dump is such that the approximation has been fixed at that location. By dragging another point of the parabola with the mouse one can now re-scale the graph. When the fixed pin is released by double clicking, then one can translate the parabola, i.e., independently change the
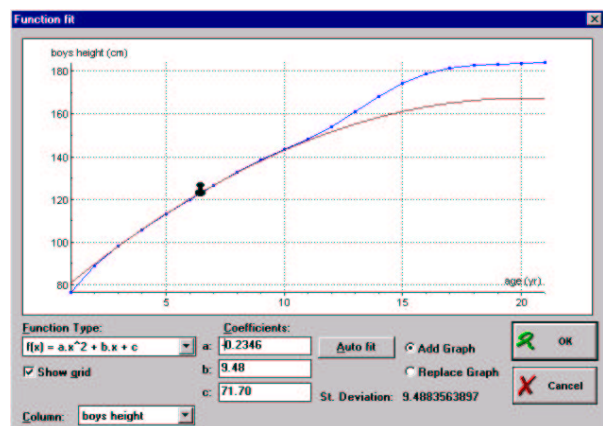
parameter c. This quadratic model is one of the regression models that can be described by a formula of type $y = \mu f(v x + \rho) + \sigma$, where f is a simple mathematical function.

The good news is that pupils have little difficulty in using this manual function fit. On the other hand, we notice that some important mathematical regression models are missing, e.g., the cubic polynomial model and the (modified) logistic model $y = a/(1 + e^{-bx-c}) + d$. This is a pity, the more so because these models are principally of the same form that is handled by the manual function fit.

By seeing various regression models in a menu, pupils feel free to try any function type in the tool, for example, to fit growth data by a modified rational function or by a modified exponential curve. When asked a simple function-fit, most pupils interpret this as a fit with a straight line; a quadratic fit is not 'simple' for them. They have difficulty in analysing a curve in parts or at least they do not do this independently. And the software does not support them much in this respect: all data are used in regression; one cannot restrict the function fit to some range of the data. This would have been useful in the study of the growth during infancy and puberty. The lack of possibilities to restrict operations to parts of data also holds for other data processing and analysis tools in Coach. In the current version, you can only copy a column in a table, discard unwanted data manually, and apply processing or analysis tools to this adjusted copy.

Investigating bridges and hanging chains

The main objectives in this practical assignment for pre-university pupils are to let them
- work with real data collected from digital images;
- apply mathematical models to investigate shapes;
- practice ICT-skills, in particular use tools to collect data from video clips and images;
- carry out practical work in which they can apply much of their mathematical knowledge.

The main task for the pupils, who are novice users of Coach, is to get familiar with the video tool and to use it to investigate the mathematical shape of bridges and hanging chains.

Let us describe briefly how the image measurement activity shown in the screen dump to the right takes place. First of all, the activity allows collection of position data from the digital image. It is possible to place the origin of the coordinate system at any desired position and to rotate the axes, if necessary. One chooses the correct scale by matching a ruler with an object in the image of known distance. coordinates are gathered by clicking on the location of points of interest. Data can be plotted and used for further analy-sis. In the lower-left window of Figure 2, the regression tool has been used to find the quadratic function that fits the data best. In the lower-right window, you see the collected points once more, together with the data plot of the difference quotients (dy/dx) of consecutive points (plotted with respect to a second vertical axis). The difference quotients lie approximately on a straight line. The best line fit can again be found with the regression tool. The third column of the table in the upper-right window shows as well clearly the pattern for the difference quotients. It follows that the shape of the arch is a well described by a parabola. But, with the regression tool one could easily discover that the sinusoidal model $y = a \sin(bx + c) + d$ works almost as good. More is needed for good understanding!

Figure 2. Analysis of the Zeeburger Bridge.

Figure 3. Analysis of a hanging chain

Figure 3 shows a screen dump of the activity in which the mathematical shape of a hanging chain is investigated. By collecting positions on the digital image and by trying a quadratic curve fit on the measured data, a pupil quickly finds out that the form of the chain is not a parabola (as Galilei erroneously claimed). At once, the simple question "How does a chain hang?" becomes a meaningful and challenging problem.
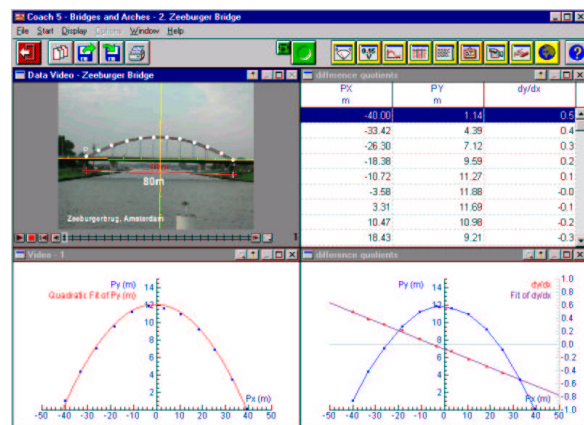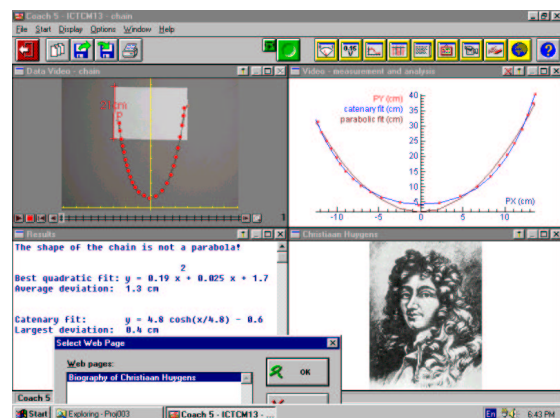
In the activity, we let the pupils study a related, but simpler problem: "How does a chain with five objects of equal weight symmetrically attached hang under gravity?" The case of weights at equal horizontal distances is investigated first. Measurements in the digital images (see Figure 4) reveal that the slopes of the right segments of the chain have a fixed ratio, viz., 1:3:5. Measuring in other images would convince pupils that this does not depend on the length of the segments or how far the suspension points are apart from each other. It turns out that one always has the following fixed ratio
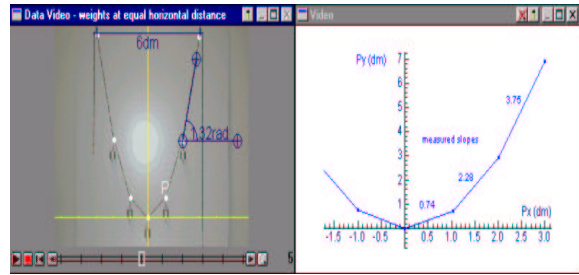

Figure 4. Measuring properties in the image.

of positive slopes: 1:3:5:7:9:… Basic physics can explain this: equilibrium of forces holds at each point of application where a weight is attached. This simple observation about slopes allows computation of the shape of the system and it explains that the points of application are necessarily on a parabola under the given circumstances. In case of weights at equal distances along the chain, one still has the above ration of slopes and one can follow in the footsteps of Huygens to conclude that the points cannot be on a parabola.

