Investigating Bridges and Hanging Chains

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Almost everywhere you can come across hanging chains and cables. Examples are necklaces, power lines, and cables that support a bridge surface. Do these cables all hang in the same mathematical shape? The first thought of many people will be: this is a parabola, isn't it? In Coach you can easily measure this on digital images. It turns out that the parabolic shape quite often occurs with bridges, but that an ordinary chain does not hang as a parabola. Can this be understood? We shall discuss our learning material and classroom experiences

Introduction

Practical assignments that pupils make by themselves with the computer form a new part of the Dutch curriculum [1]. In this work, pupils are involved in the active process of learning math & science and they use many tools: they collect and process real data, develop and run mathematical models, work with internet sources, etc. To this end, we have developed a versatile, activity-based computer learning environment, viz., Coach. Activities may contain:

- texts with activity explanations and instructions;
- pictures with illustrations of experiments, equipment, and context situation;
- video clips to illustrate phenomena or to make video-based measurements;
- measured data presented as graphs, tables, meters, or digital values;
- models (in graphical or textual mode) to describe and simulate phenomena;
- programs to control devices and to make mathematical computations;
- links to Internet sites as extra resources for students.

In addition, teachers have easy-to-use authoring tools to prepare activities. They can select and prepare texts, graphs, video clips, mathematical models, and measurement settings, and they can choose the right level according to age and skills of their pupils.

A general overview of the learning environment has been presented at ICTMT4 [2]. In this paper, we shall discuss a classroom experiment in which digital images are a basis for mathematical modelling. We believe that video and image measurement offer great opportunities to study math & science on the basis of real-world situations in challenging activities. Advantages of data video, compared to real experiments, are:

- No experimental set-up is required. This saves time and takes away many practical issues.
- Processes and objects that are difficult or impossible to measure can still be studied.
- It is not necessary to determine in advance what and how you are going to measure.
- Measurements are easy and quick. Data can be verified later on and, if desired, corrected.

The Learning Material

The learning material is designed for pupils who are in their first year of the second stage of pre-university education (age 15-16 yr.), who have no experience with practical investigation tasks, and who have not worked with Coach before. Our main objectives are to let the pupils

- work with real data collected from digital images;
- apply mathematical models to investigate shapes;
- practice ICT-skills, in particular use tools to collect data from video clips and images;
- carry out practical work in which they can apply much of their mathematical knowledge.

The main task for the pupils is to get familiar with the video tool and to use it to investigate the mathematical shape of bridges and hanging chains. Only the end of the task is a theoretical completion that involves mathematics and physics. They finish with a short investigation of a free-hanging chain, which will not curve like a parabola. English versions of assignments can be downloaded from `www.science.uva.nl/~heck/research/bridges/` Study load is 5 hours.

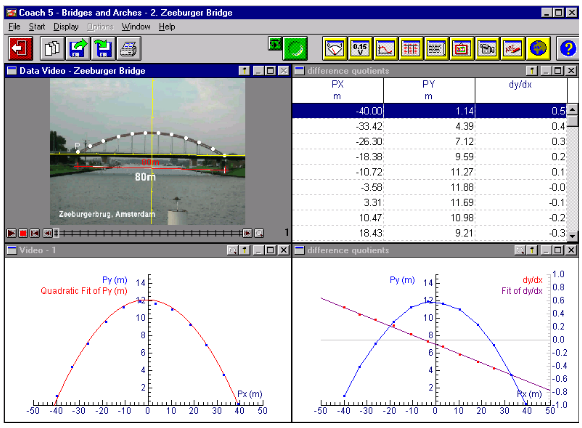

Figure 1. Analysis of the Zeeburger Bridge.

