Option Pricing and the Cost of Risk, via capital reserve and convex risk measures

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10th Winter School on Mathematical Finance, Lunteren, 2011

Outline

The market model

Let the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mu)$ be given where $T > 0$ denotes a fixed time horizon. The discounted price process is described as a $\mathbb{R}$-valued semimartingale $S = (S_t)_{t\in[0,T]}$ additional we have a set of trading strategies given by $\Pi(x)$ and a derivative $F \in \mathcal{F}_T$ which we want to price and hedge.

Pricing and hedging (x, π):

- Initial capital x.
- Trading strategy $\pi \in \Pi(x)$, such that the value of our portfolio at time T is

$$X_T^{\pi,x} := x + \int_0^T \pi_t dS_t.$$

Different pricing methods in incomplete markets

Some methods are:

- Superhedging: $\mathbb{P}(X_T^{\pi,x} \geq F) = 1$
- Mean-variance optimal: $\mathbb{E}_\mathbb{P}|X_T^{\pi,x} - F|^2$
- Utility indifference pricing:
  $u(x,F) := \sup_{\pi\in\Pi(x)} \mathbb{E}_\mathbb{P}[U(X_T^{\pi,x} + F)]$

  Buyers indifferent price: p: $u(x,0) = u(x-p,F)$
  Sellers indifferent price: s: $u(x,0) = u(x+s,-F)$
- Minimization of risk:

  Buyer: $\inf_{\pi\in\Pi(x)} \rho(F - X_T^{\pi,x})$
  Seller: $\inf_{\pi\in\Pi(x)} \rho(X_T^{\pi,x} - F)$
- ...

Trader and regulator

Model pricing and hedging of a derivative as a trade-off between trader and regulator.

- The regulator requires the traders to cover the *residual risk* via a fraction ($\varepsilon < 1$) of a risk measure. This serves as a *capital reserve* and contains the minimal amount of money needed, depending on the risk of the trader's portfolio.
- The trader tries to maximize her utility but has to put money aside to cover the riskiness of her position.

From the point of view of the trader: max (utility − capital reserve)

$$\sup_{\pi \in \Pi(x)} \left\{ \mathbb{E}[U(X_T^{\pi,x} - F)] - \varepsilon \cdot \rho(X_T^{\pi,x} - F) \right\}.$$

Duality

- E topological vector space and E' its dual space.
- The conjugate F^* and biconjugate F^{**} of a convex function $F : E \to \mathbb{R} \cup \{+\infty\}$ is given by

$$F^* : E' \to \mathbb{R} \cup \{+\infty\},\; F^*(Z) := \sup_{X \in E} \{\langle X, Z \rangle - F(X)\},$$
$$F^{**} : E \to \mathbb{R} \cup \{+\infty\},\; F^{**}(X) := \sup_{Z \in E'} \{\langle X, Z \rangle - F^*(Z)\}.$$

- If F is convex, lower semicontinuous and proper, then

Fenchel-Moreau Theorem

$$F(X) = \sup_{Z \in E'} \{\langle X, Z \rangle - F^*(Z)\}.$$

Dual operations

For functions $F_1, F_2 : E \to \mathbb{R} \cup \{+\infty\}$ we define the inf-convolution $F_1 \Box F_2 : E \to \mathbb{R} \cup \{+\infty\}$ by

Inf-convolution

$$F_1 \Box F_2(X) := \inf_{\substack{X_1 + X_2 = X \\ X_1, X_2 \in E}} \{F_1(X_1) + F_2(X_2)\}.$$

- $\big(\lambda F(X)\big)^* = \lambda F^*(\lambda^{-1}X)$ and $\big(\lambda F(\lambda^{-1}X)\big)^* = \lambda F^*(X)$.
- $\big(F_1 \Box F_2(X)\big)^* = F_1^*(X) + F_2^*(X)$.
- $\big(F_1(X) + F_2(X)\big)^* = F_1^* \Box F_2^*(X)$. This duality just holds for proper, convex and lower semicontinuous functions F_1, F_2.