What do we learn from trials of this assignment in classroom? On the one hand, pupils like this kind of practical work and they have little difficulty in learning how to handle the video and image measurement tool. On the other hand, we observe that the pupils are not much used in mathematics lessons to make by themselves choices in coordinate systems, origin, scaling, to apply symmetry, and so on. And this is an essential part of a video and image measurement activity.

The only technical problems that pupils encounter in using Coach in this task actually all have to do with the fact that the software at present only has a data video tool built-in and that still images must be treated as video clips with identical frames. This makes it, for example, cumbersome to add or insert more measurements afterwards. Once you have carried out measurements, you cannot sort the data later on and at the same time maintain the immediate link between table entries and measured points on the digital image. Some tools are missing, too: for example, you cannot measure the length of a curve or the area of some region of the image.

In the pupils' reports, the presence of units of length for slopes indicates that some pupils confuse slope and increase of a quantity. Maybe this is caused by the difference between mathematics, in which tangents are dimensionless, and science, where slope is treated as a quantity. Another interesting difference between the use of diagrams in mathematics and science pops up in the classroom experiment when pupils are making graphs invisible in a plot (by checking visibility boxes of quantities). To their surprise, some pupils get weird diagrams with no coordinate system, one axis only, or no labels near the axes. They are thinking of a graph as a representation of a function, i.e., as a representation of a single object, so that it suffices to work with one variable. This is common in mathematics. In science however, a graph represents a relation between quantities. Then one must work with at least two variables. The envisioned integrated learning environment must somehow link up with both graphical concepts.

Iterative processes

Mathematical functions are in Coach described by lists of numerical values. This has to do with the science background of physical quantities, between which a functional relationship exists. Values of quantities can be obtained in various ways:

- via a real measurement with sensors
- via measuring a video clip or digital
- manually filling out a table
- importing results from a file
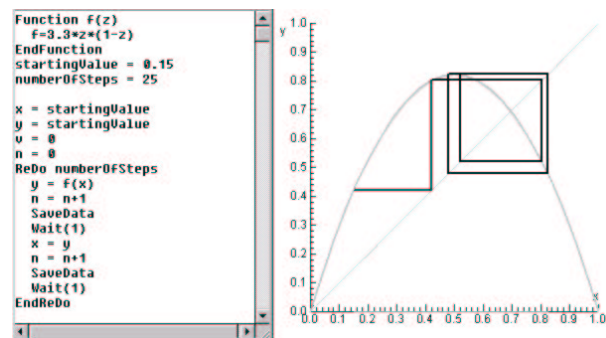- by a formula
- via a computer program

Figure 5. Animation of a cobweb diagram.

Let us have a closer look at the last possibility. Figure 5 shows a cobweb diagram of iterations of the function $f(x) = 3.3(1-x)x$. The program to make an animation is shown. A pupil only needs to enter the initial value, the number of computational steps, or the function definition to do investigations. The first three lines of the program code illustrate that the definition of a function in the software differs from the usual mathematical notation. Another remarkable point is that although Coach is list-oriented with respect to variables in graphs and tables, the system only allows the storage of a single value at a time in a computer program. The programming code of this example and other mathematical computer tasks would be a lot easier if working with lists was allowed, too.

Some other software limitations, which come out from trials of Coach for meaningful mathematics, are:

- Quite some mathematical notions, e.g., functions, equations, and recurrence relations, cannot be introduced and used in the standard mathematical way.
- A calculation window for doing numerical and symbolic computations in the same style as a calculator is missing. At present, you can only write computer programs for numerical purposes.
- Regression and simple descriptive statistics of tabular data are the only tools for doing statistics. Other statistical notions such as box plot, histogram, statistical distribution, and hypothesis test are not supported.
- Combination of diagrams originating from different sources, say combining graphs of functions, parametric curves and measured data, is cumbersome in Coach
- Sometimes the user of Coach is confronted in his/her activities with inconveniences that reveal the science roots of the software. For example, all tables in an activity have the same number of rows, viz., the number of measured data in an computer experiment. In mathematical investigations however, it is not unusual to have tabular representations with various table sizes at the same time.

Many of these limitations are not fundamental and can be or have been dealt with by the developers; they only show that the present version of Coach is not yet optimal for doing mathematics. The more crucial limitations all have to do with the issue of what kind of mathematical objects are supported by the software. Two things must be kept in mind: Firstly, we quote (Yerushalmy, 1999): "*The tool IS the design*". Coach as a science tool has been designed for collecting, processing and analysing data, and for working with computer model. These design intentions for example underpin the choice of variables as references to samples of values, either in tabular or in graphical form. In a computer measurement, one connects a column of a table with a measured quantity; derived quantities are created by menu commands or by formulas in terms of columns and/or connected quantities. So, the physical set-up of an experiment, where sensors have been connected to specific channels of a measurement panel, determines strongly the mathematical processing and analysis of data. An approach that is more detached from the physical set-up, instead identifies measured quantities by symbols, and gives the notion of variable a more central role in activities, seems to be a more fruitful approach towards an integrated math and science learning environment.

Secondly, Yerushalmy writes: "*The tool reflects curricular agenda*". In this respect, the Coach environment can be described as a set of tools to explore natural phenomena. It is an activity-based system for students to carry out practical investigation tasks and to do research work at their level: a student can use the computer environment to collect, process, and analyse data, and to report about his/her work. Depending on the subject, a student can make a choice of tools. We wish to extend this view of the use of a computer learning environment to the field of mathematics. So, the integrated learning environment must provide tools to study mathematical phenomena, methods, and techniques. For example, a natural choice would be a definition-window, a graph-window, a table-window, and a calculation-window to have access to mathematical methods, at least at the same level of functionality as a (symbolic) calculator provides.

Anyway, the development of the integrated learning environment is not only technology-driven, but can also be seen as an attempt to answer the question "What are the characteristics of investigations that can lead to good math and science problems for students and how can computers help?". (Clements, 2000) lists the following characteristics for mathematics:

- are meaningful to students;
- stimulate curiosity about a mathematical or nonmathematical domain, not just answer;
- engage knowledge that students already have, about mathematics or about the world, but challenges students to devise solutions;
- invite students to make decisions;
- lead to mathematical theories about (a) how the real world works or (b) how mathematical relationships work;
- open discussion to multiple ideas and participants; there is not a single correct response or only one thing to say;
- are amenable to continuing investigation, and generation of new problems and question.

Replace the word "mathematics" by "physics", and you get a sensible list of characteristics of good physics activities. The second part of the above question, "how can computers help with a problem-centered approach", is from software developer's point of view the most interesting one. To be better prepared for the creation of an integrated math and science learning environment, the development team must have idea of how several concepts such as the notion of variable, function, table, and graph are used in the various disciplines. As far as the mathematics component of the learning environment is concerned, one must have a clear view on what requirements physics and other sciences make for the mathematics and how this links up with the requirements from mathematics as a discipline itself. In the next three sections we shall restrict ourselves to a discussion of the concepts of variable, function, and graph.

