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# Involutive automorphisms of root systems

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## §1. Introduction and notation.

In this paper we study involutive automorphisms of reduced root systems using the following notations and definitions (patterned after those in [**3**], [**5**]).

Let $\Delta$ be a reduced root system spanning a finite-dimensional Euclidean space $E$ with Weyl group invariant inner product $(\cdot|\cdot)$. Let $\Pi$ be a fundamental system of $\Delta$. We endow the space $E$ with a partial ordering $\geqq$ with respect to $\Pi$: for $\alpha$, $\beta$ in $E$ $\alpha\geqq\beta$ if $\alpha-\beta$ is a linear combination of roots in $\Pi$ with integral non-negative coefficients. Since the Weyl group $W$ of $\Delta$ acts simply transitively on the set of fundamental systems of $\Delta$, there exists a unique element $w_\Pi$ in $W$ such that $w_\Pi(\Pi)=-\Pi$. The automorphism $\mathrm{op}_\Pi$ defined by $\mathrm{op}_\Pi:=-w_\Pi$ is called the *opposition involution* of $\Delta$ with respect to $\Pi$.

Now let $\sigma$ be an involutive automorphism of $\Delta$; denote its linearization to a transformation of $E$ by $\sigma$ too. We renorm the space $E$ in such a way that $\sigma$ extends to a congruence of $E$. We can decompose $E$ into a direct sum of subspaces $E_0:=\{\alpha\in E\mid\sigma\alpha=-\alpha\}$ and $\bar{E}:=\{\alpha\in E\mid\sigma\alpha=\alpha\}$. Let $^{-}: E\ni\alpha\mapsto\bar{\alpha}\in\bar{E}$ be the canonical projection of $E$ onto $\bar{E}$ with respect to $E_0$. We define $\Delta_0:=\Delta\cap E_0$, $\Pi_0:=\Pi\cap\Delta_0$, $\bar{\Delta}:=\{\bar{\alpha}\mid\alpha\in\Delta\setminus\Delta_0\}$ and $\bar{\Pi}:=\{\bar{\rho}\mid\rho\in\Pi\setminus\Pi_0\}$ ; $\bar{\Delta}$ is called the *system of restricted roots*. The set $\tilde{\Delta}:=\{\psi\in\bar{\Delta}\mid\psi$ is not of the form $c\eta$ with $\eta\in\bar{\Delta}$, $c\in\boldsymbol{R}$, $c>1\}$ is the *system of reduced restricted roots*. In general neither $\bar{\Delta}$ nor $\tilde{\Delta}$ is a root system.

We call $\Pi$ *$\sigma$-fundamental* if $\sigma\rho>0$ for each root $\rho$ in $\Pi\setminus\Pi_0$. Throughout this paper we will assume $\Pi$ to be a $\sigma$-fundamental system of $\Delta$ and call the corresponding partial ordering of $E$ a *$\sigma$-ordering*. In §2 we state some basic properties of $\sigma$-fundamental systems and we will also give a diagrammatic description of the action of the involutive automorphism $\sigma$ on $\Delta$ by introduction of a so-called *Satake diagram* of $\sigma$ with respect to a $\sigma$-fundamental system $\Pi$.

We define $W_\sigma:=\{w\in W\mid w\circ\sigma=\sigma\circ w\}$, $\bar{w}$ :=the restriction of an element $w$ in $W_\sigma$ to $\bar{E}$ and $\bar{W}:=\{\bar{w}\mid w\in W_\sigma\}$. Schattschneider [**6**] studied the action of a general automorphism group $G$ on the root system of a semisimple algebraic group and determined under which conditions $\tilde{\Delta}$ is a root system with Weyl group $\bar{W}$. In §3 we will give easier proofs of these results in our less general context of $G=\{1, \sigma\}$. In §4 we show that the property of $\tilde{\Delta}$ being a root system with Weyl group $\overline{W}$ is equivalent to the simple transitive action of $W_\sigma$ on the set of $\sigma$-fundamental systems of $\Delta$. This solves a problem posed by Hirai [4].

We call a Satake diagram *admissible* if it belongs to an involutive automorphism of $\Delta$ such that $\tilde{\Delta}$ is a root system with Weyl group $\overline{W}$. In §5 we classify all admissible Satake diagrams. We remark that most admissible Satake diagrams arise from so-called *$\sigma$-normal* root systems and determine which admissible Satake diagrams belong to $\sigma$-normal root systems. Our method of determining the Satake diagrams that belong to $\sigma$-normal root systems leads to a modification of Araki's method of classifying real simple Lie algebras [**1**]: instead of explicit construction of all real simple Lie algebras characterizing Satake diagrams out of the diagrams of restricted rank 1, we can use the results for restricted rank 1 merely to check if a Satake diagram belonging to a $\sigma$-normal root system characterizes a real simple Lie algebra. This last step in the classification problem is easy and left to the reader.

In general a Satake diagram of an involutive automorphism $\sigma$ with respect to a $\sigma$-fundamental system $\Pi$ depends on the choice of $\Pi$. In §6 we show that the Satake diagram does not depend on the choice of its $\sigma$-fundamental system iff the system $\tilde{\Delta}$ of reduced restricted roots is a root system (not necessarily with Weyl group $\overline{W}$). Furthermore it is proved that if $\tilde{\Delta}$ is a root system, then in almost all cases $\overline{W}$ is the Weyl group of $\tilde{\Delta}$; ○——● is the prototype of a Satake diagram for which $\overline{W}$ is not the Weyl group of $\tilde{\Delta}$.

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## §2. $\sigma$-fundamental systems and Satake diagrams.

In this section we will describe some properties of $\sigma$-fundamental systems and introduce Satake diagrams. Proofs are omitted as the reader can easily convince himself of the correctness of the results or find complete proofs in [**5**].

PROPOSITION 2.1. *The following conditions are equivalent:*

(i) *$\Pi$ is $\sigma$-fundamental.*

(ii) *If $\alpha \in E \setminus E_0$ and $\alpha > 0$, then $\sigma\alpha > 0$.*

(iii) *If $\alpha, \beta \in E \setminus E_0$ and $\alpha, \beta > 0$, then $\alpha + \beta \notin E_0$.*

(iv) *If $\alpha, \beta \in E \setminus E_0$, $\alpha - \beta \in E_0$, $\alpha > 0$ and $\beta$ is a linear combination of roots in $\Pi$ with integral coefficients, all $\geqq 0$ or all $\leqq 0$, then $\beta > 0$.*

Define $l := \dim E$, $\bar{l} := \dim \bar{E}$, $l_0 := \#\Pi_0$, $l_1 := 2\bar{l} + l_0 - l$ and $l_2 := l - l_0 - \bar{l}$.

PROPOSITION 2.2. *In a suitable numbering of the $\sigma$-fundamental system $\Pi$ the involutive automorphism $\sigma$ acts as follows:*

$$\begin{array}{ll}\sigma\rho_j=-\rho_j & \textit{for } j=1, 2, \cdots, l_0 : \ \Pi_0=\{\rho_1, \rho_2, \cdots, \rho_{l_0}\}, \\ \sigma\rho_j\equiv\rho_j \bmod \boldsymbol{Z}\Pi_0 & \textit{for } j=l_0+1, \cdots, l_0+l_1, \\ \sigma\rho_j\equiv\rho_{j+l_2} \bmod \boldsymbol{Z}\Pi_0 & \textit{for } j=l_0+l_1+1, \cdots, l_0+\bar{l}, \\ \sigma\rho_j\equiv\rho_{j-l_2} \bmod \boldsymbol{Z}\Pi_0 & \textit{for } j=l_0+\bar{l}+1, \cdots, l.\end{array}$$

PROPOSITION 2.3. *$\bar{\Pi}$ is a basis of $\bar{E}$ and each element in $\bar{\Delta}$ is a linear combination of elements in $\bar{\Pi}$ with integral coefficients, all $\geqq 0$ or all $\leqq 0$.*

