Appendix A: Malliavin Calculus on the Wiener Space

In this book we have, for several reasons, chosen to present the Malliavin calculus via chaos expansions. In the Brownian motion case this approach is basically equivalent to the construction given in the setting of the Hida white noise probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\Omega = S'(\mathbb{R})$ is the Schwartz space of tempered distributions. In the Brownian case there is an alternative setting, namely the Wiener space $\Omega = C_0[0,T]$ of continuous functions $\omega : [0,T] \to \mathbb{R}$ with $\omega(0) = 0$. We now present this approach.

Malliavin calculus was originally introduced to study the regularity of the law of functionals of the Brownian motion, in particular, of the solution of stochastic differential equations driven by the Brownian noise [158].

Shortly, the idea is as follows. Let f be a smooth function on $\mathbb{R}^d$. The crucial idea for proving the regularity of the law of an $\mathbb{R}^d$-valued functional X of the Wiener process is to express the partial derivative of f at X as a derivative of the functional $f(X)$ with respect to a *new derivation* on the Wiener space. Based on some *integration by parts formula*, this derivation should exhibit the property of fulfilling the following relation:

$$E\left[\frac{\partial^{|\alpha|} f}{\partial x^\alpha}(X)\right] = E\left[f(X)L_\alpha(X)\right],$$

where $L_\alpha(X)$ is a functional of the Wiener process not depending on f and where $\frac{\partial^{|\alpha|}}{\partial x^\alpha}$ is the partial derivative of order $|\alpha| = \alpha_1 + ... + \alpha_d$, $\alpha = (\alpha_1, ..., \alpha_d)$. Provided $L_\alpha(X)$ is sufficiently integrable, the law of X should be smooth.

Hereafter we outline the classical presentation of the Malliavin derivative on the Wiener space. For further reading, we refer to, for example, [53, 169, 212].

A.1 Preliminary Basic Concepts

Let us first recall some basic concepts from classical analysis, see, for example, [79].

Definition A.1. *Let U be an open subset of $\mathbb{R}^n$ and let $f : U \longrightarrow \mathbb{R}^m$.*

(1) We say that f has a directional derivative *at the point $x \in U$ in the direction $y \in \mathbb{R}^n$ if*

$$D_y f(x) := \lim_{\varepsilon \to 0} \frac{f(x+\varepsilon y) - f(x)}{\varepsilon} = \frac{d}{d\varepsilon}[f(x+\varepsilon y)]_{|\varepsilon=0} \tag{A.1}$$

exists. If this is the case we call the vector $D_y f(x) \in \mathbb{R}^m$ the directional derivative at x in the direction y. In particular, if we choose y to be the jth unit vector $e_j = (0, \ldots, 1, \ldots, 0)$, with 1 on jth place, we get

$$D_{\varepsilon_j} f(x) = \frac{\partial f}{\partial x_j}(x)\,,$$

the jth partial derivative *of f.*

(2) We say that f is differentiable *at $x \in U$ if there exists a matrix $A \in \mathbb{R}^{m \times n}$ such that*

$$\lim_{\substack{h \to 0 \\ h \in \mathbb{R}^n}} \frac{|f(x+h) - f(x) - Ah|}{|h|} = 0. \tag{A.2}$$

If this is the case we call A the derivative *of f at x and we write*

$$A = f'(x).$$

Proposition A.2. *The following relations between the two concepts hold true.*

(1) If f is differentiable at $x \in U$, then f has a directional derivative in all directions $y \in \mathbb{R}^n$ and

$$D_y f(x) = f'(x)y = Ay. \tag{A.3}$$

(2) Conversely, if f has a directional derivative at all $x \in U$ in all the directions $y = e_j$, $j = 1, ..., n$, and all the partial derivatives

$$D_{e_j} f(x) = \frac{\partial f}{\partial x_j}(x)$$

are continuous functions of x, then f is differentiable at all $x \in U$ and

$$f'(x) = \left[\frac{\partial f_i}{\partial x_j}(x)\right]_{\substack{1 \le i \le m \\ 1 \le j \le n}} = A \in \mathbb{R}^{m \times n}\,, \tag{A.4}$$

where f_i is component number i of f, that is, $f = (f_1, ..., f_m)^T$.

We define similar operations in a more general context. First let us recall some basic concepts from functional analysis.

Definition A.3. *Let X be a Banach space, that is, a complete, normed vector space over $\mathbb{R}$, and let $\|x\|$ denote the norm of the element $x \in X$. A* linear functional *on X is a linear map*

$$T : X \to \mathbb{R}.$$

Recall that T is called linear if $T(ax+y) = aT(x) + T(y)$ for all $a \in \mathbb{R}$, $x, y \in X$. A linear functional T is called bounded *(or* continuous*) if*

$$|||T||| := \sup_{\|x\| \leq 1} |T(x)| < \infty$$

Sometimes we write $\langle T, x\rangle$ or Tx instead of $T(x)$ and call $\langle T, x\rangle$ "the action of T on x". The set of all bounded linear functionals is called the dual *of X and is denoted by X^*. Equipped with the norm $||| \cdot |||$, the space X^* is a Banach space.*

Example A.4. $X = \mathbb{R}^n$ with the Euclidean norm $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$ is a Banach space. In this case it is easy to see that we can identify X^* with R^n.

Example A.5. Let $X = C_0([0,T])$ be the space of continuous real functions ω on $[0,T]$ such that $\omega(0) = 0$. Then

$$\|\omega\|_\infty := \sup_{t \in [0,T]} |\omega(t)|$$

is a norm on X called the *uniform norm*. With this norm, X is a Banach space and its dual X^* can be identified with the space $M([0,T])$ of all signed measures ν on $[0,T]$, with norm

$$|||\nu||| = \sup_{|f| \leq 1} \int_0^T f(t) d\nu(t) = |\nu|([0,T]).$$

Example A.6. Let $X = L^p([0,T]) = \{f : [0,T] \to \mathbb{R};\ \int_0^T |f(t)|^p dt < \infty\}$ be equipped with the norm

$$\|f\|_p = \Big[\int_0^T |f(t)|^p dt\Big]^{1/p} \qquad (1 \leq p < \infty).$$

Then X is a Banach space and its dual can be identified with $L^q([0,T])$, where

$$\frac{1}{p} + \frac{1}{q} = 1\,.$$

In particular, if $p = 2$, then $q = 2$, so $L^2([0,T])$ is its own dual.

We now extend the definitions of derivative and differentiability we had for $\mathbb{R}^n$ to arbitrary Banach spaces.

Definition A.7. *Let U be an open subset of a Banach space X and let f be a function from U into $\mathbb{R}^m$.*

(1) We say that f has a directional derivative *(or* Gateaux derivative*) $D_y f(x)$ at $x \in U$ in the direction $y \in X$ if*

$$D_y f(x) := \frac{d}{d\varepsilon}[f(x + \varepsilon y)]_{\varepsilon=0} \in \mathbb{R}^m \tag{A.5}$$

exists.

(2) We say that f is Fréchet-differentiable *at $x \in U$, if there exists a bounded linear map*

$$A : X \to \mathbb{R}^m,$$

that is, $A = (A_1, ..., A_m)^T$, with $A_i \in X^*$ for $i = 1, ..., m$, such that

$$\lim_{\substack{h \to 0 \\ h \in X}} \frac{|f(x + h) - f(x) - A(h)|}{\|h\|} = 0. \tag{A.6}$$

We write

$$f'(x) = \begin{bmatrix} f'(x)_1 \\ \vdots \\ f'(x)_m \end{bmatrix} = A \quad \in (X^*)^m. \tag{A.7}$$

for the Fréchet derivative of f at x.

Similar to the Euclidean case (see Proposition A.2) we have the following result.

Proposition A.8.

(1) *If f is Fréchet-differentiable at $x \in U \subset X$, then f has a directional derivative at x in all directions $y \in X$ and*

$$D_y f(x) = \langle f'(x), y \rangle \in \mathbb{R}^m, \tag{A.8}$$

where

$$\langle f'(x), y \rangle = (\langle f'(x)_1, y \rangle, \ldots, \langle f'(x)_m, y \rangle)^T$$

is the m-vector whose ith component is the action of the ith component $f'(x)_i$ of $f'(x)$ on y.

(2) *Conversely, if f has a directional derivative at all $x \in U$ in all directions $y \in X$ and the linear map*

$$y \to D_y f(x)\,, \qquad y \in X$$

is continuous for all $x \in U$*, then there exists an element* $\nabla f(x) \in (X^*)^m$ *such that*

$$D_y f(x) = \langle \nabla f(x), y \rangle \,.$$

If this map $x \to \nabla f(x) \in (X^*)^m$ *is continuous on* U*, then* f *is Fréchet differentiable and*

$$f'(x) = \nabla f(x) \,. \tag{A.9}$$

A.2 Wiener Space, Cameron–Martin Space, and Stochastic Derivative

We now apply these operations to the Banach space $\Omega = C_0([0,T])$ considered in Example A.5 above. This space is called the *Wiener space*, because we can regard each path

$$t \to W(t,\omega)$$

of the Wiener process starting at 0 as an element ω of $C_0([0,T])$. Thus we may identify $W(t,\omega)$ with the value $\omega(t)$ at time t of an element $\omega \in C_0([0,T])$:

$$W(t,\omega) = \omega(t).$$

The space $\Omega = C_0([0,T])$ is naturally equipped with the Borel σ-algebra generated by the topology of the uniform norm. One can prove that this σ-algebra coincides with the σ-algebra generated by the cylinder sets (see, e.g., [36]). This measurable space is equipped with the probability measure P, which is given by the probability law of the Wiener process:

$$P\big\{W(t_1) \in F_1, \dots, W(t_k) \in F_k\big\}$$
$$= \int\limits_{F_1\times\cdots\times F_k} \rho(t_1,x,x_1)\rho(t_2-t_1,x,x_2)\cdots\rho(t_k-t_{k-1},x_{k-1},x_k)dx_1,\cdots dx_k,$$

where $F_i \subset \mathbb{R}$, $0 \le t_1 < t_2 < \cdots < t_k \le T$, and

$$\rho(t,x,y) = (2\pi t)^{-1/2}\exp(-\frac{1}{2}|x-y|^2), \qquad t \in [0,T], \quad x,y \in \mathbb{R}.$$

The measure P is called the *Wiener measure* on Ω.

Just as for Banach spaces, we now give the following definition.

Definition A.9. *Let* $F : \Omega \to \mathbb{R}$ *be a random variable, choose* $g \in L^2([0,T])$*, and consider*

$$\gamma(t) = \int\limits_0^t g(s)ds \quad \in \Omega. \tag{A.10}$$

Then we define the directional derivative *of* F *at the point* $\omega \in \Omega$ *in direction* $\gamma \in \Omega$ *by*

$$D_\gamma F(\omega) = \frac{d}{d\varepsilon}[F(\omega + \varepsilon\gamma)]_{|\varepsilon=0}\,, \tag{A.11}$$

if the derivative exists in some sense (to be made precise later).

Note that we consider the derivative only in special directions, namely in the directions of elements γ of the form (A.10). The set of $\gamma \in \Omega$, which can be written on the form (A.10) for some $g \in L^2([0,T])$, is called the *Cameron–Martin space* and it is hearafter denoted by H. It turns out that it is difficult to obtain a tractable theory involving derivatives in all directions. However, the derivatives in the directions $\gamma \in H$ are sufficient for our purposes.

Definition A.10. *Assume that $F : \Omega \to \mathbb{R}$ has a directional derivative in all directions γ of the form $\gamma \in H$ in the strong sense, that is,*

$$\mathbf{D}_\gamma F(\omega) := \lim_{\varepsilon \to 0} \frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon} \tag{A.12}$$

exists in $L^2(P)$. Assume in addition that there exists $\psi(t,\omega) \in L^2(P \times \lambda)$ such that

$$\mathbf{D}_\gamma F(\omega) = \int_0^T \psi(t,\omega) g(t) dt, \qquad \text{for all } \gamma \in H\,. \tag{A.13}$$

Then we say that F is differentiable *and we set*

$$\mathbf{D}_t F(\omega) := \psi(t,\omega). \tag{A.14}$$

We call $\mathbf{D}.F \in L^2(P \times \lambda)$ the stochastic derivative *of F. The set of all differentiable random variables is denoted by $\mathcal{D}_{1,2}$.*

Example A.11. Suppose $F = \int_0^T f(s) dW(s) = \int_0^T f(s) d\omega(s)$, where $f(s) \in L^2([0,T])$. Then if $\gamma \in H$, we have

$$\begin{aligned} F(\omega + \varepsilon\gamma) &= \int_0^T f(s)(d\omega(s) + \varepsilon d\gamma(s)) \\ &= \int_0^T f(s) d\omega(s) + \varepsilon \int_0^T f(s) g(s) ds, \end{aligned}$$

and hence

$$\frac{F(\omega + \varepsilon\gamma) - F(\omega)}{\varepsilon} = \int_0^T f(s) g(s) ds$$

for all $\varepsilon > 0$. Comparing with (A.13), we see that $F \in \mathcal{D}_{1,2}$ and

$$\mathbf{D}_t F(\omega) = f(t), \qquad t \in [0,T],\ \omega \in \Omega. \tag{A.15}$$

In particular, choosing

$$f(t) = \mathcal{X}_{[0,t_1]}(t)$$

we get

$$F = \int_0^T \mathcal{X}_{[0,t_1]}(s)dW(s) = W(t_1)$$

and hence

$$\mathbf{D}_t(W(t_1)) = \mathcal{X}_{[0,t_1]}(t). \tag{A.16}$$

Let $\mathbb{P}$ denote the family of all random variables $F : \Omega \to \mathbb{R}$ of the form

$$F = \varphi(\theta_1, \dots, \theta_n),$$

where $\varphi(x_1, \dots, x_n) = \sum_\alpha a_\alpha x^\alpha$, with $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ and $\alpha = (\alpha_1, ..., \alpha_n)$, is a polynomial and $\theta_i = \int_0^T f_i(t)dW(t)$ for some $f_i \in L^2([0,T])$, $i = 1, ..., n$. Such random variables are called *Wiener polynomials*. Note that $\mathbb{P}$ is dense in $L^2(P)$.

Lemma A.12. Chain rule. *Let* $F = \varphi(\theta_1, \dots, \theta_n) \in \mathbb{P}$. *Then* $F \in \mathcal{D}_{1,2}$ *and*

$$\mathbf{D}_t F = \sum_{i=1}^n \frac{\partial \varphi}{\partial \theta_i}(\theta_1, \dots, \theta_n) \cdot f_i(t). \tag{A.17}$$

Proof Let $\psi(t)$ denote the right-hand side of (A.17). Since

$$\sup_{s \in [0,T]} E[|W(s)|^N] < \infty \qquad \text{for all } N \in \mathbb{N},$$

we see that

$$\frac{1}{\varepsilon}[F(\omega + \varepsilon\gamma) - F(\omega)] = \frac{1}{\varepsilon}\Big[\varphi\big(\theta_1 + \varepsilon(f_1, g), \dots, \theta_n + \varepsilon(f_n, g)\big) - \varphi(\theta_1, \dots, \theta_n)\Big]$$
$$\longrightarrow \sum_{i=1}^n \frac{\partial \varphi}{\partial \theta_i}(\theta_1, \dots, \theta_n) \cdot \mathbf{D}_\gamma(\theta_i), \qquad \varepsilon \to 0,$$

in $L^2(P)$. Hence F has a directional derivative in direction γ in the strong sense and by (A.15) we have

$$\mathbf{D}_\gamma F = \int_0^T \psi(t)g(t)dt.$$

By this we end the proof. □

We now introduce the norm $\|\cdot\|_{1,2}$, on $\mathcal{D}_{1,2}$:

$$\|F\|_{1,2}^2 := \|F\|_{L^2(P)}^2 + \|\mathbf{D}_t F\|_{L^2(P\times\lambda)}^2, \qquad F \in \mathcal{D}_{1,2}\,. \tag{A.18}$$

Unfortunately, it is not clear if $\mathcal{D}_{1,2}$ is closed under this norm. To avoid this difficulty we work with the following family.

Definition A.13. *We define $\mathbb{D}_{1,2}$ to be the closure of the family $\mathbb{P}$ with respect to the norm $\|\cdot\|_{1,2}$.*

Thus $\mathbb{D}_{1,2}$ consists of all $F \in L^2(P)$ such that there exists $F_n \in \mathbb{P}$ with the property that

$$F_n \longrightarrow F \qquad \text{in } L^2(P) \quad \text{as } n \to \infty \tag{A.19}$$

and

$$\{\mathbf{D}_t F_n\}_{n=1}^{\infty} \qquad \text{is convergent in } L^2(P \times \lambda).$$

If this is the case, it is tempting to *define*

$$D_t F := \lim_{n\to\infty} \mathbf{D}_t F_n \,.$$

However, for this to work we need to know that this defines $D_t F$ *uniquely.* In other words, if there is another sequence $G_n \in \mathbb{P}$ such that

$$G_n \to F \qquad \text{in } L^2(P) \text{ as } n \to \infty \tag{A.20}$$

and

$$\{\mathbf{D}_t G_n\}_{n=1}^{\infty} \qquad \text{is convergent in } L^2(P \times \lambda), \tag{A.21}$$

does it follow that $\lim_{n\to\infty} \mathbf{D}_t F_n = \lim_{n\to\infty} \mathbf{D}_t G_n$?

By considering the difference $H_n = F_n - G_n$, we see that the answer to this question is positive, in view of the following theorem.

Theorem A.14. Closability of the derivative. *The operator $\mathbf{D}_t$ is closable, that is, if the sequence $\{H_n\}_{n=1}^{\infty} \subset \mathbb{P}$ is such that*

$$H_n \to 0 \qquad \text{in } L^2(P) \text{ as } n \to \infty \tag{A.22}$$

and

$$\{\mathbf{D}_t H_n\}_{n=1}^{\infty} \qquad \text{converges in } L^2(P \times \lambda) \text{ as } n \to \infty,$$

then

$$\lim_{n\to\infty} \mathbf{D}_t H_n = 0. \tag{A.23}$$

The proof is based on the following useful result.

