What Goes Up, Must Come Down

Modeling of Tidal Movement by Students

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Abstract

Everybody knows the rise and fall of the tides. A closer look at the origin of tides and the behavior of the tides in various ports reveals a wealth of interesting phenomena that may challenge students to analyze and to model. Fortunately, many data sources are now available on the Internet for prediction and measurement of the tidal movement. Each port has its characteristic tidal spectrum. When this spectrum is converted into sound, the peculiarities of the tidal movement in this port can be heard.

In this paper we present a model for the moon-earth dynamical system, yielding the tidal movement on an all-ocean world. We also look at data sources on Internet with tidal data, and reconstruct the harmonic analysis for various ports with advanced data analysis techniques. For the well-known tidal river Thames, we model the tidal behavior as a function of the tides of the North Sea.

Tides, a fascinating phenomenon

Introduction

The attraction between the Earth and another heavenly body (like the Moon or the Sun) causes them to move in an elliptical orbit about their center of gravity with opposite phases. For the Earth-Moon system, the center of gravity Z lies inside the Earth:

$$r_E = \frac{m_M}{m_E + m_M} r = 4.67 \times 10^6 \text{ m.}$$

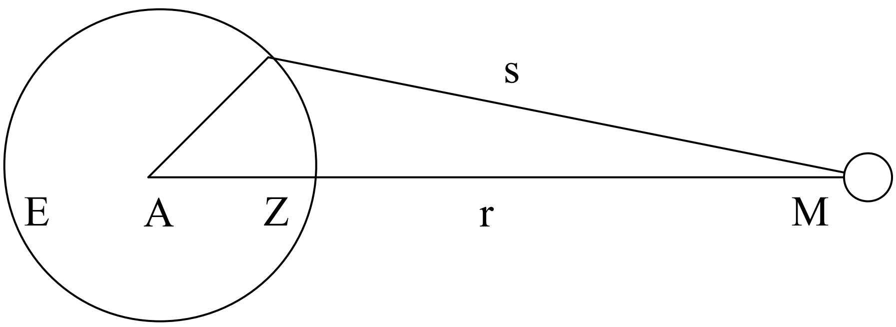
The required centripetal force for the revolution about Z at the Earth's center A is exactly produced by the gravitational force:

$$m_E \omega^2 r_E = G \frac{m_E m_M}{r^2}.$$

The angular speed ω in the above corresponds with the sidereal rotation period of the Moon of 27.32 days, the time of one complete revolution with respect to the 'fixed' stars.

Obviously, the Earth is not a point at all. Each point on the Earth moves with the same ω and r_E feeling the same centrifugal force, yet experiencing a slightly different gravitational pull from the Moon or the Sun. So, while the attractive gravitational force is exactly balanced by the centrifugal force in A, there remains for all surface points a net effect called the tidal force that by its nature only occurs for extended objects (the tidal force in A is naturally zero).

Fig. 1. Attraction of Earth surface points by the Moon, top view (the Earth seen on the North Pole).



Therefore, a surface point like a liter sea water experiences a tidal acceleration due to the discrepancy of the gravitational and the centrifugal acceleration:

$$\overrightarrow{a_{tidal}} = G \cdot m_M \left(\frac{\vec{s}}{s^3} - \frac{\vec{r}}{r^3} \right)$$

To complete this picture historically proposed and elaborated by great scientists like Newton, Bernoulli and Laplace, the Earth is simplified to an all-ocean world with the Moon in the equatorial plane. This model yields the so-called equilibrium tide that explains many tidal features such as periodicity, inequalities between successive high waters and low waters, and the occurrence of spring tides near full and new moon. Before this theory, the tides were frequently explained as being generated by the ocean breathing in and out and other unrealistic concepts.

Misconception!

