10

The Wiener–Itô Chaos Expansion

In this chapter we show that it is possible to obtain a chaos expansion similar to the one in Chap. 1 in the context of pure jump Lévy processes $\eta = \eta(t)$, $t \geq 0$. However, in this case, the corresponding iterated integrals must be taken with respect to the *compensated Poisson measure* associated with η,

$$\eta(t) = \int_0^t \int_{\mathbb{R}_0} z\tilde{N}(dt, dz), \quad t \geq 0, \tag{10.1}$$

and *not* with respect to η itself. Our method will be basically the same as for Theorem 1.10; therefore, we do not give all the details.

For a study on the construction of chaos expansions and a calculus with respect to η we can refer to e.g. [61, 71].

10.1 Iterated Itô Integrals

Let $L^2((\lambda \times \nu)^n) = L^2(([0,T] \times \mathbb{R}_0)^n)$ be the space of deterministic real functions f such that

$$\|f\|_{L^2((\lambda\times\nu)^n)} = \Big(\int_{([0,T]\times\mathbb{R}_0)^n} f^2(t_1, z_1, \ldots, t_n, z_n) dt_1 \nu(dz_1) \cdots dt_n \nu(dz_n)\Big)^{1/2} < \infty,$$

where, we recall, $\lambda(dt) = dt$ denotes the Lebesgue measure on $[0, T]$.

The *symmetrization* $\tilde{f}$ of f is defined by

$$\tilde{f}(t_1, z_1, ..., t_n, z_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, z_{\sigma_1}, ..., t_{\sigma_n}, z_{\sigma_n}),$$

the sum being taken over all permutations $\sigma = (\sigma_1, ..., \sigma_n)$ of $\{1, ..., n\}$. Note that the symmetrization is over the n pairs $(t_1, z_1), ..., (t_n, z_n)$ and not over the $2n$ variables $t_1, z_1, ..., t_n, z_n$. A function $f \in L^2((\lambda \times \nu)^n)$ is called *symmetric* if $f = \tilde{f}$. We denote the space of all symmetric functions in $L^2((\lambda \times \nu)^n)$ by $\tilde{L}^2((\lambda \times \nu)^n)$.

G.Di Nunno et al., *Malliavin Calculus for Lévy Processes with Applications to Finance*,

Define

$$G_n := \big\{(t_1, z_1, ..., t_n, z_n) \,:\, 0 \le t_1 \le ... \le t_n \le T,\ z_i \in \mathbb{R}_0,\ i = 1, ..., n\big\}$$

and let $L^2(G_n)$ be the set of the real functions g on G_n, such that

$$\|g\|_{L^2(G_n)} := \Big(\int_{G_n} g^2(t_1, z_1, ..., t_n, z_n) dt_1 \nu(dz_1) \cdots dt_n \nu(dz_n)\Big)^{1/2} < \infty.$$

Note that, for $f \in \tilde{L}^2((\lambda \times \nu)^n)$, we have $f_{|G_n} \in L^2(G_n)$ and

$$\|f\|^2_{L^2((\lambda\times\nu)^n)} = n! \|f\|^2_{L^2(G_n)}.$$

Definition 10.1. *For any $g \in L^2(G_n)$, the* n-fold iterated integral $J_n(f)$ *is the random variable in $L^2(P)$ defined as*

$$J_n(g) := \int_0^T \int_{\mathbb{R}_0} \cdots \int_0^{t_2^-} \int_{\mathbb{R}_0} g(t_1, z_1, ..., t_n, z_n) \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n).$$

We set $J_0(g) = g$ for any $g \in \mathbb{R}$.