Lemma A.15. Integration by parts formula. *Suppose $F, \varphi \in \mathcal{D}_{1,2}$ and $\gamma \in H$ with $g \in L^2([0,T])$. Then*

$$E\big[\mathbf{D}_\gamma F \cdot \varphi\big] = E\Big[F \cdot \varphi \cdot \int_0^T g(t)dW(t)\Big] - E\big[F \cdot \mathbf{D}_\gamma \varphi\big]. \tag{A.24}$$

Proof By the Cameron–Martin theorem (see, e.g., [159]) we have

$$\int_\Omega F(\omega + \varepsilon\gamma) \cdot \varphi(\omega) P(d\omega) = \int_\Omega F(\omega)\varphi(\omega - \varepsilon\gamma) Q(d\omega),$$

where

$$Q(d\omega) = \exp\Big\{\varepsilon \int_0^T g(t)dW(t) - \frac{1}{2}\varepsilon^2 \int_0^T g^2(t)dt\Big\} P(d\omega),$$

being $\omega(t) = W(t,\omega)$, $t \geq 0$, $\omega \in \Omega$, a Wiener process on the Wiener space $\Omega = C_0([0,T])$. This gives

$$\begin{aligned}
E\big[\mathbf{D}_\gamma F \cdot \varphi\big] &= \int_\Omega \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big[F(\omega+\varepsilon\gamma) - F(\omega)\Big] \cdot \varphi(\omega) P(d\omega) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega F(\omega+\varepsilon\gamma)\varphi(\omega) - F(\omega)\varphi(\omega) P(d\omega) \\
&= \lim_{\varepsilon\to 0} \frac{1}{\varepsilon} \int_\Omega F(\omega)\Big[\varphi(\omega-\varepsilon\gamma)\exp\Big\{\varepsilon \int_0^T g(t)d\omega(t) \\
&\qquad - \frac{1}{2}\varepsilon^2 \int_0^T g^2(t)dt\Big\} - \varphi(\omega)\Big] P(d\omega) \\
&= \int_\Omega F(\omega) \cdot \frac{d}{d\varepsilon}\Big[\varphi(\omega-\varepsilon\gamma)\exp(\varepsilon \int_0^T g(t)d\omega(t) \\
&\qquad - \frac{1}{2}\varepsilon^2 \int_0^T g^2(t)dt)\Big]_{|\varepsilon=0} P(d\omega) \\
&= E\Big[F\varphi \cdot \int_0^T g(t)dW(t)\Big] - E\big[F\mathbf{D}_\gamma\varphi\big].
\end{aligned}$$

By this we end the proof. □

Proof of Theorem A.14. By Lemma A.15 we get

$$E\Big[\mathbf{D}_\gamma H_n \cdot \varphi\Big] = E\Big[H_n\varphi \cdot \int_0^T g dW\Big] - E\Big[H_n \cdot \mathbf{D}_\gamma\varphi\Big] \longrightarrow 0, \qquad n \to \infty,$$

for all $\varphi \in \mathbb{P}$. Since $\{\mathbf{D}_\gamma H_n\}_{n=1}^\infty$ converges in $L^2(P)$ and $\mathbb{P}$ is dense in $L^2(P)$, we conclude that $\mathbf{D}_\gamma H_n \to 0$ in $L^2(P)$ as $n \to \infty$. Since this holds for all $\gamma \in H$, we obtain that $\mathbf{D}_t H_n \to 0$ in $L^2(P \times \lambda)$. □

In view of Theorem A.14 and the discussion preceding it, we can now make the following unambiguous definition.

Definition A.16. *Let $F \in \mathbb{D}_{1,2}$, so that there exists $\{F_n\}_{n=1}^\infty \subset \mathbb{P}$ such that*

$$F_n \to F \qquad \text{in } L^2(P)$$

and $\{\mathbf{D}_t F_n\}_{n=1}^\infty$ is convergent in $L^2(P\times\lambda)$. Then we define

$$D_t F = \lim_{n\to\infty} \mathbf{D}_t F_n \quad \text{in } L^2(P\times\lambda) \tag{A.25}$$

and

$$D_\gamma F = \int_0^T D_t F \cdot g(t)dt$$

for all $\gamma(t) = \int_0^t g(s)ds \in H$*, with* $g \in L^2([0,T])$*. We call* $D_t F$ *the* Malliavin derivative *of* F*.*

Remark A.17. Strictly speaking we now have two apparently different definitions of the derivative of F:

1. The stochastic derivative $\mathbf{D}_t F$ of $F \in \mathcal{D}_{1,2}$ given by Definition A.10.
2. The Malliavin derivative $D_t F$ of $F \in \mathbb{D}_{1,2}$ given by Definition A.16.

However, the next result shows that if $F \in \mathcal{D}_{1,2} \cap \mathbb{D}_{1,2}$, then the two derivatives coincide.

Lemma A.18. *Let* $F \in \mathcal{D}_{1,2} \cap \mathbb{D}_{1,2}$ *and suppose that* $\{F_n\}_{n=1}^\infty \subset \mathbb{P}$ *has the properties*

$$F_n \to F \quad \text{in } L^2(P) \qquad \text{and} \qquad \{\mathbf{D}_t F_n\}_{n=1}^\infty \quad \text{converges in } L^2(P \times \lambda). \tag{A.26}$$

Then

$$\mathbf{D}_t F = \lim_{n\to\infty} \mathbf{D}_t F_n \qquad \text{in } L^2(P \times \lambda)\,. \tag{A.27}$$

Hence

$$D_t F = \mathbf{D}_t F \qquad \text{for} \quad F \in \mathcal{D}_{1,2} \cap \mathbb{D}_{1,2}. \tag{A.28}$$

Proof By (A.26) we get that $\{\mathbf{D}_\gamma F_n\}_{n=1}^\infty$ converges in $L^2(P)$ for each $\gamma(t) = \int_0^t g(s)ds$ with $g \in L^2([0,T])$. By Lemma A.15 and (A.26) we get

$$E\Big[(\mathbf{D}_\gamma F_n - \mathbf{D}_\gamma F)\cdot\varphi\Big] = E\Big[(F_n - F)\cdot\varphi\cdot\int_0^t g dW\Big] - E\Big[(F_n - F)\cdot\mathbf{D}_\gamma\varphi\Big] \longrightarrow 0$$

for all $\varphi \in \mathbb{P}$. Hence $\mathbf{D}_\gamma F_n \to \mathbf{D}_\gamma F$ in $L^2(P)$ and (A.27) follows. □

In view of Lemma A.18 we now use the same symbol $D_t F$ for the derivative and $D_\gamma F$ for the directional derivative of all the elements $F \in \mathcal{D}_{1,2} \cap \mathbb{D}_{1,2}$.

Remark A.19. Note that from the definition of $\mathbb{D}_{1,2}$ follows that, if $\{F_n\}_{n=1}^\infty \in \mathbb{D}_{1,2}$ with $F_n \to F$ in $L^2(P)$ and $\{D_t F_n\}_{n=1}^\infty$ converges in $L^2(P \times \lambda)$, then

$$F \in \mathbb{D}_{1,2} \qquad \text{and} \qquad D_t F = \lim_{n\to\infty} D_t F_n\,.$$

A.3 Malliavin Derivative via Chaos Expansions

Since an arbitrary $F \in L^2(P)$ can be represented by its chaos expansion

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_n \in \widetilde{L}^2([0,T]^n)$ for all n, it is natural to ask if we can express the derivative of F (if it exists) by means of this. See Chap. 1 for the definition and properties of the Itô iterated integrals $I_n(f_n)$. Hereafer, we consider derivation according to Definition A.16 and Lemma A.18.
Let us first look at a special case.

Lemma A.20. *Suppose $F = I_n(f_n)$ for some $f_n \in \widetilde{L}^2([0,T]^n)$. Then $F \in \mathbb{D}_{1,2}$ and*

$$D_t F = n I_{n-1}(f_n(\cdot, t)), \tag{A.29}$$

where the notation $I_{n-1}(f_n(\cdot,t))$ means that the $(n-1)$-iterated Ito integral is taken with respect to the $n-1$ first variables $t_1, \dots, t_{n-1}$ of $f_n(t_1, \dots, t_{n-1}, t)$, that is, t is fixed and kept outside the integration.

Proof First consider the special case when

$$f_n = f^{\otimes n}$$

for some $f \in L^2([0,T])$, that is, when

$$f_n(t_1, \dots, t_n) = f(t_1) \dots f(t_n), \qquad (t_1, \dots, t_n) \in [0,T]^n.$$

Then exploiting the definition and properties of Hermite polynomials h_n (see (1.15)), we have

$$I_n(f_n) = \|f\|^n h_n(\frac{\theta}{\|f\|}), \tag{A.30}$$

where $\theta = \int_0^T f(t)dW(t)$. Moreover, by the chain rule (A.17) we have

$$D_t I_n(f_n) = \|f\|^n h_n'(\frac{\theta}{\|f\|}) \cdot \frac{f(t)}{\|f\|}.$$

Recall that a basic property of the Hermite polynomials is that

$$h_n'(x) = n h_{n-1}(x). \tag{A.31}$$

This gives (A.29) in this case:

$$D_t I_n(f_n) = n\|f\|^{n-1} h_{n-1}(\frac{\theta}{\|f\|}) f(t) = n I_{n-1}(f^{\otimes(n-1)}) f(t) = n I_{n-1}(f_n(\cdot, t)).$$

Next, suppose f_n has the form

$$f_n = \xi_1^{\otimes \alpha_1} \widehat{\otimes} \xi_2^{\otimes \alpha_2} \widehat{\otimes} \cdots \widehat{\otimes} \xi_k^{\otimes \alpha_k}, \qquad \alpha_1 + \cdots + \alpha_k = n, \tag{A.32}$$

where $\widehat{\otimes}$ denotes symmetrized tensor product and $\{\xi_j\}_{j=1}^{\infty}$ is an orthonormal basis for
$L^2([0,T])$. Then by an extension of (1.15) we have (see [120])

$$I_n(f_n) = h_{\alpha_1}(\theta_1) \cdots h_{\alpha_k}(\theta_k) \tag{A.33}$$

with

$$\theta_j = \int_0^T \xi_j(t) dW(t)$$

and again (A.29) follows by the chain rule (A.17). Since any $f_n \in \widetilde{L}^2([0,T]^n)$ can be approximated in $L^2([0,T]^n)$ by linear combinations of functions of the form given by (A.32), the general result follows. □

Lemma A.21. *Let $\mathbb{P}_0 \subseteq \mathbb{P}$ denote the set of Wiener polynomials of the form*

$$p_k\Big(\int_0^T \xi_1(t)dW(t), \ldots, \int_0^T \xi_k(t)dW(t)\Big),$$

where $p_k(x_1, \ldots, x_k)$ is an arbitrary polynomial in k variables and $\{\xi_1, \xi_2, \ldots\}$ is a given orthonormal basis for $L^2([0,T])$. Then $\mathbb{P}_0$ is dense in $\mathbb{P}$ in the norm $\| \cdot \|_{1,2}$.

Proof Let $q := p\Big(\int_0^T f_1(t)dW(t), \ldots, \int_0^T f_k(t)dW(t)\Big) \in \mathbb{P}$. We approximate q by

$$q^{(m)} := p\Big(\int_0^T \sum_{j=0}^{m} (f_1, \xi_j)_{L^2([0,T])} \xi_j(t) dW(t), \ldots, \int_0^T \sum_{j=0}^{m} (f_k, \xi_j)_{L^2([0,T])} \xi_j(t) dW(t)\Big).$$

Then $q^{(m)} \to q$ in $L^2(P)$ and

$$D_t q^{(m)} = \sum_{i=1}^{k} \frac{\partial p}{\partial x_i} \cdot \sum_{j=1}^{m} (f_i, \xi_j)_{L^2([0,T])} \xi_j(t) \to \sum_{i=1}^{k} \frac{\partial p}{\partial x_i} \cdot f_i(t)$$

in $L^2(P \times \lambda)$ as $m \to \infty$. □

Theorem A.22. *Let* $F = \sum_{n=0}^{\infty} I_n(f_n) \in L^2(P)$. *Then* $F \in \mathbb{D}_{1,2}$ *if and only if*

$$\sum_{n=1}^{\infty} n\, n! \|f_n\|^2_{L^2([0,T]^n)} < \infty \tag{A.34}$$

and if this is the case we have

$$D_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(\cdot, t)). \tag{A.35}$$

Proof Define $F_m = \sum_{n=0}^{m} I_n(f_n)$. Then $F_m \in \mathbb{D}_{1,2}$ and $F_m \to F$ in $L^2(P)$. Moreover, if $m > k$ we have

$$\begin{aligned}
\|D_t F_m - D_t F_k\|^2_{L^2(P\times\lambda)} &= \| \sum_{n=k+1}^{m} n I_{n-1}(f_n(\cdot, t))\|^2_{L^2(P\times\lambda)} \\
&= \int_0^T E\Big[\Big(\sum_{n=k+1}^{m} n I_{n-1}(f_n(\cdot, t))\Big)^2\Big] dt \\
&= \int_0^T \sum_{n=k+1}^{m} n^2 (n-1)! \|f_n(\cdot, t)\|^2_{L^2([0,T]^{n-1})} dt \\
&= \sum_{n=k+1}^{m} n\, n! \|f_n\|^2_{L^2([0,T]^n)},
\end{aligned} \tag{A.36}$$

by Lemma A.20 and the orthogonality of the iterated Itô integrals (see (1.12)). Hence if (A.34) holds then $\{D_t F_n\}_{n=1}^{\infty}$ is convergent in $L^2(P \times \lambda)$ and hence $F \in \mathbb{D}_{1,2}$ and

$$D_t F = \lim_{m\to\infty} D_t F_m = \sum_{n=0}^{\infty} n I_{n-1}(f_n(\cdot, t)).$$

Conversely, if $F \in \mathbb{D}_{1,2}$ then, thanks to Lemma A.21, there exist polynomials $p_k(x_1, \ldots, x_{n_k}) = \sum_{m_i : \sum m_i \le k} a_{m_1,\ldots,m_{n_k}} \prod_{i=1}^{n_k} h_{m_i}(x_i)$ of degree k for some $a_{m_1,\ldots,m_{n_k}} \in \mathbb{R}$, such that if we put $F_k = p_k(\theta_1, \ldots, \theta_{n_k})$ then $F_k \in \mathbb{P}$ and $F_k \to F$ in $L^2(P)$ and

$$D_t F_k \to D_t F \qquad \text{in } L^2(P \times \lambda) \text{ as } k \to \infty.$$

By applying (A.33) we see that there exist $f_j^{(k)} \in \widetilde{L}^2([0,T]^j)$, $1 \le j \le k$, such that

$$F_k = \sum_{j=0}^{k} I_j(f_j^{(k)}).$$

Since $F_k \to F$ in $L^2(P)$ we have

$$\sum_{j=0}^{k} j! \| f_j^{(k)} - f_j \|^2_{L^2([0,T]^j)} \leq \| F_k - F \|^2_{L^2(P)} \to 0 \qquad \text{as } k \to \infty.$$

Therefore, $\| f_j^{(k)} - f_j \|_{L^2([0,T]^j)} \longrightarrow 0$ as $k \to \infty$, for all j. This implies that

$$\| f_j^{(k)} \|_{L^2([0,T]^j)} \longrightarrow \| f_j \|_{L^2([0,T]^j)} \qquad \text{as } k \to \infty, \text{ for all } j. \tag{A.37}$$

Similarly, since $D_t F_k \to D_t F$ in $L^2(P \times \lambda)$, we get by the Fatou lemma combined with the calculation, leading to (A.36) that

$$\begin{aligned}
\sum_{j=0}^{\infty} j \cdot j! \| f_j \|^2_{L^2([0,T]^j)} &= \sum_{j=0}^{\infty} \lim_{k \to \infty} \big(j \cdot j! \| f_j^{(k)} \|^2_{L^2([0,T]^j)} \big) \\
&\leq \varliminf_{k \to \infty} \sum_{j=0}^{\infty} j \cdot j! \| f_j^{(k)} \|^2_{L^2([0,T]^j)} \\
&= \varliminf_{k \to \infty} \| D_t F_k \|^2_{L^2(P \times \lambda)} \\
&= \| D_t F \|^2_{L^2(P \times \lambda)} < \infty ,
\end{aligned}$$

where we have put $f_j^{(k)} = 0$ for $j > k$. Hence (A.34) holds and the proof is complete. □

Solutions

In this chapter we present a solution to the exercises marked with **(*)** in the book. The level of the exposition varies from fully detailed to just sketched.

Problems of Chap. 1

1.1 Solution

(a) Consider the following equalities:

$$\begin{aligned}
\exp\Big\{tx-\frac{t^2}{2}\Big\} &= \exp\Big\{\frac{1}{2}x^2\Big\}\exp\Big\{-\frac{1}{2}(x-t)^2\Big\} \\
&= \exp\Big\{\frac{1}{2}x^2\Big\}\sum_{n=0}^{\infty}\frac{t^n}{n!}\frac{d^n}{dt^n}\exp\Big\{-\frac{1}{2}(x-t)^2)\Big\}|_{t=0} \\
&= \exp\Big\{\frac{1}{2}x^2\Big\}\sum_{n=0}^{\infty}\{\frac{(-1)^n t^n}{n!}\frac{d^n}{du^n}\exp\Big\{-\frac{1}{2}u^2)\Big\}|_{u=x} \\
&= \exp\Big\{\frac{1}{2}x^2\Big\}\sum_{n=0}^{\infty}\frac{(-1)^n t^n}{n!}\frac{d^n}{dx^n}\exp\Big\{-\frac{1}{2}x^2\Big\} \\
&= \sum_{n=0}^{\infty}\frac{t^n}{n!}h_n(x),
\end{aligned}$$

with the substitution $u = x - t$.

(b) Set $u = t\sqrt{\lambda}$. Using the result in (a), we get

$$\begin{aligned}
\exp\Big\{tx-\frac{t^2\lambda}{2}\Big\} &= \exp\Big\{u\frac{x}{\sqrt{\lambda}}-\frac{u^2}{2}\Big\} \\
&= \sum_{n=0}^{\infty}\frac{u^n}{n!}h_n\Big(\frac{x}{\sqrt{\lambda}}\Big) \\
&= \sum_{n=0}^{\infty}\frac{t^n\lambda^{n/2}}{n!}h_n\Big(\frac{x}{\sqrt{\lambda}}\Big).
\end{aligned}$$

(c) If we choose $x = \theta$, $\lambda = \|g\|^2$, and $t = 1$ in (b), we get

$$\exp\Big\{\int_0^T g dW - \frac{1}{2}\|g\|^2\Big\} = \sum_{n=0}^{\infty} \frac{\|g\|^n}{n!} h_n\big(\frac{\theta}{\|g\|}\big).$$

(d) In particular, if we choose $g(s) = \chi_{[0,t]}(s)$, $s \in [0, T]$, we get

$$\exp\{W(t) - \frac{1}{2}t\} = \sum_{n=0}^{\infty} \frac{t^{n/2}}{n!} h_n\big(\frac{W(t)}{\sqrt{t}}\big). \quad \square$$

1.3 Solution

(a) $\xi = W(t) = \int_0^T \chi_{[0,t]}(s) dW(s)$, so $f_0 = 0$, $f_1 = \chi_{[0,t]}$, and $f_n = 0$ for $n \geq 2$.

(b) $\xi = \int_0^T g(s) dW(s)$, so $f_0 = 0$, $f_1 = g$, and $f_n = 0$ for $n \geq 2$.

(c) Since

$$\int_0^t \int_0^{t_2} 1\, dW(t_1) dW(t_2) = \int_0^t W(t_2) dW(t_2) = \frac{1}{2}W^2(t) - \frac{1}{2}t\,,$$

we get that

$$\begin{aligned} W^2(t) =& t + 2\int_0^t \int_0^{t_2} 1\, dW(t_1) dW(t_2) \\ =& t + 2\int_0^T \int_0^{t_2} \chi_{[0,t]}(t_1)\chi_{[0,t]}(t_2) dW(t_1) dW(t_2) = t + I_2[f_2]. \end{aligned}$$

Thus $f_0 = t$,

$$f_2(t_1, t_2) = \chi_{[0,t]}(t_1)\chi_{[0,t]}(t_2) =: \chi_{[0,t]}^{\otimes 2},$$

and $f_n = 0$ for $n \neq 2$.