The meaning of variable is variable in mathematics

In mathematics, the concept of variable has several meanings. See (Schoenfeld & Arcavi, 1988) or the following examples from school mathematics, taken from (van Etten, 1980).

$$f : x \mapsto 2x+1 \qquad\qquad A = l \times w \qquad a+b = 7$$

$$a^2 - 9 = (a-3)(a+3) \qquad x \in \mathrm{N} \qquad x+3 = x+3$$

$$n \text{ is a divisor of } 24 \qquad \sqrt{x^2} = x \qquad x+3 = 2x+8$$

$$p \text{ is a prime number} \qquad x = y \qquad A = 2\pi r$$

$$a+b = b+a \qquad\qquad a^2 = 9 \qquad S \subset \mathrm{Q}$$

$$\cos x + \sqrt{3}\sin x = 1 \qquad x < 9 \qquad x^2 + y^2 = z^2$$

$$k \text{ is parallel to the } x\text{-axis} \qquad P \text{ lies on } l$$

Even if letters are used for numbers only, different roles of letters in the algebraic context can be distinguished (Kücheman, 1981; Usiskin, 1988). It may be

- an *indeterminate*, in statements like $a^2 - 9 = (a-3)(a+3)$.

- an *unknown*, in equations such as $a + b = 7$.

- a *known number* like π.

- a *variable (generalised) number*, e.g., in $x \in \mathrm{N}$, in declaring p a prime number, and in differences like $f(a+1) - f(a)$.

- a *computable number* like A in the formula $A = 2\pi r$.

- a *placeholder*, e.g., in function definitions $f : x \mapsto 2x+1$ or $f(x) = 2x+1$.

- a *parameter*, e.g., as a label in the function definition $f_p(x) = p\,x$ to distinguish several cases.

- an *abbreviation* like $V = \{1,2,3\}$.

The multiple meanings of the term 'variable' make it hard for secondary school pupils to understand this concept. So, educators are constantly searching for ways to familiarise students with variables (e.g., see Kieran, 1997; Graham & Thomas, 2000) and to make the transition from arithmetic to algebra, or generalised arithmetic as it is commonly called, easier for them. The main strategy is to treat variables as primitive terms that are best learned by practice. If one cannot define the concept of variable rigorously, then maybe one can better show how and for what reason variables can be used. Basic idea of this approach is that pupils will learn from the examples and the exercises and that they will gradually sense the meanings of variable.

Another obstacle in the transition from arithmetic to algebra is that mathematical meaning is often determined by context rather than by formal rules and notation. For example, what does the symbolism $a(x+e)$ mean? Which of the following meanings would you choose?

- a generalized number $a \times (x+e)$. By the way, does the symbol e stand for the base of the natural logarithm?

- the function a applied to $x + e$ (or do you care that a, used as a function, is usually not in italics).

- a function in x with parameters a and e.

- a function in two or more indeterminates.

- the instruction a applied to the argument $x + e$.

Because of your training and experience, you probably answered that you could not make a choice without knowing the mathematical context or the wording used about the expression. However, for a secondary school pupil it takes time and practice to get used to the fact that a variable actually gets meaning in mathematics through its use (as indeterminate, as unknown, as parameter, etc.), through its domain of values, and through the context in which it is used.

By the word 'context' in the last sentence we also mean the context of 'doing school mathematics', which has its own mathematical conventions. For example, the word 'formula' has a special meaning in school mathematics and the role of the letters in the *formula* $y = x^2$ is not the same as in the *equation* $y - x^2 = 0$. The words 'formula' and 'equation' are used to distinguish between the case of a functional relationship between the isolated variable y that depends on the other variable and the case of a more general relationship between unknowns. For pupils it is important to make a clear distinction between these different notions. A mathematician or scientist, however, is much used to applying the same algebraic symbolism for many purposes: $y = x^2$ may stand for an equation, a function definition, an abbreviation of the expression x^2, as well as for the process of computing the value of y from the value of x.

The equal sign is anyway an intriguing symbol in mathematics. Consider the following two statements:

a) For every real number x, $\dfrac{1}{x+1} + \dfrac{1}{x+2} = \dfrac{2x+3}{x^2+3x+2}$.

b) For every real number x, $\dfrac{2x+3}{x^2+3x+2} = \dfrac{1}{x+1} + \dfrac{1}{x+2}$.

In a formal sense, these statements are equivalent. But it cannot be denied that the first statement is simply about computing the sum of two rational expressions, whereas the second one represents a partial fraction decomposition. In the first statement the equal sign is not read as "is formally equivalent to", but as "yields": it is about addition. The second statement can be interpreted as "the fraction $\dfrac{2x+3}{x^2+3x+2}$ yields $\dfrac{1}{x+1} + \dfrac{1}{x+2}$ when it is decomposed." Here, the expression refers to a process of simplification. Again, the expression is not just a mathematical object with the structure of an equation, but it has a process aspect, too.

With respect to learning algebra, much attention has been given in educational research to the dual nature of mathematical entities, which have a procedural and a structural or conceptual aspect, and to the way pupils can obtain such versatile understanding. In early algebra, a mathematical expression is often introduced as a means to describe a process of computing. But gradually, pupils are acquainted with the idea that an expression can also be viewed as a result of a computational process. An expression becomes a mathematical object on its own, which can be manipulated. In (Gray & Tall, 1994), the authors introduce the word "procept", which is a contamination of process and concept, for the combination of symbol, process, and concept, to make clear that a mathematical object never completely looses its process nature. For example, in the expression $2x+1$ the $+$ is not only a symbol that defines the object 'the sum of $2x$ and 1', but it also refers to the process of 'add 1 to the product of 2 and x'. Another example: $\lim_{n\to\infty} a_n$ represents both the process of 'tending to a limit' and the concept of the 'value of the limit'. The interested reader is referred to (Tall et al, 2001) for further discussion of the process/concept duality in algebra and its impact on learning to think mathematically. The alternative conceptions are also referred to as "process and object" (Sfard, 1991), where the word "reification" is used for the objectivation of a process, and as "procedure and structure" (Kieran, 1992).

Gradually pupils learn to see variables as a replacement not only for numbers, but also for expressions. They learn and practice the various ways in which algebraic expressions can be manipulated: combining literal terms, replacing subexpressions, factoring, completing the square in a quadratic polynomial, rationalising the denominator, subtracting the same term from both sides of an equation, solving systems of linear equations, and equivalence testing are examples of activities that are present in all mathematics curricula. In many research studies (e.g., see Kieran, 1989; Sfard at el, 1994; MacGregor and Stacey, 1997; Stacey and MacGregor, 2000; Tall et al, 2001) is reported that understanding generalised arithmetic is not easy: literal symbols are like numerals and words, yet they are different, and pupils have to learn to deal with these differences. For many a pupil, $\frac{1}{2}x + \frac{1}{2}$ relates to a straight line, but the equivalent expression $(x+1)/2$ is only considered as a rational expression, which is something completely different in their eyes. The previous examples illustrate that pupils must understand the underlying structure of algebra and become familiar with the dual character of algebraic expressions (operational/structural, process/object) to gain competence in mathematics.