If $f \in \tilde{L}^2((\lambda \times \nu)^n)$, we also define

$$I_n(f) := \int_{([0,T]\times\mathbb{R}_0)^n} f(t_1, z_1, ..., t_n, z_n) \tilde{N}^{\otimes n}(dt, dz) := n! J_n(f), \qquad (10.2)$$

where $\tilde{N}^{\otimes n}(dt, dz) = \tilde{N}(dt_1, dz_1) \cdots \tilde{N}(dt_n, dz_n)$. We also call $I_n(f)$ the n-fold iterated integral of f. For any $g \in \tilde{L}^2((\lambda \times \nu)^m)$ and $f \in \tilde{L}^2((\lambda \times \nu)^n)$, we have the following relations

$$E[I_m(g) I_n(f)] = \begin{cases} 0, & n \ne m \\ (g, f)_{L^2((\lambda\times\nu)^n)}, & n = m \end{cases} \quad (m, n = 1, 2, ...), \qquad (10.3)$$

where

$$(g, f)_{L^2((\lambda\times\nu)^n)} := \int_{([0,T]\times\mathbb{R}_0)^n} g(t_1, z_1, ..., t_n, z_n) f(t_1, z_1, ..., t_n, z_n) dt_1 \nu(dz_1) \cdots dt_n \nu(dz_n).$$

10.2 The Wiener–Itô Chaos Expansion

We can now formulate the chaos expansion with respect to the Poisson random measure. See [121].

Theorem 10.2. Wiener–Itô chaos expansion for Poisson random measures. *Let $F \in L^2(P)$ be a $\mathcal{F}_T$-measurable random variable. Then F admits the representation*

$$F = \sum_{n=0}^{\infty} I_n(f_n) \tag{10.4}$$

via a unique sequence of elements $f_n \in \tilde{L}^2((\lambda \times \nu)^n)$, $n = 1, 2, ...$ *Here we set* $I_0(f_0) := f_0$ *for the constant values* $f_0 \in \mathbb{R}$. *Moreover, we have that*

$$\|F\|^2_{L^2(P)} = \sum_{n=0}^{\infty} n! \|f_n\|^2_{L^2((\lambda\times\nu)^n)}. \tag{10.5}$$

Proof The proof follows the lines of the proof of Theorem 1.10; here we sketch only the arguments.

By the Itô representation theorem, there exists a predictable process $\psi_1(t_1, z_1)$, $(t_1, z_1) \in [0, T] \times \mathbb{R}_0$, such that

$$F = E[F] + \int_0^T \int_{\mathbb{R}_0} \Psi_1(t_1, z_1) \tilde{N}(dt_1, dz_1),$$

where

$$\|F\|^2_{L^2(P)} = \big(E[F]\big)^2 + E\Big[\int_0^T \int_{\mathbb{R}_0} \Psi_1^2(t_1, z_1) dt_1 \nu(dz_1)\Big] < \infty.$$

Hence for almost all $(t_1, z_1) \in [0, T] \times \mathbb{R}_0$, there exists a predictable process $\Psi_2(t_1, z_1, t_2, z_2)$, $(t_2, z_2) \in [0, t_1] \times \mathbb{R}_0$, such that

$$\Psi(t_1, z_1) = E[\Psi(t_1, z_1)] + \int_0^T \int_{\mathbb{R}_0} \Psi_2(t_1, z_1, t_2, z_2) \tilde{N}(dt_2, dz_2).$$

This gives

$$\begin{aligned} F = E[F] &+ \int_0^T \int_{\mathbb{R}_0} E\big[\Psi_1(t_1, z_1)\big] \tilde{N}(dt_1, dz_1) \\ &+ \int_0^T \int_{\mathbb{R}_0} \int_0^{t_1^-} \int_{\mathbb{R}_0} \Psi_2(t_1, z_1, t_2, z_2) \tilde{N}(dt_2, dz_2) \tilde{N}(dt_1, dz_1). \end{aligned}$$

Define

$$\begin{aligned} &g_0 := E[F] \\ &g_1(t_1, z_1) := E[\Psi_1(t_1, z_1)], \quad (t_1, z_1) \in [0, T] \times \mathbb{R}_0. \end{aligned}$$