Convex risk measures on L^p-spaces

Föllmer, Schied (2002) for L^∞, *Biagini, Frittelli (2009)* for L^p

Definition

A L^p-convex risk measure $p \in [0, \infty]$ is a mapping $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ satisfying the following properties:

- Monotonicity: If $X \leq Y$, then $\rho(X) \geq \rho(Y)$.
- Translation invariance: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.
- Convexity: $\rho(\lambda X + (1 - \lambda)Y) \leq \lambda\rho(X) + (1 - \lambda)\rho(Y)$, for $0 \leq \lambda \leq 1$.
- Lower semicontinuity w.r.t $\|\cdot\|_p$.
- Normality: $\rho(0) = 0$.

Convex risk measures on L^p-spaces

Dual representation

Suppose $\rho : L^p \to \mathbb{R} \cup \{+\infty\}$ is a convex risk measure. Then ρ admits the following dual representation

$$\rho(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_\rho(\mathbb{P}) \right\}.$$

Inf-convolution of risk measures

Barrieu, El Karoui (2005) for L^∞, *Toussaint, Sircar* (2009) for L^2, *Arai* (2010) for L^Φ.

Definition

Let ρ_1, ρ_2 be L^p-convex risk measure. We define the inf-convolution of ρ_1 and ρ_2 as

$$\rho_1 \Box \rho_2(X) := \inf_{\substack{X_1, X_2 \in L^p \\ X_1 + X_2 = X}} \{\rho_1(X_1) + \rho_2(X_2)\}.$$

Inf-convolution of risk measures

Dual representation

Suppose that ρ_1 and ρ_2 are L^p-convex risk measure.
Then the inf-convolution $\rho_1 \Box \rho_2$ is a (proper) convex risk measure and admits the dual representation

$$\rho_1 \Box \rho_2(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_{\rho_1 \Box \rho_2}(\mathbb{P}) \right\}$$

with penalty function

$$\alpha_{\rho_1 \Box \rho_2}(\mathbb{P}) = \alpha_{\rho_1}(\mathbb{P}) + \alpha_{\rho_2}(\mathbb{P}).$$

Dilated risk measures

Barrieu, El Karoui (2005)

Definition

Let ρ be a convex risk measure with penalty function α_ρ. The associated dilated risk measure ρ_β is defined by

$$\rho_\beta(X) := \frac{1}{\beta}\rho(\beta X) \quad \text{with} \quad \alpha_{\rho_\beta}(\mathbb{P}) = \frac{1}{\beta}\alpha_\rho(\mathbb{P}),$$

where $\beta > 0$ is the risk aversion coefficient.

Example: entropic risk measure.

Trader and regulator

Assume that U is a monetary concave utility. Then we can reformulate our problem using the one-to-one correspondence between risk measures and monetary concave utilities and basic duality.

$$\sup_{\pi \in \Pi(x)} \left\{ \mathbb{E}[U(X_T^{\pi,x} - F)] - \varepsilon \cdot \rho(X_T^{\pi,x} - F) \right\}$$
$$= - \inf_{\pi \in \Pi(x)} \left\{ \rho_1(X_T^{\pi,x} - F) + \varepsilon \cdot \rho_2(X_T^{\pi,x} - F) \right\}$$

Sum of risk measures

This leads to the problem

$$(\lambda_1 + \lambda_2) \cdot \phi(X) := \lambda_1 \rho_1(X) + \lambda_2 \rho_2(X),$$

for $\lambda_1, \lambda_2 > 0$ and ρ_1, ρ_2 are risk measures.

- Is ϕ a risk measure?
- If yes, how can we characterize α_ϕ?