(d) By Problem 1.1 (c) and (1.15), we have

$$\begin{aligned} \xi &= \exp\Big\{\int_0^T g(s) dW(s)\Big\} \\ &= \exp\{\frac{1}{2}\|g\|^2\} \sum_{n=0}^{\infty} \frac{\|g\|^n}{n!} h_n\big(\frac{\theta}{\|g\|}\big) \\ &= \exp\{\frac{1}{2}\|g\|^2\} \sum_{n=0}^{\infty} J_n[g^{\otimes n}] = \sum_{n=0}^{\infty} \frac{1}{n!} \exp\{\frac{1}{2}\|g\|^2\} I_n[g^{\otimes n}]. \end{aligned}$$

Hence

$$f_n = \frac{1}{n!}\exp(\frac{1}{2}\|g\|^2)g^{\otimes n}, \qquad n = 0, 1, 2, \ldots,$$

where

$$g^{\otimes n}(x_1, \ldots, x_n) := g(x_1)g(x_2)\cdots g(x_n).$$

(e) We have the following equalities:

$$\begin{aligned} \xi &= \int_0^T g(s)W(s)\,ds \\ &= \int_0^T g(s)\int_0^s 1dW(t)\,ds \\ &= \int_0^T \int_t^T g(s)ds\,dW(t) \\ &= I_1(f_1), \end{aligned}$$

where $f_1(t) := \int_t^T g(s)ds$, $t \in [0, T]$. □

1.4 Solution

(a) Since $\int\limits_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T$, we have

$$F = W^2(T) = T + 2\int\limits_0^T W(t)dW(t).$$

Hence $E[F] = T$ and $\varphi(t) = 2W(t)$, $t \in [0, T]$.

(b) Define $M(t) = \exp\{W(t) - \frac{1}{2}t\}$, $t \in [0, T]$. Then by the Itô formula

$$dM(t) = M(t)dW(t)$$

and therefore

$$M(T) = 1 + \int\limits_0^T M(t)dW(t).$$

Moreover,

$$F = \exp\{W(T)\} = \exp\{\frac{T}{2}\} + \exp\{\frac{T}{2}\}\int\limits_0^T \exp\{W(t) - \frac{1}{2}t\}dW(t).$$

Hence

$$E[F] = \exp\{\frac{T}{2}\} \quad \text{and} \quad \varphi(t) = \exp\{W(t) + \frac{T-t}{2}\}, \quad t \in [0, T].$$

(c) Integration by parts (application of the Itô formula) gives

$$F = \int_0^T W(t)dt = TW(T) - \int_0^T t dW(t) = \int_0^T (T-t)dW(t).$$

Hence, $E[F] = 0$ and $\varphi(t) = T - t$, $t \in [0,T]$.

(d) By the Itô formula

$$dW^3(t) = 3W^2(t)dW(t) + 3W(t)dt.$$

Hence

$$F = W^3(T) = 3\int_0^T W^2(t)dW(t) + 3\int_0^T W(t)dt.$$

Therefore, by (c) we get

$$E[F] = 0 \quad \text{and} \quad \varphi(t) = 3W^2(t) + 3T(1-t), \quad t \in [0,T].$$

(e) Put $X(t) = e^{\frac{1}{2}t}$, $Y(t) = \cos W(t)$, $N(t) = X(t)Y(t)$, $t \in [0,T]$. Then we have

$$\begin{aligned} dN(t) &= X(t)dY(t) + Y(t)dX(t) + dX(t)dY(t) \\ &= e^{\frac{1}{2}t}[-\sin W(t)dW(t) - \frac{1}{2}\cos W(t)dt] + \cos W(t)\, e^{\frac{1}{2}t}\,\frac{1}{2}dt \\ &= -e^{\frac{1}{2}t}\sin W(t)dW(t). \end{aligned}$$

Hence

$$e^{\frac{1}{2}T}\cos W(T) = 1 - \int_0^T e^{\frac{1}{2}t}\sin W(t)dW(t)$$

and also

$$F = \cos W(T) = e^{-\frac{1}{2}T} - e^{-\frac{1}{2}T}\int_0^T e^{\frac{1}{2}t}\sin W(t)dW(t).$$

Hence $E[F] = e^{-\frac{1}{2}T}$ and $\varphi(t) = -e^{\frac{1}{2}(t-T)}\sin W(t)$, $t \in [0,T]$. □

1.5 Solution

(a) By Itô formula and Kolmogorov backward equation we have

$$
\begin{aligned}
dY(t) &= \frac{\partial g}{\partial t}(t, X(t))dt + \frac{\partial g}{\partial x}(t, X(t))dX(t) + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X(t))(dX(t))^2 \\
&= \frac{\partial}{\partial t}[P_{T-t}f(\xi)]_{|\xi=X(t)}dt + \sigma(X(t))\frac{\partial}{\partial \xi}[P_{T-t}f(\xi)]_{|\xi=X(t)}dW(t) \\
&\qquad +\{b(X(t))\frac{\partial}{\partial \xi}[P_{T-t}f(\xi)]_{|\xi=X(t)} + \frac{1}{2}\sigma^2(X(t))\frac{\partial^2}{\partial \xi^2}[P_{T-t}f(\xi)]_{|\xi=X(t)}\}dt \\
&= \frac{\partial}{\partial t}[P_{T-t}f(\xi)]_{|\xi=X(t)}dt + \sigma(X(t))\frac{\partial}{\partial \xi}[P_{T-t}f(\xi)]_{|\xi=X(t)}dW(t) \\
&\qquad + \frac{\partial}{\partial u}[P_u f(\xi)]_{\substack{|\xi=X(t)\\ |u=T-t}} dt \\
&= \sigma(X(t))\frac{\partial}{\partial \xi}[P_{T-t}f(\xi)]_{|\xi=X(t)}dW(t).
\end{aligned}
$$

Hence

$$
Y(T) = Y(0) + \int_0^T [\sigma(x)\frac{\partial}{\partial \xi}P_{T-t}f(\xi)]_{|\xi=X(t)}dW(t).
$$

Since $Y(T) = g(T, X(T)) = [P_0 f(\xi)]_{|\xi=X(T)} = f(X(T))$ and $Y(0) = g(0, X(0)) = P_T f(X)$, (1.29) follows.

(b.1) If $F = W^2(T)$, we apply (a) to the case when $f(\xi) = \xi^2$ and $X(t) = x + W(t)$ (assuming $W(0) = 0$ as before). This gives

$$
P_s f(\xi) = E^\xi[f(X(x))] = E^\xi[X^2(s)] = \xi^2 + s
$$

and hence

$$
E[F] = P_T f(x) = x^2 + T
$$

and

$$
\varphi(t) = [\frac{\partial}{\partial \xi}(\xi^2 + s)]_{|\xi=x+W(t)} = 2W(t) + 2x.
$$

(b.2) If $F = W^3(T)$, we choose $f(\xi) = \xi^3$ and $X(t) = x + W(t)$ and get

$$
P_s f(\xi) = E^\xi[X^3(s)] = \xi^3 + 3s\xi\,.
$$

Hence

$$
E[F] = P_T f(x) = x^3 + 3Tx
$$

and

$$
\varphi(t) = [\frac{\partial}{\partial \xi}(\xi^3 + 3(T-t)\xi)]_{|\xi=x+W(t)} = 3(x + W(t))^2 + 3(T-t).
$$

(b.3) In this case $f(\xi) = \xi$, so

$$
P_s f(\xi) = E^\xi[X(s)] = \xi e^{\rho s}
$$

and so

$$
E[F] = P_T f(x) = x e^{\rho T}
$$

and

$$\begin{aligned}\varphi(t) &= [\alpha\xi\frac{\partial}{\partial\xi}\big(\xi e^{\rho(T-t)}\big)]_{|\xi=X(t)}\\ &= \alpha X(t)\exp(\rho(T-t))\\ &= \alpha x\exp\big\{\rho T-\frac{1}{2}\alpha^2 t+\alpha W(t)\big\}.\end{aligned}$$

(c) We proceed as in (a) and put

$$Y(t)=g(t,X(t)),\quad t\in[0,T],\qquad \text{with}\quad g(t,x)=P_{T-t}f(x)$$

and

$$dX(t)=b(X(t))dt+\sigma(X(t))dW(t)\,;\qquad X(0)=x\in\mathbb{R}^n,$$

where

$$b:\mathbb{R}^n\to\mathbb{R}^n,\quad \sigma:\mathbb{R}^n\to\mathbb{R}^{n\times m}\qquad \text{and}\quad W(t)=(W_1(t),\ldots,W_m(t))$$

is the m-dimensional Wiener process. Then by Itô formula and (1.31), we have

$$\begin{aligned}dY(t) &= \frac{\partial g}{\partial t}(t,X(t))dt+\sum_{i=1}^{n}\frac{\partial g}{\partial x_i}(t,X(t))dX_i(t)\\ &\quad +\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 g}{\partial x_i\partial x_j}(t,X(t))dX_i(t)dX_j(t)\\ &= \frac{\partial}{\partial t}[P_{T-t}f(\xi)]_{|\xi=X(t)}dt+[\sigma^T(\xi)\nabla_\xi(P_{T-t}f(\xi))]_{|\xi=X(t)}dW(t)\\ &\quad +[L_\xi(P_{T-t}f(\xi))]_{|\xi=X(t)}dt,\end{aligned}$$

where

$$L_\xi=\sum_{i=1}^{n}b_i(\xi)\frac{\partial}{\partial\xi_i}+\frac{1}{2}\sum_{i,j=1}^{n}(\sigma\sigma^T)_{ij}(\xi)\frac{\partial^2}{\partial\xi_i\partial\xi_j}$$

is the generator of the Itô diffusion $X(t)$, $t\geq 0$. So by the Kolmogorov backward equation we get

$$dY(t)=[\sigma^T(\xi)\nabla_\xi(P_{T-t}f(\xi)]_{|\xi=X(t)}dW(t)$$

and hence, as in (a),

$$Y(T)=f(X(T))=P_Tf(x)+\int_0^T[\sigma^T(\xi)\nabla_\xi(P_{T-t}f(\xi))]_{|\xi=X(t)}dW(t),$$

which gives, with $F=f(X(T))$,

$$E[F]=P_Tf(x)\quad\text{and}\quad \varphi(t)=[\sigma^T(\xi)\nabla_\xi(P_{T-t}f(\xi))]_{|\xi=X(t)},\quad t\in[0,T].\quad\square$$

Problems of Chap. 2

2.4 Solution

(a) Since $W(t)$, $t \in [0,T]$, is $\mathbb{F}$-adapted, we have

$$\int_0^T W(t)\delta W(t) = \int_0^T W(t)dW(t) = \frac{1}{2}W^2(T) - \frac{1}{2}T.$$

(b) $\int_0^T \Big(\int_0^T g(s)dW(s)\Big)\delta W(t) = \int_0^T I_1[f_1(\cdot,t)]\delta W(t) = I_2[\widetilde{f}_1]$, where $f_1(t_1,t) = g(t_1)$, $t \in [0,T]$. This gives

$$\widetilde{f}_1(t_1,t) = \frac{1}{2}\big[g(t_1) + g(t)\big], \quad t \in [0,T],$$

and hence

$$\begin{aligned} I_2[\widetilde{f}_1] &= 2\int_0^T \int_0^{t_2} \widetilde{f}_1(t_1,t_2)dW(t_1)dW(t_2) \\ &= \int_0^T\int_0^{t_2} g(t_1)dW(t_1)dW(t_2) + \int_0^T\int_0^{t_2} g(t_2)dW(t_1)dW(t_2) \qquad \text{(S.1)} \\ &= \int_0^T\int_0^{t_2} g(t_1)dW(t_1)dW(t_2) + \int_0^T W(t_2)g(t_2)dW(t_2). \end{aligned}$$

Using integration by parts (i.e., the Itô formula), we see that

$$\begin{aligned} \Big(\int_0^T g(t_1)dW(t_1)\Big)W(T) &= \int_0^T\int_0^{t_2} g(t_1)dW(t_1)dW(t_2) \\ &\quad + \int_0^T g(t_2)W(t_2)dW(t_2) + \int_0^T g(t)dt. \end{aligned} \qquad \text{(S.2)}$$

Combining (S.1) and (S.2), we get

$$\int_0^T \Big(\int_0^T g(s)dW(s)\Big)\delta W(t) = \Big(\int_0^T g(t)dW(t)\Big)W(T) - \int_0^T g(t)dt.$$

(c) By Problem 1.3 (c) we have

$$\int_0^T W^2(t_0)\delta W(t) = \int_0^T \big(t_0 + I_2[f_2(\cdot,t)]\big)\delta W(t),$$

where

$$f_2(t_1,t_2,t) = \chi_{[0,t_0]}(t_1)\chi_{[0,t_0]}(t_2), \quad t \in [0,T].$$

Now

$$\begin{aligned}
\widetilde{f_2}(t_1,t_2,t) &= \frac{1}{3}\big[f_2(t_1,t_2,t) + f_2(t,t_2,t_1) + f_2(t_1,t,t_2)\big] \\
&= \frac{1}{3}\big[\chi_{[0,t_0]}(t_1)\chi_{[0,t_0]}(t_2) + \chi_{[0,t_0]}(t)\chi_{[0,t_0]}(t_2) + \chi_{[0,t_0]}(t_1)\chi_{[0,t_0]}(t)\big] \\
&= \frac{1}{3}\big[\chi_{\{t_1,t_2<t_0\}} + \chi_{\{t,t_2<t_0\}} + \chi_{\{t_1,t<t_0\}}\big] \\
&= \chi_{\{t,t_1,t_2<t_0\}} + \frac{1}{3}\chi_{\{t_1,t_2<t_0<t\}} + \frac{1}{3}\chi_{\{t,t_2<t_0<t_1\}} + \frac{1}{3}\chi_{\{t,t_1<t_0<t_2\}}
\end{aligned}$$

and hence, using (1.15),

$$\begin{aligned}
\int_0^T W^2(t_0)\delta W(t) &= t_0 W(T) + \int_0^T I_2[f_2(\cdot,t)]\delta W(t) \\
&= t_0 W(T) + I_3[\widetilde{f_2}] = t_0 W(T) + 6J_3[\widetilde{f_2}] \\
&= t_0 W(T) \\
&\quad + 6\int_0^T\int_0^{t_3}\int_0^{t_2} \chi_{[0,t_0]}(t_1)\chi_{[0,t_0]}(t_2)\chi_{[0,t_0]}(t_3)dW(t_1)dW(t_2)dW(t_3) \\
&\quad + 6\int_0^T\int_0^{t_3}\int_0^{t_2} \frac{1}{3}\chi_{\{t_1,t_2<t_0<t_3\}}dW(t_1)dW(t_2)dW(t_3) \\
&= t_0 W(T) + t_0^{3/2}h_3\Big(\frac{W(t_0)}{\sqrt{t_0}}\Big) \\
&\quad + 2\int_{t_0}^T\int_0^{t_0}\int_0^{t_2} dW(t_1)dW(t_2)dW(t_3) \\
&= t_0 W(T) + t_0^{3/2}\Big(\frac{W^3(t_0)}{t_0^{3/2}} - 3\frac{W(t_0)}{\sqrt{t_0}}\Big) \\
&\quad + 2\int_{t_0}^T \Big(\frac{1}{2}W^2(t_0) - \frac{1}{2}t_0\Big)dW(t_3) \\
&= t_0 W(T) + W^3(t_0) - 3t_0 W(t_0) + \big(W^2(t_0) - t_0\big)\big(W(T) - W(t_0)\big) \\
&= W^2(t_0)W(T) - 2t_0 W(t_0).
\end{aligned}$$

(d) By Problem 1.3 (d) and (1.15) we get

$$\int_0^T \exp(W(T))\delta W(t) = \int_0^T \sum_{n=0}^{\infty} \frac{1}{n!} e^{T/2} I_n[1]\, \delta W(t)$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!} e^{T/2} I_{n+1}[1]$$
$$= e^{T/2} \sum_{n=0}^{\infty} \frac{1}{n!} T^{\frac{n+1}{2}} h_{n+1}\big(\frac{W(T)}{\sqrt{T}}\big).$$

(e) Using Problem 1.3 (e) we have that

$$F = \int_0^T F\delta W(t) = I_1(f_1),$$

with $f_1(t) := \int_t^T g(s)ds$, $t \in [0,T]$. Hence

$$\int_0^T F\delta W(t) = I_2(\widetilde{f}_1),$$

where

$$\widetilde{f}_1(t_1,t_2) = \frac{1}{2}\Big(\int_{t_1}^T g(s)ds + \int_{t_2}^T g(s)ds\Big).$$

This gives

$$\begin{aligned}
F =& \int_0^T F\delta W(t) \\
=& J_2\Big(\int_{t_1}^T g(s)ds\Big) + J_2\Big(\int_{t_2}^T g(s)ds\Big) \\
=& \int_0^T \int_0^{t_2} \Big(\int_{t_1}^T g(s)ds\Big) dW(t_1)dW(t_2) \\
&+ \int_0^T \Big(\int_0^{t_2} 1dW(t_1)\Big)\Big(\int_{t_2}^T g(s)ds\Big) dW(t_2) \\
=& \int_0^T \Big(\int_0^{t_2} g(s)W(s)ds\Big) dW(t_2) + \int_0^T 2W(t_2)\Big(\int_{t_2}^T g(s)ds\Big) dW(t_2). \quad \square
\end{aligned}$$

Problems of Chap. 3

3.2 Solution

(a) $D_t W(T) = \chi_{[0,T]}(t) = 1$, $t \in [0,T]$, by (3.8).