PROPOSITION 2.4. *Let $\Pi$, $\Pi'$ be $\sigma$-fundamental systems of $\Delta$. Then $\Pi=\Pi'$ iff $\Pi_0=\Pi_0'$ and $\bar{\Pi}=\bar{\Pi}'$.*

PROPOSITION 2.5. *$\Delta_0$ is a root system with fundamental system $\Pi_0$.*

Let $W_0$ be the Weyl group of the root system $\Delta_0$ and consider $W_0$ as a subgroup of the Weyl group $W$ of $\Delta$. Number the $\sigma$-fundamental system $\Pi$ of $\Delta$ as indicated in Proposition 2.2. In the sequel $w_0$ instead of $w_{\Pi_0}$ denotes the unique element in $W_0$ such that $w_0\Pi_0=-\Pi_0$. Define $\tilde{\sigma}:=w_0\circ\sigma$; then $\tilde{\sigma}\in\mathrm{Aut}(\Pi)$, $\tilde{\sigma}^2=1$, $\tilde{\sigma}$ leaves $\Pi_0$ invariant and the action of $\tilde{\sigma}$ on $\Pi\setminus\Pi_0$ is given by

$$\begin{array}{ll}\tilde{\sigma}\rho_j=\rho_j & \text{for } j=l_0+1, \cdots, l_0+l_1, \\ \tilde{\sigma}\rho_j=\rho_{j+l_2} & \text{for } j=l_0+l_1+1, \cdots, l_0+\bar{l}, \\ \tilde{\sigma}\rho_j=\rho_{j-l_2} & \text{for } j=l_0+\bar{l}+1, \cdots, l.\end{array}$$

We define the *Satake diagram* of $\sigma$ with respect to the $\sigma$-fundamental system $\Pi$ as the triple $(\Pi, \Pi_0, \tilde{\sigma})$. It is pictured as the Dynkin diagram of $\Pi$ (using white circles for the vertices) in which vertices that represent roots in $\Pi_0$ are coloured black and the action of $\tilde{\sigma}$ on $\Pi\setminus\Pi_0$ is marked by arrows, i.e. if $\rho$, $\rho'\in\Pi\setminus\Pi_0$ and $\tilde{\sigma}\rho=\rho'$, then the action of $\tilde{\sigma}$ is indicated by arrows $\underset{\rho}{\circ}\overset{\frown}{\longleftrightarrow}\underset{\rho'}{\circ}$ (cf. [5], [8]). We call two Satake diagrams $(\Pi, \Pi_0, \tilde{\sigma})$ and $(\Pi', \Pi_0', \tilde{\sigma}')$ isomorphic if there exists a map $\phi : \Pi\rightarrow\Pi'$ satisfying the following conditions:

( i ) $\phi$ is an isomorphism between the Dynkin diagrams of $\Pi$ and $\Pi'$.

(ii) $\phi\Pi_0=\Pi_0'$.

(iii) $\phi\circ\tilde{\sigma}=\tilde{\sigma}'\circ\phi$.

This is equivalent to the existence of an isomorphism $\phi$ from $\Pi$ to $\Pi'$ such that $\phi\circ\sigma=\sigma'\circ\phi$.

## § 3. The restriction $\bar{W}$ of the Weyl group.

Schattschneider [6] studied the action of a general automorphism group $G$ on the root system $\Delta$ of a semisimple algebraic group. In the Euclidean space $E$ spanned by the roots she defines the subspaces $E_0:=\{\alpha\in E \mid \sum_{\sigma\in G}\sigma\alpha=0\}$ and $\bar{E}:=\{\alpha\in E \mid \sigma\alpha=\alpha \text{ for all } \sigma \text{ in } G\}$. The definitions of the system $\bar{\Delta}$ of restricted roots and the system $\tilde{\Delta}$ of reduced restricted roots are the same as given in the first section. A total ordering $\geqq$ of $E$ is called an $E_0$-adapted ordering if $\alpha$, $\beta \in E \setminus E_0$ and $\alpha, \beta > 0$ imply $\alpha + \beta \notin E_0$ (cf. Proposition 2.1, (iii)). The pair $(E, E_0)$ is called admissible if for each $E_0$-adapted ordering of $E$ with corresponding fundamental system $\Pi$ the restricted fundamental system $\bar{\Pi}$ is a basis of $\bar{E}$ (note that $\sigma$-fundamental systems according to our definition satisfy the latter condition by Proposition 2.3). $\bar{W}$ is the group of restrictions to $\bar{E}$ of elements in the Weyl group $W$ of $\Delta$ that leave $\bar{E}$ invariant. For admissible pairs $(E, E_0)$ Schattschneider determines necessary and sufficient conditions for $\tilde{\Delta}$ to be a root system with Weyl group $\bar{W}$. We will give easier proofs of these results in our less general context of $G = \{1, \sigma\}$.

PROPOSITION 3.1. $\bar{W} \cong W_\sigma / W_0$.

PROOF. In view of the first isomorphism theorem it is sufficient to show that $W_0$ is the kernel of the surjective homomorphic map $^-: W_\sigma \ni w \mapsto \bar{w} \in \bar{W}$. Since $(\bar{E} \mid E_0) = \{0\}$, $W_0$ acts trivially on $\bar{E}$ and hence $W_0 \subseteqq$ the kernel of the map $^-$. Now let $w$ be an element in $W_\sigma$ such that $\bar{w} = 1_{\bar{E}}$. Since $w$ leaves each vector in $\bar{E}$ invariant, $w$ can be written as a product of reflections in roots orthogonal to $\bar{E}$ (Bourbaki [3], ch. V, § 3, Proposition 1). So $w$ can be written as a product of reflections in roots in $E_0$, i.e. $w \in W_0$. Thus the kernel of the map $^- \subseteqq W_0$.

For $\gamma$ in $\bar{\Pi}$ we define $\Pi^\gamma := \{\rho \in \Pi \setminus \Pi_0 \mid \bar{\rho} = \gamma\}$, $\Pi_\gamma := \Pi^\gamma \cup \Pi_0$ and $\Delta_\gamma :=$ $\Delta \cap$(linear span of $\Pi_\gamma$). Let $W_\gamma$ be the Weyl group of the root system $\Delta_\gamma$, understood as a subgroup of the Weyl group $W$ of $\Delta$, and write $w_\gamma$ instead of $w_{\Pi_\gamma}$.

LEMMA 3.2. *The following conditions are equivalent for* $\gamma$ *in* $\bar{\Pi}$:

( i ) $\bar{W}$ *contains the reflection* $S_\gamma$ *in the hyperplane orthogonal to* $\gamma$.

(ii) $w_\gamma \in W_\sigma$ *and* $\bar{w}_\gamma = S_\gamma$.

(iii) *The opposition involution* $\mathrm{op}_{\Pi_\gamma}$ *of* $\Delta_\gamma$ *with respect to* $\Pi_\gamma$ *leaves* $\Pi_0$ *invariant.*