Sum of risk measures

Dual representation

Let ρ_1 and ρ_2 be two convex risk measures from $L^p \to \mathbb{R} \cup \{\infty\}$. And

$$(\lambda_1 + \lambda_2) \cdot \phi(X) := \lambda_1 \rho_1(X) + \lambda_2 \rho_2(X).$$

Then ϕ is a convex risk measure and

$$\phi(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_\phi(\mathbb{P}) \right\}$$

with the penalty function

$$\alpha_\phi(\mathbb{P}) := \inf_{\substack{\mathbb{P}_1, \mathbb{P}_2 \in \mathcal{P} \\ \frac{\lambda_1}{\lambda_1+\lambda_2}\mathbb{P}_1 + \frac{\lambda_2}{\lambda_1+\lambda_2}\mathbb{P}_2 = \mathbb{P}}} \left\{ \frac{\lambda_1}{\lambda_1 + \lambda_2} \alpha_{\rho_1}(\mathbb{P}_1) + \frac{\lambda_2}{\lambda_1 + \lambda_2} \alpha_{\rho_2}(\mathbb{P}_2) \right\}.$$

Proof

The sum of risk measures multiplied with positive scalars is

- monotone,
- convex,
- lower semi continuous,
- normal.

Translation invariance follows from scaling.

Dual representation

$$\phi(X) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[-X] - \alpha_{\phi}(\mathbb{P}) \right\}.$$

Proof

Use dual operations to derive the penalty function.

$$
\begin{aligned}
\phi^*(X) &= \left(\lambda_1 \rho_1(X) + \lambda_2 \rho_2(X)\right)^* \\
&= \inf_{X_1 + X_2 = X} \left\{ \left(\lambda_1 \rho_1(X_1)\right)^* + \left(\lambda_2 \rho_2(X_2)\right)^* \right\} \\
&= \inf_{X_1 + X_2 = X} \left\{ \lambda_1 \rho_1^*(\lambda_1^{-1} X_1) + \lambda_2 \rho_2^*(\lambda_2^{-1} X_2) \right\} \\
&= \inf_{\lambda_1 X_1 + \lambda_2 X_2 = X} \left\{ \lambda_1 \rho_1^*(X_1) + \lambda_2 \rho_2^*(X_2) \right\}.
\end{aligned}
$$

Sum of entropic risk measures

Problem 1

$$\rho_{\beta_1}(X - F) + \varepsilon \cdot \rho_{\beta_2}(X - F) = (1 + \varepsilon) \cdot \phi(X - F)$$

- $u(x) = -e^{-\beta_1 x}$,
- $\mathbb{E}[U(X)] = u^{-1}(\mathbb{E}[u(X)]) = -\frac{1}{\beta_1} \log \mathbb{E}[e^{-\beta_1 X}] = -\rho_{\beta_1}$,
- $\rho_{\beta_1}, \rho_{\beta_2}$ entropic risk measure with $\lambda_1 = 1, \lambda_2 = \varepsilon$.

$$\phi(X - F) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[F - X] - \alpha_\phi(\mathbb{P}) \right\}$$

with optimum $\frac{d\mathbb{P}^1}{d\mu} = \left(\frac{d\mathbb{P}_2}{d\mu}\right)^{\beta_1/\beta_2} \Big/ \mathbb{E}\left[\left(\frac{d\mathbb{P}_2}{d\mu}\right)^{\beta_1/\beta_2}\right]$ for the penalty function. Use indifference pricing.

Optimal design for entropic risk measures

Problem 2

$$\rho_{\beta_1}(X - F) + \varepsilon \cdot \phi(X - F) = (1 + \varepsilon) \cdot \rho_{\beta_2}(X - F)$$

- $\rho_{\beta_1}, \rho_{\beta_2}$ entropic risk measure with $\beta_2 > \beta_1$.

$$\phi(X - F) = \sup_{\mathbb{P} \in \mathcal{P}} \left\{ \mathbb{E}_{\mathbb{P}}[F - X] - \alpha_\phi(\mathbb{P}) \right\}$$

with penalty function $\alpha_\phi(\mathbb{P}) = ?$

Concluding remarks

Next steps:

- Optimal design.
- Numerical results.
- Dynamic formulation.
- Closure property. Find a parametric family of risk measures $\rho_\beta \in S$ for all $\beta > 0$ such that

$$\frac{\lambda_1}{\lambda_1 + \lambda_2}\rho_{\beta_1} + \frac{\lambda_2}{\lambda_1 + \lambda_2}\rho_{\beta_2} \in S.$$