(b) By (3.8) we get

$$D_t \int_0^T s^2 dW(s) = t^2.$$

(c) By (3.2) we have

$$\begin{aligned} D_t \int_0^T \int_0^{t_2} \cos(t_1 + t_2) dW(t_1) dW(t_2) &= D_t\big(\frac{1}{2} I_2[\cos(t_1 + t_2)]\big) \\ &= \frac{1}{2}\, 2\, I_1[\cos(\cdot + t)] \\ &= \int_0^T \cos(t_1 + t) dW(t_1). \end{aligned}$$

(d) By the chain rule, we get

$$\begin{aligned} D_t\big(3W(s_0)W^2(t_0) + \log(1 + W^2(s_0))\big) = &\Big[3W^2(t_0) + \frac{2W(s_0)}{1 + W^2(s_0)}\Big]\, \chi_{[0,s_0]}(t) \\ &+ 6W(s_0)W(t_0)\chi_{[0,t_0]}(t). \end{aligned}$$

(e) By Problem 2.4 (b) we have

$$\begin{aligned} D_t \int_0^T W(t_0)\delta W(t) &= D_t\big(W(t_0)W(T) - t_0\big) \\ &= W(t_0)\, \chi_{[0,T]}(t) + W(T)\chi_{[0,t_0]}(t) \\ &= W(t_0) + W(T)\chi_{[0,t_0]}(t). \quad \square \end{aligned}$$

3.3 Solution

(a) By Problem 1.3 (d) and (3.2), we have

$$\begin{aligned} D_t \exp\Big\{\int_0^T g(s) dW(s)\Big\} &= D_t \sum_{n=0}^{\infty} I_n[f_n] = \sum_{n=1}^{\infty} n I_{n-1}[f_n(\cdot, t)] \\ &= \sum_{n=1}^{\infty} n\, \frac{1}{n!} \exp\big\{\frac{1}{2}\|g\|^2\big\} I_{n-1}\big[g(t_1) \dots g(t_{n-1}) g(t)\big] \\ &= g(t) \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \exp\big\{\frac{1}{2}\|g\|^2\big\} I_{n-1}\big[g^{\otimes(n-1)}\big] \\ &= g(t) \exp\Big\{\int_0^T g(s) dW(s)\Big\}. \end{aligned}$$

(b) The suggested chain rule and (3.8) give

$$D_t \exp\Big\{\int_0^T g(s)dW(s)\Big\} = \exp\Big\{\int_0^T g(s)dW(s)\Big\} D_t\Big(\int_0^T g(s)dW(t)\Big)$$
$$= g(t)\exp\Big\{\int_0^T g(s)dW(s)\Big\}.$$

(c) The points above together with Corollary 3.13 give

$$D_t \exp\{W(t_0)\} = \exp\{W(t_0)\}\,\chi_{[0,t_0]}(t). \quad \square$$

Problems of Chap. 4

4.1 Solution

(a) If $s > t$ we have

$$\begin{aligned} E_Q[\widetilde{W}(s)|\mathcal{F}_t] &= \frac{E[Z(T)\widetilde{W}(s)|\mathcal{F}_t]}{E[Z(T)|\mathcal{F}_t]} \\ &= \frac{E[Z(T)\widetilde{W}(s)|\mathcal{F}_t]}{Z(t)} \qquad \text{(S.3)} \\ &= Z^{-1}(t)E[E[Z(T)\widetilde{W}(s)|\mathcal{F}_s]|\mathcal{F}_t] \\ &= Z^{-1}(t)E[\widetilde{W}(s)E[Z(T)|\mathcal{F}_s]|\mathcal{F}_t] \\ &= Z^{-1}(t)E[\widetilde{W}(s)Z(s)|\mathcal{F}_t]. \qquad \text{(S.4)} \end{aligned}$$

Applying Itô formula to $Y(t) := Z(t)\widetilde{W}(t)$, we get

$$\begin{aligned} dY(t) &= Z(t)d\widetilde{W}(t) + \widetilde{W}(t)dZ(t) + d\widetilde{W}(t)dZ(t) \\ &= Z(t)[\theta(t)dt + dW(t)] + \widetilde{W}(t)[-\theta(t)Z(t)dW(t)] - \theta(t)Z(t)dt \\ &= Z(t)[1 - \theta(t)\widetilde{W}(t)]dW(t), \end{aligned}$$

and hence $Y(t)$ is an $\mathcal{F}_t$-martingale (with respect to P). Therefore, by (S.3),

$$E_Q[\widetilde{W}(s)|\mathcal{F}_t] = Z^{-1}(t)E[Y(s)|\mathcal{F}_t] = Z^{-1}(t)Y(t) = \widetilde{W}(t).$$

(b) We apply the Girsanov theorem to the case with $\theta(t) = a$. Then X is a Wiener process with respect to the measure Q defined by

$$Q(d\omega) = Z(T,\omega)P(d\omega) \qquad \text{on } \mathcal{F}_T,$$

where

$$Z(t) = \exp\Big\{-aW(t) - \frac{1}{2}a^2 t\Big\}, \qquad 0 \le t \le T.$$

(c) In this case we have

$$\beta(t) = bY(t), \quad \alpha(t) = aY(t), \quad \gamma(t) = cY(t)$$

and hence we put

$$\theta = \frac{\beta(t) - \alpha(t)}{\gamma(t)} = \frac{b-a}{c}$$

and

$$Z(t) = \exp\big\{-\theta W(t) - \frac{1}{2}\theta^2 t\big\}, \qquad 0 \le t \le T.$$

Then

$$\widetilde{W}(t) := \theta t + W(t), \quad 0 \le t \le T,$$

is a Wiener process with respect to the measure Q defined by $Q(d\omega) = Z(T,\omega)P(d\omega)$ on $\mathcal{F}_T$ and

$$dY(t) = bY(t)dt + cY(t)[d\widetilde{W}(t) - \theta dt] = aY(t)dt + cY(t)d\widetilde{W}(t). \quad \square$$

4.2 Solution

(a) $F = W(T)$ implies $D_t F = \chi_{[0,T]}(t) = 1$, for $t \in [0,T]$, and hence

$$E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dW(t) = \int_0^T 1\,dW(t) = W(T) = F.$$

(b) $F = \int_0^T W(s)ds$ implies $D_t F = \int_0^T D_t W(s)ds = \int_0^T \chi_{[0,s]}(t)ds = \int_t^T ds = T - t$, which gives

$$\begin{aligned} E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dW(t) &= \int_0^T (T-t)dW(t) \\ &= \int_0^T W(s)dW(s) = F, \end{aligned}$$

using integration by parts.

(c) $F = W^2(T)$ implies $D_t F = 2W(T)\,D_t W(T) = 2W(T)$. Hence

$$\begin{aligned} E[F] + \int_0^T E[D_t F|\mathcal{F}_t]dW(t) &= T + \int_0^T E[2W(T)|\mathcal{F}_t]dW(t) \\ &= T + 2\int_0^T W(t)dW(t) \\ &= T + W^2(T) - T = W^2(T) = F. \end{aligned}$$

(d) $F = W^3(T)$ implies $D_tF = 3W^2(T)$. Hence, by Itô formula,

$$E[F] + \int_0^T E[D_tF|\mathcal{F}_t]dW(t) = \int_0^T E[3W^2(T)|\mathcal{F}_t]dW(t)$$

$$= 3\int_0^T E[(W(T) - W(t))^2 + 2W(t)W(T) - W^2(t)|\mathcal{F}_t]dW(t)$$

$$= 3\int_0^T (T-t)dW(t) + 6\int_0^T W^2(t)dW(t) - 3\int_0^T W^2(t)dW(t)$$

$$= 3\int_0^T W^2(t)dW(t) - 3\int_0^T W(t)dt = W^3(T).$$

(e) $F = \exp\{W(T)\}$ implies $D_tF = \exp\{W(T)\}$. Hence

$$RHS = E[F] + \int_0^T E[D_tF|\mathcal{F}_t]dW(t)$$

$$= e^{T/2} + \int_0^T E[\exp\{W(T)\}|\mathcal{F}_t]dW(t)$$

$$= e^{T/2} + \int_0^T E[\exp\big\{W(T) - \frac{1}{2}T\big\}\, e^{T/2}|\mathcal{F}_t]dW(t)$$

$$= e^{T/2} + \exp\{\frac{1}{2}T\}\int_0^T \exp\big\{W(t) - \frac{1}{2}t\big\}dW(t). \tag{S.5}$$

Here we have used that

$$M(t) := \exp\Big\{W(t) - \frac{1}{2}t\Big\}$$

is a martingale. In fact, by Itô formula we have $dM(t) = M(t)dW(t)$. Combined with (S.5) this gives

$$RHS = \exp\big\{\frac{1}{2}T\big\} + \exp\big\{\frac{1}{2}T\big\}(M(T) - M(0)) = \exp W(T) = F.$$

(f) $F = (W(T) + T)\exp\big\{-W(T) - \frac{1}{2}T\big\}$ implies $D_tF = \exp\big\{-W(T) - \frac{1}{2}T\big\}\big[1 - W(T) - T\big]$. Note that

$$Y(t) := \big(W(t) + t\big)N(t), \quad \text{with} \quad N(t) = \exp\big\{-W(t) - \frac{1}{2}t\big\}$$

is a martingale, since

$$dY(t) = (W(t)+t)N(t)(-dW(t)) + N(t)(dW(t)+dt) - N(t)dt$$
$$= N(t)[1 - t - W(t)]dW(t).$$

Therefore,

$$\begin{aligned} E[F] + \int_0^T E[D_tF|\mathcal{F}_t]dW(t) &= \int_0^T E[N(T)(1-(W(T)+T))|\mathcal{F}_t]dW(t) \\ &= \int_0^T N(t)(1-(W(t)+t))dW(t) \\ &= \int_0^T 1\, dY(t) = Y(T) - Y(0) \\ &= (W(T)+T)\exp\Big\{-W(T) - \frac{1}{2}T\Big\} = F. \quad \square \end{aligned}$$

4.3 Solution

(a) $\widetilde{\varphi}(t) = E_Q[D_tF - F\int_t^T D_t\theta(s)d\widetilde{W}(s)|\mathcal{F}_t]$. If $\theta(s)$, $t \in [0,T]$, is deterministic, then $D_t\theta = 0$ and hence

$$\begin{aligned} \widetilde{\varphi}(t) &= E_Q[D_tF|\mathcal{F}_t] = E_Q[2W(T)|\mathcal{F}_t] \\ &= E_Q[2\widetilde{W}(T) - 2\int_0^T \theta(s)ds|\mathcal{F}_t] \\ &= 2\widetilde{W}(t) - 2\int_0^T \theta(s)ds \\ &= 2W(t) - 2\int_t^T \theta(s)ds. \end{aligned}$$

(b) By application of the generalized Clark–Ocone formula, we have

$$\begin{aligned} \widetilde{\varphi}(t) &= E_Q[D_tF|\mathcal{F}_t] \\ &= E_Q\Big[\exp\Big\{\int_0^T \lambda(s)dW(s)\Big\}\lambda(t)|\mathcal{F}_t\Big] \\ &= \lambda(t)E_Q\Big[\exp\Big\{\int_0^T \lambda(s)d\widetilde{W}(s) - \int_0^T \lambda(s)\theta(s)ds\Big\}|\mathcal{F}_t\Big] \end{aligned}$$

$$= \lambda(t) \exp\Big\{ \int_0^T (\frac{1}{2}\lambda^2(s) - \lambda(s)\theta(s))ds \Big\} E_Q\Big[\exp\Big\{ \int_0^T \lambda(s) d\widetilde{W}(s) - \frac{1}{2}\int_0^T \lambda^2(s)ds \Big\} | \mathcal{F}_t \Big]$$

$$= \lambda(t) \exp\Big\{ \int_0^T \lambda(s)(\frac{1}{2}\lambda(s) - \theta(s))ds \Big\} \exp\Big\{ \int_0^t \lambda(s) d\widetilde{W}(s) - \frac{1}{2}\int_0^t \lambda^2(s)ds \Big\}$$

$$= \lambda(t) \exp\Big\{ \int_0^t \lambda(s) dW(s) + \int_t^T \lambda(s)(\frac{1}{2}\lambda(s) - \theta(s))ds \Big\}. \tag{S.6}$$

(c) By application of the generalized Clark–Ocone formula, we have

$$\begin{aligned} \widetilde{\varphi}(t) &= E_Q[D_t F - F \int_t^T D_t\theta(s) d\widetilde{W}(s) | \mathcal{F}_t] \\ &= E_Q[\lambda(t) F | \mathcal{F}_t] - E_Q[F \int_t^T d\widetilde{W}(s) | \mathcal{F}_t] \\ &=: A - B. \end{aligned} \tag{S.7}$$

Now $\widetilde{W}(t) = W(t) + \int_0^t \theta(s)ds = W(t) + \int_0^t W(s)ds$ or

$$dW(t) + W(t)dt = d\widetilde{W}(t).$$

We solve this equation for $W(t)$ by multiplying by the "integrating factor" e^t and get

$$d(e^t W(t)) = e^t d\widetilde{W}(t).$$

Hence

$$W(u) = e^{-u} \int_0^u e^s d\widetilde{W}(s) \tag{S.8}$$

or

$$dW(u) = -e^{-u} \int_0^u e^s d\widetilde{W}(s)du + d\widetilde{W}(u). \tag{S.9}$$

Using (S.9) we may rewrite F as follows:

$$\begin{aligned}
F &= \exp\Big\{\int_0^T \lambda(s)dW(s)\Big\} \\
&= \exp\Big\{\int_0^T \lambda(s)d\widetilde{W}(s) - \int_0^T \lambda(u)e^{-u}(\int_0^u e^s d\widetilde{W}(s))du\Big\} \\
&= \exp\Big\{\int_0^T \lambda(s)d\widetilde{W}(s) - \int_0^T (\int_0^T \lambda(u)e^{-u}du)e^s d\widetilde{W}(s)\Big\} \\
&= K(T)\ \exp\Big\{\frac{1}{2}\int_0^T \xi^2(s)ds\Big\},
\end{aligned}$$

where

$$\xi(s) = \lambda(s) - e^s \int_s^T \lambda(u)e^{-u}du \tag{S.10}$$

and

$$K(t) = \exp\Big\{\int_0^t \xi(s)d\widetilde{W}(s) - \frac{1}{2}\int_0^t \xi^2(s)ds\Big\}, \qquad 0 \le t \le T. \tag{S.11}$$

Hence

$$\begin{aligned}
A &= E_Q[\lambda(t)F|\mathcal{F}_t] && \text{(S.12)} \\
&= \lambda(t)\exp\Big\{\frac{1}{2}\int_0^T \xi^2(s)ds\Big\}E[K(T)|\mathcal{F}_t] \\
&= \lambda(t)\exp\Big\{\frac{1}{2}\int_0^T \xi^2(s)ds\Big\}K(t). && \text{(S.13)}
\end{aligned}$$

Moreover, if we put

$$H := \exp\Big\{\frac{1}{2}\int_0^T \xi^2(s)ds\Big\}, \tag{S.14}$$

we get

$$\begin{aligned}
B &= E_Q[F(\widetilde{W}(T) - \widetilde{W}(t))|\mathcal{F}_t] \\
&= H\, E_Q[K(T)(\widetilde{W}(T) - \widetilde{W}(t))|\mathcal{F}_t]
\end{aligned}$$

$$= H\,E_Q\Big[K(t)\,\exp\Big\{\int_t^T \xi(s)d\widetilde{W}(s) - \frac{1}{2}\int_t^T \xi^2(s)ds\Big\}(\widetilde{W}(T) - \widetilde{W}(t))|\mathcal{F}_t\Big]$$

$$= H\,K(t)E_Q\Big[\exp\Big\{\int_t^T \xi(s)d\widetilde{W}(s) - \frac{1}{2}\int_t^T \xi^2(s)ds\Big\}(\widetilde{W}(T) - \widetilde{W}(t))\Big]$$

$$= H\,K(t)E\Big[\exp\Big\{\int_t^T \xi(s)dW(s) - \frac{1}{2}\int_t^T \xi^2(s)ds\Big\}(W(T) - W(t))\Big]. \quad \text{(S.15)}$$

This last expectation can be evaluated by using the Itô formula. Put

$$X(t) = \exp\Big\{\int_{t_0}^t \xi(s)dW(s) - \frac{1}{2}\int_{t_0}^t \xi^2(s)ds\Big\}$$

and

$$Y(t) = X(t)\,(W(t) - W(t_0)).$$

Then

$$\begin{aligned} dY(t) &= X(t)dW(t) + (W(t) - W(t_0))dX(t) + dX(t)dW(t) \\ &= X(t)[1 + (W(t) - W(t_0))\xi(t)]dW(t) + \xi(t)X(t)dt \end{aligned}$$

and hence

$$\begin{aligned} E[Y(T)] &= E[Y(t_0)] + E[\int_{t_0}^T \xi(s)X(s)ds] \\ &= \int_{t_0}^T \xi(s)E[X(s)]ds \qquad \text{(S.16)} \\ &= \int_{t_0}^T \xi(s)ds. \qquad \text{(S.17)} \end{aligned}$$

Combining (S.7) and (S.10)–(S.16), we conclude that

$$\begin{aligned} \widetilde{\varphi}(t) &= \lambda(t)HK(t) - H\,K(t)\int_t^T \xi(s)ds \\ &= \exp\Big\{\frac{1}{2}\int_0^T \xi^2(s)ds\Big\}\,\exp\Big\{\int_0^t \xi(s)d\widetilde{W}(s) \\ &\qquad -\frac{1}{2}\int_0^t \xi^2(s)ds\Big\}\Big[\lambda(t) - \int_t^T \xi(s)ds\Big]. \qquad \square \end{aligned}$$

4.4 Solution

(a) Since $u = \frac{\mu-\rho}{\sigma}$ is constant we get using (4.27)

$$\theta_1(t) = e^{\rho t}\sigma^{-1}S^{-1}(t)E_Q[e^{-\rho T}D_t W(T)|\mathcal{F}_t] = e^{\rho(t-T)}\sigma^{-1}S^{-1}(t).$$

(b) Here $\mu = 0$, $\sigma(s) = c\,S^{-1}(s)$ and hence

$$u(s) = \frac{\mu-\rho}{\sigma} = -\frac{\rho}{c}S(s) = -\rho(W(s)+S(0)).$$

Hence

$$\int_t^T D_t u(s)d\widetilde{W}(s) = \rho\big[\widetilde{W}(t) - \widetilde{W}(T)\big].$$

Therefore,

$$B := E_Q\Big[F\int_t^T D_t u(s)d\widetilde{W}(s)|\mathcal{F}_t\Big] = \rho E_Q[e^{-\rho T}W(T)(\widetilde{W}(t)-\widetilde{W}(T))|\mathcal{F}_t]. \tag{S.18}$$

To proceed further, we need to express W in terms of $\widetilde{W}$: since

$$\widetilde{W}(t) = W(t) + \int_0^t u(s)ds = W(t) - \rho S(0)t - \rho\int_0^t W(s)ds,$$

we have

$$d\widetilde{W}(t) = dW(t) - \rho W(t)dt - \rho S(0)dt$$

or

$$e^{-\rho t}dW(t) - e^{-\rho t}\rho W(t)dt = e^{-\rho t}(d\widetilde{W}(t) + \rho S(0)dt)$$

or

$$d(e^{-\rho t}W(t)) = e^{-\rho t}d\widetilde{W}(t) + \rho e^{-\rho t}S(0)dt.$$

Hence

$$W(t) = S(0)[e^{\rho t}-1] + e^{\rho t}\int_0^t e^{-\rho s}d\widetilde{W}(s). \tag{S.19}$$

Substituting this in (S.18) we get

$$\begin{aligned} B &= \rho E_Q[\int_0^T e^{-\rho s}d\widetilde{W}(s)\,(\widetilde{W}(t)-\widetilde{W}(T))|\mathcal{F}_t] \\ &= \rho E_Q[\int_0^t e^{-\rho s}d\widetilde{W}(s)\,(\widetilde{W}(t)-\widetilde{W}(T))|\mathcal{F}_t] \end{aligned}$$

$$+\rho E_Q[\int_t^T e^{-\rho s} d\widetilde{W}(s)\,(\widetilde{W}(t) - \widetilde{W}(T))|\mathcal{F}_t]$$

$$= \rho E_Q[\int_t^T e^{-\rho s} d\widetilde{W}(s)\,(\widetilde{W}(t) - \widetilde{W}(T))]$$

$$= \rho \int_t^T e^{-\rho s}(-1)ds = e^{-\rho T} - e^{-\rho t}.$$

Hence

$$\begin{aligned}\theta_1(t) &= e^{\rho t} c^{-1}(E_Q[D_t(e^{-\rho T} W(T))|\mathcal{F}_t] - B) \\ &= e^{\rho t} c^{-1}(e^{-\rho T} - e^{-\rho T} + e^{-\rho t}) = c^{-1},\end{aligned}$$

as expected.