PROOF. (i)⇒(ii): let $w$ be an element in $W_\sigma$ such that $\bar{w} = S_\gamma$. Then $w$ leaves all vectors in $\bar{E} \cap \gamma^\perp$ invariant and consequently $w$ can be written as a product of reflections in roots orthogonal to $\bar{E} \cap \gamma^\perp$. $(\bar{E} \cap \gamma^\perp)^\perp = E_0 + \boldsymbol{R}\gamma$ and $(E_0 + \boldsymbol{R}\gamma) \cap \Delta = \Delta_\gamma$. Therefore $w \in W_\gamma$. Since $w\Pi_0$ and $-\Pi_0$ are fundamental systems of the root system $\Delta_0$, there exists a unique element $w'$ in $W_0$ such that $w'w\Pi_0 = -\Pi_0$. Then $w'w\Pi_\gamma$ and $-\Pi_\gamma$ are $\sigma$-fundamental systems of $\Delta_\gamma$ satisfying $(w'w\Pi_\gamma)_0 = (-\Pi_\gamma)_0$ and $\overline{(w'w\Pi_\gamma)} = \overline{(-\Pi_\gamma)}$. By Proposition 2.4 $w'w\Pi_\gamma = -\Pi_\gamma$, whence $w'w = w_\gamma$. In particular $w_\gamma \in W_\sigma$ and $\bar{w}_\gamma = S_\gamma$. The implication (ii)⇒(i) is obvious. (ii)⇒(iii): if $w_\gamma \in W_\sigma$, then $w_\gamma \Pi_0 = -\Pi_0$ and therefore $\Pi_0$ is invariant under the action of the opposition involution $\mathrm{op}_{\Pi_\gamma}$ $(= -w_\gamma)$. (iii)⇒(ii): suppose that $\Pi_0$ is invariant under the action of the opposition involution $\mathrm{op}_{\Pi_\gamma}$, i.e. $w_\gamma \Pi_0 = -\Pi_0$. We need to show that $w_\gamma$ and $\sigma$ commute or, equivalently, that $w_\gamma$ leaves $\bar{E}$ invariant. Since $\Delta_\gamma = \Delta \cap [(\bar{E} \cap \gamma^\perp)^\perp]$, $w_\gamma$ leaves $\bar{E} \cap \gamma^\perp$ invariant and therefore it suffices to prove the $w_\gamma$-invariance of $\boldsymbol{R}\gamma$. Define $E':=$linear span of $\Delta_\gamma$ and $E'_0:=\{\alpha\in E'\mid\sigma\alpha=-\alpha\}$. We show that $w_\gamma$ leaves $E'_0$ invariant for then $\boldsymbol{R}\gamma$ is $w_\gamma$-invariant too. We distinguish two cases: (a) $\sharp\Pi^\gamma=1$: $E'_0$ is the linear span of $\Pi_0$ and so $E'_0$ is $w_\gamma$-invariant. (b) $\sharp\Pi^\gamma=2$, say $\Pi^\gamma=\{\rho, \rho'\}$: then $E'_0=\boldsymbol{R}(\rho-\rho')+$the linear span of $\Pi_0$. Since $w_\gamma\Pi_0=-\Pi_0$ and $w_\gamma(\rho-\rho')=\pm(\rho-\rho')$, $E'_0$ is $w_\gamma$-invariant.

We now come to the main theorem on the restriction $\overline{W}$ of the Weyl group.

THEOREM 3.3. *The following conditions are equivalent:*

(i) *The opposition involution* $\mathrm{op}_{\Pi_\gamma}$ *of* $\Delta_\gamma$ *with respect to* $\Pi_\gamma$ *leaves* $\Pi_0$ *invariant for all* $\gamma$ *in* $\overline{\Pi}$.

(ii) *If* $\phi\in\tilde{\Delta}$, *then* $S_\phi\in\overline{W}$.

(iii) $\overline{W}$ *is generated by* $\{S_\gamma\mid\gamma\in\overline{\Pi}\}$.

(iv) $\tilde{\Delta}$ *is a root system in* $\overline{E}$ *with Weyl group* $\overline{W}$ *and fundamental system* $\overline{\Pi}$.

PROOF. We only prove the implication (ii)$\Rightarrow$(iii) for the rest of the proof is merely a slight modification of Schattschneider's proof of Theorem 2.6 in [**6**]. (ii)$\Rightarrow$(iii): suppose $S_\phi\in\overline{W}$ for all $\phi$ in $\tilde{\Delta}$; then in particular $S_\gamma\in\overline{W}$ for all $\gamma$ in $\overline{\Pi}$. By Lemma 3.2 $w_\gamma\in W_\sigma$ for all $\gamma$ in $\overline{\Pi}$. Let $W'_\sigma$ be the subgroup of $W_\sigma$ generated by $\{w_\gamma\mid\gamma\in\overline{\Pi}\}\cup\{S_\rho\mid\rho\in\Pi_0\}$. According to Proposition 3.1 it is sufficient to prove $W_\sigma=W'_\sigma$, so what remains is the proof of the inclusion $W_\sigma\subseteqq W'_\sigma$. Let $w\in W_\sigma$; by induction on the length $l(w)$ we show $w\in W'_\sigma$ (for properties of the length function see Bourbaki [**3**], ch. IV, § 1). If $l(w)>0$, then there exists a root in $\Pi$ that is mapped by $w$ into a negative root. We distinguish two cases: (a) There exists an element $\rho$ in $\Pi_0$ such that $w\rho<0$: define $w_1:=S_\rho$; then $l(ww_1)=l(w)-1<l(w)$. (b) There exists no root in $\Pi_0$ that is mapped by $w$ into a negative root: this implies that $w\Pi_0=\Pi_0$. Let $\gamma$ be an element in $\overline{\Pi}$ and $\rho$ a root in $\Pi^\gamma$ such that $w\rho<0$. First we show that $w\tilde{\rho}<0$ for each root $\tilde{\rho}$ in $\Pi^\gamma$; for this we only have to consider the case $\sharp\Pi^\gamma=2$, say $\Pi^\gamma=\{\rho, \rho'\}$ and $w\rho<0$. From $w\in W_\sigma$, $w\Pi_0=\Pi_0$ and $\rho'\equiv\sigma\rho \bmod \boldsymbol{Z}\Pi_0$ follows that $w\rho'\equiv\sigma(w\rho) \bmod \boldsymbol{Z}\Pi_0$. Since $w\rho\notin\Delta_0$, $w\rho<0$ and $<$ is a $\sigma$-ordering of $E$, $\sigma(w\rho)<0$ and hence $w\rho'<0$. So $w\in W_\sigma$, $w\Pi_0=\Pi_0$ and $w\rho<0$ for all $\rho$ in $\Pi^\gamma$. Now define $w_1:=w_0w_\gamma$; then $w_1\in W'_\sigma$, $l(w_1)>0$, $w_1$ leaves $\Pi_\gamma$ invariant and $l(ww_1)<l(w)$:

$$\begin{aligned}l(ww_1)&=\sharp\{\alpha\in\Delta^+\mid ww_1\alpha<0\}=\sharp\{\alpha\in\Delta^+\setminus\Delta_\gamma\mid ww_1\alpha<0\}+\sharp\{\alpha\in\Delta_\gamma^+\mid ww_1\alpha<0\}\\&=\sharp\{\alpha\in\Delta^+\setminus\Delta_\gamma\mid ww_1\alpha<0\}=\sharp\{\alpha\in\Delta^+\setminus\Delta_\gamma\mid w\alpha<0\}\\&=\sharp\{\alpha\in\Delta^+\mid w\alpha<0\}-\sharp\{\alpha\in\Delta_\gamma^+\mid w\alpha<0\}=l(w)-\sharp\{\alpha\in\Delta_\gamma^+\mid w\alpha<0\}<l(w).\end{aligned}$$

In both cases (a) and (b) we have defined an element $w_1$ in $W'_\sigma$ such that $l(ww_1)<l(w)$ and $ww_1\in W_\sigma$. By induction $ww_1\in W'_\sigma$ and so $w\in W'_\sigma$.

If $\tilde{\Delta}$ is a root system with Weyl group $\overline{W}$, the next problem is the determination of its type. Of course direct projection of all roots in $\Delta\setminus\Delta_0$ onto elements of $\bar{\Delta}$ and further restriction to the system $\tilde{\Delta}$ of reduced restricted roots is one way of determining the type of root system $\tilde{\Delta}$, but this brute force method is less attractive than the method described by Borel and Tits [2], ch. 6.

## § 4. **Hirai's problem.**

Hirai [4] has posed the question whether $W_\sigma$ acts simply transitively on the set of $\sigma$-fundamental systems of $\Delta$. In general the answer will be negative, but we will determine necessary and sufficient conditions for simple transitivity of $W_\sigma$.