We repeat the same argument for (t_2, z_2) – almost all the random variables of the integrand process $\Psi_2(t_1, z_1, t_2, z_2)$ and again for the new integrands. This yields

$$F = \sum_{n=0}^{k-1} J(g_n) + \int_{G_k} \Psi_k(t_1, z_1, ..., t_k, z_k) \tilde{N}^{\otimes k}(dt, dz).$$

Proceeding as in the proof of Theorem 1.10, we can see that the residual term here above vanishes, that is,

$$\int_{G_k} \Psi_k \tilde{N}^{\otimes k}(dt, dz) \longrightarrow 0, \quad k \to \infty,$$

with the convergence in $L^2(P)$. This gives the chaos expansion

$$F = \sum_{n=0}^{\infty} J_n(g_n)$$

in $L^2(P)$ with $g_n \in L^2(G_n)$, $n = 1, 2, ...$ Extend the function g_n on the whole $([0,T] \times \mathbb{R}_0)^n$ by putting $g_n := 0$ on $([0,T] \times \mathbb{R}_0)^n \setminus G_n$ and define $f_n := \tilde{g}_n$. Then

$$I_n(f_n) = n! J_n(f_n) = n! J_n(\tilde{g}_n) = J_n(g_n).$$

Thus we obtain the claim

$$F = \sum_{n=0}^{\infty} I_n(f_n).$$

Moreover, the isometry (10.5) is obtained directly from (10.3). □

Example 10.3. Let $F = \eta^2(T)$. From (9.34) we have

$$\begin{aligned}
\eta^2(T) &= T \int_{\mathbb{R}_0} z^2 \nu(dz) + \int_0^T \int_{\mathbb{R}_0} \big(2\eta(t^-)z + z^2\big) \tilde{N}(dt, dz) \\
&= T \int_{\mathbb{R}_0} z^2 \nu(dz) + \int_0^T \int_{\mathbb{R}_0} z_1^2 \tilde{N}(dt_1, dz_1) \\
&\quad + \int_0^T \int_{\mathbb{R}_0} \int_0^{t_2^-} \int_{\mathbb{R}_0} 2 z_1 z_2 \tilde{N}(dt_2, dz_2) \tilde{N}(dt_1, dz_1) \\
&= \sum_{n=0}^{2} I_n(f_n),
\end{aligned} \tag{10.6}$$

with

$$\begin{aligned}
f_0 &:= T \int_{\mathbb{R}_0} z^2 \nu(dz) \\
f_1(t_1, z_1) &:= z_1^2 \\
f_2(t_1, z_1, t_2, z_2) &= z_1 z_2.
\end{aligned}$$

Example 10.4. Let $F = Y(T)$, where

$$\begin{aligned}
Y(t) = \exp\Big\{ &\int_0^t \int_{\mathbb{R}_0} h(s) z \tilde{N}(ds, dz) \\
&- \int_0^t \int_{\mathbb{R}_0} \big(e^{h(s)z} - 1 - h(s)z\big) \nu(dz) ds \Big\}, \quad t \in [0,T],
\end{aligned}$$

with $h \in L^2([0,T])$ is a càglàd real function. Then by the Itô formula we have

$$dY(t) = Y(t^-) \int_{\mathbb{R}_0} \big(e^{h(t)z} - 1\big)\tilde{N}(dt,dz)$$

and thus

$$Y(T) = 1 + \int_0^T \int_{\mathbb{R}_0} Y(t^-)\big(e^{h(t)z} - 1\big)\tilde{N}(dt,dz).$$