(c) Here $\sigma = c\,S^{-1}(t)$ and hence

$$u(s) = \frac{\mu - \rho}{c} S(s) = \frac{\mu - \rho}{c}\Big[e^{\mu s} S(0) + c\int_0^s e^{\mu(s-r)} dW(r)\Big].$$

So

$$D_t u(s) = (\mu - \rho) e^{\mu(s-t)} \chi_{[0,s]}(t).$$

Hence

$$\theta_1(t) = e^{\rho t} c^{-1} E_Q[(D_t(e^{-\rho T} W(T)) - e^{-\rho T} W(T) \int_t^T D_t u(s) d\widetilde{W}(s)|\mathcal{F}_t]$$

$$= e^{\rho(t-T)} c^{-1}(1 - (\mu - \rho) E_Q[W(T) \int_t^T e^{\mu(s-t)} d\widetilde{W}(s)|\mathcal{F}_t]). \quad \text{(S.20)}$$

Again we try to express W in terms of $\widetilde{W}$: since

$$\begin{aligned}d\widetilde{W}(t) &= dW(t) + u(t)dt \\ &= dW(t) + \frac{\mu - \rho}{c}[e^{\mu t} S(0) + c\int_0^t e^{\mu(t-r)} dW(r)]dt,\end{aligned}$$

we have

$$e^{-\mu t} d\widetilde{W}(t) = e^{-\mu t} dW(t) + [\frac{\mu - \rho}{c} S(0) + (\mu - \rho)\int_0^t e^{-\mu r} dW(r)]dt. \quad \text{(S.21)}$$

If we put

$$X(t) = \int_0^t e^{-\mu r} dW(r), \qquad \widetilde{X}(t) = \int_0^t e^{-\mu r} d\widetilde{W}(r),$$

(S.21) can be written as

$$d\widetilde{X}(t) = dX(t) + \frac{\mu - \rho}{c} S(0)dt + (\mu - \rho)X(t)dt$$

or

$$d(e^{(\mu-\rho)t}X(t)) = e^{(\mu-\rho)t} d\widetilde{X}(t) - \frac{\mu-\rho}{c} S(0)e^{(\mu-\rho)t}dt$$

or

$$\begin{aligned} X(t) &= e^{(\rho-\mu)t} \int_0^t e^{-\rho s} d\widetilde{W}(s) - \frac{\mu-\rho}{c} S(0)e^{(\rho-\mu)t} \int_0^t e^{(\mu-\rho)s} ds \\ &= e^{(\rho-\mu)t} \int_0^t e^{-\rho s} d\widetilde{W}(s) - \frac{S(0)}{c}[1 - e^{(\rho-\mu)t}]. \end{aligned}$$

From this we get

$$\begin{aligned} e^{-\mu t}dW(t) = e^{(\rho-\mu)t}e^{-\rho t}d\widetilde{W}(t) + (\rho-\mu)e^{(\rho-\mu)t}(\int_0^t e^{-\rho s}d\widetilde{W}(s))dt \\ + \frac{S(0)}{c}(\rho-\mu)e^{(\rho-\mu)t}dt \end{aligned}$$

or

$$dW(t) = d\widetilde{W}(t) + (\rho-\mu)e^{\rho t}(\int_0^t e^{-\rho s} d\widetilde{W}(s))dt + \frac{S(0)}{c}(\rho-\mu)e^{\rho t}dt.$$

In particular,

$$W(T) = \widetilde{W}(T) + (\rho-\mu)\int_0^T e^{\rho s}(\int_0^s e^{-\rho r} d\widetilde{W}(r))ds + \frac{S(0)}{\rho c}(\rho-\mu)(e^{\rho T} - 1). \tag{S.22}$$

Substituted in (S.20) this gives

$$\begin{aligned} \theta_1(t) = e^{\rho(t-T)}c^{-1}\{1 - (\mu-\rho)E_Q[\widetilde{W}(T) \int_t^T e^{\mu(s-t)} d\widetilde{W}(s)|\mathcal{F}_t] \\ +(\mu-\rho)^2 E_Q[\int_0^T e^{\rho s}(\int_0^s e^{-\rho r} d\widetilde{W}(r))ds \int_t^T e^{\mu(s-t)} d\widetilde{W}(s)|\mathcal{F}_t]\} \end{aligned}$$

$$= e^{\rho(t-T)}c^{-1}\{1-(\mu-\rho)\int_t^T e^{\mu(s-t)}ds$$

$$+(\mu-\rho)^2\int_t^T e^{\rho s}E_Q[(\int_t^s e^{-\rho r}d\widetilde{W}(r))\,(\int_t^T e^{\mu(r-t)}d\widetilde{W}(r))|\mathcal{F}_t]ds\}$$

$$= e^{\rho(t-T)}c^{-1}\{1-\frac{\mu-\rho}{\mu}(e^{\mu(T-t)}-1)+(\mu-\rho)^2\int_t^T e^{\rho r}(\int_t^s e^{-\rho r}\,e^{\mu(r-t)}dr)ds\}$$

$$= e^{\rho(t-T)}c^{-1}\{1-\frac{\mu-\rho}{\rho}(e^{\rho(T-t)}-1)\}. \quad \square$$

Problems of Chap. 5

5.5 Solution

(a) We have the following equations:

$$\int_0^T W(T)\delta W(t) = \int_0^T W(T)\diamond \overset{\bullet}{W}(t)dt$$

$$= W(T)\diamond\int_0^T \overset{\bullet}{W}(t)dt$$

$$= W(T)\diamond W(T) = W^2(T)-T,$$

by (5.65).

(b) We have the following equations:

$$\int_0^T \Big(\int_0^T gdW\Big)\diamond \overset{\bullet}{W}(t)dt = \Big(\int_0^T gdW\Big)\diamond\int_0^T \overset{\bullet}{W}(t)dt$$

$$= (\int_0^T gdW)\diamond W(T)$$

$$= \Big(\int_0^T gdW\Big)W(T) - \int_0^T g(s)ds,$$

by (5.62).

(c) We have the following equations:

$$\begin{aligned}\int_0^T W^2(t_0)\delta W(t) &= \int_0^T (W^{\diamond 2}(t_0)+t_0)\delta W(t)\\ &= W^{\diamond 2}(t_0)\diamond W(T)+t_0W(T)\\ &= W^{\diamond 2}(t_0)\diamond (W(T)-W(t_0))\\ &\quad +W^{\diamond 2}(t_0)\diamond W(t_0)+t_0W(T)\\ &= W^{\diamond 2}(t_0)\,(W(T)-W(t_0))+W^{\diamond 3}(t_0)+t_0W(T)\\ &= (W^2(t_0)-t_0)\,(W(T)-W(t_0))+W^3(t_0)\\ &\quad -3t_0W(t_0)+t_0W(T)\\ &= W^2(t_0)W(T)-2t_0W(t_0),\end{aligned}$$

where we have used (5.40) and (5.65).

(d) We have the following equations:

$$\begin{aligned}\int_0^T \exp(W(T))\delta W(t) &= \exp(W(T))\diamond\int_0^T \overset{\bullet}{W}(t)dt\\ &= \exp(W(T))\diamond W(T)\\ &= \exp^{\diamond}(W(T)+\frac{1}{2}T)\diamond W(T)\\ &= \exp(\frac{1}{2}T)\sum_{n=0}^{\infty}\frac{1}{n!}W(T)^{\diamond(n+1)}\\ &= \exp(\frac{1}{2}T)\sum_{n=0}^{\infty}\frac{T^{\frac{n+1}{2}}}{n!}h_{n+1}\left(\frac{W(T)}{\sqrt{T}}\right). \quad \square\end{aligned}$$

Problems of Chap. 6

6.4 Solution

Since $\overset{\bullet}{W}(s)=\sum_{i=1}^{\infty}e_i(s)H_{\epsilon^{(i)}}$, the expansion (6.8) for $D_t\overset{\bullet}{W}(s)$ is

$$\begin{aligned}D_t\overset{\bullet}{W}(s) &= \sum_{i,k=1}^{\infty}e_i(s)e_k(t)H_{\epsilon^{(i)}-\epsilon^{(k)}}\chi_{\{i=k\}}\\ &= \sum_{i=1}^{\infty}e_i(s)e_i(t),\end{aligned}$$

which is not convergent. Hence, for all $s\in\mathbb{R}$, we have $\overset{\bullet}{W}(s)\notin Dom(D_t)$. $\square$

Problems of Chap. 7

7.2 Solution

By Proposition 7.2 we have

$$g(e^Y) = \int_{\mathbb{R}} g(t^y) \frac{1}{\sqrt{2\pi v}} \exp^\diamond \Big\{ -\frac{(y-Y)^{\diamond 2}}{2v} \Big\} dy.$$

Substituting $e^y = z$ in the integral, we obtain

$$g(Z) = g(e^Y) = \int_0^\infty g(z) \frac{1}{\sqrt{2\pi v}} \exp^\diamond \Big\{ -\frac{(\log z - \log Z)^{\diamond 2}}{2v} \Big\} \frac{dz}{z}.$$

Hence the Donsker delta function of Z is

$$\delta_Z(z) = \frac{1}{\sqrt{2\pi v}} \frac{1}{z} \exp^\diamond \Big\{ -\frac{(\log z - \log Z)^{\diamond 2}}{2v} \Big\} \chi_{(0,\infty)}(z),$$

as claimed. □

Problems of Chap. 8

8.2 Solution

(a) $\int_0^T W(T) d^- W(t) = W(T) \int_0^T dW(t) = W^2(T)$

(b) Consider the following equations:

$$\begin{aligned}
&\int_0^T W(t)\big[W(T) - W(t)\big] d^- W(t) \\
&= \int_0^T W(t)W(T) d^- W(t) - \int_0^T W^2(t) d^- W(t) \\
&= W(T) \int_0^T W(t) dW(t) - \big[\frac{1}{3} W^3(T) - \int_0^T W(t) dt\big] \\
&= W(T)\big[\frac{1}{2} W^2(T) - \frac{1}{2} T\big] - \big[\frac{1}{3} W^3(T) - \int_0^T W(t) dt\big] \\
&= \frac{1}{6} W^3(T) - \frac{1}{2} T W(T) + \int_0^T W(t) dt.
\end{aligned}$$

(c) Consider the following equations:

$$\begin{aligned}
&\int_0^T \Big(\int_0^T g(s) dW(s) \Big) d^- W(t) \\
&= \int_0^T g(s) dW(s) \int_0^T d^- W(t) \\
&= W(T) \int_0^T g(s) dW(s). \quad \square
\end{aligned}$$

Problems of Chap. 9

9.1 Solution

(a) By Theorem 9.4 we have

$$\begin{aligned} dY(t) =& 2Y(t)\big[\alpha(t)dt + \beta(t)dW(t)\big] + \beta^2(t)dt \\ &+ \int_{\mathbb{R}_0} \big[(X(t)+\gamma(t,z))^2 - X^2(t) - 2X(t)\gamma(t,z)\big]\nu|(dz)dt \\ &+ \int_{\mathbb{R}_0} \big[(X(t^-)+\gamma(t,z))^2 - X^2(t^-)\big]\widetilde{N}(dt,dz) \\ =& \big(2\alpha(t)Y(t) + \beta^2(t) + \int_{\mathbb{R}_0} \gamma^2(t,z)\nu(dz)\big)dt + 2\beta(t)Y(t)dW(t) \\ &+ \int_{\mathbb{R}_0} \big[2X(t^-)\gamma(t,z) + \gamma^2(t,z)\big]\widetilde{N}(dt,dz). \end{aligned}$$

(b) By Theorem 9.4 we have

$$\begin{aligned} dY(t) =& Y(t)\big[\alpha(t)dt + \beta(t)dW(t)\big] + \frac{1}{2}Y(t)\beta^2(t)dt \\ &+ \int_{\mathbb{R}_0} \big[\exp\{X(t)+\gamma(t,z)\} - \exp\{X(t)\} - \exp\{X(t)\}\gamma(t,z)\big]\nu(dz)dt \\ &+ \int_{\mathbb{R}_0} \big[\exp\{X(t^-)+\gamma(t,z)\} - \exp\{X(t^-)\big]\widetilde{N}(dt,dz) \\ =& Y(t^-)\Big[\Big(\alpha(t) + \frac{1}{2}\beta^2(t) + \int_{\mathbb{R}_0} \big[\exp\{\gamma(t,z)\} - 1 - \gamma(t,z)\big]\nu(dz)\Big)dt\Big] \\ &+ \beta(t)dW(t) + \int_{\mathbb{R}_0} \big[\exp\{\gamma(t,z) - 1\big]\widetilde{N}(dt,dz). \end{aligned}$$

(c) By Theorem 9.4 we have

$$\begin{aligned} dY(t) =& -\sin X(t)\,\big[\alpha(t)dt + \beta(t)dW(t)\big] - \frac{1}{2}\cos X(t)\,\beta^2(t)dt \\ &+ \int_{\mathbb{R}_0} \big[\cos\big(X(t)+\gamma(t,z)\big) - \cos X(t) + \sin X(t)\,\gamma(t,z)\big]\nu(dz)dt \\ &+ \int_{\mathbb{R}_0} \big[\cos\big(X(t^-)+\gamma(t,z)\big) - \cos X(t^-)\big]\widetilde{N}(dt,dz) \\ =& \big[-\alpha(t)\sin X(t) - \frac{1}{2}\beta^2(t)\cos X(t)\big] \\ &+ \cos X(t)\int_{\mathbb{R}_0} \big[\cos\gamma(t,z) - 1\big]\nu(dz) \\ &+ \sin X(t)\int_{\mathbb{R}_0} \big[\gamma(t,z) - \sin\gamma(t,z)\big]\nu(dz)dt - \beta(t)\sin X(t)dW(t) \\ &+ \int_{\mathbb{R}_0} \big[\cos X(t^-)\big(\cos\gamma(t,z) - 1\big) - \sin X(t^-)\sin\gamma(t,z)\big]\widetilde{N}(dt,dz). \ \square \end{aligned}$$

9.2 Solution
Applying Problem 9.1 (b) to the case

$$\alpha(t) = -\int_{\mathbb{R}_0} \left[e^{h(t)z} - 1 - h(t)z\right]\nu(dz),$$
$$\beta(t) = 0$$
$$\gamma(t,z) = h(t)z,$$

we obtain

$$dY(t) = Y(t)\Big\{\Big(-\int_{\mathbb{R}_0}\left[e^{h(t)z} - 1 - h(t)z\right]\nu(dz) + \int_{\mathbb{R}_0}\left[e^{h(t)z} - 1 - h(t)z\right]\nu(dz)\Big)dt$$
$$+ \int_{\mathbb{R}_0}\big[e^{h(t)z} - 1big]\widetilde{N}(dt,dz)\Big\}$$
$$= Y(t)\int_{\mathbb{R}_0}\big[e^{h(t)z} - 1big]\widetilde{N}(dt,dz). \quad \square$$

9.6 Solution

(a) Since $d\big(t\eta(t)\big) = td\eta(t) + \eta(t)dt$, we have

$$F = \int_0^T \eta(t)dt$$
$$= T\eta(T) - \int_0^T t d\eta(t)$$
$$= T\int_0^T\int_{\mathbb{R}_0} z\widetilde{N}(dt,dz) - \int_0^T t\int_{\mathbb{R}_0} z\widetilde{N}(dt,dz)$$
$$= \int_0^T\int_{\mathbb{R}_0}(T-t)z\widetilde{N}(dt,dz).$$

Hence F is replicable, with replicating portfolio $\varphi(t) = T - t$, $t \in [0,T]$.

(c) Define

$$Y(t) = \exp\Big\{\eta(t) - \int_0^t\int_{\mathbb{R}_0}\big(e^z - 1 - z\big)\nu(dz)ds\Big\}.$$

Then, by Problem 9.2 we have

$$dY(t) = Y(t^-)\int_{\mathbb{R}_0}\big(e^z - 1\big)\widetilde{N}(dt,dz).$$

Hence

$$Y(T) = 1 + \int_0^T\int_{\mathbb{R}_0} Y(t^-)\big(e^z - 1\big)\widetilde{N}(dt,dz).$$

Therefore,

$$\begin{aligned} e^{\eta(T)} =& Y(T)M \\ =& M + \int_0^T \int_{\mathbb{R}_0} MY(t^-)\big(e^z - 1\big)\widetilde{N}(dt, dz), \end{aligned}$$

where

$$M = \exp\Big\{T \int_{\mathbb{R}_0} \big(e^z - 1 - z\big)\nu(dz)\Big\}.$$

Hence $F = \exp\{\eta(T)\}$ is not replicable unless $\nu(dz) = \lambda\delta_{z_0}(dz)$ is a point mass at some point $z_0 \neq 0$. In this case, the process $\eta(t)$, $t \in [0, T]$, corresponds to the compensated Poisson process with jump size z_0 and intensity $\lambda > 0$. □

Problems of Chap. 10

10.1 Solution

(a) By the Itô formula we have

$$\begin{aligned} \eta^3(T) = & \int_0^T \int_{\mathbb{R}_0} \big[(\eta(t) + z)^3 - \eta^3(t) - 3\eta^2(t)z\big]\nu(dz)dt \\ & + \int_0^T \int_{\mathbb{R}_0} \big[(\eta(t) + z)^3 - \eta^3(t)\big]\widetilde{N}(dt, dz) \\ = & \int_0^T \int_{\mathbb{R}_0} \big[3\eta(t)z^2 + z^3\big]\nu(dz)dt \\ & + \int_0^T \int_{\mathbb{R}_0} \big[3\eta^2(t)z + 3\eta(t)z^2 + z^3\big]\widetilde{N}(dt, dz) \\ = & \int_0^T \int_{\mathbb{R}_0} z^3\nu(dz)dt + \int_0^T \int_{\mathbb{R}_0} 3\eta(t)z^2\nu(dz)dt \\ & + \int_0^T \int_{\mathbb{R}_0} z^3\widetilde{N}(dz, dt) + \int_0^T \int_{\mathbb{R}_0} 3\eta(t)z^2\widetilde{N}(dt, dz) \\ & + \int_0^T \int_{\mathbb{R}_0} 3\eta^2(t)z\widetilde{N}(dt, dz) \\ = & \int_0^T \int_{\mathbb{R}_0} z^3\nu(dz)dt + \int_0^T \int_{\mathbb{R}_0} 3\Big(\int_0^t \int_{\mathbb{R}_0} z_1\widetilde{N}(ds, dz_1)\Big)z^2\nu(dz)dt \\ & + \int_0^T \int_{\mathbb{R}_0} z^3\widetilde{N}(dt, dz) + \int_0^T \int_{\mathbb{R}_0} \Big(\int_0^t \int_{\mathbb{R}_0} 3z_1\widetilde{N}(dt, dz_1)\Big)z^2\widetilde{N}(dt, dz) \\ & + \int_0^T \int_{\mathbb{R}_0} 3\eta^2(t)z\widetilde{N}(dt, dz). \end{aligned}$$