LEMMA 4.1. *Let $w$ be an element in $W$ such that $w\Pi$ is a $\sigma$-fundamental system of $\Delta$. Then $\Pi$ is $w^{-1}\sigma w$-fundamental and the Satake diagrams $(w\Pi, (w\Pi)_0, \tilde{\sigma})$ and $(\Pi, \Pi_0, \widetilde{w^{-1}\sigma w})$ are isomorphic.*

PROOF. If $\rho\in\Pi$ and $(w^{-1}\sigma w)\rho\neq-\rho$, then $\sigma w\rho\neq-w\rho$. Since $w\Pi$ is $\sigma$-fundamental, $\sigma w\rho>0$ (with respect to the partial ordering corresponding to $w\Pi$). So $(w^{-1}\sigma w)\rho>0$ (with respect to the partial ordering corresponding to $\Pi$). Thus $\Pi$ is $w^{-1}\sigma w$-fundamental. Clearly the Satake diagrams $(w\Pi, (w\Pi)_0, \tilde{\sigma})$ and $(\Pi, \Pi_0, \widetilde{w^{-1}\sigma w})$ are isomorphic by the map $w^{-1}: w\Pi\to\Pi$.

Consider a fixed $\sigma$-fundamental system $\Pi$ of $\Delta$. Define $\Sigma_\Pi:=\{\sigma'\in\mathrm{Aut}(\Delta)\mid(\sigma')^2=1,\ \Pi$ is $\sigma'$-fundamental$\}$ and $K_\sigma:=\{w^{-1}\sigma w\mid w\in W\}$. It follows from Lemma 4.1 that the set of Satake diagrams of $\sigma$ with respect to one or another $\sigma$-fundamental system equals the set of Satake diagrams of involutions in $\Sigma_\Pi\cap K_\sigma$ with respect to the fixed $\sigma$-fundamental system $\Pi$.

LEMMA 4.2. *$W_\sigma$ acts simply transitively on the set of $\sigma$-fundamental systems iff $\Sigma_\Pi\cap K_\sigma=\{\sigma\}$.*

PROOF. This is an immediate consequence of the simple transitive action of the Weyl group $W$ on the set of fundamental systems, Lemma 4.1 and the definitions of $\Sigma_\Pi$ and $K_\sigma$.

LEMMA 4.3. *If $\Sigma_\Pi\cap K_\sigma=\{\sigma\}$, then $\tilde{\Delta}$ is a root system with Weyl group $\overline{W}$.*

PROOF. First we show that $\Pi$ is $w_\gamma^{-1}\sigma w_\gamma$-fundamental for all $\gamma$ in $\bar{\Pi}$. Let $\rho\in\Pi\setminus\Pi_\gamma$. Then $w_\gamma\rho\equiv\rho \bmod \boldsymbol{Z}\Pi_\gamma$. As a consequence of the $\sigma$-invariance of $\Delta_\gamma$ $\sigma w_\gamma\rho\equiv\sigma\rho \bmod \boldsymbol{Z}\Pi_\gamma$, whence $(w_\gamma^{-1}\sigma w_\gamma)\rho\equiv\sigma\rho \bmod \boldsymbol{Z}\Pi_\gamma$; $\sigma\rho\notin$ linear span of $\Pi_\gamma$ and $\sigma\rho>0$, so $(w_\gamma^{-1}\sigma w_\gamma)\rho>0$. Now let $\rho\in\Pi_\gamma$. Then $w_\gamma\Pi_\gamma=-\Pi_\gamma$ implies $w_\gamma\rho<0$. Since $\Pi$ is $\sigma$-fundamental, $(\sigma w_\gamma)\rho<0$ if $w_\gamma\rho\notin\Delta_0$ and $(\sigma w_\gamma)\rho>0$ if $w_\gamma\rho\in\Delta_0$; in the latter case $\sigma w_\gamma\rho=-w_\gamma\rho$. Thus $(w_\gamma^{-1}\sigma w_\gamma)\rho>0$ if $w_\gamma\rho\notin\Delta_0$ and $(w_\gamma^{-1}\sigma w_\gamma)\rho=-\rho$ if $w_\gamma\rho\in\Delta_0$. We conclude that $(w_\gamma^{-1}\sigma w_\gamma)\rho>0$ or $(w_\gamma^{-1}\sigma w_\gamma)\rho=-\rho$ for all $\rho$ in $\Pi$ and $\gamma$ in $\bar{\Pi}$. So $\Pi$ is $w_\gamma^{-1}\sigma w_\gamma$-fundamental for all $\gamma$ in $\bar{\Pi}$. If $\Sigma_\Pi\cap K_\sigma=\{\sigma\}$, then $w_\gamma^{-1}\sigma w_\gamma=\sigma$, i.e. $w_\gamma\in W_\sigma$ for all $\gamma$ in $\bar{\Pi}$. Lemma 3.2 and Theorem 3.3 imply that $\tilde{\Delta}$ is a root system with Weyl group $\overline{W}$.

LEMMA 4.4. *If $\tilde{\Delta}$ is a root system with Weyl group $\overline{W}$, then $W_\sigma$ acts simply transitively on the set of $\sigma$-fundamental systems of $\Delta$.*

PROOF. Let $\Pi$, $\Pi'$ be $\sigma$-fundamental systems of $\Delta$. $\overline{\Pi}$ and $\overline{\Pi}'$ are fundamental systems of the root system $\tilde{\Delta}$ and so there exists an element $w$ in $W_\sigma$ such that $\bar{w}\overline{\Pi}=\overline{\Pi}'$. According to Proposition 3.1 $w$ is determined up to a product with an element of $W_0$. Let $w'$ be the unique element in $W_0$ such that $w'(w\Pi)_0=\Pi'_0$. Then $w'w$ is the unique element of $W_\sigma$ such that $(w'w\Pi)_0=\Pi'_0$ and $\overline{(w'w\Pi)}=\overline{\Pi}'$. Proposition 2.4 implies $w'w\Pi=\Pi'$.

The following theorem, which is an immediate consequence of the preceding lemmas, gives a solution to Hirai's problem:

THEOREM 4.4. *The following conditions are equivalent:*
(i) $\Sigma_\Pi \cap K_\sigma = \{\sigma\}$.
(ii) *$W_\sigma$ acts simply transitively on the set of $\sigma$-fundamental systems of $\Delta$.*
(iii) *$\tilde{\Delta}$ is a root system with Weyl group $\overline{W}$.*

PROPOSITION 4.5. *If $\tilde{\Delta}$ is a root system with Weyl group $\overline{W}$ and fundamental system $\overline{\Pi}$, then the Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ is up to isomorphism independent of the choice of the $\sigma$-fundamental system $\Pi$.*

PROOF. Let $\Pi$, $\Pi'$ be $\sigma$-fundamental systems of $\Delta$. By Theorem 4.4 there exists an element $w$ in $W_\sigma$ such that $w\Pi=\Pi'$. The restriction of $w$ to $\Pi$ gives an isomorphism between the Satake diagrams $(\Pi, \Pi_0, \tilde{\sigma})$ and $(\Pi', \Pi'_0, \tilde{\sigma}')$.

In section 6 we will extend the result of Proposition 4.5 and determine necessary and sufficient conditions for the Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ to be independent of the choice of $\Pi$.

## § 5. Admissible Satake diagrams.

We call the root system $\Delta$ *$\sigma$-irreducible* if it cannot be decomposed into two disjoint nonempty orthogonal $\sigma$-invariant subsets. In a classification of Satake diagrams we may restrict ourselves to $\sigma$-irreducible root systems. We call a Satake diagram *admissible* if it belongs to an involutive automorphism $\sigma$ of $\Delta$ such that $\tilde{\Delta}$ is a root system with Weyl group $\overline{W}$ or, equivalently, if it satisfies condition (i) of Theorem 3.3. According to Proposition 4.5 the admissible Satake diagram does not depend up to isomorphism on the choice of its $\sigma$-fundamental system. We will give a classification of $\sigma$-irreducible admissible Satake diagrams. The following proposition describes the situation in which $\Delta$ is a $\sigma$-irreducible, but not irreducible root system.

