3

Malliavin Derivative via Chaos Expansion

3.1 The Malliavin Derivative

The Malliavin calculus (see [158], see also, for example, [53, 72, 160, 169, 212]) was originally created as a tool for studying the regularity of densities of solutions of stochastic differential equations. Subsequently, partly due to the papers [173] and [174], the significance of Malliavin calculus in finance became clear. This triggered a tremendous interest in the subject, also among economists. Today the range of applications has extended even further to include numerical methods, stochastic control, and insider trading, not just for systems driven by Brownian motion, but for systems driven by general Lévy processes. These applications will be covered later in this book.

There are many ways of introducing the Malliavin derivative. The original construction was given on the Wiener space $\Omega = C_0([0,T])$ consisting of all continuous functions $\omega : [0,T] \longrightarrow \mathbb{R}$ with $\omega(0) = 0$. This construction is outlined in Appendix A.

In this book, we mainly use an approach based on chaos expansions. We give a presentation in this chapter. In the Brownian motion case this approach is basically equivalent to the construction of the Malliavin derivative as a stochastic gradient on the space $\Omega = \mathcal{S}'(\mathbb{R})$. This last approach has the advantage of being more intuitive. Moreover, it opens for a useful combination with Hida white noise calculus, which turns out to be a useful framework for both Malliavin calculus, Skorohod integrals, and anticipative calculus in general. We discuss this in Chap. 6.

Definition 3.1. *Let $F \in L^2(P)$ be $\mathcal{F}_T$-measurable with chaos expansion*

$$F = \sum_{n=0}^{\infty} I_n(f_n),$$

where $f_n \in \widetilde{L}^2([0,T]^n)$, $n = 1, 2, \ldots$.

G.Di Nunno et al., *Malliavin Calculus for Lévy Processes with Applications to Finance,*

(i) We say that $F \in \mathbb{D}_{1,2}$ *if*

$$\|F\|^2_{\mathbb{D}_{1,2}} := \sum_{n=1}^{\infty} nn! \|f_n\|^2_{L^2([0,T]^n)} < \infty. \tag{3.1}$$

(ii) If $F \in \mathbb{D}_{1,2}$ *we define the* Malliavin derivative D_tF of F at time t *as the expansion*

$$D_tF = \sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot, t)), \qquad t \in [0,T], \tag{3.2}$$

where $I_{n-1}(f_n(\cdot,t))$ *is the* $(n-1)$*-fold iterated integral of* $f_n(t_1, ..., t_{n-1}, t)$ *with respect to the first* $n-1$ *variables* $t_1, ..., t_{n-1}$ *and* $t_n = t$ *left as parameter.*

Remark 3.2. Note that if (3.1) holds, then

$$\begin{aligned}\|D.F\|^2_{L^2(P\times\lambda)} &= E\Big[\int_0^T (D_tF)^2 dt\Big] = \sum_{n=1}^{\infty} \int_0^T n^2(n-1)! \|f_n(\cdot,t)\|^2_{L^2([0,T]^n)} dt \\ &= \sum_{n=1}^{\infty} nn! \|f_n\|^2_{L^2([0,T]^n)} = \|F\|^2_{\mathbb{D}_{1,2}} < \infty,\end{aligned} \tag{3.3}$$

so $D.F = D_tF$, $t \in [0,T]$, is well defined as an element of $L^2(P\times\lambda)$.

We first establish the following fundamental result.

Theorem 3.3. Closability of the Malliavin derivative. *Suppose* $F \in L^2(P)$ *and* $F_k \in \mathbb{D}_{1,2}$, $k = 1, 2, ...$, *such that*

(i) $F_k \longrightarrow F$, $k \to \infty$, *in* $L^2(P)$
(ii) $\{D_tF_k\}_{k=1}^{\infty}$ *converges in* $L^2(P\times\lambda)$.

Then $F \in \mathbb{D}_{1,2}$ *and* $D_tF_k \longrightarrow D_tF$, $k \to \infty$, *in* $L^2(P\times\lambda)$.

Proof Let $F = \sum_{n=0}^{\infty} I_n(f_n)$ and $F_k = \sum_{n=0}^{\infty} I_n(f_n^{(k)})$, $k = 1, 2, \dots$. Then by (i)

$$f_n^{(k)} \longrightarrow f_n, \quad k \to \infty, \quad \text{in } L^2(\lambda^n)$$

for all n. By (ii) we have

$$\sum_{n=1}^{\infty} nn! \|f_n^{(k)} - f_n^{(j)}\|^2_{L^2(\lambda^n)} = \|D_tF_k - D_tF_j\|^2_{L^2(P\times\lambda)} \longrightarrow 0, \quad j,k \to \infty.$$

Hence by the Fatou lemma,

$$\lim_{k\to\infty} \sum_{n=1}^{\infty} nn! \|f_n^{(k)} - f_n\|^2_{L^2(\lambda^n)} \le \lim_{k\to\infty} \varliminf_{j\to\infty} \sum_{n=1}^{\infty} nn! \|f_n^{(k)} - f_n^{(j)}\|^2_{L^2(\lambda^n)} = 0.$$

This implies that $F \in \mathbb{D}_{1,2}$ and

$$D_tF_k \longrightarrow D_tF, \quad k \to \infty, \quad \text{in } L^2(P\times\lambda). \quad \square$$

3.2 Computation and Properties of the Malliavin Derivative

In this section we proceed presenting a collection of results that constitute the rules of calculus of the Malliavin derivatives.