Now we also have

$$\begin{aligned}\int_0^T\int_{\mathbb{R}_0} 3\eta^2(t)z\widetilde{N}(dt,dz) =& \int_0^T\int_{\mathbb{R}_0} 3\Big(t\int_{\mathbb{R}_0}\zeta^2\nu(d\zeta)\Big)z\widetilde{N}(dt,dz)\\ &+\int_0^T\int_{\mathbb{R}_0} 3\Big(\int_0^t\int_{\mathbb{R}_0} z_1^2\widetilde{N}(dt,dz_1)\Big)z\widetilde{N}(dt,dz)\\ &+\int_0^T\int_{\mathbb{R}_0} 6\Big(\int_0^t\int_{\mathbb{R}_0}\int_0^{t_2}\int_{\mathbb{R}_0} z_1z_2\widetilde{N}(dt_1,dz_1)\widetilde{N}(dt_2,dz_2)\Big)z\widetilde{N}(dt,dz)\\ =& I_1\big(3t_1z_1\int_{\mathbb{R}_0}\zeta^2\nu(d\zeta)\big) + J_2\big(3z_1^2z_2\big) + J_3\big(6z_1z_2z_3\big).\end{aligned}$$

Summing up we get

$$\begin{aligned}\eta^3(T) =& Tm_3 + I_1\big(3Tm_2z_1 + z_1^3\big) + J_2\big(3z_1z_2^2 + 3z_1^2z_2\big) + J_3\big(6z_1z_2z_3\big)\\ =& Tm_3 + I_1\big(3Tm_2z_1 + z_1^3\big) + I_2\big(\frac{3}{2}(z_1z_2^2 + z_1^2z_2)\big) + I_3\big(z_1z_2z_3\big),\end{aligned}$$

where $m_1 = \int_{\mathbb{R}_0}\zeta^i\nu(d\zeta)$, $i = 2, 3, ...$

(b) By Example 10.4 we have that

$$F_0 := \exp\Big\{\int_0^T\int_{\mathbb{R}_0} z\widetilde{N}(dt,dz) - \int_0^T\int_{\mathbb{R}_0}\big(e^z - 1 - z\big)\nu(dz)dt\Big\}$$

has chaos expansion $F_0 = \sum_{n=0}^{\infty} I_n(f_n)$ given by (10.7):

$$f_n = \frac{1}{n!}\big(e^z - 1\big)^{\otimes n}(t_1, z_1, ..., t_n, z_n).$$

It follows that F has the expansion

$$F = \sum_{n=0}^{\infty} I_n(Kf_n) \quad \text{where} \quad K := \exp\Big\{T\int_{\mathbb{R}_0}\big(e^z - 1 - z\big)\nu(dz)\Big\}.$$

(c) We have the following equalities:

$$F = \int_0^T g(s)d\eta(s) = \int_0^T\int_{\mathbb{R}_0} g(s)z\widetilde{N}(ds,dz) = I_1(f_1),$$

where $f_1(s,z) = g(s)z$, $s \in [0,T]$, $z \in \mathbb{R}_0$.

(d) We have the following equalities:

$$\begin{aligned}F =& \int_0^T g(s)\eta(s) = \int_0^T g(s)\int_0^s\int_{\mathbb{R}_0} z\widetilde{N}(ds,dz)\\ =& \int_0^T\int_{\mathbb{R}_0}\Big(\int_t^T g(s)ds\Big)z\widetilde{N}(dt,dz) = I_1(f_1)\end{aligned}$$

with $f_1(t,z) = z\int_t^T g(s)ds$, $t \in [0,T]$, $z \in \mathbb{R}_0$. □

Problems of Chap. 11

11.2 Solution

(a) Since

$$\int_0^T g(s)d\eta(s) = \int_0^T \int_{\mathbb{R}_0} g(s)z\widetilde{N}(dt,dz) = I_1(g(t_1)z_1),$$

we get

$$\begin{aligned}
\int_0^T \Big(\int_0^T g(s)d\eta(s)\Big) f(t)\delta\eta(t) &= \int_0^T \int_{\mathbb{R}_0} I_1(g(t_1)z_1) f(t_z) z_2 \widetilde{N}(dt_2,dz_2)\\
=& I_2\big(\frac{1}{2}\big(g(t_1)f(t_2)+g(t_2)f(t_1)\big)z_1z_2\big)\\
=& \int_0^T\int_{\mathbb{R}_0}\Big(int_0^{t_2^-}\int_{\mathbb{R}_0}\frac{1}{2}\big(g(t_1)f(t_2)+g(t_2)f(t_1)\big)z_1z_2\widetilde{N}(dt_1,dz_1)\Big)\widetilde{N}(dt_2,dz_2)\\
=& \int_0^T\int_{\mathbb{R}_0}\big[f(t_2)\int_0^{t_2^-} g(t_1)d\eta(t_1) + g(t_2)\int_0^{t_2^-} f(t_1)d\eta(t_1)\big] z_2\widetilde{N}(dt_2,dz_2)\\
=& \int_0^T\Big(\int_0^{t_2^-} g(t_1)d\eta(t_1)\Big) f(t_2)d\eta(t_2) + \int_0^T\Big(\int_0^{t_2^-} f(t_1)d\eta(t_1)\Big) g(t_2)d\eta(t_2).
\end{aligned} \tag{S.23}$$

(b) Using the computation in (S.23), but with f and g interchanged, we get

$$\int_0^T \Big(\int_0^T f(t)d\eta(t)\Big) g(s)\delta\eta(s) = I_2\big(\frac{1}{2}\big(f(t_1)g(t_2)+f(t_2)g(t_1)\big)z_1z_2\big),$$

which is the same as we obtained in (a).

(c) This is a direct consequence of (a) and (b). □

Problems of Chap. 12

12.1 Solution

(a) By Problem 10.1 we have the expansion

$$\eta^3(T) = Tm_3 + I_1\big(3Tm_2z_1 + z_1^3\big) + I_2\big(\frac{3}{2}\big(z_1z_2^2 + z_1^2z_2\big)\big) + I_3\big(z_1z_2z_3\big),$$

where $m_i = \int_{\mathbb{R}_0}\zeta^i\nu(d\zeta)$, $i = 1, 2, ...$ This gives

$$\begin{aligned}
D_{t,z}\eta^3(T) =& 3Tm_2z + z^3 + 3I_1(z_1z^2 + z_1^2z) + 3I_2(z_1z_2z)\\
=& 3Tm_2z + z^3 + 3z^2\eta(T) + 3zI_1(z_1^2) + 3zI_2(z_1z_2).
\end{aligned}$$

If we use that

$$\eta^2(T) = Tm_2 + I_1(z^2) + I_2(z_1 z_2)$$

(see Example 12.9), we can see that the above expression can be written

$$D_{t,z}\eta^3(T) = 3\eta^2(T)z + 3\eta(T)z^2 + z^3.$$

(b) By Problem 10.1 we have the expansion

$$e^{\eta(T)} = \sum_{n=0}^{\infty} I_n(g_n),$$

where

$$g_n = Kf_n \quad \text{with} \quad f_n = \frac{1}{n!}(e^z - 1)^{\otimes n}, \quad n = 1, 2, ...$$
$$\text{and} \quad K = \exp\Big\{T \int_{\mathbb{R}_0} \big(e^z - 1 - z\big)\nu(dz)\Big\}.$$

This gives

$$\begin{aligned} D_{t,z}e^{\eta(T)} &= \sum_{n=1}^{\infty} nI_{n-1}(g_n(\cdot, t.z)) \\ &= \sum_{n=1}^{\infty} nI_{n-1}\big(\frac{K}{n!}(e^z - 1)^{\otimes n-1}\big)(e^z - 1) \\ &= \sum_{n=1}^{\infty} I_{n-1}\big(\frac{K}{(n-1)!}(e^z - 1)^{\otimes n-1}\big)(e^z - 1) \\ &= e^{\eta(T)}(e^z - 1). \quad \square \end{aligned}$$

12.2 Solution
The direct application of Theorem 12.8 yields

(a) $D_{t,z}\eta^3(T) = (\eta(T) + z)^3 - \eta^3(T) = 3\eta^2(T)z + 3\eta(T)z^2 + z^3$.
(b) $D_{t,z}e^{\eta(T)} = e^{\eta(T)+z} - e^{\eta(T)} = e^{\eta(T)}(e^z - 1)$.

Compare with the solution of Problem 12.1. □

Problems of Chap. 13

13.1 Solution
Recall that

$$\eta(t) = m_2 \sum_{i=1}^{\infty} \Big(\int_0^t e_i(s)ds\Big) K_{\varepsilon^{(i,1)}}, \quad t \in \mathbb{R},$$

and

$$\overset{\bullet}{\eta}(t) = m_2 \sum_{i=1}^{\infty} e_i(t) K_{\varepsilon^{(i,1)}}, \quad t \in \mathbb{R},$$

where $\varepsilon^{(i,j)} = \varepsilon^{(\kappa(i,j))}$. Hence

$$\frac{\eta(t+h) - \eta(t)}{h} - \overset{\bullet}{\eta}(t) = m_2 \sum_{i=1}^{\infty} \Big(\frac{1}{h} \int_t^{t+h} \big[e_i(s) - e_i(t) \big] ds \Big) K_{\varepsilon^{(i,1)}}.$$

By (13.11) we have

$$\kappa(i,1) = 1 + \frac{i(i-1)}{2}.$$

Therefore, if we put

$$a_i(h) := \frac{1}{h} \int_t^{t+h} \big[e_i(s) - e_i(t) \big] ds,$$

we have

$$\begin{aligned} \Big\| \frac{\eta(t+h) - \eta(t)}{h} - \overset{\bullet}{\eta}(t) \Big\|_{-q}^2 &= m_2^2 \sum_{i=1}^{\infty} |a_i(h)|^2 \varepsilon^{(i,1)}!(2\mathbb{N})^{-q\varepsilon^{(i,1)}} \\ &= m_2^2 \sum_{i=1}^{\infty} |a_i(h)|^2 (2\kappa(i,1))^{-q} \\ &= m_2^2 \sum_{i=1}^{\infty} |a_i(h)|^2 (2 + i(i-1))^{-q} \end{aligned}$$

by (13.25). Since

$$\sup_{t \in \mathbb{R}} |e_k(t)| = \mathcal{O}(k^{-1/2})$$

(see [105]), we see that

$$\sup \big\{ |a_i(h)| \ h \in [0,1], i = 1, 2, ... \big\} < \infty.$$

Moreover, since

$$a_i(h) \longrightarrow 0, \quad h \to 0 \quad (i = 1, 2, ...),$$

we can conclude that

$$\Big\| \frac{\eta(t+h) - \eta(t)}{h} - \overset{\bullet}{\eta}(t) \Big\|_{-q}^2 \longrightarrow 0, \quad h \to 0,$$

for all $q \geq 1$, by bounded convergence. This implies that

$$\frac{d}{dt} \eta(t) = \overset{\bullet}{\eta}(t) \quad \text{in } (\mathcal{S})^*. \quad \square$$

Problems of Chap. 15

15.1 Solution

(a) By Lemma 15.5 we have

$$\begin{aligned}X(t) &= \int_0^t \int_{\mathbb{R}_0} \eta(T) z \widetilde{N}(\delta s, dz)\\ &= \int_0^t \int_{\mathbb{R}_0} \big(\eta(T)z + D_{t+,z}\eta(T)z\big)\widetilde{N}(\delta s, dz) - \int_0^t \int_{\mathbb{R}_0} z^2 \widetilde{N}(ds, dz)\\ &= \int_0^t \int_{\mathbb{R}_0} \eta(T) z \widetilde{N}(d^- s, dz) - \int_0^T \int_{\mathbb{R}_0} z^2 \nu(dz) ds - \int_0^t \int_{\mathbb{R}_0} z^2 \widetilde{N}(ds, dz)\\ &= \eta(t)\eta(T) - \int_0^t \int_{\mathbb{R}_0} z^2 N(ds, dz), \quad t \in [0, T].\end{aligned}$$

(b) Since

$$D_{t+,z}\eta(t) = D_{t+,z}\Big(\int_0^t \int_{\mathbb{R}_0} z^2 N(ds, dz)\Big) = 0,$$

by applications of the chain rule we get that $D_{t+,z}X(t) = \eta(t)z$.

(c) By (15.8), we have that

$$\theta(t, z) := S_{t,z}\gamma(t, z) = zS_{t,z}\eta(T) = z(\eta(T) - z).$$

(d) First note that by the above we have

$$\begin{aligned}A(t, z) :=& D_{t+,z}\Big(2X(t^-)z(\eta(T) - z) + z^2(\eta(T) - z)^2\Big)\\ =& 2X(t^-)z^2 + z(\eta(T) - z)2\eta(t^-)z + 2\eta(t^-)z^3\\ &+ z^2(\eta^2(T) - (\eta(T) - z)^2)\\ =& 2X(t^-)z^2 + 2\eta(t^-)\eta(T)z^2 + 2\eta(T)z^3 - z^4.\end{aligned}$$

Therefore, the Itô formula for Skorohod integrals gives

$$\begin{aligned}\delta X^2(t) =& \int_{\mathbb{R}_0} \Big[(X(t^-) + \theta(t, z))^2 - X^2(t^-) + A(t, z)\Big]\widetilde{N}(\delta t, dz)\\ &+ \int_{\mathbb{R}_0} \Big[(X(t) + \theta(t, z))^2 - X^2(t) - 2X(t)\theta(t, z) - A(t, z)\\ &- f'(X(t))D_{t+,z}\theta(t, z)\Big]\nu(dz)dt\\ =& \int_{\mathbb{R}_0} \Big[2X(t^-)z(\eta(T) - z) + z^2(\eta(T) - z)^2 + A(t, z)\Big]\widetilde{N}(\delta t, dz)\\ &+ \int_{\mathbb{R}_0} \Big[z^2(\eta(T) - z)^2 + A(t, z) - 2X(t)z^2\Big]\nu(dz)dt. \quad \square\end{aligned}$$

Problems of Chap. 16

16.1 Solution

(a) By the chain rule and (16.5) we have

$$D_t F_\pi(T) = D_t \frac{1}{X_\pi(T)} = -\frac{1}{X_\pi^2(T)}\, D_t X_\pi(T) = -\frac{1}{X_\pi(T)}\, \sigma(t)\pi(t)$$

and

$$\begin{aligned} D_{t,z} F_\pi(T) &= \frac{1}{X_\pi(T) + D_{t,z} X_\pi(T)} - \frac{1}{X_\pi(T)} \\ &= \frac{1}{X_\pi(T)(1+\pi(t)\theta(t,z))} - \frac{1}{X_\pi(T)} \\ &= \frac{-\pi(t)\theta(t,z)}{X_\pi(T)(1+\pi(t)\theta(t,z))}. \end{aligned}$$

(b) The equation for the optimal deterministic portfolio π is obtained by choosing $\mathcal{E}_t = \mathcal{F}_0$ for all $t \in [0,T]$ in (16.12). This gives

$$\mu(s) - \rho(s) - 2\sigma^2(s)\pi(s) - \int_{\mathbb{R}_0} \frac{\pi(s)\theta^2(s,z)(2+\pi(s)\theta(s,z))}{(1+\pi(s)\theta(s,z))^2} \nu(dz) = 0. \quad \square$$

References

1. K. Aase, T. Bjuland, and B. Øksendal. Transfinite mean value interpolation. Eprint in Mathematics, University of Oslo, 2007.
2. K. Aase, B. Øksendal, N. Privault, and J. Ubøe. White noise generalizations of the Clark-Haussmann-Ocone theorem with application to mathematical finance. *Finance Stoch.*, 4(4):465–496, 2000.
3. K. Aase, B. Øksendal, and J. Ubøe. Using the Donsker delta function to compute hedging strategies. *Potential Analysis*, 14(4):351–374, 2001.
4. S. Albeverio, Y. Hu, and X.Y. Zhou. A remark on non-smoothness of the self-intersection local time of planar Brownian motion. *Statist. Probab. Lett.*, 32(1):57–65, 1997.
5. E. Alòs and D. Nualart. An extension of Itô's formula for anticipating processes. *J. Theoret. Probab.*, 11(2):493–514, 1998.
6. J. Amendinger, P. Imkeller, and M. Schweizer. Additional logarithmic utility of an insider. *Stochastic Process. Appl.*, 75(2):263–286, 1998.
7. D. Applebaum. Covariant Poisson fields in Fock space. In *Analysis, geometry and probability*, volume 10 of *Texts Read. Math.*, pages 1–15. Hindustan Book Agency, Delhi, 1996.
8. D. Applebaum. *Lévy Processes and Stochastic Calculus*, volume 93 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2004.
9. J. Asch and J. Potthoff. Itô's lemma without nonanticipatory conditions. *Probab. Theory Related Fields*, 88(1):17–46, 1991.
10. M. Avellaneda and R. Gamba. Conquering the Greeks in Monte Carlo: efficient calculation of the market sensitivities and hedge-ratios of financial assets by direct numerical simulation. In *Mathematical finance—Bachelier Congress, 2000 (Paris)*, Springer Finance, pages 93–109. Springer, Berlin, 2002.
11. K. Back. Insider trading in continuous time. *Review of Financial Studies*, 5:387–409, 1992.
12. V. Bally. On the connection between the Malliavin covariance matrix and Hörmander's condition. *J. Funct. Anal.*, 96(2):219–255, 1991.
13. V. Bally. Integration by parts formula for locally smooth laws and applications to equations with jumps I. Technical report, Mittag-Leffler Institut, 2007.
14. V. Bally. Integration by parts formula for locally smooth laws and applications to equations with jumps II. Technical report, Mittag-Leffler Institut, 2007.

15. V. Bally, M.-P. Bavouzet, and M. Messaoud. Integration by parts formula for locally smooth laws and applications to sensitivity computations. *Ann. Appl. Probab.*, 17(1):33–66, 2007.
16. V. Bally, L. Caramellino, and A. Zanette. Pricing and hedging American options by Monte Carlo methods using a Malliavin calculus approach. *Monte Carlo Methods Appl.*, 11(2):97–133, 2005.
17. O.E. Barndorff-Nielsen. Processes of normal inverse Gaussian type. *Finance Stoch.*, 2(1):41–68, 1998.
18. O.E. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 63(2):167–241, 2001.
19. R.F. Bass and M. Cranston. The Malliavin calculus for pure jump processes and applications to local time. *Ann. Probab.*, 14(2):490–532, 1986.
20. D.R. Bell. *The Malliavin Calculus*, volume 34 of *Pitman Monographs and Surveys in Pure and Applied Mathematics*. Longman Scientific & Technical, Harlow, 1987.
21. E. Benhamou. Optimal Malliavin weighting function for the computation of the Greeks. *Math. Finance*, 13(1):37–53, 2003.
22. A. Bensoussan. *Stochastic Control of Partially Observable Systems*. Cambridge University Press, Cambridge, 1992.
23. F.E. Benth. Integrals in the Hida distribution space $(\mathcal{S})^*$. In *Stochastic analysis and related topics (Oslo, 1992)*, volume 8 of *Stochastics Monogr.*, pages 89–99. Gordon and Breach, Montreux, 1993.
24. F.E. Benth, L.O. Dahl, and K.H. Karlsen. Quasi Monte-Carlo evaluation of sensitivities of options in commodity and energy markets. *Int. J. Theor. Appl. Finance*, 6(8):865–884, 2003.
25. F.E. Benth, G. Di Nunno, A. Løkka, B. Øksendal, and F. Proske. Explicit representation of the minimal variance portfolio in markets driven by Lévy processes. *Math. Finance*, 13(1):55–72, 2003.
26. F.E. Benth and J. Gjerde. A remark on the equivalence between Poisson and Gaussian stochastic partial differential equations. *Potential Anal.*, 8(2):179–193, 1998.
27. F.E. Benth, M. Groth, and P.C. Kettler. A quasi-Monte Carlo algorithm for the normal inverse Gaussian distribution and valuation of financial derivatives. *Int. J. Theor. Appl. Finance*, 9(6):843–867, 2006.
28. F.E. Benth, M. Groth, and O. Wallin. Derivative-free Greeks for the Barndorff-Nielsen and Shephard stochastic volatility model. Eprint in Mathematics, University of Oslo, 2007.
29. F.E. Benth and A. Løkka. Anticipative calculus for Lévy processes and stochastic differential equations. *Stoch. Stoch. Rep.*, 76(3):191–211, 2004.
30. M.A. Berger. A Malliavin-type anticipative stochastic calculus. *Ann. Probab.*, 16(1):231–245, 1988.
31. M.A. Berger and V.J. Mizel. An extension of the stochastic integral. *Ann. Probab.*, 10(2):435–450, 1982.
32. J. Bertoin. *Lévy Processes*, volume 121 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1996.
33. F. Biagini and B. Øksendal. A general stochastic integral approach to insider trading. *Appl. Math. Optim.*, 52(4):167–181, 2005.