At the side facing the Moon or the Sun, the attractive force wins from the centrifugal force while at the opposite side the centrifugal force is largest. Although on the one hand this explains that a particular surface point meets two high waters a day by the Earth's rotation, on the other it also suggests the widely spread misconception (even amongst physicists) that high water should be the result of the Moon's gravitational pull. But how could the *radial* component of the tidal acceleration (about $10^{-7} \cdot g$) pull this off against *g* of the Earth itself? Obviously it will disappear completely in the balance of the dominant normal (or buoyant) force and gravity. By contrast, tidal waves on Earth are determined by the remaining *tangential* components of the tidal force field called the *tractive* force. The dominant component after decomposition of $\overrightarrow{a_{tidal}}$ into radial and tangential components is calculated to be

$$a_{\varphi} = -\frac{3}{2} \frac{G \cdot m_M}{r^3} \cdot R \cdot \sin 2\varphi$$

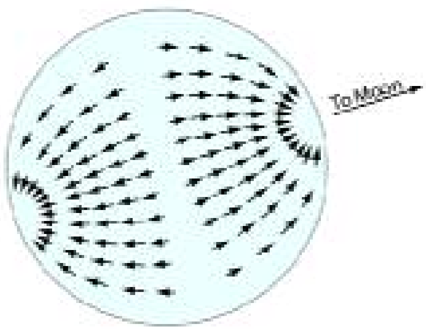
(with *R* the Earth's radius) which is maximal for $\varphi = 45^{\circ}, 135^{\circ}$. With respect to the Moon, the angle φ varies with a period given by the relative angular speed: $\varphi = \omega t = \frac{2\pi t}{T}$. As a result, the tidal force varies by

$$2\varphi = \frac{2\pi t}{\frac{1}{2} T}$$

and so by half the period *T*, explaining the periodicity of the semidiurnal (two-daily) M2 tide:

$$T = \frac{1}{2} \left(\frac{1}{23.93} - \frac{1}{27.32 \times 23.93} \right)^{-1} = 12.42 \text{ h} = 12 \text{ h } 25 \text{ min.}$$

Fig. 2. Tractive force field (image from Navy Operational Ocean Circulation and Tide Models at **www.oc.nps.navy.mil/nom/day1/partc.html**)



However, objects like the moons of Jupiter *do* experience a strong 'massage' of the radial force, which explains the existence of volcanism (Io), the presence of subterranean liquid water and even the possibility of life (Europa). If a satellite approaches its planet close enough for the radial tidal force to equal the satellite's gravity, it will be torn apart by tidal disruption. The rings of Saturn lie inside this Roche limit:

$$R_c = R_{planet}\sqrt[3]{\frac{2\rho_{planet}}{\rho_{satellite}}}\ .$$

Towards a model

We assume that for an equatorial sea channel, the work done on the water fully originates from the tidal acceleration:

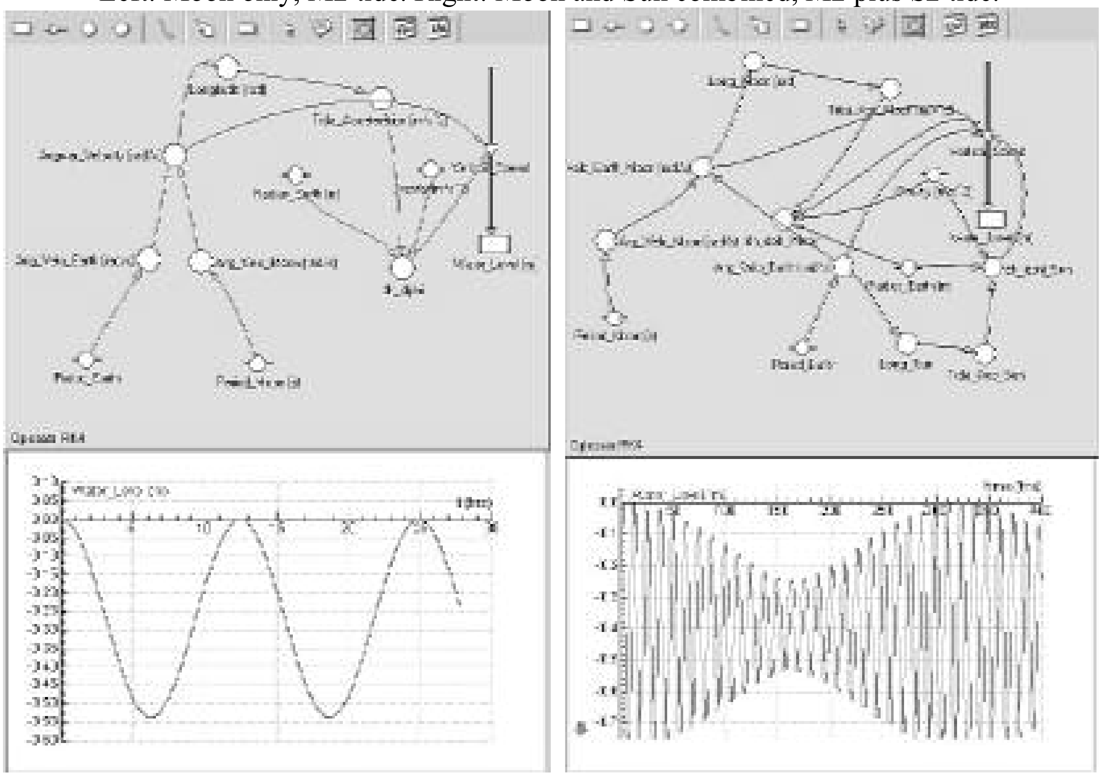
$$a_\varphi R\mathrm{d}\varphi = g\mathrm{dh}\ .$$