PROPOSITION 5.1. *If $\Delta$ is $\sigma$-irreducible, but not irreducible, then $\Delta$ is a union of irreducible root systems $\Delta_1$, $\Delta_2$ with $\Delta_1 \cong \Delta_2$ and $\sigma(\Delta_1)=\Delta_2$. $\tilde{\Delta}$ is a root system isomorphic to $\Delta_1$ (and $\Delta_2$) with Weyl group $\overline{W}$ isomorphic to the Weyl group of*

*$\Delta_1$ (and $\Delta_2$).*

The easy proof is omitted.

PROPOSITION 5.2. *If $(\Pi, \Pi_0, \tilde{\sigma})$ is an admissible Satake diagram, then the opposition involution $\mathrm{op}_\Pi$ of $\Delta$ with respect to $\Pi$ commutes with the involution $\sigma$.*

PROOF. If $\Pi$ is $\sigma$-fundamental, then $-\Pi$ is $\sigma$-fundamental too. By Theorem 4.4 there exists an element $w$ in $W_\sigma$ such that $w\Pi=-\Pi$. However $-w$ equals the opposition involution $\mathrm{op}_\Pi$ and therefore $\mathrm{op}_\Pi$ and $\sigma$ commute.

We now state sufficient criteria to classify admissible Satake diagrams.

CRITERION 5.3. *An admissible Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ is invariant under the opposition involution $\mathrm{op}_\Pi$.*

PROOF. This is an immediate consequence of Proposition 5.2.

CRITERION 5.4. *Let $(\Pi, \Pi_0, \tilde{\sigma})$ be an admissible Satake diagram. If $\Pi'$ is a subset of $\Pi$ such that $\Pi_0 \subseteqq \Pi'$ and the root system $\Delta'$ spanned by $\Pi'$ is $\sigma$-invariant, then the Satake diagram $(\Pi', \Pi_0, \widetilde{(\sigma|_{\Delta'})})$ is admissible.*

PROOF. Condition (i) of Theorem 3.3 remains true for the restricted fundamental system $\bar{\Pi}'$.

CRITERION 5.5. *By removal of a white circle and if possible its partner in an admissible Satake diagram a new admissible Satake diagram is constructed.*

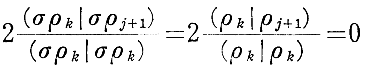
PROOF. This criterion follows from Criterion 5.4.

CRITERION 5.6. *If a $\sigma$-irreducible root system consists of two disjoint nonempty orthogonal root systems $\Delta_1$ and $\Delta_2$, then $\sigma$ is an isomorphism between the root systems $\Delta_1$ and $\Delta_2$.*

PROOF. This is a re-formulation of Proposition 5.1.

The following three criteria hold for any Satake diagram.

CRITERION 5.7. *Suppose the Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ contains the following subgraph*



*with $j \in \boldsymbol{N} \cup \{0\}$, $\sim$ = —— or ⇛ and $\rho_1, \rho_2, \cdots, \rho_j$ are not joined to other vertices in the graph. If $\sigma\rho_{j+1} \equiv \rho_{j+1} \bmod \boldsymbol{Z}\Pi_0$, then $j \in \{0, 1\}$ and if $\sigma\rho_{j+1} \not\equiv \rho_{j+1} \bmod \boldsymbol{Z}\Pi_0$, then $j=0$.*

PROOF. Let $\sigma\rho_{j+1}=\rho_{j+1}+x_1\rho_1+\cdots+x_j\rho_j+\alpha$ with $\alpha \in$ linear span of $\Pi_0 \setminus \{\rho_1, \rho_2, \cdots, \rho_j\}$ and $x_k \in \boldsymbol{N} \cup \{0\}$ for all $k$ in $\{1, 2, \cdots, j\}$. From

$$2\frac{(\sigma\rho_k|\sigma\rho_{j+1})}{(\sigma\rho_k|\sigma\rho_k)}=2\frac{(\rho_k|\rho_{j+1})}{(\rho_k|\rho_k)}=0$$

for all $k$ in $\{1, 2, \cdots, j-1\}$ and

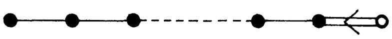
$$2\frac{(\sigma\rho_j|\sigma\rho_{j+1})}{(\sigma\rho_j|\sigma\rho_j)}=2\frac{(\rho_j|\rho_{j+1})}{(\rho_j|\rho_j)}=-1$$

we get the following system of linear equations: $2x_1-x_2=0$, $2x_k-x_{k-1}-x_{k+1}=0$ $(k=2, 3, \cdots, j-1)$, $2x_j-x_{j-1}-1=1$. Its solution is $x_k=2k/(j+1)$ for $k=1, 2, \cdots, j$. So $x_1\in\boldsymbol{N}\cup\{0\}$ implies $j=0$ or $j=1$. Now let $\sigma\rho_{j+1}=\rho'_{j+1}+x_1\rho_1+\cdots+x_j\rho_j+\alpha$ with $\alpha\in$linear span of $\Pi_0\setminus\{\rho_1, \rho_2, \cdots, \rho_j\}$, $\rho'_{j+1}\in\Pi\setminus(\{\rho_{j+1}\}\cup\Pi_0)$ and $x_k\in\boldsymbol{N}\cup\{0\}$ for all $k$ in $\{1, 2, \cdots, j\}$. Analogously to the former case we get the following system of linear equations: $2x_1-x_2=0$, $2x_k-x_{k-1}-x_{k+1}=0$ $(k=2, 3, \cdots, j-1)$, $2x_j-x_{j-1}=1$. Its solution is $x_k=k/(j+1)$ for $k=1, 2, \cdots, j$. So $x_1\in\boldsymbol{N}\cup\{0\}$ implies $j=0$.

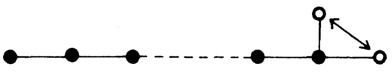
CRITERION 5.8. *Let $\rho$ be a root in $\Pi$ such that $\sigma\rho\equiv\rho \bmod \boldsymbol{Z}\Pi_0$. Define $\Pi':=\Pi'_1\cup\Pi'_2\cup\cdots\cup\Pi'_m\cup\{\rho\}$ with $\Pi'_1, \Pi'_2, \cdots, \Pi'_m$ the distinct irreducible components of $\Pi_0$ not orthogonal to $\rho$. If the only automorphism of $\Pi'$ leaving $\rho$ and all components $\Pi'_k$ $(k=1, 2, \cdots, m)$ invariant is trivial, then the opposition involution $\mathrm{op}_{\Pi'_k}$ of the root system $\Delta'_k$ corresponding to $\Pi'_k$ is trivial for all $k$ in $\{1, 2, \cdots, m\}$.*

PROOF. For each index $k$ in $\{1, 2, \cdots, m\}$ $w_k$ instead of $w_{\Pi'_k}$ denotes the unique element in the Weyl group of root system $\Delta'_k$ such that $w_k\Pi'_k=-\Pi'_k$. For all $k$ in $\{1, 2, \cdots, m\}$ $w_1w_2\cdots w_m\sigma$ transforms roots in $\{\rho\}\cup\Pi'_k$ into non-negative linear combinations of elements in $\{\rho\}\cup\Pi'_k$ and $\rho$ is transformed into $\rho+$(a non-negative linear combination of elements in $\Pi'_k$). This implies that $w_1w_2\cdots w_m\sigma$ leaves $\{\rho\}\cup\Pi'_k$ invariant for all $k$ in $\{1, 2, \cdots, m\}$ and $(w_1w_2\cdots w_m\sigma)\rho=\rho$. Since the only automorphism of $\Pi'$ leaving $\rho$ and all components $\Pi'_k$ invariant is trivial $w_1w_2\cdots w_m\sigma=1$. Hence $\mathrm{op}_{\Pi'_k}=-w_k=(w_1w_2\cdots w_m\sigma)|_{\Delta'_k}=1_{\Delta'_k}$ for all $k$ in $\{1, 2, \cdots, m\}$.