34. P. Biane. Calcul stochastique non-commutatif. In *Lectures on probability theory (Saint-Flour, 1993)*, volume 1608 of *Lecture Notes in Math.*, pages 1–96. Springer, Berlin, 1995.
35. K. Bichteler, J.-B. Gravereaux, and J. Jacod. *Malliavin Calculus for Processes with Jumps*, volume 2 of *Stochastics Monographs*. Gordon and Breach Science Publishers, New York, 1987.
36. P. Billingsley. *Convergence of Probability Measures*. John Wiley & Sons Inc., New York, second edition, 1999.
37. J.-M. Bismut. Martingales, the Malliavin calculus and hypoellipticity under general Hörmander's conditions. *Z. Wahrsch. Verw. Gebiete*, 56(4):469–505, 1981.
38. J.-M. Bismut. Calcul des variations stochastique et processus de sauts. *Z. Wahrsch. Verw. Gebiete*, 63(2):147–235, 1983.
39. F. Black and M. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81:637–654, 1973.
40. C. Blanchet-Scalliet, N. El Karoui, M. Jeanblanc, and L. Martinelli. Optimal investment and consumption decisions when time-horizon is uncertain. Technical report, 2002.
41. B. Bouchard and H. Pham. Wealth-path dependent utility maximization in incomplete markets. *Finance Stoch.*, 8(4):579–603, 2004.
42. N. Bouleau and F. Hirsch. Propriétés d'absolue continuité dans les espaces de Dirichlet et application aux équations différentielles stochastiques. In *Séminaire de Probabilités, XX, 1984/85*, volume 1204 of *Lecture Notes in Math.*, pages 131–161. Springer, Berlin, 1986.
43. M. Broadie and P. Glasserman. Estimating security price derivatives using simulation. *Management Science*, 42:169–285, 1996.
44. E.A. Carlen and É. Pardoux. Differential calculus and integration by parts on Poisson space. In *Stochastics, algebra and analysis in classical and quantum dynamics (Marseille, 1988)*, volume 59 of *Math. Appl.*, pages 63–73. Kluwer Acad. Publ., Dordrecht, 1990.
45. N. Chen and P. Glasserman. Malliavin Greeks without Malliavin calculus. Eprint, Columbia University, 2006.
46. J.M.C. Clark. The representation of functionals of Brownian motion by stochastic integrals. *Ann. Math. Statist.*, 41:1282–1295, 1970.
47. J.M.C. Clark. Correction to: "The representation of functionals of Brownian motion by stochastic integrals" (Ann. Math. Statist. **41** (1970), 1282–1295). *Ann. Math. Statist.*, 42:1778, 1971.
48. R. Cont and P. Tankov. *Financial Modelling with Jump Processes*. Chapman & Hall/CRC Financial Mathematics Series. Chapman & Hall/CRC, Boca Raton, FL, 2004.
49. J.M. Corcuera, P. Imkeller, A. Kohatsu-Higa, and D. Nualart. Additional utility of insiders with imperfect dynamical information. *Finance Stoch.*, 8(3):437–450, 2004.
50. R. Coviello and F. Russo. Modeling financial assets without semimartingales. Technical report, BiBoS, Bielefeld, 2006.
51. D. Cuoco and J. Cvitanić. Optimal consumption choices for a "large" investor. *J. Econom. Dynam. Control*, 22(3):401–436, 1998.
52. M. Curran. Strata gems. *RISK*, 7:70–71, 1994.

53. G. Da Prato. *Introduction to Stochastic Analysis and Malliavin Calculus*, volume 6 of *Appunti. Scuola Normale Superiore di Pisa*. Edizioni della Normale, Pisa, 2007.
54. M.H.A. Davis and M.P. Johansson. Malliavin Monte Carlo Greeks for jump diffusions. *Stochastic Process. Appl.*, 116(1):101–129, 2006.
55. V. Debelley and N. Privault. Sensitivity analysis of European options in jump diffusion models via the Malliavin calculus on the Wiener space. Eprint, Université de la Rochelle, 2004.
56. F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Math. Ann.*, 300(3):463–520, 1994.
57. L. Denis. A criterion of density for solutions of Poisson-driven SDEs. *Probab. Theory Related Fields*, 118(3):406–426, 2000.
58. A. Dermoune, P. Krée, and L. Wu. Calcul stochastique non adapté par rapport à la mesure aléatoire de Poisson. In *Séminaire de Probabilités, XXII*, volume 1321 of *Lecture Notes in Math.*, pages 477–484. Springer, Berlin, 1988.
59. G. Di Nunno. Random fields evolution: non-anticipating integration and differentiation. *Teor. Ĭmovīr. Mat. Stat.*, 66:82–94, 2002.
60. G. Di Nunno. Stochastic integral representations, stochastic derivatives and minimal variance hedging. *Stoch. Stoch. Rep.*, 73(1-2):181–198, 2002.
61. G. Di Nunno. On orthogonal polynomials and the Malliavin derivative for Lévy stochastic measures. *Séminaires et Congrès*, 16:55–69, 2007.
62. G. Di Nunno. Random fields: non-anticipating derivative and differentiation formulas. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 10(3):465–481, 2007.
63. G. Di Nunno, A. Kohatsu-Higa, T. Meyer-Brandis, B. ksendal, F. Proske, and A Sulem. Anticipative stochastic control for Lévy processes with application to insider trading. *Handbook in Mathematical Sciences, Publisher: ELSEVIER, Editors: Alain Bensoussan and Qiang Zhang*, 2008.
64. G. Di Nunno, T. Meyer-Brandis, B. Øksendal, and F. Proske. Malliavin calculus and anticipative Itô formulae for Lévy processes. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 8(2):235–258, 2005.
65. G. Di Nunno, T. Meyer-Brandis, B. Øksendal, and F. Proske. Optimal portfolio for an insider in a market driven by Lévy processes. *Quant. Finance*, 6(1):83–94, 2006.
66. G. Di Nunno and B. Øksendal. Optimal portfolio, partial information and Malliavin calculus. *Eprint in Mathematics*, University of Oslo, 2006.
67. G. Di Nunno and B. Øksendal. The Donsker delta function, a representation formula for functionals of a Lévy process and application hedging in incomplete markets. *Séminaires et Congrès*, 16:71–82, 2007.
68. G. Di Nunno and B. Øksendal. A representation theorem and a sensitivity result for functionals of jump diffusions. In *Mathematical analysis of random phenomena*, pages 177–190. World Sci. Publ., Hackensack, NJ, 2007.
69. G. Di Nunno, B. Øksendal, and F. Proske. White noise analysis for Lévy processes. *J. Funct. Anal.*, 206(1):109–148, 2004.
70. G. Di Nunno and Yu.A. Rozanov. On stochastic integration and differentiation. *Acta Appl. Math.*, 58(1-3):231–235, 1999.
71. G. Di Nunno and Yu.A. Rozanov. Stochastic integrals and adjoint derivatives. In *Stochastic Analysis and its Applications*, volume 2 of *Abel Symposia*, pages 265–307. Springer, Heidelberg, 2007.

72. A.A. Dorogovtsev. Elements of stochastic differential calculus. In *Mathematics today '88 (Russian)*, pages 105–131. "Vishcha Shkola", Kiev, 1988.
73. D. Duffie. *Dynamic Asset Pricing Theory.* Princeton University Press, Princeton, second edition, 1992.
74. E.B. Dynkin. *Markov Processes. Vols. I, II.* Academic Press Inc., Publishers, New York, 1965.
75. E. Eberlein and S. Raible. Term structure models driven by general Lévy processes. *Math. Finance*, 9(1):31–53, 1999.
76. R.J. Elliott, H. Geman, and B.M. Korkie. Portfolio optimization and contingent claim pricing with differential information. *Stochastics Stochastics Rep.*, 60(3-4):185–203, 1997.
77. R.J. Elliott and M. Jeanblanc. Incomplete markets with jumps and informed agents. *Mathematical Methods of Operations Research*, 50:475–492, 1998.
78. K.D. Elworthy and X.-M. Li. Formulae for the derivatives of heat semigroups. *J. Funct. Anal.* 125(1):252–286, 1994.
79. G.B. Folland. *Real Analysis.* Pure and Applied Mathematics (New York). John Wiley & Sons Inc., New York, second edition, 1999.
80. E. Fournié, J.-M. Lasry, J. Lebuchoux, and P.-L. Lions. Applications of Malliavin calculus to Monte-Carlo methods in finance. II. *Finance Stoch.*, 5(2):201–236, 2001.
81. E. Fournié, J.-M. Lasry, J. Lebuchoux, P.-L. Lions, and N. Touzi. Applications of Malliavin calculus to Monte Carlo methods in finance. *Finance Stoch.*, 3(4):391–412, 1999.
82. N. Fournier. Smoothness of the law of some one-dimensional jumping S.D.E.s with non-constant rate of jump. *Electron. J. Probab.*, 13:no. 6, 135–156, 2008.
83. N. Fournier and J.-S. Giet. Existence of densities for jumping stochastic differential equations. *Stochastic Process. Appl.*, 116(4):643–661, 2006.
84. U. Franz, R. Léandre, and R. Schott. Malliavin calculus for quantum stochastic processes. *C. R. Acad. Sci. Paris Sér. I Math.*, 328(11):1061–1066, 1999.
85. U. Franz, N. Privault, and R. Schott. Non-Gaussian Malliavin calculus on real Lie algebras. *J. Funct. Anal.*, 218(2):347–371, 2005.
86. L. Gawarecki and V. Mandrekar. Itô-Ramer, Skorohod and Ogawa integrals with respect to Gaussian processes and their interrelationship. In *Chaos expansions, multiple Wiener-Itô integrals and their applications (Guanajuato, 1992)*, Probab. Stochastics Ser., pages 349–373. CRC, Boca Raton, FL, 1994.
87. I.M. Gel'fand and N.Ya. Vilenkin. *Generalized Functions. Vol. 4.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1964 [1977].
88. H. Gjessing, H. Holden, T. Lindstrøm, B. Øksendal, J. Ubøe, and T.-S. Zhang. The Wick product. In *Frontiers in Pure and Applied Probability*, volume 1, pages 29–67. TVP Publishers, Moscow, 1993.
89. P.W. Glynn. Optimization of stochastic systems via simulation. In *Proceedings of the 1989 Winter Simulation Conference, Society for Computer Simulation*, pages 90–105, New York, 1989. ACM.
90. E. Gobet and A. Kohatsu-Higa. Computation of Greeks for barrier and lookback options using Malliavin calculus. *Electron. Comm. Probab.*, 8:51–62 (electronic), 2003.
91. A. Grorud. Asymmetric information in a financial market with jumps. *Int. J. Theor. Appl. Finance*, 3(4):641–659, 2000.
92. A. Grorud and M. Pontier. Probabilités neutres au risque et asymétrie d'information. *C. R. Acad. Sci. Paris Sér. I Math.*, 329(11):1009–1014, 1999.

93. A. Grorud and M. Pontier. Asymmetrical information and incomplete markets. *Int. J. Theor. Appl. Finance*, 4(2):285–302, 2001.
94. A. Grorud and M. Pontier. Financial market model with influential informed investors. *Int. J. Theor. Appl. Finance*, 8(6):693–716, 2005.
95. M. Grothaus, Y.G. Kondratiev, and L. Streit. Complex Gaussian analysis and the Bargmann-Segal space. *Methods Funct. Anal. Topology*, 3(2):46–64, 1997.
96. M. Grothaus, Yu. G. Kondratiev, and G. F. Us. Wick calculus for regular generalized stochastic functionals. *Random Oper. Stochastic Equations*, 7(3):263–290, 1999.
97. P.R. Halmos. *Measure Theory.* D. Van Nostrand Company, Inc., New York, N. Y., 1950.
98. U.G. Haussmann. On the integral representation of functionals of Itô processes. *Stochastics*, 3(1):17–27, 1979.
99. T. Hida. *Brownian Motion.* Springer-Verlag, New York, 1980.
100. T. Hida. Generalized Brownian functionals. In *Theory and application of random fields (Bangalore, 1982)*, volume 49 of *Lecture Notes in Control and Inform. Sci.*, pages 89–95. Springer, Berlin, 1983.
101. T. Hida and N. Ikeda. Analysis on Hilbert space with reproducing kernel arising from multiple Wiener integral. In *Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66). Vol. II: Contributions to Probability Theory, Part 1*, pages 117–143. Univ. California Press, Berkeley, Calif., 1967.
102. T. Hida, H.-H. Kuo, J. Potthoff, and L. Streit. *White Noise.* Kluwer Academic Publishers Group, Dordrecht, 1993.
103. T. Hida and J. Potthoff. White noise analysis—an overview. In *White noise analysis (Bielefeld, 1989)*, pages 140–165. World Sci. Publishing, River Edge, NJ, 1990.
104. C. Hillairet. Existence of an equilibrium with discontinuous prices, asymmetric information, and nontrivial initial σ-fields. *Math. Finance*, 15(1):99–117, 2005.
105. E. Hille and R.S. Phillips. *Functional Analysis and Semi-Groups.* American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957.
106. H. Holden and B. Øksendal. A white noise approach to stochastic differential equations driven by Wiener and Poisson processes. In *Nonlinear theory of generalized functions (Vienna, 1997)*, volume 401 of *Chapman & Hall/CRC Res. Notes Math.*, pages 293–313. Chapman & Hall/CRC, Boca Raton, FL, 1999.
107. H. Holden, B. Øksendal, J. Ubøe, and T. Zhang. *Stochastic Partial Differential Equations.* Second Ed. Springer. To appear 2008/2009.
108. Y. Hu. Itô-Wiener chaos expansion with exact residual and correlation, variance inequalities. *J. Theoret. Probab.*, 10(4):835–848, 1997.
109. Y. Hu and B. Øksendal. Wick approximation of quasilinear stochastic differential equations. In *Stochastic analysis and related topics, V (Silivri, 1994)*, pages 203–231. Birkhäuser Boston, Boston, MA, 1996.
110. Y. Hu and B. Øksendal. Chaos expansion of local time of fractional Brownian motions. *Stochastic Anal. Appl.*, 20(4):815–837, 2002.
111. Y. Hu and B. Øksendal. Optimal smooth portfolio selection for an insider. *J. Appl. Probab.*, 44(3):742–752, 2007.
112. S. Huddart, J.S. Hughes, and C.B. Levine. Public disclosure and dissimulation of insider trades. *Econometrica*, 69(3):665–681, 2001.

113. F. Huehne. A Clark-Ocone-Haussmann formula for optimal portfolios under Girsanov transformed pure-jump Lévy processes. Technical report, 2005.
114. N. Ikeda and S. Watanabe. An Introduction to Malliavin's Calculus. In *Stochastic analysis (Katata/Kyoto, 1982)*, volume 32 of *North-Holland Math. Library*, pages 1–52. North-Holland, Amsterdam, 1984.
115. N. Ikeda and S. Watanabe. *Stochastic Differential Equations and Diffusion Processes*, volume 24 of *North-Holland Mathematical Library*. North-Holland Publishing Co., Amsterdam, second edition, 1989.
116. P. Imkeller. Malliavin's calculus in insider models: additional utility and free lunches. *Math. Finance*, 13(1):153–169, 2003.
117. P. Imkeller, M. Pontier, and F. Weisz. Free lunch and arbitrage possibilities in a financial market model with an insider. *Stochastic Process. Appl.*, 92(1):103–130, 2001.
118. Y. Ishikawa and H. Kunita. Malliavin calculus on the Wiener-Poisson space and its application to canonical SDE with jumps. *Stochastic Process. Appl.*, 116(12):1743–1769, 2006.
119. K. Itô. On stochastic processes. I. (Infinitely divisible laws of probability). *Jap. J. Math.*, 18:261–301, 1942.
120. K. Itô. Multiple Wiener integral. *J. Math. Soc. Japan*, 3:157–169, 1951.
121. K. Itô. Spectral type of the shift transformation of differential processes with stationary increments. *Trans. Amer. Math. Soc.*, 81:253–263, 1956.
122. K. Itô. Extension of stochastic integrals. In *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)*, pages 95–109, New York, 1978. Wiley.
123. J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*, volume 288 of *Grundlehren der Mathematischen Wissenschaften*. Springer-Verlag, Berlin, second edition, 2003.
124. Ju.M. Kabanov. A generalized Itô formula for an extended stochastic integral with respect to Poisson random measure. *Uspehi Mat. Nauk*, 29(4(178)):167–168, 1974.
125. Ju.M. Kabanov. Extended stochastic integrals. *Teor. Verojatnost. i Primenen.*, 20(4):725–737, 1975.
126. A.B. Kaminsky. Extended stochastic calculus for the Poisson random measures. *Nats. Akad. Nauk Ukraïn. Īnst. Mat. Preprint*, 15:i+16, 1996.
127. A.B. Kaminsky. An integration by parts formula for the Poisson random measures and applications. *Nats. Akad. Nauk Ukraïn. Īnst. Mat. Preprint*, 9:i+20, 1996.
128. A.B. Kaminsky. A white noise approach to stochastic integration for a Poisson random measure. *Teor. Ĭmovīr. Mat. Stat.*, 57:41–50, 1997.
129. I. Karatzas and S.E. Shreve. *Brownian Motion and Stochastic Calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991.
130. I. Karatzas and S.E. Shreve. *Methods of Mathematical Finance*. Springer-Verlag, New York, 1998.
131. A. Kohatsu-Higa. Enlargement of filtrations and models for insider trading. In *Stochastic processes and applications to mathematical finance*, pages 151–165. World Sci. Publ., River Edge, NJ, 2004.
132. A. Kohatsu-Higa. Models for insider trading with finite utility. In *Paris-Princeton Lectures on Mathematical Finance 2004*, volume 1919 of *Lecture Notes in Math.*, pages 103–171. Springer, Berlin, 2007.