This immediately yields:

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \omega\frac{\mathrm{d}h}{\mathrm{d}\varphi} = \omega\frac{a_\varphi R}{g}.$$

Together with the expression for a_φ given above, this equation can now be used in a dynamical system model as shown below in Figure 3. This over-simplified all-ocean model turns out to give an M2 tidal amplitude of 27 cm. A similar model including the Sun as well yields interference effects as neap and spring tides (when Sun and Moon align).

Fig. 3. Graphical models of the all-ocean equilibrium tide.
Left: Moon only, M2 tide. Right: Moon and Sun combined, M2 plus S2 tide.



Tidal friction

The Earth does not respond immediately to the Moon's location, as energy is dissipated in making the tidal bulges. Tidal friction, prolonging the Earth day by 1.6 ms each century (dissipation of 2×10^{12} W), is strongly influenced by the distribution of land and water and thus by the continental drift. The 'flood crests' are carried along by the planet's rotation; the instantaneous tidal forces partially counteract, causing a dragging effect. The Moon withdraws from the Earth for this reason by 3.7 cm a year, due to conservation of angular momentum: as the Earth's spin diminishes, the orbital angular momentum should increase.

This process will continue until the spin and orbital periods are equal and month and day coincide! (This future month-day will actually become some 50 of our present days.)

By the same effect, the spin of the Moon itself equals its revolution period. Therefore, the Moon continuously faces the planet with the same (heavy) side, like almost every other satellite (and Mercury towards the Sun) in our solar system. Tidal dissipation thus accounts for the spin and rotation states of all celestial binaries. The endpoint of tidal evolution is a circular, synchronous orbital motion (Pluto and Charon).

Shortcomings of the model

Deviations from the above model arise because the Moon usually lies outside the equatorial plane, exhibiting a declination instead. As a result, the two height maxima during a natural day will have different amplitude: the daily inequality. In addition, the Moon moves in an elliptical rather than in a circular orbit, causing a variation in r of 10% and thus in r^3 of 30%.

The presence of land not only makes high and low water observational, it also changes the local phases and amplitudes in a dramatic way, both at sea and over land. Only the period is directly recognizable while many higher harmonics like M4, S4, and so on are produced. To describe and accurately predict tidal effects at seaports harmonic analysis is indispensable. To study the change in phase and amplitude as the tide runs up from a seaport into a convergent estuary, the behavior of a tidal wave must be modeled.