3.2.1 Chain Rules for Malliavin Derivative

We proceed to prove a useful chain rule for Malliavin derivatives. First let us consider the case when $f_n = f^{\otimes n}$ for some $f \in L^2([0,T])$, that is,

$$f_n(t_1, ..., t_n) = f(t_1) \cdots f(t_n).$$

Then by (1.15) we have

$$I_n(f_n) = \|f\|^n h_n\Big(\frac{\theta}{\|f\|}\Big), \tag{3.4}$$

where $\|f\| = \|f\|_{L^2([0,T])}$, $\theta = \int_0^T f(t)dW(t)$ and h_n is the Hermite polynomial of order n. Then by (3.2) we have

$$\begin{aligned} D_t I_n(f_n) &= n I_{n-1}(f_n(\cdot, t)) \\ &= n I_{n-1}(f^{\otimes(n-1)}) f(t) \\ &= n\|f\|^{n-1} h_{n-1}\Big(\frac{\theta}{\|f\|}\Big) f(t). \end{aligned} \tag{3.5}$$

A basic property of the Hermite polynomials is that

$$h_n'(x) = n h_{n-1}(x). \tag{3.6}$$

Combining this with (3.4) and (3.5) we get

$$D_t h_n\Big(\frac{\theta}{\|f\|}\Big) = h_n'\Big(\frac{\theta}{\|f\|}\Big) \frac{f(t)}{\|f\|}. \tag{3.7}$$

In particular, choosing $n = 1$, we get

$$D_t \int_0^T f(s)dW(s) = f(t). \tag{3.8}$$

Similarly, by (3.6) and induction, for $n = 2, 3, ...$, we have

$$D_t\Big(\int_0^T f(s)dW(s)\Big)^n = n\Big(\int_0^T f(s)dW(s)\Big)^{n-1} f(t). \tag{3.9}$$

Let $\mathbb{D}_{1,2}^0$ be the set of all $F \in L^2(P)$ whose chaos expansion has only finitely many terms. Then we have the following result.

Theorem 3.4. Product rule for Malliavin derivative. *Suppose* $F_1, F_2 \in \mathbb{D}^0_{1,2}$*. Then* $F_1, F_2 \in \mathbb{D}_{1,2}$ *and also* $F_1 F_2 \in \mathbb{D}_{1,2}$ *with*

$$D_t(F_1 F_2) = F_1 D_t F_2 + F_2 D_t F_1. \tag{3.10}$$

Proof Being $F_1, F_2 \in \mathbb{D}^0_{1,2}$, clearly $F_1, F_2 \in \mathbb{D}_{1,2}$ and, since the Gaussian random variables have all finite moments, we also have that $F_1 F_2 \in L^2(P)$. First of all let us consider the random variables $F_k^{(n)}$ $(n = 1, 2, ..., k = 1, 2)$ as linear combination of iterated integrals of tensor products of functions ξ_i in an orthogonal basis $\{\xi_j\}_{j=1}^\infty$ of $L^2([0, T])$. Thanks to the structure of the Hermite polynomials, the argument above together with (1.14) shows that $F_1^{(n)}, F_2^{(n)}$ and $F_1^{(n)} F_2^{(n)}$ are in $\mathbb{D}_{1,2}$ for all n, with

$$D_t(F_1^{(n)} F_2^{(n)}) = F_1^{(n)} D_t F_2^{(n)} + F_2^{(n)} D_t F_1^{(n)}. \tag{3.11}$$

We can choose the two sequences so that $F_k^{(n)} \longrightarrow F_k$ in $L^2(P)$ and $D_t F_k^{(n)} \longrightarrow D_t F_k$ in $L^2(P \times \lambda)$, for $n \to \infty$ $(k = 1, 2)$. Then, being $F_1 F_2 \in \mathbb{D}^0_{1,2}$, we have that $F_1^{(n)} F_2^{(n)} \longrightarrow F_1, F_2$ in $L^2(P)$ and also $\{D_t(F_1^{(n)} F_2^{(n)})\}_{n=1}^\infty$ converges in $L^2(P \times \lambda)$. Hence we can conclude by Theorem 3.3. □

See also Problem 3.1.

A version of the chain rule can be formulated as follows, see also [169].