Let us return to the concept of variable. That it is almost impossible to rigorously define the term does not mean that one cannot classify the various appearances of variables in mathematics. The following three uses of variable are distinguished in (Freudenthal, 1983):

1. as a *placeholder*, which denotes the places in an expression where the same object is meant.

Variables as placeholders mostly occur in function definitions: for example, the definitions $f(x, y) = \sqrt{x^2 + y^2}$ and $f(a,b) = \sqrt{a^2 + b^2}$ both define one and the same function, viz., the norm in two-dimensional Euclidean space. One also refers to these placeholders as dummy variables: they do not indicate objects anymore, but rather the locations for replacements with certain kind of objects. If other variables are present in a function definition, e.g., in $f(x) = a\,x^2 + b\,x + c$, they are distinguished from the dummy variables and they are called parameters. At first sight this is an easy distinction, but the use of parameters is in practice more complicated and more difficult to master (e.g., see Furinghetti & Paola, 1994; Drijvers & van Herwaarden, 2000). One of the greatest obstacles for a pupil to be able to handle a parameter is that (s)he must see the structure of the formula, for instance see that $2-x$ and $2x+1$ are both examples of the linear expression $a\,x+b$.

2. as a *polyvalent name*, i.e., a name for an object that can take a multitude of values.

If n is a divisor of 6, the letter stands for any of the numbers 1, 2, 3, and 6. In the statement that we have a real number x such that $x^2 + x - 2 = 0$, the letter x refers to a number that is yet unknown, but can be computed, and that has the property that the sum of this number, its square, and -2 is equal to 0. Without knowing its exact

value, one can deduce that for this number holds $2x = x^3 + x^2$. Solving the equation means finding the x for which the statement is true. A priori x is indeterminate, a posteriori x can take two values.

3. as a *variable object*, i.e., a symbol for an object with varying value.

In mathematics, the object to be thought of can be a number whose value may change like in the locutions "2^n for n from 1 upward" and "a_n converges to 0 as n goes to infinity". Of course, one can replace these locutions with 2^n for $n \in \mathrm{N}$ and $\lim_{n \to \infty} a_n = 0$, but then one looses the kinematic aspect of the variable. However, this kind of variable very often occurs in mathematical models that are constructed to study real-world phenomena. The object can be a physical quantity such as time, position, and temperature, or an economic quantity such as price, capital, and income. Clearly quantities, often depending on time, with varying values.

A variable object may be related with others. One speaks of independent variables, whose values one is free to choose, and of dependent variables, whose values one can compute given the values of the independent variables. Many applications can be given: the position of a moving object depending on time, the room temperature depending on the time of the day, the temperature of a long rod as a function of place, and so on. The roles of independent and dependent variables are often not fixed during a computation. For example, studying the motion of a sprinter, one may on the one hand consider acceleration as a function of time, but on the other hand describe it as a function of the velocity of the sprinter. One of the big ideas in calculus, and in mathematics in general, is the freedom of choosing independent and dependent variables.

The above distinction of three uses of variable can be applied to parameter as well: in the role of a placeholder, the parameter has one value at a time. For example, in the formula $T = 2\pi\sqrt{l/g}$ for the period of the mathematical pendulum, the letter g stands for the gravitational constant. You may study the motion of the pendulum on earth and as well as on moon, but always stands g for one value only. The given formula expresses the relationship between the period of the pendulum (T) and the length of the pendulum (l). So the letter l plays the role of a variable object. Rephrasing Freudenthal, one speaks about "the use of a secondary – as it were sleeping – independent variable, which, if need be, can be accounted for – as it were wakened up – for instance in order to get a system T_l of periods by the variability of the parameter l." Thirdly, as a polyvalent name, a parameter allows you to write general formulas or to distinguish various cases: the label p in the function definition $f_p(x) = p\,x$ is used to distinguish several cases. Coming back to our example of the pendulum: ω (angular frequency), and ϕ (phase) are used in the formula $u = A\sin(\omega t + \phi)$ to describe harmonic motion mathematically. ω and ϕ are according to their origin dependent variables (dependent on the pendulum and its position at a certain time), but according to their appearance independent variables and they serve to describe the general periodic motion of the pendulum. Another example is the role of the initial height y_0, the initial velocity v_0, and the release angle α in the mathematical formula $y = y_0 + v_0 t \sin(\alpha) - \tfrac{1}{2} g t^2$ that gives the vertical position of an object that is thrown away. Typically, one keeps values of all parameters except one fixed and draws graphs for several values of the free parameter in one picture, i.e., the parameter in its generalising role is used to distinguish several cases.

Freudenthal mentions another occurrence of parameter, which does not seem to fit so well in the previous three roles of parameters at first sight, viz., the parameter representation of curves and surfaces. For example, the unit sphere in the x-y plane is parameterised by means of the arc length s from a fixed point as $x = \cos(s), y = \sin(s)$. The parameter s arises as dependent variable (arc length), dependent on the point of the curve. A posteriori it is used as independent variable in order to represent the curve. Thus, the parameter in the parameter representation of a curve is mainly used as a variable object.

Besides, parameter representation of curves and surfaces, there exist many more applications of the concept of parameter in school mathematics. We mention three applications, in which software can help to visualize parameters and to study the effect of changing parameter values (e.g., see van der Giessen, 2001):

*   curve fitting: linear fit ($y = a\,x + b$), exponential fit ($y = y_0\,g^x$), and sinusoidal fit ($y = a\sin(b\,x + c) + d$).

*   exploring transformations of a function: e.g., starting from a standard function like $f(x) = x^2$ investigate the behaviour of $f(x) + a, a \cdot f(x), f(x + a)$ and $f(a \cdot x)$.

*   Solving problems numerically or graphically: e.g., "find a value of the parameter a such that the graph of $y = \sin(x + a)$ passes through the point (0,1) or that the graph of $y = a^x$ and its derivative are equal."

Variables in physics

In (Vredenduin, 1979), the author discusses some of the differences in terminology and notational systems between physics and mathematics. Some of his essential differences have to do with the use of variables and have been incorporated in Table 1. Below we work out the details.

Mathematics	Physics
Generalized arithmetic with dimensionless variables is used.	Quantity arithmetic with its own rules and use of units is dominant.
Names of variables are free to choose and changing names in an expression does not change its meaning	A variable is often related with some physics concept and its name is an abbreviation of this notion.
Irrational numbers like $\sqrt{2}$, π and e are important; floating-point numbers are without accuracy: 1.0 = 1.000.	In measurements, only natural numbers and floating-point numbers with accuracy occur: $1.0 \neq 1.000$
There is a strong focus on special properties of functions, e.g., on asymptotic behaviour.	Properties and assumptions may rule out parts of mathematical interest.
Words like "big", "small", "negligible" have little meaning.	A small change of a quantity Q is also a quantity ΔQ with its own arithmetic.

Table 1. Some essential differences between the use of variables in mathematics and physics.