Harmonic Analysis of Tides by Students

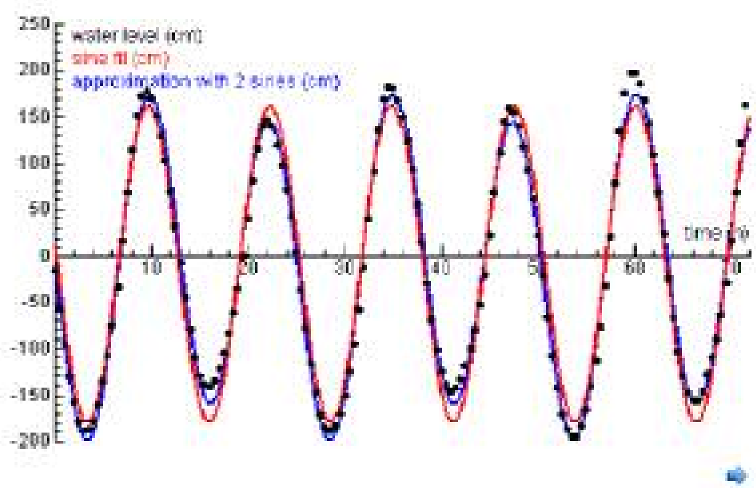
Introduction

Because of the geographic position and the size of the Netherlands every inhabitant is familiar with tides. Sea level rises and falls cyclically on a twice-daily basis and the graph of a tidal motion, measured or computed for a coastal place or an oil platform, is periodic. A periodic function can be described with sine functions. Tidal curves are always approximated in Dutch mathematics textbooks by single sine functions. Students search on Internet for tidal data of a coastal town on a certain day or the yearly average, and then they try to match the data found with a good sine fit using graphical software or a graphing calculator. It's true that limitations of this simple mathematical model are briefly discussed and that in particular the asymmetry between low and high tide is pointed at, but textbook authors do not go further than a short reference to harmonic analysis and a pointer for background information at a website like www.getij.nl. They suggest students to explore tidal motion further in practical work or a research project, but they do not give any clue of how to do this with a chance of success. A citation of Jan de Lange (2000), in a paper on studying tidal motion in the classroom, is hardly encouraging: "The students don't have the tools to find a better way of coping with the lack of symmetry of the real graph." As we shall see, the situation in 2006 has changed: in the Coach 6 environment (Mioduszewska & Ellermeijer, 2001) students have access to a state-of-the-art signal analysis tool that allows a more realistic description of tidal motion.

Harmonic analysis of tidal motion

The tides in the North Sea are semidiurnal, that is, you have (more or less) two cycles per day, the two low waters of each tidal day are almost equal in height, and the same holds for the two high waters. See the screen shot below (Fig. 4) of a diagram with the predicted tidal data (black dots) at Flushing from May 21 till May 23, 2006. In the same diagram are shown two regression curves of these data: a sine fit (red) and a better approximation with a sum of two sine functions (blue), which is determined via the Prony method for spectral analysis (Mackisack et al, 1994).

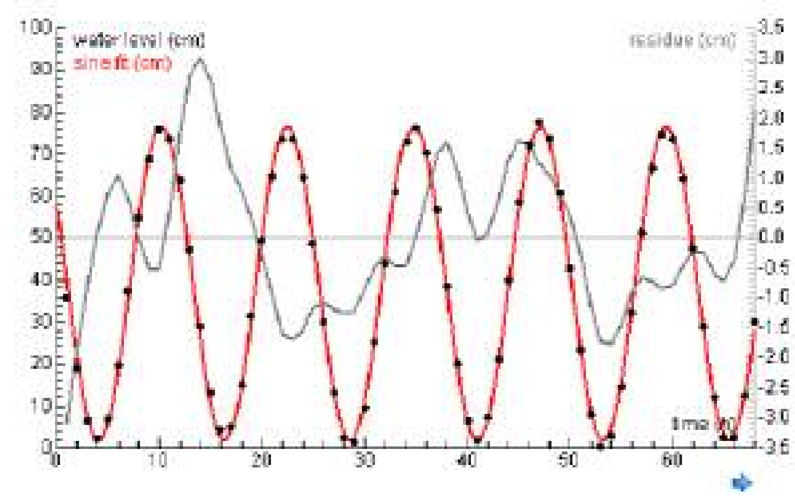
Fig. 4. Predicted tides at Flushing from May 21 till May 23, 2006, and two approximations of the tidal curve.



Tidal currents are not everywhere on earth of the same type: there are coastal areas with a diurnal cycle, i.e., with a period of approximately 24 hours, and there are locations on earth where you have two cycles per day, but the two high waters and the two low waters have marked differences in their heights.