Theorem 3.5. Chain rule. *Let* $G \in \mathbb{D}_{1,2}$ *and* $g \in C^1(\mathbb{R})$ *with bounded derivative. Then* $g(G) \in \mathbb{D}_{1,2}$ *and*

$$D_t g(G) = g'(G) D_t G. \tag{3.12}$$

Here $g'(x) = \frac{d}{dx} g(x)$.

Proof The result can be derived as a corollary to a forthcoming general result. See Theorem 6.3 and Corollary 6.4. □

Remark 3.6. Another chain rule requiring only the Lipschitz continuity of φ can be found in [169, Proposition 1.2.4].

3.2.2 Malliavin Derivative and Conditional Expectation

We now present some preliminary results on conditional expectations.

Definition 3.7. *Let* G *be a Borel set in* $[0, T]$*. We define* $\mathcal{F}_G$ *to be the completed* σ*-algebra generated by all random variables of the form*

$$F = \int_0^T \chi_A(t) dW(t),$$

for all Borel sets $A \subseteq G$.

Thus if $G = [0, t]$, for any $t \in [0, T]$ fixed, we have that $\mathcal{F}_{[0,t]} = \mathcal{F}_t$. Note that if G_1, G_2 are Borel sets in $[0, T]$, then $\mathcal{F}_{G_1} \cap \mathcal{F}_{G_2} = \mathcal{F}_{G_1 \cap G_2}$.

Lemma 3.8. *For any $g \in L^2([0, T])$ we have*

$$E\Big[\int_0^T g(t)dW(t)|\mathcal{F}_G\Big] = \int_0^T \chi_G(t)g(t)dW(t).$$

Proof By definition of conditional expectation, it is sufficient to verify that the random variable

$$\int_0^T \chi_G(t)g(t)dW(t) \qquad \text{is } \mathcal{F}_G\text{-measurable} \tag{3.13}$$

and that

$$E\Big[F\int_0^T g(t)dW(t)\Big] = E\Big[F\int_0^T \chi_G(t)g(t)dW(t)\Big] \tag{3.14}$$

for all bounded $\mathcal{F}_G$-measurable random variables F.
To prove (3.13) we may assume that g is continuous, because the continuous functions are dense in $L^2([0, T])$. If g is continuous, then

$$\int_0^T \chi_G(t)g(t)dW(t) = \lim_{\Delta t_i \to 0} \sum_{i=0}^{n} g(t_i) \int_{t_i}^{t_{i+1}} \chi_G(t)dW(t),$$

where the limit is in $L^2(P)$ for the vanishing mesh Δt_i of the partitions $0 = t_0 < ... < t_n = T$. Since each term in the sum is $\mathcal{F}_G$-measurable, the sum is also $\mathcal{F}_G$-measurable. Then by taking a subsequence converging P-a.s. we conclude that the limit represents an $\mathcal{F}_G$-measurable random variable.

To prove (3.14) we may assume $F = \int_0^T \chi_A(t)dW(t)$ for some $A \subseteq G$. Then by the Itô isometry we have

$$E\Big[F\int_0^T g(t)dW(t)\Big] = E\Big[\int_0^T \chi_A(t)g(t)dt\Big],$$

and also

$$E\Big[F\int_0^T \chi_G(t)g(t)dW(t)\Big] = E\Big[\int_0^T \chi_A(t)\chi_G(t)g(t)dt\Big] = E\Big[\int_0^T \chi_A(t)g(t)dt\Big].$$

Then the proof can be completed by a density argument. □

Lemma 3.9. *Let $G \subseteq [0,T]$ be a Borel set and $v = v(t)$, $t \in [0,T]$, be a stochastic process such that*

(1) *for all t, $v(t)$ is measurable with respect to $\mathcal{F}_t \cap \mathcal{F}_G$*

(2) $E\big[\int_0^T v^2(t)dt\big] < \infty$.

Then

$$\int_G v(t)dW(t) \qquad \text{is } \mathcal{F}_G\text{-measurable.}$$

Proof By a standard approximation procedure it is sufficient to consider v to be an elementary process of the form

$$v(t) = \sum_{i=1}^n v_i \chi_{(t_i,t_{i+1}]}(t),$$

where $0 = t_0 < t_1 < \cdots < t_n = T$ and v_i are $\mathcal{F}_{t_i} \cap \mathcal{F}_G$-measurable random variables such that (2) is satisfied. For such v we have

$$\int_G v(t)dW(t) = \sum_{i=1}^n v_i \int_{G\cap(t_i,t_{i+1}]} 1\, dW(t),$$

which is a sum of products of $\mathcal{F}_G$-measurable functions and hence $\mathcal{F}_G$-measurable. $\square$

Lemma 3.10. *Let $u = u(t)$, $t \in [0,T]$, be an $\mathbb{F}$-adapted stochastic process in $L^2(P \times \lambda)$. Then*

$$E\Big[\int_0^T u(t)dW(t)|\mathcal{F}_G\Big] = \int_G E[u(t)|\mathcal{F}_G]dW(t).$$