Repeating the argument, we get

$$\begin{aligned} Y(T) = 1 &+ \int_0^T \int_{\mathbb{R}_0} \Big(1 + \int_0^{t_1^-} \int_{\mathbb{R}_0} Y(t_2^-)\big(e^{h(t_2)z_2} - 1\big)\tilde{N}(dt_2,dz_2)\Big) \\ &\cdot \big(e^{h(t_1)z_1} - 1\big)\tilde{N}(dt_1,dz_1) \\ = 1 &+ \int_0^T \int_{\mathbb{R}_0} \big(e^{h(t_1)z_1} - 1\big)\tilde{N}(dt_1,dz_1) \\ &+ \int_0^T \int_{\mathbb{R}_0} \int_0^{t_1^-} \int_{\mathbb{R}_0} Y(t_2^-)\big(e^{h(t_2)z_2} - 1\big)\big(e^{h(t_1)z_1} - 1\big)\tilde{N}(dt_2,dz_2)\tilde{N}(dt_1,dz_1). \end{aligned}$$

Proceeding by iteration, we obtain

$$\begin{aligned} Y(T) &= \sum_{n=0}^{k-1} I_n(f_n) \\ &\quad + \int_0^T \int_{\mathbb{R}_0} \cdots \int_0^{t_2^-} \int_{\mathbb{R}_0} Y(t_1^-) \prod_{i=1}^{k} \big(e^{h(t_i)z_i} - 1\big)\tilde{N}(dt_1,dz_1) \cdots \tilde{N}(dt_k,dz_k), \end{aligned}$$

where

$$\begin{aligned} f_n(t_1,z_1,...,t_n,z_n) :=& \frac{1}{n!} \prod_{i=1}^{n} \big(e^{h(t_i)z_i} - 1\big) \\ =& \frac{1}{n!} \big(e^{h(t)z} - 1\big)^{\otimes n}(t_1,z_1,...,t_n,z_n), \end{aligned} \qquad (10.7)$$

which leads to the chaos expansion

$$Y(T) = \sum_{n=0}^{\infty} I_n(f_n),$$

with convergence in $L^2(P)$. To prove this we need to verify that

$$E\Big[\Big(\int_0^T \int_{\mathbb{R}_0} \cdots \int_0^{t_2^-} \int_{\mathbb{R}_0} Y(t_1^-)\big(e^{h(t)z} - 1\big)^{\otimes k} \tilde{N}^{\otimes k}(dt,dz)\Big)^2\Big] \longrightarrow 0, \quad k \to \infty.$$

This follows from the estimate

$$\int_0^T \int_{\mathbb{R}_0} \cdots \int_0^{t_2} \int_{\mathbb{R}_0} E\big[Y^2(t_1^-)\big] \big|\big(e^{h(t)z} - 1\big)^{\otimes k}\big|^2 dt_1 \nu(dz_1) \cdots dt_k \nu(dz_k)$$

$$\leq E\big[Y^2(T)\big] \frac{1}{k!} \Big(\int_0^T \int_{\mathbb{R}_0} \big(e^{h(t_1)z_1} - 1\big)^2 dt_1 \nu(dz_1) \Big)^k \longrightarrow 0, \quad k \to \infty.$$

10.3 Exercises

Problem 10.1. (*) Let

$$\eta(t) = \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(ds, dz), \quad t \in [0,T].$$

Find the chaos expansion of

(a) $F = \eta^3(T)$
(b) $F = \exp \eta(T)$
(c) $F = \int_0^T g(s) d\eta(s)$, where $g \in L^2([0,T])$
(d) $F = \int_0^T g(s)\eta(s) ds$, where $g \in L^2([0,T])$

Problem 10.2. With η as in Problem 10.1, let $\mathbb{F}^\eta = \{\mathcal{F}_t^\eta,\ t \in [0,T]\}$ be the P-augmented filtration of η. Find $F \in L^2(P)$, $\mathcal{F}_T^\eta$-measurable, for which it is *not* possible to write a chaos expansion in terms of $d\eta$, that is,

$$F = E\big[F\big] + \sum_{n=1}^{\infty} n! \int_0^T \int_0^{t_n} \cdots \int_0^{t_2} f_n(t_1, ..., t_n) d\eta(t_1) \cdots d\eta(t_n)$$

for a sequence $f_n \in \tilde{L}^2([0,T]^n)$, $n = 1, 2, ...$ [*Hint.* See Example 9.15].