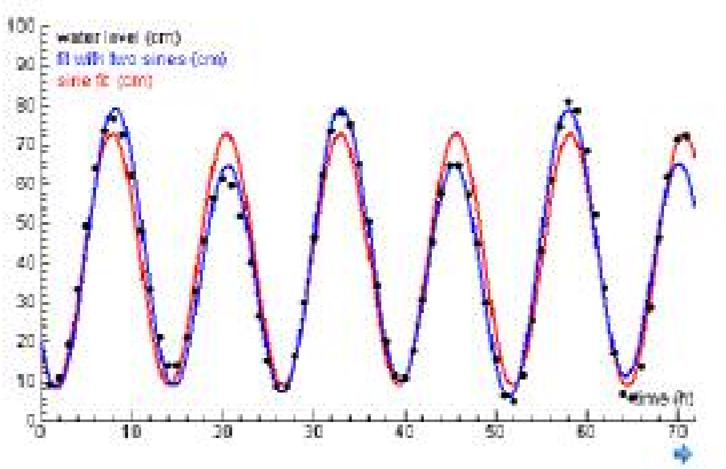
Official tide tables for almost all coastal areas on earth can be found on Internet. This enables students to find places and time periods for which the tidal model of a single sinusoid works well. You can find for example on the website www.tidesandcurrents.noaa.gov of the National Oceanic & Atmospheric Administration in the United States of America the tidal data for Sewells Point, Hampton Roads in Virginia, from March 13 till March 15, 2006. The screen shot below (Fig. 5) shows the predicted tidal data (black dots) and the best sinusoidal model (red). The graph of the differences between predicted water levels and the outcomes of the sinusoidal model (residual graph in gray, with its own coordinate system) reveals that the simple sinusoidal model is accurate up to 3 cm.

Fig. 5. Tides at Sewells Point (Virginia) from March 13 till March 15, 2006, and the tidal graph approximated with a sinusoidal model.



Do not draw the wrong conclusion that the tides at Sewells Point in Virginia are always well described by a sinusoidal model. Figure 6 shows the tidal data from March 23 till 25, 2006, an approximation with a single sine function (red) and a better approximation with a sum of two sine functions (blue), which is determined via the R-ESPRIT method for spectral analysis (Mahata, 2003).

Fig. 6. Tides at Sewells Point (Virginia) from March 23 till March 25, 2006, and two approximations with sine functions.



In order to get an adequate mathematical model of the tides with sine functions for a longer time period you must add more terms ('harmonic constituents') of the form $H\sin\left(\omega t+\phi\right)$. In Coach this can be done best via the R-ESPRIT method. In this method you can set two parameters: the snapshot dimension, which determines the length of the sequence of consecutive data that is used in the harmonic analysis, and the number of sine functions present in the mathematical model. If you model the predicted tides at Sewells Point in 2005 with 8 sine functions, then you get with the following formula with the automatically chosen snapshot dimension equal to 84:

$$\begin{aligned}&41.11+35.12\sin(28.982\,t-75.637)+6.38\sin(30.002\,t+6.794)+\\&4.97\sin(15.052t-152.698)+4.95\sin(13.946t+78.075)+\\&0.45\sin(0.200\,t+106.750)+0.96\sin(27.990\,t+92.143)+\\&0.02\sin(31.472\,t-24.033)+0.02\sin(33.216t+103.765),\end{aligned}$$

where t is the time (in hours) from the beginning of 2005 and the speed of each constituent (if you wish, the frequency of each sine function) has the in tidal analysis commonly used unit of degrees per hour. The standard deviation turns out to be about 8 cm. You can extend this model to 18 harmonic constituents by applying the same spectral analysis to the difference of the predicted tides and the approximation already found. The next eight constituents are:

$$\begin{aligned}&0.11+7.70\sin(28.438t+60.437)+4.06\sin(0.049t-177.360)+\\&1.30\sin(15.067t-37.467)+0.38\sin(57.963t-79.633)+\\&0.06\sin(29.475t-77.609)+0.05\sin(30.721t+106.212)+\\&0.02\sin(13.569t-164.1467)+0.01\sin(43.806t-43.641)\end{aligned}$$

The standard deviation of the model with 16 sine functions is about 5 cm. In Table 1 we compare the five most important contributions in our Coach model with literature values of the harmonic constituents:

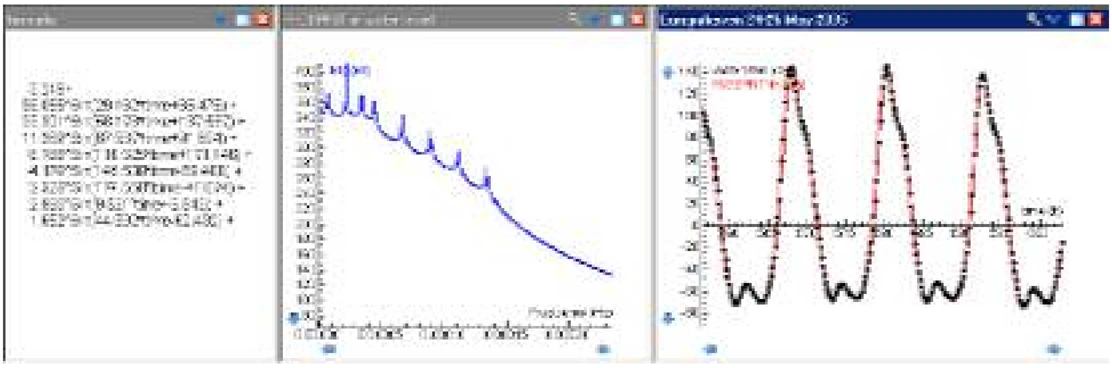
Table 1. Comparison of the principle harmonic constituents at Sewells Point (Virginia) found in Coach and provided by the National Oceanic & Atmospheric Administration in the USA.

Coach model		*Literature data*		
speed (°/h)	amplitude (cm)	speed (°/h)	amplitude (cm)	label
28.982	35.12	28.984	36.6	M2
28.438	7.70	28.440	8.1	N2
30.002	6.38	30.000	6.4	S2
15.052	4.97	15.041	4.9	K1
13.946	4.95	13.943	4.2	O1

You may call this agreement between the Coach model and the literature values astonishing. The sixth most important contribution in the model with speed 0.049 degrees per hour and amplitude 4.06 probably corresponds with the harmonic constituents labeled SA and SSA, which describe the yearly meteorological variations and their influence on the sea level. The labels M2 and S2 belong to the harmonic constituents that are linked with the motion of the moon around the earth and the motion of the earth around the sun, respectively, and which cause the semidiurnal tide. The N2 constituent takes into account the effect that the orbit of the moon around the earth is in reality not a circle but an ellipse. The diurnal constituents K1 and O1 take into account (amongst other things) the inclination of the earth's equatorial plane with respect to the plane of the moon's orbit. Most (if not all) harmonic constituents can be related to astronomical phenomena and therefore a tidal prediction is often referred to as astronomical tide. In practice, the number of constituents needed for accurate tidal predication and the amplitude, speed and phase of each harmonic constituent are often determined from tidal records of three consecutive years. As a matter of fact, the speeds are always fixed and only the amplitudes and phases of the strongest tidal constituents that have propagated to a point of interest must be determined by regression methods. Short term tidal constituents (say for data periods up to one month) are also determined by the Fourier harmonic analysis method that we use in our work. We refer to the NOAA report 'Tidal Current Analysis Procedures and Associated Computer Programs' (Zervas, 1999) for detailed information.

Tidal analysis becomes more interesting for students when they can investigate tidal motion closer at home. Figure 7 is a Coach screen shot of a tidal analysis of two consecutive days at Europahaven (May 24-25, 2005). It shows that the phenomenon of double low water (also known as agger), which means that the low water consists of two minima separated by a relatively small elevation, can be modeled well by harmonic analysis. Actually the model consists of a fundamental tidal speed and overtides, i.e., harmonic constituents with a speeds that are an exact multiple of the fundamental constituent.

Fig. 7. Screen shot of modeling double low water at Europahaven (May 24-25, 2005).



In order to get good results with the R-ESPRIT method, which are in good agreement with recorded data or official tidal predictions the choice of the snapshot dimension is important. You must determine this parameter by trial and error in case no suitable value less than or equal to 100 has been found automatically. For example, we have found for the location Roompot-Buiten in the Oosterschelde the following six constituents on the basis of tidal data from 2005 (Table 2).

The astonishing agreement between our spectral analysis with Coach and the literature data of RIKZ (2006) has been accomplished by choosing a snapshot dimension of 521 for a model with eight sine functions. The best snapshot dimension less than or equal to 100 leads to the first two constituents only. Admittedly, it is a bit of a puzzle to find a suitable snapshot dimension, but in this way students might learn to look critically at mathematical models and computed results presented to them in future.

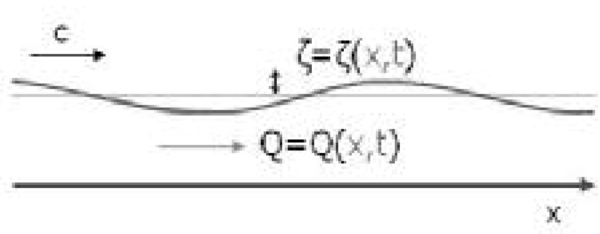
Table 2. Comparison of the principle harmonic constituents at Roompot-Buiten found with Coach and provided by the National Institute for Coastal and Marine Management.