Proof Lemma 3.9 guarantees that $\int_G E[u(t)|\mathcal{F}_G]dW(t)$ is $\mathcal{F}_G$-measurable. Then it suffices to verify that

$$E\Big[F\int_0^T u(t)dW(t)\Big] = E\Big[F\int_G E[u(t)|\mathcal{F}_G]dW(t)\Big]$$

for all F of the form $F = \int_A dW(t)$, where $A \subseteq G$ is a Borel set. In this case we obtain by the Itô isometry that

$$E\Big[F\int_0^T u(t)dW(t)\Big] = E\Big[\int_0^T \chi_A(t)u(t)dt\Big] = \int_A E[u(t)]dt$$

and

$$
\begin{aligned}
E\Big[F\int_G E[u(t)|\mathcal{F}_G]dW(t)\Big] &= E\Big[\int_0^T \chi_A(t)\chi_G(t)E\big[u(t)|\mathcal{F}_G\big]dt\Big] \\
&= \int_0^T \chi_A(t)E\big[E[u(t)|\mathcal{F}_G]\big]dt \\
&= \int_A E[u(t)]dt.
\end{aligned}
$$

A density argument completes the proof. □

Proposition 3.11. *Let $f_n \in \widetilde{L}^2([0,T]^n)$, $n = 1, 2, \ldots$. Then*

$$
E[I_n(f_n)|\mathcal{F}_G] = I_n[f_n\chi_G^{\otimes n}], \tag{3.15}
$$

where $(f_n\chi_G^{\otimes n})(t_1,\ldots,t_n) = f_n(t_1,\ldots,t_n)\chi_G(t_1)\cdots\chi_G(t_n)$.

Proof We proceed by induction on n. For $n = 1$ we have

$$
E[I_1(f_1)|\mathcal{F}_G] = E[\int_0^T f_1(t_1)dW(t_1)|\mathcal{F}_G] = \int_0^T f_1(t_1)\chi_G(t_1)dW(t_1) = I_1\big[f_1\chi_G^{\otimes 1}\big]
$$

by Lemma 3.10. Assume that (3.15) holds for $n = k$. Then, again by Lemma 3.10, we have

$$
\begin{aligned}
&E[I_{k+1}(f_{k+1})|\mathcal{F}_G] \\
&= (k+1)!E\Big[\int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} f_{k+1}(t_1,\ldots,t_{k+1})dW(t_1)\cdots dW(t_k)dW(t_{k+1})|\mathcal{F}_G\Big] \\
&= (k+1)!\int_0^T E\Big[\int_0^{t_{k+1}} \cdots \int_0^{t_2} f_{k+1}(t_1,\ldots,t_{k+1})dW(t_1)\cdots dW(t_k)|\mathcal{F}_G\Big] \\
&\qquad \cdot \chi_G(t_{k+1})dW(t_{k+1}) \\
&= \ldots = (k+1)!\int_0^T \int_0^{t_{k+1}} \cdots \int_0^{t_2} f_{k+1}(t_1,\ldots,t_{k+1})\chi_G(t_1)\cdots\chi_G(t_{k+1})dW(t_1)\cdots dW(t_{k+1}) \\
&= I_{k+1}[f_{k+1}\chi_G^{\otimes(k+1)}],
\end{aligned}
$$

and the proof is complete. □

Proposition 3.12. *If $F \in \mathbb{D}_{1,2}$, then $E[F|\mathcal{F}_G] \in \mathbb{D}_{1,2}$ and*

$$
D_tE[F|\mathcal{F}_G] = E[D_tF|\mathcal{F}_G]\chi_G(t).
$$

Proof First assume that $F = I_n(f_n)$ for some $f_n \in \widetilde{L}^2([0,T]^n)$. By Proposition 3.11 we have

$$\begin{aligned} D_t E[F|\mathcal{F}_G] &= D_t E[I_n(f_n)|\mathcal{F}_G] \\ &= D_t I_n(f_n \chi_G^{\otimes n}) \\ &= n I_{n-1}[f_n(\cdot,t)\chi_G^{\otimes(n-1)}(\cdot)\chi_G(t)] \\ &= n I_{n-1}[f_n(\cdot,t)\chi_G^{\otimes(n-1)}(\cdot)]\chi_G(t) \\ &= E[D_t F|\mathcal{F}_G]\chi_G(t). \end{aligned} \tag{3.16}$$

Next, let $F = \sum_{n=0}^{\infty} I_n(f_n)$ belong to $\mathbb{D}_{1,2}$. Let $F_k = \sum_{n=0}^{k} I_n(f_n)$. Then

$$F_k \to F \quad \text{in} \quad L^2(\Omega) \quad \text{and} \quad D_t F_k \to D_t F \quad \text{in} \quad L^2(P \times \lambda)$$

as $k \to \infty$. By (3.16) we have

$$D_t E[F_k|\mathcal{F}_G] = E[D_t F_k|\mathcal{F}_G]\chi_G(t),$$

for all k, and taking the limit with convergence in $L^2(P\times\lambda)$ of this, as $k \to \infty$, we obtain the result. □

Corollary 3.13. *Let $u = u(s)$, $s \in [0,T]$, be an $\mathbb{F}$-adapted stochastic process and assume that $u(s) \in \mathbb{D}_{1,2}$ for all s. Then*

(i) $D_t u(s)$, $s \in [0,T]$, is $\mathbb{F}$-adapted for all t;
(ii)$D_t u(s) = 0$, for $t > s$.