Let us describe briefly how the image measurement activity shown in the screen dump to the right goes. First of all, the activity allows collection of position data from the digital image. It is possible to place the origin of the coordinate system at any desired position and to rotate the axes, if necessary. You choose the correct scale by matching a ruler with a known distance in the image. Data are gathered by clicking on the location of points of interest. Data can be plotted and used for further analysis. In the lower-left window of Figure 1, the regression tool has been used to find the quadratic function that fits the data best. In the lower-right window, you see the collected points once more, together with the data plot of the difference quotients (dy/dx) of consecutive points (plotted with respect to a second vertical axis). The difference quotients lie approximately on a straight line. The best line fit can be found with the regression tool. The third column of the table in the upper-right window also shows clearly the pattern for the difference quotients. It follows that the shape of the arch is a well described by a parabola. But, with the regression tool you easily discover that the sinusoidal model $y = a\sin(bx + c) + d$ works as good. More is needed for good understanding!

Mathematics of the Hanging Chain

Figure 2 shows a screen dump of the activity in which the mathematical shape of a hanging chain is investigated. By collecting positions on the digital image and by trying a quadratic curve fit on the measured data, a pupil quickly finds out that the form of the chain is not a parabola (as Galilei erroneously claimed). At once, the simple question "How does a chain hang?" becomes a challenging problem.

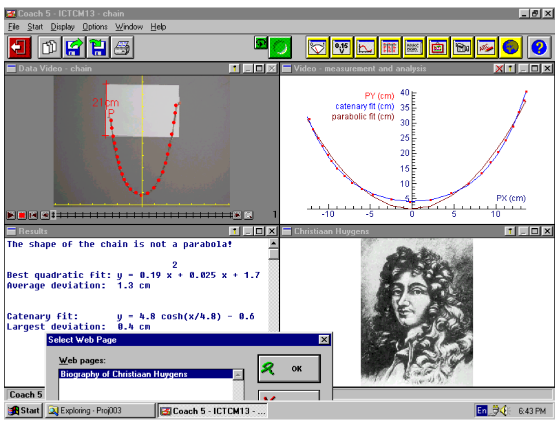

Figure 2. Analysis of a hanging chain

Pupils can follow in the footsteps of Huygens, Bernoulli, and Leibniz, who solved the problem of the catenary at the end of the 17th century, when differential calculus was discovered [3]. The function $y(x)$ that describes the vertical position of a point on the chain as a function of the horizontal displacement x satisfies the following differential equation: $cy''=\sqrt{1+(y')^2}$, for some positive constant c. If the coordinate system is chosen such that the origin equals the lowest point of the chain, the solution is as follows:

$$y(x)=c\left(\cosh(\frac{x}{c})-1\right)=c\left(\frac{e^{x/c}+e^{-x/c}}{2}-1\right).$$

Of course, pupils could try to match the data with this formula. But this would not lead to deeper understanding. They could have validated as well that the 'ideal' chain hanging under gravity with suspension points (-1,1) and (1,1), and with its minimum at (0,0) can be approximated almost perfectly by the rational formula $y=9x^2/(11-2x^2)$.

Instead, we prefer for our pupils a different approach to investigate the catenary. We let them study a related, but simpler problem: "How does a chain with five objects of equal weight symmetrically attached hang under gravity?" The case of weights at equal horizontal distances is investigated first. Measurements in the digital images reveal that the slopes of the right segments of the chain have a fixed ratio, viz., 1:3:5. Measuring in other images would convince pupils that this does not depend on the length of the segments or how far the suspension points are apart from each other. It turns out that one always has the following fixed ratio of positive slopes: 1:3:5:7:9:… Basic physics can explain this: equilibrium of forces holds at each point of application where a weight is attached. This simple observation about slopes allows computation of the shape of the system and it explains that the points of application are necessarily on a parabola.

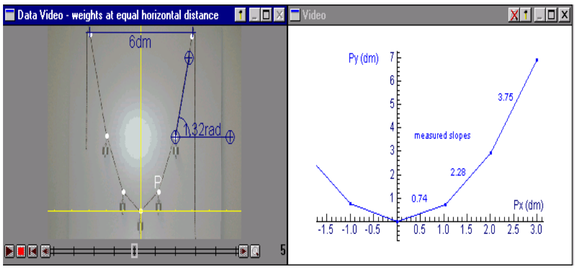

Figure 3. Measuring properties in the image.