Coach model		*Literature data*		
speed (°/h)	amplitude (cm)	speed (°/h)	amplitude (cm)	label
28.980	131.1	28.984	135	M2
29.996	36.3	30.000	36	S2
28.457	15.3	28.440	22	N2
30.098	10.5	30.082	10	K2
13.956	10.5	13.943	11	O1
27.995	4.9	27.968	11	μ2

Tidal waves in convergent estuaries

The dominant influence on the tides in estuaries is the change of water depth and estuary width as the tide propagates up the estuary. The shoaling and narrowing of the estuary slows the progress of the tidal wave, increasing its amplitude. At the same time, there is a damping effect of channel friction balancing the tidal amplification in shallow estuaries (Friedrichs & Aubrey, 1994; de Swart, 2006).

Fig. 8. Description of a tidal wave by the relative height ζ and the flow Q.



Tidal waves may be described by the de Saint-Venant equations. Using the Lorentz linearization procedure for the quadratic friction (Lorentz, 1922; Labeur, 2006), they can be simplified to a telegraphist's equation:

$$\frac{\partial^2\zeta}{\partial t^2} - c_0^2\frac{\partial^2\zeta}{\partial x^2} + \kappa\frac{\partial\zeta}{\partial t} = 0.$$

Here, $c_0 = \sqrt{gh}$ is the frictionless wave speed and κ the linearized friction factor. In a prismatic channel (constant width and depth), this yields a simple traveling wave as a solution:

$$\zeta(x,t) = A\,e^{-\mu x}\cos(\omega t - kx + \varphi).$$

Here,

$$\mu = k\tan\delta$$

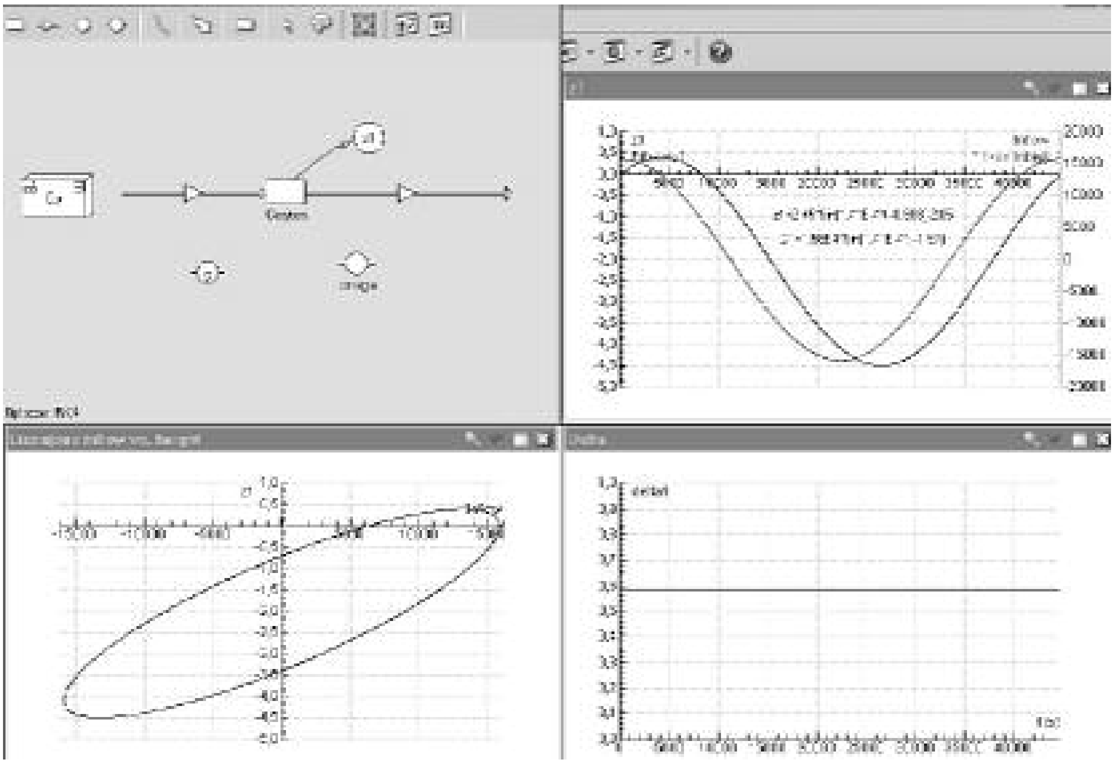
is related to a 'friction angle'

$$\delta = \tfrac{1}{2}\tan^{-1}\left(\kappa/\omega\right)$$

that in turn is determined by the friction factor κ.