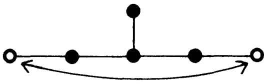
CRITERION 5.9. *The following graphs are no Satake diagrams:*

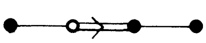


*of type $C_l$ with $l\geqq3$*



*of type $D_l$ with $l\geqq4$*





PROOF. It can easily be seen that there does not exist an involutive automorphism that has one of the above graphs as its Satake diagram.

The above criteria 5.3 up to 5.9 enable us to find all admissible Satake diagrams in a straightforward manner. This does not solve the existence problem of Satake diagrams, but actually it is a not too difficult problem to find an involutive automorphism $\sigma$ that suits a given diagram $(\Pi, \Pi_0, \tilde{\sigma})$. At the end of this section in table I we present all irreducible admissible Satake diagrams together with their involutions in case $\sigma \neq \pm 1$.

Most admissible Satake diagrams arise from so-called $\sigma$-normal root systems. A root system $\Delta$ with involutive automorphism $\sigma$ is called *$\sigma$-normal* if $\sigma\alpha - \alpha \notin \Delta$ for each root $\alpha$ in $\Delta$. The following proposition, due to Araki [**1**], shows that a Satake diagram corresponding to a $\sigma$-normal root system is admissible; the converse statement does not hold as can be seen in the counter-example ●⇒○.

PROPOSITION 5.10. *If $\Delta$ is $\sigma$-normal, then $\bar{\Delta}$ is a (possibly non-reduced) root system with Weyl group $\overline{W}$.*

PROOF. See Araki [**1**], Proposition 2.1.

Now we present the results of the classification of irreducible admissible Satake diagrams. For each irreducible root system type we determine all admissible Satake diagrams $(\Pi, \Pi_0, \tilde{\sigma})$ and corresponding involutions $\sigma$. For each Satake diagram we check if it belongs to a $\sigma$-normal root system; if $\Delta$ is $\sigma$-normal we determine the root system type of $\bar{\Delta}$ by the method of Borel and Tits [**2**], ch. 6, if not the root system type of $\tilde{\Delta}$ is determined. In table I $\sigma$-fundamental systems are numbered according to the convention of Bourbaki [**3**], planches I-IX; we also use Bourbaki's description of root systems of classical type in order to get simpler expressions for the involution $\sigma$. For root systems of type $A_l$ $(l \geqq 2)$, $D_l$ $(l \geqq 4)$ and $E_6$ $\mathcal{A}$ denotes the following automorphism:

$$A_l: \rho_k \longleftrightarrow \rho_{l+1-k} \quad \text{for } k=1, 2, \cdots, l$$

$$D_l: \rho_{l-1} \longleftrightarrow \rho_l, \quad \rho_k \longmapsto \rho_k \quad \text{for } k=1, 2, \cdots, l-2$$

$$E_6: \rho_1 \longleftrightarrow \rho_6, \quad \rho_3 \longleftrightarrow \rho_5, \quad \rho_2 \longmapsto \rho_2, \quad \rho_4 \longmapsto \rho_4 .$$

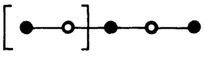
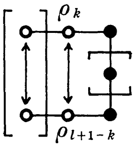
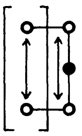
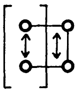
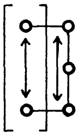
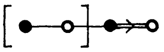
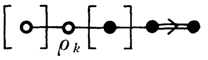
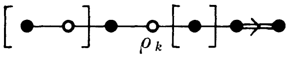
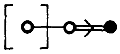
In table I $\mathscr{B}$ and $\mathscr{D}$ denote the following expressions:

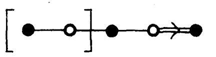
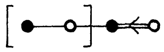
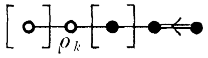
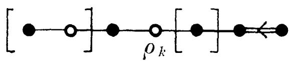
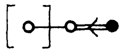
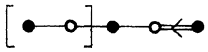
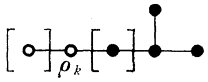
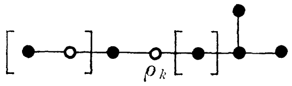
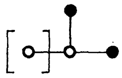
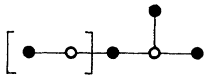
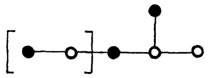
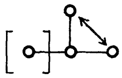
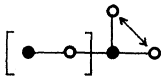
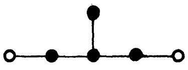
$$\mathscr{B} := \prod_{j=0}^{(l-k-2)/2} (S_{\varepsilon_{k+2j+1}-\varepsilon_{k+2j+2}} S_{\varepsilon_{k+2j+1}+\varepsilon_{k+2j+2}}),$$

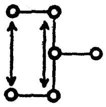
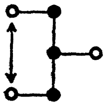
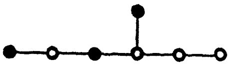
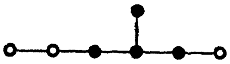
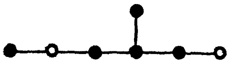
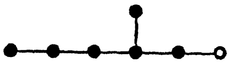
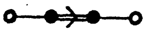
$$\mathscr{D} := \prod_{j=0}^{(l-k-3)/2} (S_{\varepsilon_{k+2j+1}-\varepsilon_{k+2j+2}} S_{\varepsilon_{k+2j+1}+\varepsilon_{k+2j+2}}).$$

We adapt the graphical representation of a Satake diagram of classical type as follows: portions of the graph that are repeated zero or more times are drawn once and are enclosed by brackets; by giving the type of the root system and marking some vertices in the graph by their corresponding fundamental roots we get a compact diagram that unambiguously describes the Satake diagram (dotted lines possibly leading to pictorial misinterpretations are not needed!). In table I the trivial cases $\sigma=1$ and $\sigma=-1$ are not included.

Table I: Irreducible admissible Satake diagrams $(\Pi, \Pi_0, \tilde{\sigma})$ with $\sigma \neq \pm 1$

| type $\Delta$ | graph representing $(\Pi, \Pi_0, \tilde{\sigma})$ | $\sigma$-normal | type $\bar{\Delta}$ or $\tilde{\Delta}$ | $\sigma$ |
|---|---|---|---|---|
| $A_l$ |  | + | $A_{(l-1)/2}$ | $\prod_{j=0}^{(l-1)/2} S_{\rho_{2j+1}}$ |
| $A_l$ |  | + | $BC_k$ | $\begin{cases} \mathcal{A}(\prod_{j=k+1}^{l/2} S_{\varepsilon_j-\varepsilon_{l+2-j}}) & \text{if } l \text{ even} \\ \mathcal{A}(\prod_{j=k+1}^{(l+1)/2} S_{\varepsilon_j-\varepsilon_{l+2-j}}) & \text{if } l \text{ odd} \end{cases}$ |
| $A_l$ |  | + | $BC_{(l-1)/2}$ | $\mathcal{A}S_{\rho_{(l+1)/2}}$ |
| $A_l$ |  | + | $BC_{l/2}$ | $\mathcal{A}$ |
| $A_l$ |  | + | $C_{(l+1)/2}$ | $\mathcal{A}$ |
| $B_l$ |  | − | $B_{l/2}$ | $\prod_{j=0}^{(l-2)/2} S_{\rho_{2j+1}}$ |
| $B_l$ |  | + | $B_k$ | $\begin{cases} \mathcal{B} & \text{if } l-k \text{ even} \\ \mathcal{D}S_{\rho_l} & \text{if } l-k \text{ odd} \end{cases}$ |
| $B_l$ |  | − | $B_{k/2}$ | $\begin{cases} (\prod_{j=0}^{(k-2)/2} S_{\rho_{2j+1}})\mathcal{B} & \text{if } l-k \text{ even} \\ (\prod_{j=0}^{(k-2)/2} S_{\rho_{2j+1}})\mathcal{D}S_{\rho_l} & \text{if } l-k \text{ odd} \end{cases}$ |
| $B_l$ |  | + | $B_{l-1}$ | $S_{\rho_l}$ |