In physics, a variable is most often used as a name for a quantity that can vary (often with respect to time) and that in many cases can be measured. Physical quantities can be magnitudes such as temperature, mass, and length, or vector quantities such as velocity, acceleration, and force. One striking feature of the symbolic writings in physics is that function, sample of function values, and single function value are mixed up easily and apparently without much harm: a physical quantity is sometimes a function of time, in other occasions a finite sample of values measured at different times, and sometimes a function value at a certain fixed time. As concrete example we consider Boyle's law $p\,V = \text{constant}$, for pressure p and volume V. The variables are in fact functions of time, viz., $p : t \mapsto p(t)$ and $V : t \mapsto V(t)$. Boyle's law says that the product of these two functions is a constant function, i.e., $p(t) \times V(t) = \text{constant}$, at every time *t*. In an experiment, one verifies or rediscovers the relationship by measuring p and V under different conditions. Here, one compares samples of function values. But in a school problem like "suppose that $V = 10\,\text{dl}$ when $p = 2\,\text{bar}$, how much is V when $p = 4\,\text{bar}$?", p and V do not represent functions or samples of function values anymore, but they represent single function values, viz., the pressure and volume at a certain fixed time. In mathematics, this ambiguous use of notation for function, sample of function values, and function value rarely occurs.

In mathematics it is often allowed to change names in an expression without changing the meaning: for example, $\{(x, y)\,|\, x\,y = 1\}$ is the same set of ordered pairs as $\{(a,b)\,|\, a\,b = 1\}$. There only exist some generally agreed conventions about the use of letters such as *x* and *y* in order that almost every mathematically trained person reads $y = a\,x + b$ as a functional relationship between these two variables with parameters *a* and *b*. This conventional use of letters makes a discussion about a mathematical problem easier, because it takes away the need to explain every time the notation in use. In physics, replacements of letters are almost always forbidden: replacing traditional names for energy *E*, mass *m*, and the velocity of light *c* in Einstein's law $E = m\,c^2$ will ruin it. The reason is simple: most variables in physics are not meaningless but deal with concepts in physics that have the property of quantity. For this reason, one often chooses the first character of the name of the concept as the name of the variable: *F* for force, *m* for mass, and *a* for acceleration in Newton's law $F = m\,a$ is one example of many.

Physical quantities can often be measured, but the measured values are always natural numbers (in counting processes) or floating-point numbers, possibly with margins of error. Irrational numbers like √2 or π do not occur in measurements. Nevertheless, physicists mostly compute with quantities as if they take real values. But they treat floating-point numbers differently than most mathematicians do. For many a mathematician, the number 1.23000 is the same as the number 1.23 without the trailing zeros. However, in numerical analysis and in physics, the notation 1.23000 implies that the number is known more accurately than 1.23. If 1.23000 is a measured value of a quantity, you know it is between 1.229995 and 1.230005.

A value of a physical quantity actually consists of three parts, viz., the numerical value (a number), the precision (the number of significant decimals or the margins of error), and the unit that is used to measure the quantity. This makes quantity arithmetic more difficult to learn and to use than reference-free number arithmetic. The following example, taken from (van der Kooij 1999), illustrates this. Compare the given answers to the following problem:

"Peter and John walk in a straight line in the same direction from the same starting point, with the same speed of 2 m/s. Peter starts first at time t=0 and John 3 seconds later. Give the formula for the distance s walked by John after t seconds, for t > 3 sec."

Answer 1: $s = 2\,(t - 3)$.   Answer 2: $s = 2t - 6$.

In generalized arithmetic, the two expressions are equivalent: a factored and expanded form. In quantity arithmetic, which takes dimensions into consideration, the first answer $s = 2(t-3)$ represents a time-approach of the problem via a *distance = speed × time* formula and the second answer represents a distance-approach $s = 2t - 6$ via a *distance = distance − distance* formula. So, thinking about dimensions and units of measurement makes clear that in real life problems not only the variables but also the numbers have contextual meanings. Especially the number 1 is tricky because it is always left out of expression: no one writes the formula $N = N_0 2^t$ of exponential growth as $N = N_0 2^{(t/1)}$ to show that the doubling time is 1 time unit and that dimensions are actually correct in this formula. When manipulating the formulas in a real world context, a pupil better keeps the dimensions in mind to verify work. Physics teachers have good reasons to request from their pupils that they check the dimensions and the units of measurement in their answers.

The focus of mathematical thinking may differ radically from the one in a study of a real world problem. For example, when one encounters in mathematical work a rational function of the form $\dfrac{a\,x}{x+b}$ the attention goes almost automatically to the singular behaviour as x approaches $-b$. Compare this with the study of enzyme kinetics, where the Michaelis-Menten expression for the initial rate of transformation of a substrate, S, by an enzyme, is $v = \dfrac{V_{\max}[S]}{K_m + [S]}$, where $[S]$ is the concentration of S, $V_{\max}$ is the maximum rate, and K_m is constant called the Michaelis-Menten constant. Since the parameters and concentrations are positive, the singularity is never encountered and the common mathematical analysis is irrelevant.

In physics, words like "big", "small", "relatively small", "negligible" can be used while talking about quantities. A small change of a quantity Q is also given a name, such as ΔQ, and one manipulates it as any other variable, except that one often ignores higher order terms like $(\Delta Q)^2$ to get a simpler model description. Going from calculus of small changes to infinitesimal change and calculus with differentials is then a natural step. The following example shows how it works. We look at the formula $s = a\,t^2$ for a moving body with s being the distance travelled as function of time t. Suppose that one is interested at the speed of the object during its fall. The change in distance travelled $\Delta s(t_1)$ during a small time interval $[t_1, t_1 + \Delta t]$ is given by $\Delta s(t_1) = a(t_1 + \Delta t)^2 - a\,t_1^2 = 2a\,t_1 \Delta t + a(\Delta t)^2$. A physicist will say that, when Δt is small, the term with $(\Delta t)^2$ can be neglected. So, for the rate of change one has $\Delta s(t_1)/\Delta t \approx 2a\,t_1$. In the limit case of infinitesimal changes, one has $ds/dt\,(t_1) = 2a\,t_1$. So, the speed at time t_1 is equal to $2a\,t_1$. The mathematician, on the other hand, is not happy with the sentence "when Δt is small, you can neglect the term with $(\Delta t)^2$". He or she will say that $\Delta s(t_1)/\Delta t = 2a\,t_1 + a\,\Delta t$ and that therefore $\lim_{\Delta t \to 0} \Delta s(t_1)/\Delta t = 2a\,t_1$. The same conclusion, but a different style of working.

The above differences add a good deal to the understanding of the difficulties that pupils have in relating the mathematical methods and techniques that are used in physics to what is learned in mathematics lessons. In mathematics, they use variables mostly as placeholders and polyvalent names. Emphasis is on generalised pure arithmetic, with reference-free numbers, and on the concept of function defined as a special kind of correspondence between nonempty sets A and B which assigns to each element in A one and only one element in B, i.e., on the Dirichlet approach to the concept of function. In physics, the third kind of variable, viz., the variable object, comes into play. One is involved with functional relationships between varying quantities, in which one distinguishes between dependent and independent variables. In working with variable objects and relationships between them one uses mainly the theory and practice of solving equations in known and unknown quantities and one uses the calculus of change.