In a model, we want to study the progression of the tidal amplitude and phase along an estuary like the river Thames. Following the stock-flow philosophy of graphical modeling, it seems logical to introduce bathtubs of 1m length of Thames water with given in- and outflows. The instantaneous water volume divided by the width then directly yields the height as a function of time. Not surprisingly, this height lags behind the inflow by the friction angle δ. First, we study the situation at Coryton:

Fig. 9. Graphical model of the tidal wave at Coryton.

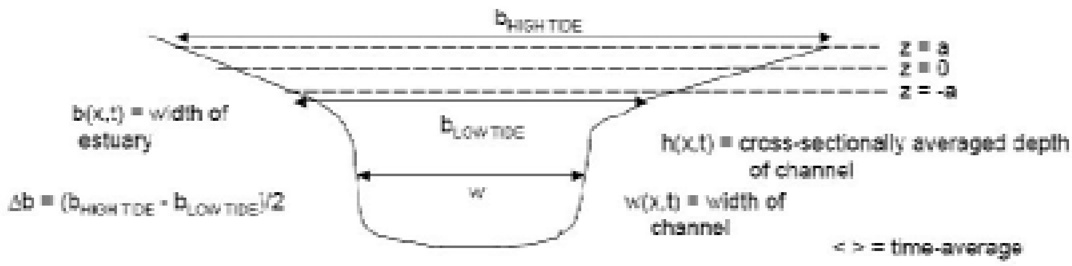


To continue, we assume an exponential behavior for the width and the average depth of the Thames as a function of distance along the river:

$$B(x) = 4000\exp(-x/25) \text{ and } h(x) = 12.5\exp(-x/79).$$

Furthermore, the river is subdivided into seven prismatic sections, each with a constant width and depth determined by the one meter bathtub in the middle. To achieve a more quantitative picture, some assumptions on the inclination of the banks are made:

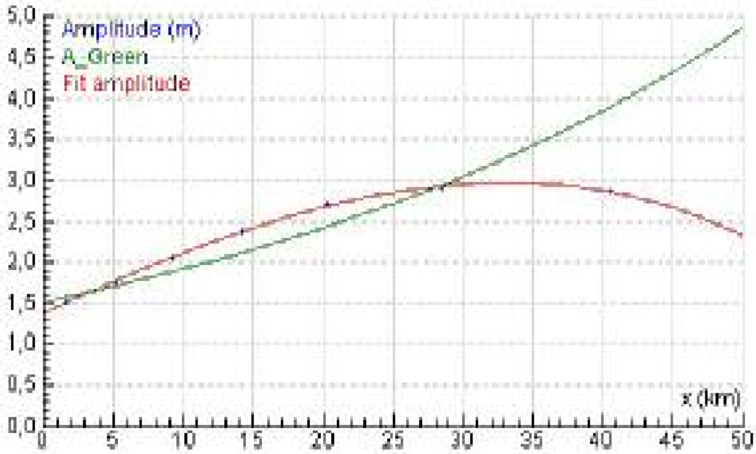
Fig. 10. Modeling of the river cross section.



Due to the flattening of the banks, the quantity ζ deviates only slightly from a pure sinusoidal function. Fitting ζ to a sine function with ω fixed to the M2 frequency yields the requested phases and amplitudes.

The full model is a straightforward extension of the above, each section being a copy of the former with corresponding changes in distance, width and depth. Obviously, phase and amplitude matching has to be applied at the transition of each section. As can be seen from Ffigure 11, both the amplitudes and the phases now turn out to correspond reasonably well with experiment. Please note that the amplitude almost linearly increases during the first 20 km, reaches a maximum at 33 km from the mouth (actually at the Tidal Barrier at 40 km) and then decreases again. The data are compared to the results of Green's classical theory for amplitude amplification, that does not take into account frictional effects.

Fig. 11. Tidal amplitudes of the river Thames as a function of distance.



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