| | | | | |
|---|---|---|---|---|
| $B_l$ |  | $-$ | $B_{(l-1)/2}$ | $\prod_{j=0}^{(l-1)/2} S_{\rho_{2j+1}}$ |
| $C_l$ |  | $+$ | $C_{l/2}$ | $\prod_{j=0}^{(l-2)/2} S_{\rho_{2j+1}}$ |
| $C_l$ |  | $-$ | $B_k$ | $\begin{cases}\mathcal{B} & \text{if } l-k \text{ even}\\ \mathcal{D}S_{\rho_l} & \text{if } l-k \text{ odd}\end{cases}$ |
| $C_l$ |  | $+$ | $BC_{k/2}$ | $\begin{cases}(\prod_{j=0}^{(k-2)/2} S_{\rho_{2j+1}})\mathcal{B} & \text{if } l-k \text{ even}\\ (\prod_{j=0}^{(k-2)/2} S_{\rho_{2j+1}})\mathcal{D}S_{\rho_l} & \text{if } l-k \text{ odd}\end{cases}$ |
| $C_l$ |  | $-$ | $B_{l-1}$ | $S_{\rho_l}$ |
| $C_l$ |  | $+$ | $BC_{(l-1)/2}$ | $\prod_{j=0}^{(l-1)/2} S_{\rho_{2j+1}}$ |
| $D_l$ |  | $+$ | $B_k$ | $\begin{cases}\mathcal{B} & \text{if } l-k \text{ even}\\ \mathcal{D}\mathcal{A} & \text{if } l-k \text{ odd}\end{cases}$ |
| $D_l$ |  | $+$ | $BC_{k/2}$ | $\begin{cases}(\prod_{j=0}^{(k-2)/2} S_{\rho_{2j+1}})\mathcal{B} & \text{if } l-k \text{ even}\\ (\prod_{j=0}^{(k-2)/2} S_{\rho_{2j+1}})\mathcal{D}\mathcal{A} & \text{if } l-k \text{ odd}\end{cases}$ |
| $D_l$ |  | $+$ | $B_{l-2}$ | $S_{\rho_{l-1}}S_{\rho_l}$ |
| $D_l$ |  | $+$ | $BC_{(l-2)/2}$ | $(\prod_{j=0}^{(l-4)/2} S_{\rho_{2j+1}})S_{\rho_{l-1}}S_{\rho_l}$ |
| $D_l$ |  | $+$ | $B_{l/2}$ | $\prod_{j=0}^{(l-2)/2} S_{\rho_{2j+1}}$ |
| $D_l$ |  | $+$ | $B_{l-1}$ | $\mathcal{A}$ |
| $D_l$ |  | $+$ | $BC_{(l-1)/2}$ | $\mathcal{A}(\prod_{j=0}^{(l-3)/2} S_{\rho_{2j+1}})$ |
| $E_6$ |  | $+$ | $A_2$ | $S_{\rho_5}S_{\rho_2+\rho_3+2\rho_4+\rho_5}S_{\rho_2}S_{\rho_3}$ |

| | | | | |
|---|---|---|---|---|
| $E_6$ |  | + | $BC_1$ | $\mathcal{A}S_{\rho_1+\rho_3+\rho_4+\rho_5+\rho_6}S_{\rho_3+\rho_4+\rho_5}S_{\rho_4}$ |
| $E_6$ |  | + | $F_4$ | $\mathcal{A}$ |
| $E_6$ |  | + | $BC_2$ | $\mathcal{A}S_{\rho_3+\rho_4+\rho_5}S_{\rho_4}$ |
| $E_7$ |  | + | $F_4$ | $S_{\rho_2}S_{\rho_5}S_{\rho_7}$ |
| $E_7$ |  | + | $C_3$ | $S_{\rho_5}S_{\rho_2+\rho_3+2\rho_4+\rho_5}S_{\rho_2}S_{\rho_3}$ |
| $E_7$ |  | + | $BC_2$ | $S_{\rho_7}S_{\rho_5}S_{\rho_2+\rho_3+2\rho_4+\rho_5}S_{\rho_2}S_{\rho_3}$ |
| $E_7$ |  | + | $BC_1$ | $-S_{123\overset{2}{4}32}$ |
| $E_8$ |  | + | $F_4$ | $S_{\rho_5}S_{\rho_2+\rho_3+2\rho_4+\rho_5}S_{\rho_2}S_{\rho_3}$ |
| $E_8$ |  | + | $BC_2$ | $-S_{2345\overset{3}{6}42}S_{0123\overset{2}{4}32}$ |
| $E_8$ |  | + | $BC_1$ | $-S_{2345\overset{3}{6}42}$ |
| $F_4$ |  | − | $B_2$ | $S_{\rho_3}S_{\rho_2+\rho_3}$ |
| $F_4$ |  | + | $BC_1$ | $S_{\rho_1}S_{\rho_1+2\rho_2+2\rho_3}S_{\rho_3}$ |
| $F_4$ |  | − | $A_1$ | $S_{\rho_4}S_{\rho_2+2\rho_3+\rho_4}S_{\rho_2}$ |
| $G_2$ |  | − | $A_1$ | $S_{\rho_2}$ |
| $G_2$ |  | − | $A_1$ | $S_{\rho_1}$ |

## § 6. The system $\tilde{\Delta}$ of reduced restricted roots.

In section 3 we have determined necessary and sufficient conditions for $\tilde{\Delta}$ to be a root system with Weyl group $\overline{W}$. Now we will weaken these conditions and examine the situation in which $\tilde{\Delta}$ is merely a root system, not necessarily with Weyl group $\overline{W}$.

Theorem 6.1. *$\tilde{\Delta}$ is a root system in $\bar{E}$ iff each $\sigma$-irreducible component of the Satake diagram is admissible or isomorphic to* ○——●.

Proof. We consider the case that $\Delta$ is irreducible. First we look at the case $\dim \bar{E}=1$. If the opposition involution $\mathrm{op}_{\Pi}$ of $\Delta$ with respect to $\Pi$ leaves $\Pi_0$ invariant, then by Theorem 3.3 $\tilde{\Delta}$ is a root system in $\bar{E}$ with Weyl group $\overline{W}$. Since the opposition involution $\mathrm{op}_{\Pi}$ is nontrivial iff $\Delta$ is a root system of type $A_l$ ($l \geqq 2$), $D_l$ ($l \geqq 4$, $l$ odd) or $E_6$, only for these types $\tilde{\Delta}$ might be a root system without Weyl group $\overline{W}$ and in this case $\mathrm{op}_{\Pi}$ must not leave $\Pi_0$ invariant. For each of these root system types we determine all non-admissible Satake diagrams $(\Pi, \Pi_0, \tilde{\sigma})$ with $\dim \bar{E}=1$. We use criteria 5.7 and 5.8 and number the $\sigma$-fundamental system $\Pi$ according to Bourbaki [3], planches I–IX. $A_l$ ($l \geqq 2$): we may assume that in the Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ no arrows are present, for otherwise $\Pi_0$ is $\mathrm{op}_{\Pi}$-invariant (and hence $(\Pi, \Pi_0, \tilde{\sigma})$ is admissible). Let $\rho_j$ ($j \in \{1, 2, \cdots, l\}$) be the root corresponding to the white circle in the Satake diagram; then Criterion 5.7 implies $l=2$ and ($j=1$ or $2$). So $(\Pi, \Pi_0, \tilde{\sigma})$ is isomorphic to ○——●. $D_l$ ($l \geqq 4$, $l$ odd): in this case $\mathrm{op}_{\Pi}$ does not leave $\Pi_0$ invariant iff no arrows are present in the Satake diagram and $\rho_{l-1}$ or $\rho_l$ is the white circle. However by Criterion 5.8 for $\rho=\rho_{l-1}$ or $\rho_l$ no such Satake diagram exists. $E_6$: in this case $\mathrm{op}_{\Pi}$ does not leave $\Pi_0$ invariant iff no arrows are present in the Satake diagram and $\rho_2, \rho_4 \in \Pi_0$. So the only possible non-admissible Satake diagrams consist of black circles and one white circle representing a root $\rho$ in $\{\rho_1, \rho_3, \rho_5, \rho_6\}$. Application of Criterion 5.8 for each possible root $\rho$ shows that no such Satake diagram exists. We conclude that in the case $\dim \bar{E}=1$ the only non-admissible Satake diagram is isomorphic to ○——●. Now we will consider the case $\dim \bar{E}>1$; we shall see that then $\tilde{\Delta}$ is not a root system if the Satake diagram is not admissible. Let $(\Pi, \Pi_0, \tilde{\sigma})$ be a non-admissible Satake diagram. According to Theorem 3.3 there should exist an element $\gamma$ in $\bar{\Pi}$ such that $\Pi_0$ is not left invariant by the opposition involution $\mathrm{op}_{\Pi_\gamma}$ of $\Delta_\gamma$ with respect to $\Pi_\gamma$. Since $\sharp \bar{\Pi}_\gamma=1$, the first part of the proof shows that the Satake diagram is of the form