Different contexts for graphing in mathematics and physics

Why do we ask pupils to make graphs? The answer to this question differs from discipline to discipline, but the reasons for using graphs are commonly divided into two classes: analysis and communication (Friel et al, 2001). For example, a physics teacher (Barton, 1998) may say that the graph is simply a means to an end: plotting graphs helps to interpret measured data. A diagram gives an overview of the measured data and from its shape one may get a clue about the possible relationship between the physical quantities in which one is interested. In order to better see or verify these relationships all kinds of scaling of graphs are at hand, such as (semi-)logarithmic and double-logarithmic plots. Derived quantities can be introduced to make the relationship clearer: for example, if a quantity Q_2 is inversely proportional to Q_1, and therefore the quantities Q_2 and $1/Q_1$ are proportional, it may be wise to plot Q_2 against $1/Q_1$ and see if it gives a straight line through the origin. This

helps a secondary school pupil to learn that relationships between quantities do not go without saying in physics and that people to better understand or to predict relationships have invented the physical laws. Another purpose of a physics graph is that it allows easier discussion about the physics problem and that it provides a good means for presenting results to other interested persons.

For science, one could certainly say that graphing is not a context-independent skill. Rather, competencies with respect to graph interpretation are highly contextual and are a function of the scientists' familiarity with the phenomena to which a graph pertains and their understanding and familiarity with representation practices. (Roth & Bowen, 2001) take this as a starting-point for implications for teaching graphing in school mathematics and science settings. The authors take a sociocultural orientation toward graphing as practice (by professionals), instead of the cognitive-psychological perspective, in which problems and misconceptions in students' reading and interpreting graphs are identified (e.g., Leinhardt et al, 1990). Their analysis of how professionals read graphs shows that competent graphing practices are related to understanding of both the real world phenomena and the structure of signifying domain, familiarity with conventions underlying these two domains, and familiarity with translating between the two domains. To read a graph competently, one needs more than instruction on the mechanical aspect of producing graphs. One must also be familiar with representation-producing mechanisms, data-collection devices, feature-enhancing techniques, etc. In education, extensive interaction with phenomena and representational means seems to be a prerequisite for graph sense. And this is where ICT is expected to contribute to the learning process.

In mathematics, drawing the graph of a function has not much to do with finding a relationship between quantities; in most cases, the function is already given by a formula or a table of function values, and has nothing to do with a real world context. One does not have to look for the mathematical object of study and one makes the graph mainly to present a single view on various properties of the mathematical object. From the graph of a function one can get an idea about the number and location of zeros, maxima, and minima, about the asymptotic behaviour of the function, about points of interest such as discontinuities and bending points, about increase and decrease, and so on. The shape of a curve is important issue in mathematics lessons. Calculus helps to make graphical observations more precise: e.g., computing the derivative and making a sign diagram of it gives precise information about change, and solving equations provides appropriate answers to questions about zero's and extrema of the function and about specific function values. Comparison of graphs can be done to illustrate the effect of transformations of functions. Because a diagram is in the field of mathematics essentially a set of numeric 2-tuples, it is easy to combine graphs of different origin in one picture. For example, one can draw without difficulty in one picture a graph of the function $y = x^2$, the graph of the inverse $y = \sqrt{x}$, the upper part of the unit circle via the parameter representation $x = \cos(s), y = \sin(s)$, and the straight line going through (0,0) and (1,1). In all cases, a mathematical graph is used to show various aspects of a function in a single picture and it is one of the means to solve mathematical problems.

The graphs in mathematics and physics also differ from technical point of view. A pupil studying a Dutch mathematics textbook can easily recognize when (s)he is dealing with a mathematical graph or with a graph coming from an application in a real world context. In a mathematical graph, the coordinate system consists of two perpendicular number lines of the same kind intersecting at (0,0), and the horizontal and vertical axes are labelled by x and y, respectively. Unless otherwise defined, the domain and range of a function are infinite. Accordingly, a pupil is free to choose the linear scale of the number lines such that all interesting parts of the graph of a function are visible, but always 0 appears on the lines. The introduction of the graphing calculator, which allows non-linear scaling of axes and a plotting area that does not contain (0,0), has not brought much change in the mathematical graphs in the Dutch textbooks. Often textbook authors make a distinction between 'plotting a graph of a function', when the calculator is used, and 'drawing the graph of a function', in case the pupils has to do it by pencil and paper. In a graph coming from a real context, the axes represent quantities and the names of these quantities appear at the axes together with the chosen units. An arrow represents the positive direction of the axis, because this is not fixed anymore in applications. Frequently, physical quantities have by origin a limited range of possible values. Then these ranges determine the plot range of a graph and accordingly the origin can be different from (0,0). As was said before, scaling is a matter of choice and does not have to be linear: if it is more convenient to choose another one, say a logarithmic scale, to clarify a relationship between quantities, then one a free to do so.

The quantities represented by the axes in a physics graph are not the only quantities presented by the diagram. Also the ratio between proportional quantities, the gradient of a function, and the area between the graph of the function and the horizontal axis represent quantities. For example, in a position-time graph, the gradient represents speed (or velocity if direction of motion is taken into account) and the area gives the distance travelled during some time. The ratio between mass and volume of matter is a derived quantity, viz., the mass density. In a mathematical graph, the directional coefficient of a straight line is a dimensionless number as well as the slope of a graph in a given point, which only has a geometric interpretation as the limit of a difference quotient or the slope of the tangent line in that point.

We summarise our findings so far in Table 2.

	Mathematics	*Physics*
Graph	Represents a single object, viz., a function. Main purpose is to give a single view of various aspects of a given function.	Represents a relationship between two quantities. Main purpose is to explore or to present the relationship between quantities.
Axes	Dimensionless numbers are represented. Scaling is by default linear.	Values of quantities are expressed in some unit. Scaling is a matter of choice and may be non-linear.
Origin	(0,0) is the fixed position.	Arbitrary position.
Plot range	In principle infinite	Determined by the ranges of the quantities.
Slope/ Gradient	Dimensionless number having a geometric interpretation only.	Represents the change of a quantity with respect to another and is on its turn a quantity with a unit.

Table 2. Different contexts of graphing in mathematics and physics.

Graphs in mathematics and physics do not only differ in their construction or purpose; also reading of graphs is different and this is reflected in the language that is commonly used. We give some examples. In mathematics, one speaks about the origin of a coordinate system and about the zero on the number line, in the sense of the point that represents the integer 0 with the property of the unit element in an additive group. In physics one does not speak about the origin of quantity values, but instead one uses the wording 'zero'. For example, temperature can be expressed in various units depending on the zero that one chooses: 0 Kelvin refers to the lowest temperature that an object can have, whereas 0 °C refers to the temperature of melting ice. Another common example of the freedom to choose the zero of a quantity in a physics problem setting is the potential of a force field, say an electric or gravitational field.

A mathematical graph represents a single object, the graph of a given function. Accordingly one speaks about 'the graph of the function y', 'the graph of $y(x)$', or 'the x-y graph'. A physics graph represents a relationship between two quantities and one generally speaks about 'the graph of one quantity versus the other'. In this wording one uses the name of the quantity represented by the vertical axis first, especially when the horizontal axis represents time. For example, one speaks about a v-t diagram or an x-t diagram. This is opposite ordering of names used in mathematics. It might help pupils to better understand graphing if physics textbooks and teachers would regularly use the terms '$v(t)$ diagram' and '$x(t)$ diagram', and if mathematics textbooks and teachers would regularly use the terminology 'graph of y as function of x'.