Proof By Proposition 3.12 we have that

$$D_t u(s) = D_t E[u(s)|\mathcal{F}_s] = E[D_t u(s)|\mathcal{F}_s]\chi_{[0,s]}(t) = E[D_t u(s)|\mathcal{F}_s]\chi_{[t,T]}(s),$$

from which (i) and (ii) follow immediately. □

3.3 Malliavin Derivative and Skorohod Integral

3.3.1 Skorohod Integral as Adjoint Operator to the Malliavin Derivative

The following result shows that the Malliavin derivative is the adjoint operator of the Skorohod integral.

Theorem 3.14. Duality formula. *Let $F \in \mathbb{D}_{1,2}$ be $\mathcal{F}_T$-measurable and let u be a Skorohod integrable stochastic process. Then*

$$E\Big[F \int_0^T u(t)\delta W(t)\Big] = E\Big[\int_0^T u(t) D_t F dt\Big]. \tag{3.17}$$

Proof Let $F = \sum_{n=0}^{\infty} I_n(f_n)$ and, for all t, $u(t) = \sum_{k=0}^{\infty} I_k(g_k(\cdot,t))$ be the chaos expansions of F and $u(t)$, respectively. Then

$$\begin{aligned} E\Big[F\int_0^T u(t)\delta W(t)\Big] &= E\Big[\sum_{n=0}^{\infty} I_n(f_n)\int_0^T \sum_{k=0}^{\infty} I_k(g_k(\cdot,t))\delta W(t)\Big] \\ &= E\Big[\sum_{n=0}^{\infty} I_n(f_n)\sum_{k=0}^{\infty} I_{k+1}(\widetilde{g}_k)\Big] \\ &= E\Big[\sum_{k=0}^{\infty} I_{k+1}(f_{k+1})I_{k+1}(\widetilde{g}_k)\Big] \\ &= \sum_{k=0}^{\infty}(k+1)!\int_{[0,T]^{k+1}} f_{k+1}(x)\widetilde{g}_k(x)dx \\ &= \sum_{k=0}^{\infty}(k+1)!\big(f_{k+1},\widetilde{g}_k\big)_{L^2([0,T]^{k+1})}, \end{aligned} \tag{3.18}$$

where $\widetilde{g}_k$ is the symmetrization of $g_k(x_1,...,x_n,t)$ as a function of $n+1$ variables (see (2.1)). On the other side we have

$$\begin{aligned} E\Big[\int_0^T u(t)D_tF dt\Big] &= E\Big[\int_0^T\Big(\sum_{k=0}^{\infty} I_k(g_k(\cdot,t))\Big)\Big(\sum_{n=1}^{\infty} nI_{n-1}(f_n(\cdot,t))\Big)dt\Big] \\ &= \int_0^T\sum_{k=0}^{\infty} E\big[(k+1)I_k(g_k(\cdot,t))I_k(f_{k+1}(\cdot,t))\big]dt \\ &= \int_0^T\sum_{k=0}^{\infty}(k+1)k!\big(f_{k+1}(\cdot,t),g_k(\cdot,t)\big)_{L^2([0,T]^k)}dt \\ &= \sum_{k=0}^{\infty}(k+1)!\big(f_{k+1},g_k\big)_{L^2([0,T]^{k+1})}. \end{aligned} \tag{3.19}$$

Now

$$\begin{aligned} \big(f_{k+1},\widetilde{g}_k\big)_{L^2([0,T]^{k+1})} &= \int_0^T \big(f_{k+1}(\cdot,t),\widetilde{g}_k(\cdot,t)\big)_{L^2([0,T]^k)}dt \\ &= \frac{1}{k+1}\sum_{j=1}^{k+1}\int_0^T \big(f_{k+1}(\cdot,t_j),g_k(\cdot,t_j)\big)_{L^2([0,T]^k)}dt_j \\ &= \int_0^T \big(f_{k+1}(\cdot,t),g_k(\cdot,t)\big)_{L^2([0,T]^k)}dt \\ &= \big(f_{k+1},g_k\big)_{L^2([0,T]^{k+1})}. \end{aligned} \tag{3.20}$$