$$- - - \overset{\rho_1}{\circ}\!\!-\!\!\!-\!\!\!-\!\!\overset{\rho_2}{\bullet} - - -$$

with $\bar{\rho}_1=\gamma$ and $\rho_1, \rho_2$ not joined to other black vertices. Then $\sigma\rho_1=\rho_1+\rho_2$ and therefore $\bar{\rho}_1=\rho_1+(1/2)\rho_2$. Since $\dim \bar{E}>1$ and $\Delta$ is supposed to be irreducible, $\rho_1$

and/or $\rho_2$ are joined to a white circle. We distinguish two cases: (i) $\rho_1$ is joined to a white circle $\rho_3$: then

$$2\frac{(\bar{\rho}_1|\bar{\rho}_3)}{(\bar{\rho}_1|\bar{\rho}_1)}=2\frac{(\bar{\rho}_1|\rho_3)}{(\bar{\rho}_1|\bar{\rho}_1)}=\frac{4}{3}\cdot 2\frac{(\rho_1|\rho_3)}{(\rho_1|\rho_1)}.$$

Since $2\frac{(\rho_1|\rho_3)}{(\rho_1|\rho_1)}\in\{-1, -2, -3\}$, $2\frac{(\bar{\rho}_1|\bar{\rho}_3)}{(\bar{\rho}_1|\bar{\rho}_1)}\notin\{0, -1, -2, -3\}$. It follows that $\tilde{\Delta}$ is not a root system in $\bar{E}$. (ii) $\rho_2$ is joined to a white circle $\rho_3$: then

$$2\frac{(\bar{\rho}_1|\bar{\rho}_3)}{(\bar{\rho}_1|\bar{\rho}_1)}=2\frac{(\bar{\rho}_1|\rho_3)}{(\bar{\rho}_1|\bar{\rho}_1)}=\frac{2}{3}\cdot 2\frac{(\rho_2|\rho_3)}{(\rho_2|\rho_2)}.$$

Since $2\frac{(\rho_2|\rho_3)}{(\rho_2|\rho_2)}\in\{-1, -2, -3\}$ and $2\frac{(\rho_2|\rho_3)}{(\rho_2|\rho_2)}=-3$ iff $\Delta$ is of type $G_2$, $2\frac{(\bar{\rho}_1|\bar{\rho}_3)}{(\bar{\rho}_1|\bar{\rho}_1)}$ $\notin\{0, -1, -2, -3\}$. It follows that $\tilde{\Delta}$ is not a root system in $\bar{E}$. We conclude that in the case $\dim\bar{E}>1$ each Satake diagram with root system $\tilde{\Delta}$ is in fact admissible. So if $\tilde{\Delta}$ is a root system in $\bar{E}$, then $\tilde{\Delta}$ has Weyl group $\overline{W}$ (and hence the Satake diagram is admissible) or the Satake diagram is isomorphic to ○——●. The converse statement is obvious.

As Sugiura [7] already remarked, the Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ may depend on the choice of the $\sigma$-fundamental system $\Pi$, i.e. if $\Pi$ and $\Pi'$ are $\sigma$-fundamental systems of root system $\Delta$, the Satake diagrams $(\Pi, \Pi_0, \tilde{\sigma})$ and $(\Pi', \Pi'_0, \tilde{\sigma}')$ need not be isomorphic. We will show that a Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ up to isomorphism does not depend on the choice of $\Pi$ iff $\tilde{\Delta}$ is a root system. We need the following lemma and criterion.

LEMMA 6.2. *Let $(\Pi, \Pi_0, \tilde{\sigma})$ be a Satake diagram of the form* ---$\overset{\rho_1}{\circ}$——$\overset{\rho_2}{\bullet}$--- *with $\rho_2$ not joined to any other black vertex and $\rho_2$ the only black vertex joined to $\rho_1$. Then there exists a $\sigma$-fundamental system $\Pi'$ with Satake diagram $(\Pi', \Pi'_0, \tilde{\sigma}')$ constructed from the original Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ by switching the colours of the vertices $\rho_1$ and $\rho_2$.*

PROOF. Define $\Pi':=(S_{\rho_1+\rho_2}S_{\rho_1})\Pi$. Then $\Pi'$ is a $\sigma$-fundamental system that has the right properties.

CRITERION 6.3. *By removal of a white circle and if possible its partner in a Satake diagram a new Satake diagram is constructed.*

We omit the proof of this obvious statement.

THEOREM 6.4. *$\tilde{\Delta}$ is a root system iff the Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ up to isomorphism does not depend on the choice of $\Pi$.*

PROOF. We may restrict ourselves to a $\sigma$-irreducible root system $\Delta$. Moreover Proposition 5.1 permits us to consider only the case that $\Delta$ is irreducible. If $\tilde{\Delta}$ is a root system, then by Theorem 6.1 $\overline{W}$ is the Weyl group of $\tilde{\Delta}$ (and hence the Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ is admissible) or the Satake diagram

$(\Pi, \Pi_0, \tilde{\sigma})$ is isomorphic to ○——●. If $(\Pi, \Pi_0, \tilde{\sigma})$ is admissible, then by Proposition 4.5 the Satake diagram is independent of the choice of $\Pi$. If $(\Pi, \Pi_0, \tilde{\sigma})$ is isomorphic to ○——●, then the reader can easily convince himself that $(\Pi, \Pi_0, \tilde{\sigma})$ does not depend on the choice of $\Pi$. Now we prove the converse statement: if an irreducible Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ does not depend on the choice of $\Pi$, then $\tilde{\Delta}$ is a root system. According to Theorem 6.1 we must prove that an irreducible Satake diagram that does not depend on the choice of its $\sigma$-fundamental system is admissible or isomorphic to ○——●. If the root system $\Delta$ is not of type $A_l$ $(l \geqq 2)$, $D_l$ $(l \geqq 4)$ or $E_6$, then $\mathrm{Aut}(\Delta)=$Weyl group $W$ of $\Delta$. In this case the independence of the Satake diagram $(\Pi, \Pi_0, \tilde{\sigma})$ on the choice of $\Pi$ is equivalent to the simple transitive action of $W_\sigma$ on the set of $\sigma$-fundamental systems. By Theorem 4.4 $\tilde{\Delta}$ is a root system with Weyl group $\overline{W}$; so $(\Pi, \Pi_0, \tilde{\sigma})$ is admissible. What remains are the three cases of a root system $\Delta$ of type $A_l$ $(l \geqq 2)$, $D_l$ $(l \geqq 4)$ and $E_6$. For each type we can use criteria 5.6 up to 5.9 and 6.3 as well as Lemma 6.2 to construct all Satake diagrams that do not depend on the choice of the $\sigma$-fundamental system; the result of this straightforward determination is a set of admissible Satake diagrams and a Satake diagram isomorphic to ○——●. Thus the proof of the theorem is completed.

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