The word 'range' has a different meaning in mathematics and physics. In the latter field, it refers to the values that a physical quantity can take. So, it is legitimate to talk about the range of the quantities at the horizontal and vertical axes of a graph. In mathematics, the words 'domain' and 'range' are used in connection with the notion of function and they specify for what values the function is in principle defined and to what set the function values belong.

In many physics textbooks, the difference between steepness of a straight line and its slope is stressed. The reason lies in the use of dimensions: a quantity like speed is connected with the steepness of the tangent line at points in the x-t diagram. The steepness of a tangent line can be approximated by a difference quotient, i.e., the quotient of two differences of quantities. In mathematics, which uses dimensionless numbers, the slope of a tangent line or the coefficient of direction, as it is also called, is mainly interpreted geometrically as the arctangent of the directional angle of the line, but only in case equal horizontal and vertical scales are used. The slope has to do with angle in a geometrical picture and it is not a physical quantity.

The above differences in the use of language for graphing, which is summarized in Table 3, may contribute to the difficulties that pupils have in using graphs similarly, but yet so differently in mathematics and physics.

Mathematics	*Physics*
Tangent line, slope.	Gradient, steepness.
Origin (of coordinate system).	Zero (of a quantity).
Domain and range (of a function).	Range (of quantity values).
Graph of $y(x)$, x-y graph.	v-t diagram, x-t diagram.
Set of 2-tuples (x,y)	Plot of y vs. x

Table 3. Some different languages for graphs.

Consequences for the design of an integrated learning environment.

What can be learned from the discussion of mathematical notions in the last three sections? Certainly, analysing differences between the use of variables, functions, and graphs in mathematics and physics is not enough when it comes to the design of an integrated learning environment. One thing to keep constantly in mind is that doing mathematics on a computer differs essentially from tradition pencil-and-paper work. In (Heck, 2001), the author discusses the differences between the concept of variable in mathematics and science and in computer algebra. He lists the following properties of computer algebra variables that make them different from variables in mathematics:

- A computer algebra variable always points to a value, which can be almost anything.
- Manipulation of computer algebra variables has its own rules, in which internal storage of expressions, automatic simplification, ordering of commands, and evaluation scheme play a role.
- An expression can represent a mathematical object as well as describe a particular process to be carried out. Both notions are frequently used.
- Some variables have special meaning distinct from standard mathematics.
- Although modern computer algebra systems try to mimic mathematical notation as much as possible, their users still have to translate to and from standard notation on many occasions.
- In computer algebra, there is a strong focus on solving generic problems, i.e., special cases such as special values of parameters are not taken into account.

Being aware of such differences between traditional work and computer-based work, what kind of suggestions and recommendations can we make to designers of educational software environments? First of all, a lesson from the past: many successful education software environments like dynamic geometry systems have been primarily designed on the basis of what *should* be done in education instead of on the basis of what *can* be done technically. In other words, the activities of students and teachers have been chosen as the basis to work from, not the knowledge that apparently must be incorporated. In short, we would like to advocate the following primary design goals:

- to provide a useful, powerful, and scalable set of facilities for students and teachers;
- with an easy to use and easy to learn, appealing user interface;
- ideal for teaching and learning activities;
- that runs on moderate and affordable equipment;
- allows easy exchange with other software products such as spreadsheets, text processors, presentation tools;
- supports easy network communications such as internet browsing and distance learning.

We envision a scenario of a teacher and students using a set of tools for the study of natural and mathematical phenomena. This set of tools is integrated in one open environment designed for the educational setting. Such environment is open in the sense that it is

- a flexible and customisable multi-purpose tool;
- an environment for solving open problems that need definition, set-up, exploration, etc., i.e., a thinking tool;
- free of didactic context or principles, i.e., it less considered as a pedagogical tool and more as mathematical and scientific tool.

Some technical recommendations are:

- provide many useful representations of mathematical objects;
- allow flexible notation of mathematical expressions and user-defined objects;
- allow more than one way of creating and manipulating objects;
- make tools customisable to users.
- provide an easy to learn and easy to use programming language for writing computer model and extending the built-in library of user commands.

Let us be a little bit more specific and technical about the above guideline for the development of an integrated learning environment that allows the user to express easily mathematical and scientific thinking. Firstly, what kind of model for the calculus of variables and expressions do we envision and what do we learn from our analysis of the use of variables? Whatever the answer is, the model should support the objects and operations from mathematical and scientific contexts in a natural way. In other words, the way that the software treats them should be consistent with the expectations of a user. For a software system that is meant for education, this is even more important, since the software representations will influence the formation of concepts.

In general, one can say that the more structure the model has, the more support it can give. But a stricter structure also implies less freedom and introduces the need of more conversions of one structure type to another. Current mathematical software systems take different approaches towards this design choice. Matlab supposes that everything is a matrix. Axiom uses strong typing, together with a system for type inference based on mathematical categories. Derive, Maple and *Mathematica* have no strong typing of objects, and give the user an

enormous freedom. Such a freedom has a price; the interested reader is referred for a discussion about Derive to (Artigue, 1997), (Guin & Trouche, 1998), and (Drijvers & van Herwaarden, 2000). A price to pay is in general that although a user may enter an expression that is mathematically unsound, it can very well be syntactically sound. In this case the user will only be notified by a runtime error, or will not be notified at all. Moreover, more structural commands are needed in the language to make up for the lack of structure in the model.

We are of opinion that one can follow here an intermediate route and choose for a strong typing, but of a very simple nature. Few atomic objects like numbers, booleans and strings are required, and there must be a way of creating indexed lists of objects. In most contexts, the types of the objects will follow immediately from the context; one does not have to declare them beforehand. In this way, we hope and expect to have found a proper balance between power in expressiveness and structural support.

Variables are ubiquitous in science and mathematics. In general, one thinks of a variable as something with a symbol as name and a value. The symbol may be used in computations and mathematical expressions as if it were a number. The value may be not determined, it may change, and it may vary. This description is still somewhat vague, and in textbooks the concept is often ill defined or not defined at all. We find Freudenthal's classification of the various appearances of the concept of variable very useful, even under the constraint that variables always need a finite representation on a computer. The finite representation of the variable poses for the use of a variable as a placeholder or polyvalent name not a particular problem. It is a symbol without a particular value, or with a value that can be changed. The meaning of the symbol depends on the mathematical context: expressing statements, solving equations, defining functions, and so on. But in the third type of use, i.e., the variable as variable object, the finiteness of any representation on a computer does impose restrictions. We see for an object with varying value two possibilities:

- a finite representation with a finite indexed listing of values;
- a finite representation with an algorithm expressed in finitely many terms.