Therefore, by (3.20) combined with (3.18) and (3.19) the result follows. □

3.3.2 An Integration by Parts Formula and Closability of the Skorohod Integral

Theorem 3.15. Integration by parts. *Let $u(t)$, $t \in [0,T]$, be a Skorohod integrable stochastic process and $F \in \mathbb{D}_{1,2}$ such that the product $Fu(t)$, $t \in [0,T]$, is Skorohod integrable. Then*

$$F \int_0^T u(t)\delta W(t) = \int_0^T Fu(t)\delta W(t) + \int_0^T u(t) D_t F dt. \tag{3.21}$$

Proof First assume that $F \in \mathbb{D}_{1,2}^0$ (see Theorem 3.4). Choose $G \in \mathbb{D}_{1,2}^0$. By Theorem 3.14 and Theorem 3.4 we get

$$\begin{aligned} E\Big[G \int_0^T Fu(t)\delta W(t)\Big] &= E\Big[\int_0^T Fu(t) D_t G dt\Big] \\ &= E\Big[GF \int_0^T u(t)\delta W(t)\Big] - E\Big[G \int_0^T u(t) D_t F dt\Big]. \end{aligned}$$

Since the set of all $G \in \mathbb{D}_{1,2}^0$ is dense in $L^2(P)$, it follows that

$$F \int_0^T u(t)\delta W(t) = \int_0^T Fu(t)\delta W(t) + \int_0^T u(t) D_t F dt \quad P-a.s.$$

Then the result follows for general $F \in \mathbb{D}_{1,2}$ by approximating F by $F^{(n)} \in \mathbb{D}_{1,2}^0$ such that $F^{(n)} \longrightarrow F$ in $L^2(P)$ and $D_t F^{(n)} \longrightarrow D_t F$ in $L^2(P \times \lambda)$, for $n \to \infty$. □

Remark 3.16. The arguments of the proof of Theorem 3.15 actually show that the assumption of the Skorohod integrability of Fu can be replaced by requiring the existence of the integrals

$$F \int_0^T u(t)\delta W(t) \quad \text{and} \quad \int_0^T u(t) D_t F dt$$

in $L^2(P)$.

We can now use the duality formula to prove the following important result.

Theorem 3.17. Closability of the Skorohod integral. *Suppose that $u_n(t)$, $t \in [0,T]$, $n = 1,2,...$, is a sequence of Skorohod integrable stochastic processes and that the corresponding sequence of Skorohod integrals*

$$\delta(u_n) := \int_0^T u_n(t)\delta W(t), \quad n = 1,2,...$$

converges in $L^2(P)$. Moreover, suppose that

$$\lim_{n\to\infty} u_n = 0 \quad \text{in } L^2(P \times \lambda).$$

Then

$$\lim_{n\to\infty} \delta(u_n) = 0 \quad \text{in } L^2(P).$$

Proof By Theorem 3.14, we have that

$$\big(\delta(u_n), F\big)_{L^2(P)} = \big(u_n, D.F\big)_{L^2(P\times\lambda)} \longrightarrow 0, \quad n \to \infty,$$

for all $F \in \mathbb{D}_{1,2}$. We conclude that $\delta(u_n) \longrightarrow 0$ weakly in $L^2(P)$. Since $\{\delta(u_n)\}_{n=0}^{\infty}$ is convergent in $L^2(P)$, we obtain that $\delta(u_n) \longrightarrow 0$ in $L^2(P)$.

□

3.3.3 A Fundamental Theorem of Calculus

The next result gives a useful connection between differentiation and Skorohod integration.

Theorem 3.18. The fundamental theorem of calculus. *Let $u = u(s)$, $s \in [0,T]$, be a stochastic process such that*

$$E\Big[\int_0^T u^2(s)ds\Big] < \infty \tag{3.22}$$

and assume that, for all $s \in [0,T]$, $u(s) \in \mathbb{D}_{1,2}$ and that, for all $t \in [0,T]$, $D_t u \in Dom(\delta)$. Assume also that

$$E\Big[\int_0^T \big(\delta(D_t u)\big)^2 dt\Big] < \infty. \tag{3.23}$$

Then $\int_0^T u(s)\delta W(s)$ is well-defined and belongs to $\mathbb{D}_{1,2}$ and

$$D_t\Big(\int_0^T u(s)\delta W(s)\Big) = \int_0^T D_t u(s)\delta W(s) + u(t). \tag{3.24}$$