133. A. Kohatsu-Higa and A. Sulem. A large trader-insider model. In *Stochastic processes and applications to mathematical finance*, pages 101–124. World Sci. Publ., Hackensack, NJ, 2006.
134. A. Kohatsu-Higa and A. Sulem. Utility maximization in an insider influenced market. *Math. Finance*, 16(1):153–179, 2006.
135. Yu.G. Kontratiev. *Generalized functions in problems of infinitedimensional analysis*. PhD thesis, Kiev University, 1970.
136. D. Kramkov and W. Schachermayer. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *Ann. Appl. Probab.*, 13(4):1504–1516, 2003.
137. O.M. Kulik. Malliavin calculus for Lévy processes with arbitrary Lévy measures. *Teor. Ĭmovīr. Mat. Stat.*, 72:67–83, 2005.
138. H. Kunita. *Stochastic Flows and Stochastic Differential Equations*, volume 24 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.
139. H.-H. Kuo. Donsker's delta function as a generalized Brownian functional and its application. In *Theory and application of random fields (Bangalore, 1982)*, volume 49 of *Lecture Notes in Control and Inform. Sci.*, pages 167–178. Springer, Berlin, 1983.
140. H.-H. Kuo. *White Noise Distribution Theory*. Probability and Stochastics Series. CRC Press, Boca Raton, FL, 1996.
141. S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. I. In *Stochastic analysis (Katata/Kyoto, 1982)*, volume 32 of *North-Holland Math. Library*, pages 271–306. North-Holland, Amsterdam, 1984.
142. S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 32(1):1–76, 1985.
143. S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. III. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 34(2):391–442, 1987.
144. A.S. Kyle. Continuous auctions and insider trading. *Econometrica*, 53:1315–1335, 1985.
145. A. Lanconelli and F. Proske. On explicit strong solution of Itô-SDE's and the Donsker delta function of a diffusion. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 7(3):437–447, 2004.
146. Y.-J. Lee and H.-H. Shih. Donsker's delta function of Lévy process. *Acta Appl. Math.*, 63(1-3):219–231, 2000.
147. J.A. León, R. Navarro, and D. Nualart. An anticipating calculus approach to the utility maximization of an insider. *Math. Finance*, 13(1):171–185, 2003.
148. J.A. León, J.L. Solé, F. Utzet, and J. Vives. On Lévy processes, Malliavin calculus and market models with jumps. *Finance Stoch.*, 6(2):197–225, 2002.
149. D. Lépingle and J. Mémin. Sur l'intégrabilité uniforme des martingales exponentielles. *Z. Wahrsch. Verw. Gebiete*, 42(3):175–203, 1978.
150. T. Lindstrøm, B. Øksendal, and J. Ubøe. Stochastic differential equations involving positive noise. In *Stochastic analysis (Durham, 1990)*, volume 167 of *London Math. Soc. Lecture Note Ser.*, pages 261–303. Cambridge Univ. Press, Cambridge, 1991.
151. T. Lindstrøm, B. Øksendal, and J. Ubøe. Stochastic modelling of fluid flow in porous media. In *Control theory, stochastic analysis and applications (Hangzhou, 1991)*, pages 156–172. World Sci. Publishing, River Edge, NJ, 1991.

152. T. Lindstrøm, B. Øksendal, and J. Ubøe. Wick multiplication and Itô-Skorohod stochastic differential equations. In *Ideas and methods in mathematical analysis, stochastics, and applications (Oslo, 1988)*, pages 183–206. Cambridge Univ. Press, Cambridge, 1992.
153. M. Loève. *Probability Theory. I and II.* Springer-Verlag, New York, fourth edition, 1977, 1978.
154. A. Løkka. Martingale representation of functionals of Lévy processes. *Stochastic Anal. Appl.*, 22(4):867–892, 2004.
155. A. Løkka, B. Øksendal, and F. Proske. Stochastic partial differential equations driven by Lévy space-time white noise. *Ann. Appl. Probab.*, 14(3):1506–1528, 2004.
156. A. Løkka and F.N. Proske. Infinite dimensional analysis of pure jump Lévy processes on the Poisson space. *Math. Scand.*, 98(2):237–261, 2006.
157. S. Luo and Q. Zhang. Dynamic insider trading. In *Applied probability (Hong Kong, 1999)*, volume 26 of *AMS/IP Stud. Adv. Math.*, pages 93–104. Amer. Math. Soc., Providence, RI, 2002.
158. P. Malliavin. Stochastic calculus of variation and hypoelliptic operators. In *Proceedings of the International Symposium on Stochastic Differential Equations (Res. Inst. Math. Sci., Kyoto Univ., Kyoto, 1976)*, pages 195–263, New York, 1978. Wiley.
159. P. Malliavin. *Integration and Probability*, volume 157 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.
160. P. Malliavin. *Stochastic Analysis*, volume 313 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1997.
161. P. Malliavin and A. Thalmaier. *Stochastic Calculus of Variations in Mathematical Finance*. Springer Finance. Springer-Verlag, Berlin, 2006.
162. S. Mataramvura, B. Øksendal, and F. Proske. The Donsker delta function of a Lévy process with application to chaos expansion of local time. *Ann. Inst. H. Poincaré Probab. Statist.*, 40(5):553–567, 2004.
163. M. Mensi and N. Privault. Conditional calculus on Poisson space and enlargement of filtration. *Stochastic Anal. Appl.*, 21(1):183–204, 2003.
164. R. Merton. The theory of rational option pricing. *Bell Journal of Economics and Management Science*, 4:141–183, 1973.
165. P.-A. Meyer and J.A. Yan. Distributions sur l'espace de Wiener (suite) d'après I. Kubo et Y. Yokoi. In *Séminaire de Probabilités, XXIII*, volume 1372 of *Lecture Notes in Math.*, pages 382–392. Springer, Berlin, 1989.
166. T. Meyer-Brandis and F. Proske. On the existence and explicit representability of strong solutions of Lévy noise driven SDE's with irregular coefficients. *Commun. Math. Sci.*, 4(1):129–154, 2006.
167. E. Nicolato and E. Venardos. Option pricing in stochastic volatility models of the Ornstein-Uhlenbeck type. *Math. Finance*, 13(4):445–466, 2003.
168. J. Norris. Simplified Malliavin calculus. In *Séminaire de Probabilités, XX, 1984/85*, volume 1204 of *Lecture Notes in Math.*, pages 101–130. Springer, Berlin, 1986.
169. D. Nualart. *The Malliavin Calculus and Related Topics*. Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.
170. D. Nualart and É. Pardoux. Stochastic calculus with anticipating integrands. *Probab. Theory Related Fields*, 78(4):535–581, 1988.

171. D. Nualart and W. Schoutens. Chaotic and predictable representations for Lévy processes. *Stochastic Process. Appl.*, 90(1):109–122, 2000.
172. D. Nualart and J. Vives. Anticipative calculus for the Poisson process based on the Fock space. In *Séminaire de Probabilités, XXIV, 1988/89*, volume 1426 of *Lecture Notes in Math.*, pages 154–165. Springer, Berlin, 1990.
173. D. Ocone. Malliavin's calculus and stochastic integral representations of functionals of diffusion processes. *Stochastics*, 12(3-4):161–185, 1984.
174. D.L. Ocone and I. Karatzas. A generalized Clark representation formula, with application to optimal portfolios. *Stochastics Stochastics Rep.*, 34(3-4):187–220, 1991.
175. S. Ogawa. Quelques propriétés de l'intégrale stochastique du type noncausal. *Japan J. Appl. Math.*, 1(2):405–416, 1984.
176. S. Ogawa. The stochastic integral of noncausal type as an extension of the symmetric integrals. *Japan J. Appl. Math.*, 2(1):229–240, 1985.
177. B. Øksendal. Stochastic partial differential equations—a mathematical connection between macrocosmos and microcosmos. In *Analysis, algebra, and computers in mathematical research (Luleå, 1992)*, volume 156 of *Lecture Notes in Pure and Appl. Math.*, pages 365–385. Dekker, New York, 1994.
178. B. Øksendal. An Introduction to Malliavin Calculus with Applications to Economics. Technical report, Norwegian School of Economics and Business Administration, Bergen, 1996.
179. B. Øksendal. *Stochastic Differential Equations*. Universitext. Springer-Verlag, Berlin, sixth edition, 2003.
180. B. Øksendal. A universal optimal consumption rate for an insider. *Math. Finance*, 16(1):119–129, 2006.
181. B. Øksendal and F. Proske. White noise of Poisson random measures. *Potential Anal.*, 21(4):375–403, 2004.
182. B. Øksendal and A. Sulem. Partial observation control in an anticipating environment. *Uspekhi Mat. Nauk*, 59(2(356)):161–184, 2004.
183. B. Øksendal and A. Sulem. *Applied Stochastic Control of Jump Diffusions*. Springer-Verlag, Berlin, second edition, 2007.
184. Y.Y. Okur. White noise generalization of the Clark-Ocone formula under change of measure. Eprint in Mathematics, University of Oslo, 2007.
185. Y.Y. Okur. An extension of the Clark-Ocone formula under change of measure for Lévy processes. Eprint in Mathematics, University of Oslo, 2008.
186. Y.Y. Okur, F. Proske, and H.B. Salleh. SDE solutions in the space of smooth random variables. Eprint in Mathematics, University of Oslo, 2008.
187. É. Pardoux and S.G. Peng. Adapted solution of a backward stochastic differential equation. *Systems Control Lett.*, 14(1):55–61, 1990.
188. K.R. Parthasarathy. *An Introduction to Quantum Stochastic Calculus*, volume 85 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.
189. J. Picard. Formules de dualité sur l'espace de Poisson. *Ann. Inst. H. Poincaré Probab. Statist.*, 32(4):509–548, 1996.
190. J. Picard. On the existence of smooth densities for jump processes. *Probab. Theory Related Fields*, 105(4):481–511, 1996.
191. I. Pikovsky and I. Karatzas. Anticipative portfolio optimization. *Adv. in Appl. Probab.*, 28(4):1095–1122, 1996.
192. M. Pontier. [Essai de panorama de la] modélisation et [de la] détection du délit d'initié. *Matapli*, 77:58–75, 2005.

193. J. Potthoff and L. Streit. A characterization of Hida distributions. *J. Funct. Anal.*, 101(1):212–229, 1991.
194. J. Potthoff and M. Timpel. On a dual pair of spaces of smooth and generalized random variables. *Potential Anal.*, 4(6):637–654, 1995.
195. N. Privault. An extension of stochastic caluculus to certain non-Markovian processes. Technical report, 1997.
196. N. Privault. Equivalence of gradients on configuration spaces. *Random Oper. Stochastic Equations*, 7(3):241–262, 1999.
197. N. Privault. Independence of a class of multiple stochastic integrals. In *Seminar on Stochastic Analysis, Random Fields and Applications (Ascona, 1996)*, volume 45 of *Progr. Probab.*, pages 249–259. Birkhäuser, Basel, 1999.
198. N. Privault. Connections and curvature in the Riemannian geometry of configuration spaces. *J. Funct. Anal.*, 185(2):367–403, 2001.
199. N. Privault, J.L. Solé, and J. Vives. Chaotic Kabanov formula for the Azéma martingales. *Bernoulli*, 6(4):633–651, 2000.
200. N. Privault and X. Wei. A Malliavin calculus approach to sensitivity analysis in insurance. *Insurance Math. Econom.*, 35(3):679–690, 2004.
201. N. Privault and J.-L. Wu. Poisson stochastic integration in Hilbert spaces. *Ann. Math. Blaise Pascal*, 6(2):41–61, 1999.
202. F. Proske. The stochastic transport equation driven by Lévy white noise. *Commun. Math. Sci.*, 2(4):627–641, 2004.
203. F. Proske. Stochastic differential equations – some new ideas. *Stochastics*, 79(6):563–600, 2007.
204. P.E. Protter. *Stochastic Integration and Differential Equations*. Springer-Verlag, Berlin, 2005. Second edition.
205. M. Reed and B. Simon. *Methods of Modern Mathematical Physics. I.* Academic Press Inc., New York, second edition, 1980.
206. L.C.G. Rogers and D. Williams. *Diffusions, Markov Processes, and Martingales. Vol. 2.* Cambridge University Press, Cambridge, 2000.
207. Yu.A. Rozanov. *Innovation Processes.* V. H. Winston & Sons, Washington, D. C., 1977. Translated from the Russian, Preface by translation editor A. V. Balakrishnan, Scripta Series in Mathematics.
208. W. Rudin. *Functional Analysis.* International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, second edition, 1991.
209. F. Russo and P. Vallois. Forward, backward and symmetric stochastic integration. *Probab. Th. Rel. Fields*, 93(4):403–421, 1993.
210. F. Russo and P. Vallois. The generalized covariation process and Itô formula. *Stoch. Proc. Appl.*, 59(4):81–104, 1995.
211. F. Russo and P. Vallois. Stochastic calculus with respect to continuous finite quadratic variation processes. *Stoch. Stoch. Rep.*, 70(4):1–40, 2000.
212. M. Sanz-Solé. *Malliavin Calculus*. Fundamental Sciences. EPFL Press, Lausanne, 2005.
213. K. Sato. *Lévy Processes and Infinitely Divisible Distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1999.
214. W. Schachermayer. *Portfolio optimization in incomplete financial markets.* Cattedra Galileiana. Scuola Normale Superiore, Classe di Scienze, Pisa, 2004.
215. W. Schoutens. *Stochastic Processes and Orthogonal Polynomials*, volume 146 of *Lecture Notes in Statistics*. Springer-Verlag, New York, 2000.

216. I. Shigekawa, Stochastic Analysis. Translation of Mathematical Monographs 214. American Mathematical Society, 2004.
217. A.V. Skorohod. On a generalization of the stochastic integral. *Teor. Verojatnost. i Primenen.*, 20(2):223–238, 1975.
218. D.W. Stroock. The Malliavin calculus and its application to second order parabolic differential equations. I. *Math. Systems Theory*, 14(1):25–65, 1981.
219. D.W. Stroock. The Malliavin calculus and its application to second order parabolic differential equations. II. *Math. Systems Theory*, 14(2):141–171, 1981.
220. D. Surgailis. On multiple Poisson stochastic integrals and associated Markov semigroups. *Probab. Math. Statist.*, 3(2):217–239, 1984.
221. A. Takeuchi. The Malliavin calculus for SDE with jumps and the partially hypoelliptic problem. *Osaka J. Math.*, 39(3):523–559, 2002.
222. S. Thangavelu. *Lectures on Hermite and Laguerre expansions*, volume 42 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1993.
223. A. S. Üstünel. Representation of the distributions on Wiener space and stochastic calculus of variations. *J. Funct. Anal.*, 70(1):126–139, 1987.
224. A.S. Üstünel. *An Introduction to Analysis on Wiener Space*, volume 1610 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1995.
225. S. Watanabe. *Lectures on Stochastic Differential Equations and Malliavin Calculus*, volume 73 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Published for the Tata Institute of Fundamental Research, Bombay, 1984.
226. G.C. Wick. The evaluation of the collision matrix. *Physical Rev. (2)*, 80:268–272, 1950.
227. N. Wiener. The homogeneous chaos. *Amer. J. Math.*, 60(4):897–936, 1938.
228. A.L. Yablonski. The Malliavin calculus for processes with conditionally independent increments. In *Stochastic analysis and applications*, volume 2 of *Abel Symp.*, pages 641–678. Springer, Berlin, 2007.
229. J. Yong and X.Y. Zhou. *Stochastic Controls*, volume 43 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 1999.
230. M. Zakai. The Malliavin calculus. *Acta Appl. Math.*, 3(2):175–207, 1985.
231. T. Zhang. Characterizations of the white noise test functionals and Hida distributions. *Stochastics Stochastics Rep.*, 41(1-2):71–87, 1992.

Notation and Symbols

Numbers

$\mathbb{N}$	The natural numbers
$\mathbb{Z}$	The integer numbers
$\mathbb{Q}$	The rational numbers
$\mathbb{R}$	The real numbers
$\mathbb{R}_0$	p. 162
$\mathbb{C}$	The complex numbers
$\mathbb{C}^{\mathbb{N}}$	The set of all sequences of complex numbers
T	p. 7
$\mathcal{J}$	p. 66
$\left(2\mathbb{N}\right)^{\alpha}$	p. 68
$\mathbb{K}_q(R)$	p. 74
$\mathbb{C}_c^{\mathbb{N}}$	p. 75
m_2	p. 217

Measures

P	pp. 7, 64, 215, 238
P^W	pp. 197, 238
$P^{\widetilde{N}}$	pp. 197, 238
$\lambda = dt$	Lebesgue measure
ν	p. 162
ρ	p. 217

Operations

$f \otimes g$	p. 11
$f \hat{\otimes} g$	p. 11
$W^{\otimes(n+1)}$	p. 13
$\langle \omega, \phi \rangle$	p. 64
$X \diamond Y$	pp. 70, 221

Spaces and Norms

Filtrations and σ-Algebras

Functions, Random Variables, and Transforms

Processes and Fields

Integrals and Differentials

$J_n(f)$	pp. 8, 178
$I_n(g)$	pp. 10, 178
$\delta(u)$	pp. 20, 183
$\widetilde{N}(\delta t, dz)$	p. 183
$\delta\eta(t)$	pp. 20, 184
$d^- W(s)$	p. 134
$\widetilde{N}(d^- t, dz)$	p. 267

Derivatives

$\partial^\alpha,$	p. 64
$D_t F$	pp. 28, 88, 89, 360
$D_\gamma F$	pp. 87, 357
$\mathbf{D}_\gamma F$	p. 358
$\mathbf{D}_t F$	p. 358
$D_{t,z} F$	pp. 188, 230
$D_{t^+}\varphi(t)$	p. 140
$D_{t^+,z}\theta(t,z)$	p. 268
$\mathcal{D}_t F$	p. 240
$D^j_{t_1,\dots,t_j,}$	p. 343
$D_y f$	pp. 354, 356

Admissible Controls

$\mathcal{A}$	p. 170
$\mathcal{A}_{\mathbb{F}}$	pp. 131, 301,
$\mathcal{A}_{\mathbb{G}}$	pp. 145, 132
$\mathcal{A}_{\mathbb{G},\mathcal{Q}}$	p. 150
$\mathcal{A}_{\mathbb{E}}$	pp. 200, 278, 291, 296
$\mathcal{A}_{\mathbb{H}}$	pp. 305, 311, 322

Notations

M^T	Transpose of a matrix M
$P \sim Q$	Measure P is equivalent to measure Q
$E[F]$	(generalized) Expectation w.r.t. measure P
$E_Q[F]$	Expectation w.r.t. measure Q
$E[F\|\mathcal{F}_t]$	(generalized) Conditional expectation
càdlàg	Right continuous with left limits
càglàd	Left continuous with right limits
a.a., a.e., a.s.	Almost all, almost everywhere, almost surely
s.t.	Such that
w.r.t.	With respect to
SDE	Stochastic differential equation
BSDE	Backward stochastic differential equation
:=	Equal to by definition

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