The next step is to realise that it also follows that the curve is not a parabola in case the weights are attached at equal distances *along* the chain. Finally, the fixed ratio can be utilised to approximate the free-hanging 'ideal' chain by modelling it as a string of equidistant beads. Simulation runs with various values of the slope between the lowest bead and its right neighbour can be used to produce a curve that fits the hanging chain.

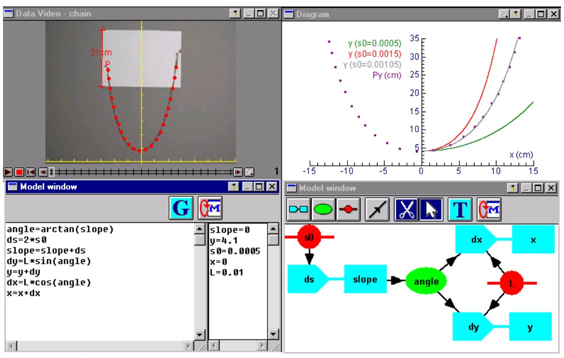

Figure 4. Modelling the hanging chain.

Classroom Experiences

The learning material has been tested in a class of 24 pupils. They worked by themselves in pairs, mainly in a computer lab, and they had to hand in the report of their work (written with a text editor), a questionnaire, and a diskette with their Coach activities and results. Together with the classroom observations and video recordings (including video capturing of computer work), these materials give us an impression of what the pupils actually do and how they experience the work. Below, we list our main findings (see the ICTMT5 CD-rom for details).

Most pupils report that they like the practical work. For example, Jordi and Wester write: “It is more fun than the regular lessons, because the method is different from usual and more varied. Here you have to discover more by yourself.” Bianca and Alexandra comment as follows: “It is fun and it makes a change. It was not really difficult, not even the theoretical part. You must ‘see it’, and then it is easy. Please, do this more often.”

Our classroom observations and the submitted reports confirm the positive reactions of the pupils. Their enthusiasm is great. Most pupils work without stops and we do not notice any negative attitude to work. They do their best to get accurate data in measurements. Pupils collaborate well and work rather efficiently. They get quickly familiar with Coach and the video tool. Working with graphs and measuring positions, angles and slopes in digital images cause no problems

Some pupils are a bit disappointed about the mathematical contents. They find that the assignments focus too much on Coach or on physics, and that the tasks have less to do with mathematics. In retrospect, we must admit that they are right. We could have gone further into mathematical modelling and we could have paid more attention to the role of symmetry and the coordinate system in the problems. Especially the effect that a change of the coordinate system has on the data and on the formulas would have been an interesting topic.

Some pupils misunderstand the question about the pattern in the angles and/or slopes of the chain segments. They write down that larger angle implies larger slope, and that the tangent of the angle is equal to the slope. They are right of course, but this was not the authors’ intention. Most pupils do not find by themselves the pattern from their collected data. They find the pattern first in the standard parabola $y = x^2$, and then check if this also occurs in the measured data. Most pupils need help with the theoretical explanation.

In summary, we are quite satisfied by the motivation and performance of the pupils. We also think that the discrete modelling approach offers an opening to the investigation of other systems of masses acting under gravity on a rope and that the scope of investigation can be broadened to anchor catenaries [4], shapes of suspension bridges, and to architectural structures. Such activities would illustrate the use of common mathematical shapes and functions and it would reinforce some of the ideas of calculus. But maybe more important, it would bring the real world into mathematics lessons in an attractive way.

References

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[2] A. Heck. Coach: an environment where mathematics meets science and technology. In: *Proceedings ICTMT4*, Plymouth, 1999, W. Maull & J. Sharp (eds.). CD-ROM published by the Univ. of Plymouth, 2000. URL: `http://www.tech.plymouth.ac.uk/maths/CTMHOME/ictmt4/P75_Heck.pdf`

[3] H.J.M. Bos. *The Calculus in the Eighteenth Century II: Techniques and Applications.* The Open University Press, 1975.

[4] J. Lamb. Use only as much chain as you need. *Practical BoatOwner* 407: 117-119, 2000.