Proof First assume that

$$u(s) = I_n(f_n(\cdot, s)),$$

where $f_n(t_1,\dots,t_n,s)$ is symmetric with respect to $t_1,\dots,t_n$. Then

$$\int_0^T u(s)\delta W(s) = I_{n+1}[\widetilde{f}_n],$$

where

$$\widetilde{f}_n(x_1,\dots,x_{n+1}) = \frac{1}{n+1}\Big[f_n(\cdot,x_1) + \ldots + f_n(\cdot,x_{n+1})\Big]$$

is the symmetrization of f_n as a function of all its $n+1$ variables. Hence

$$D_t\Big(\int_0^T u(s)\delta W(s)\Big) = (n+1)I_n[\widetilde{f}_n(\cdot,t)], \tag{3.25}$$

where

$$\widetilde{f}_n(\cdot,t) = \frac{1}{n+1}\Big[f_n(t,\cdot,x_1) + \ldots + f_n(t,\cdot,x_n) + f_n(\cdot,t)\Big] \tag{3.26}$$

(since f_n is symmetric with respect to its first n variables, we may choose t to be the first of them, in the first n terms on the right-hand side). Combining (3.25) with (3.26) we get

$$\begin{aligned} D_t\Big(\int_0^T u(s)\delta W(s)\Big) &= I_n\Big[f_n(t,\cdot,x_1) + \ldots + f_n(t,\cdot,x_n) + f_n(\cdot,t)\Big] \\ &= I_n\Big[f_n(t,\cdot,x_1) + \ldots + f_n(t,\cdot,x_n)\Big] + u(t) \end{aligned} \tag{3.27}$$

(the integration in I_n is with respect to $(x_1,\ldots,x_n)$). To compare this with the right-hand side of (3.24) we consider

$$\begin{aligned} \delta(D_t u) &= \int_0^T D_t u(s)\delta W(s) \\ &= \int_0^T n I_{n-1}[f_n(\cdot,t,s)]\delta W(s) \\ &= n I_n[\widehat{f}_n(\cdot,t,\cdot)], \end{aligned} \tag{3.28}$$

where

$$\widehat{f}_n(x_1,\ldots,x_{n-1},t,x_n) = \frac{1}{n}\Big[f_n(t,\cdot,x_1) + \ldots + f_n(t,\cdot,x_n)\Big]$$

is the symmetrization of $f_n(x_1,\ldots,x_{n-1},t,x_n)$ with respect to $x_1,\ldots,x_n$. Then, from (3.28) we get

$$\int_0^T D_t u(s)\delta W(s) = I_n\Big[f_n(t,\cdot,x_1) + \ldots + f_n(t,\cdot,x_n)\Big]. \tag{3.29}$$

Comparing (3.27) and (3.29) we obtain (3.24).

Next, consider the general case when

$$u(s) = \sum_{n=0}^{\infty} I_n[f_n(\cdot,s)].$$

Define

$$u_m(s) = \sum_{n=0}^{m} I_n[f_n(\cdot, s)], \qquad m = 1, 2, \ldots.$$

By (3.22) we have $\|u - u_m\|^2_{L^2(P\times\lambda)} \longrightarrow 0, m \to \infty$.
Then by the above argument we have

$$D_t(\delta(u_m)) = \delta(D_t u_m) + u_m(t), \quad \text{for all } m. \tag{3.30}$$

By (3.28) we see that (3.23) is equivalent to saying that

$$\begin{aligned} E\Big[\int_0^T (\delta(D_t u))^2 dt\Big] &= \sum_{n=1}^{\infty} n^2 n! \int_0^T \|\widehat{f}_n(\cdot, t, \cdot)\|^2_{L^2([0,T]^n)} dt \\ &= \sum_{n=1}^{\infty} n^2 n! \|\widehat{f}_n\|^2_{L^2([0,T]^{n+1})} < \infty, \end{aligned} \tag{3.31}$$

since $D_t u \in Dom(\delta)$. Hence, for $m \to \infty$,

$$\|\delta(D_t u) - \delta(D_t u_m)\|^2_{L^2(P\times\lambda)} = \sum_{n=m+1}^{\infty} n^2 n! \|\widehat{f}_n\|^2_{L^2([0,T]^{n+1})} \longrightarrow 0. \tag{3.32}$$

Therefore, by (3.30)

$$D_t(\delta(u_m)) \to \delta(D_t u) + u(t), \quad m \to \infty,$$

in $L^2(P \times \lambda)$. Note that

$$(n+1)\widetilde{f}_n(\cdot, t) = n\widehat{f}_n(\cdot, t, \cdot) + f_n(\cdot, t)$$

and hence

$$(n+1)!\|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} \le \frac{2n^2 n!}{n+1}\|\widehat{f}_n\|^2_{L^2([0,T]^{n+1})} + \frac{2n!}{n+1}\|f_n\|^2_{L^2([0,T]^{n+1})}.$$