It is the first possibility of a notion of variable with indices to express the variability that leads us to a type for variables: e.g., the type `float[i,j]` will mean that the variable has as values floating point numbers that depend on the indices `i` and `j`. So, such a variable has a symbol (name) and a type associated to it, possibly together with a value that can be a number, a list of numbers, a list of lists of numbers, etc. To keep track of the origin of the data, variables will have a source attribute that contains a reference to the instance (file, URL, sensor, GUI-element, etc.) that actually sets the values of the variable. Although such a typing system allows structured support for the user and a short and therefore powerful language, it is the purpose of the model to ensure that most users do not have to set the types explicitly. In fact, only advanced users will have to understand the typing system.

The second possibility is about algorithms. This leads us at first to formulas: mathematical expressions containing arithmetic operations, function calls, variables and indexed lists. Formulas are in some way the easiest algorithms, but one will soon feel the need for other algorithms, e.g., with repetitions and conditionals, or operations in which also the indices are involved. This is where the programming language comes in. Expressions can be formed in terms of variables, indexed lists and operations on those. A prototype for internal use by software developers at the AMSTEL institute shows that this model for calculating with variables of different types is feasible.

The second major theme of our discussion about consequences for the design of an integrated learning environment is graphics and its role in mathematics education. First of all, there seems to be general consensus about some starting-points of teaching and learning calculus:

- equally important role of graph, table, formula, and context (certainly in the beginning).
- use of multiple representations of the same mathematical object.
- stress on relational understanding instead of instrumental understanding: the difference is in understanding *why* and *how*. For example, drawing a graph of a function is of instrumental nature, whereas curve fitting requires knowledge about properties of functions.
- use of real contexts for concept building and application.
- importance of dynamics for deepening mathematical insight.

Experimental research of recent years (e.g., Hennessy, 1999) supports the assumption that graphing calculators can already stimulate

- the use of realistic contexts and bring mathematical modeling and interpretation of results of simulations within pupils' reach;
- the exploratory and dynamic approach of mathematics: think of pupils' activities like investigating and classifying parabola, and graphical techniques like zooming;
- a more integrated view of mathematics: pupils can use multiple representations of mathematical objects;
- a more flexible problem solving behavior: new strategies like solving equations graphically are possible.

The question is not *whether* technology can contribute, but more *how* it can contribute and *what requirements* can be made for an integrated environment to support this view on calculus education. We limit ourselves to role of graph, table, formula, and situation.

Graph and table are mathematical instruments to represent data in a clear structured way. They are in general more than an ordered set of point and numbers: they tell a story or represent a process. For example, in an interpretation of a velocity-time diagram qualitative issues such as change of a graph and asymptotic behavior are often more important than the quantitative aspects. To facilitate discussion about a diagram and to make it possible to construct diagrams in the same format as one sees in textbooks, it is necessary that one can make diagrams more beautiful. Use of different colors, different line styles, various characters, annotations and legends, auxiliary points and lines, and so on, must be possible. Various zooming facilities such as automatic zooming or the use of a (dedicated) zoom box allow the user to focus on interesting parts of a diagram. To better understand the role of a parameter in a mathematical model or to carry out a sensibility analysis, a lot is gained when a range of parameter values is available via graphical user interface elements like a slider bar. This immediate graphical response to changing a parameter is expected to contribute to mathematical understanding.

Construction of graphs and table should be easy and natural. It must be easy to comply with a request like "give me a straight line", "give me a parabola", or more generally, "give me a graph of a function of this well-known type", and provide a graph together with handles for further adjustment. Good-quality freehand sketches of graphs may be difficult to make, but standard techniques like the use of Bezier curves may already suffice for making a sketch. Also, when a mathematical function f is defined in a standard way, say $f(x) = x^2 - 3x + 1$, dragging and dropping the symbol f into a diagram window may already produce the graph of f for some default domain. Similarly, a definition like $y := x^2 - 3x + 1$ can be dragged and dropped into a diagram window to get the isolated variable y plotted against the independent variable x. The same holds for dragging a symbol or definition into a table window.

Graphs and columns in tables do not need to come from formulas exclusively. For example, to create an increase diagram of some functional relationship or to filter some data it is enough to select some appropriate (menu) command. Certainly, in case you want to be able to carry out operations on graphs of functions *before* the required algebraic manipulation skills have been learned and practiced by students, this kind of graph manipulation without a formula is fruitful. A more general recommendation to software developers is to de-emphasize algebraic input via keyboard or palette, but to allow other ways of creating mathematical objects.

One of the behaviors associated with graph sense is the ability to understand the relationships among a table, a graph, and the function or data being analyzed. Many research studies (e.g., Kaput, 1992) report about the possible contribution of the use of multiple, linked representations of mathematical object to mathematical thinking. The main rationale of linked representations is to give students the opportunity to explore ideas from a variety of directions. Let us have a brief look at what this means for learning the concept of function. The matrix in Table 4, developed in (Janvier, 1978), gives an overview of the transitions between table, graph, formula, and situation.

from → to	situation	table	graph	formula
situation	restructuring, searching relevant data	deriving data from the situation, e.g., via data collection	sketching of graph form verbal description	modelling, finding a formula from a verbal description
table	reading and interpreting	changing one table into another, e.g., adding an increase column	plotting ordered pairs of numbers in a grid	finding a formula for tabular data
graph	interpreting charac-teristics of functions	reading off coordinates	making one graph from another	curve fitting
formula	formula recognition	computing	sketching a curve from a formula	simplifying and re-writing a formula

Table 4. Transitions between situation, table, graph, and formula.

In mathematics lessons, the most used transitions are t → g, g → t, f → g, f → t, and f → f , where t, g, f stand for table, graph and formula. From an integrated learning environment one may expect that it makes such transitions simple to do and that is allows student to concentrate on mathematical and scientific issues, instead on technical details. After all, the technology must solve problems for its user and not create additional problems. But one may expect more: transitions involving situation are possible in video and digital image measurement. And manipulations of graphs can be dealt with in a dynamic way.

We give the above long and yet incomplete list of recommended facilities for graphing because they play a major role in the setting in which graphs are represented, used, or learned. They have much to do with the following list of features that contribute to students' success in graphing (Ainly, 2000):

- The presentation of a complete image allows students to take a global view of the graph, rather than focussing on separate components. So, zooming facilities are important.
- The use of a number of similar graphs encourages students to focus on similarities and differences between examples and sharpens their discrimination. The easier to graph, the more examples pupils can study.
- The ability to manipulate the graph and change its experience is closely related to the previous point. Re-representing the same data in different ways helps to emphasize features of the data.
- A familiar and/or meaningful context allows students to feel ownership of the data and to make sense of it. This will obviously be true when they have collected and recorded data themselves.
- A purposeful task in which the graph is used to solve a problem encourages active use of graphs, in which it is natural to work on reading the graph and on manipulating the graph to make it more readable, rather than seeing it as an illustration.

Conclusion

From an integrated learning environment for math and science one may expect that students and teachers can
- use a rich variety of tools to express mathematical and scientific ideas in a concrete form;
- explore ideas from various directions
- be explicit about why they are using a particular tool and method.

However, to implement these wishes is nontrivial. A complicating factor is that mathematical concepts are not always used the same in mathematics and science.

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