Therefore,

$$\begin{aligned} \|\delta(u)\|^2_{\mathbb{D}_{1,2}} &= \sum_{n=0}^{\infty} (n+1)(n+1)!\|\widetilde{f}_n\|^2_{L^2([0,T]^{n+1})} \\ &\le \sum_{n=0}^{\infty} \big[2n^2 n!\|\widehat{f}_n\|^2_{L^2([0,T]^{n+1})} + 2n!\|f_n\|^2_{L^2([0,T]^{n+1})}\big] \\ &\le 2\|\delta(D_t u)\|^2_{L^2(P\times\lambda)} + 2\|u\|^2_{L^2(P\times\lambda)} < \infty, \end{aligned}$$

by (3.31) and (3.22). Then we conclude that $\delta(u)$ is well-defined and belongs to $\mathbb{D}_{1,2}$. By similar computations, we obtain

$$\begin{aligned}
&\|D_t\Big(\int_0^T u(s)\delta W(s)\Big) - D_t\Big(\int_0^T u_m(s)\delta W(s)\Big)\|^2_{L^2(P\times\lambda)} \\
&= \|\sum_{n=m+1}^{\infty}(n+1)I_n(\widetilde{f}_n(\cdot,t))\|^2_{L^2(P\times\lambda)} \\
&= \int_0^T \sum_{n=m+1}^{\infty}(n+1)^2 n!\|\widetilde{f}_n(\cdot,t)\|^2_{L^2([0,T]^n)}dt \\
&\leq 2\sum_{n=m+1}^{\infty}\big[n^2 n!\|\widehat{f}_n\|^2_{L^2([0,T]^{n+1})} + n!\|f_n\|^2_{L^2([0,T]^{n+1})}\big],
\end{aligned} \tag{3.33}$$

which vanishes when $m \to \infty$. Hence given (3.32) and (3.33), we obtain (3.24):

$$D_t(\delta(u)) = \delta(D_t u) + u(t),$$

by letting $m \to \infty$ in (3.30). □

Corollary 3.19. *Let u be as in Theorem 3.18 and assume in addition that $u(s)$, $s \in [0,T]$, is $\mathbb{F}$-adapted. Then*

$$D_t\Big(\int_0^T u(s)dW(s)\Big) = \int_t^T D_t u(s)dW(s) + u(t). \tag{3.34}$$

Proof This is an immediate consequence of Theorem 3.18 and Corollary 3.13. □

3.4 Exercises

Problem 3.1. Let ξ, ζ be orthonormal functions in $L^2([0,T])$. Using the properties of Hermite polynomials compute directly the following:

(a) $I_1(\xi)I_2(\zeta^{\otimes 2})$
(b) $I_3(\xi\hat{\otimes}\zeta^{\otimes 2})$
(c) $D_t I_3(\xi\hat{\otimes}\zeta^{\otimes 2})$ [*Hint.* Use (1.14), (3.5)–(3.9)].

Using the chain rule compute:

(d) $D_t(I_1(\xi)I_2(\zeta^{\otimes 2})$.

Compare the results in (c) and (d).

Problem 3.2. (*) Find the Malliavin derivative $D_t F$ of the following random variables:

(a) $F = W(T)$.

(b) $F = \int_0^T s^2 dW(s)$.

(c) $F = \int_0^T \int_0^{t_2} \cos(t_1 + t_2) dW(t_1) dW(t_2)$.

(d) $F = 3W(s_0)W^2(t_0) + \log(1 + W^2(s_0))$, for given $s_0, t_0 \in [0, T]$.

(e) $F = \int_0^T W(t_0)\delta W(t)$, for a given $t_0 \in [0, T]$. [*Hint*. Use Problem 2.4 (b).]

Problem 3.3. (*)

(a) Find the Malliavin derivative $D_t F$, when

$$F = e^G \quad \text{with} \quad G = \int_0^T g(s) dW(s), \quad g \in L^2([0, T]),$$

by using that $F = \sum_{n=0}^{\infty} I_n[f_n]$, with

$$f_n(t_1, \dots, t_n) = \frac{1}{n!} \exp\left\{\frac{1}{2}\|g\|^2_{L^2([0,T])}\right\} g(t_1) \dots g(t_n)$$

(see Problem 1.1 and Problem 1.3 (d)).

(b) Verify that the result in (a) can be expressed in terms of the chain rule: $D_t e^G = e^G D_t G$.

(c) Find the Malliavin derivative of $F = e^G$ with $G = W(t_0)$, for a given $t_0 \in [0, T]$.

Problem 3.4. Use the integration by parts formula (Theorem 3.15) to compute the Skorohod integrals

$$\int_0^T F \delta W(t),$$

for the random variables F given in Problem 3.2 and in Problem 3.3.

Problem 3.5. Use the integration by parts formula to compute the Skorohod integrals in Problem 2.4.

Problem 3.6. Let $u = u(t)$, $t \in [0, T]$, be a stochastic process such that

$$E\Big[\int_0^T u^2(t) dt\Big] < \infty.$$

Suppose that there exists a constant K (which can depend on u) such that

$$\Big|E\Big[\int_0^T D_t F u(t) dt\Big]\Big| \le K\|F\|_{L^2(P)}, \quad \text{for all } F \in \mathbb{D}_{1,2}.$$

Show that u is Skorohod integrable.