10th Winter School– Energy Markets

Lecture 5
Emission Trading Schemes

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1 Permit Price Analysis
- Review Cap-and-Trade
- Calculating Permit Prices
- Permit prices for different approximation approaches
- Theoretical discussion of permit price slump in 2006
- Extensions: Hybrid systems

2 Dynamics of CO2 permit prices
- An Equilibrium Model
- Central Planner and Equilibrium

3 Reduced Form Models
- Motivation
- Dynamics of the permit process in the reduced form model
- Option Pricing

Agenda

1. Permit Price Analysis
   - Review Cap-and-Trade
   - Calculating Permit Prices
   - Permit prices for different approximation approaches
   - Theoretical discussion of permit price slump in 2006
   - Extensions: Hybrid systems
2. Dynamics of CO2 permit prices
3. Reduced Form Models

Basic idea of cap-and-trade systems

- **At the beginning** of the compliance period, the regulator **allocates** permits to the companies
- **During** the compliance period, the companies can **trade** permits among each other
- **At the end** of the compliance period, a regulated company has to **hand in** one permit or **pay a penalty fee** per unit of emission

Permit price in the EU ETS during the first phase

Figure: EUA-Dec07 futures price (22 April 2005 - 17 December 2007).

Permit price in the EU ETS during the first phase

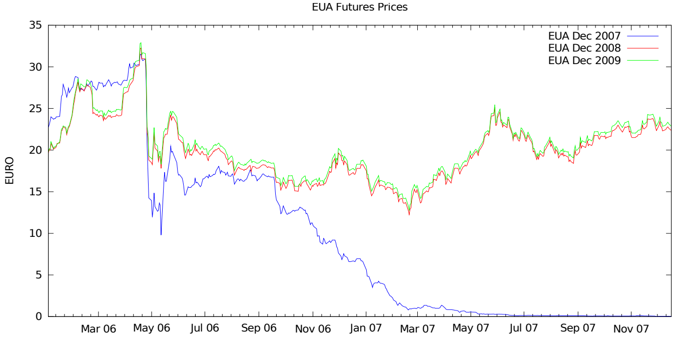

Figure: EUA-Dec07/08/09 futures price (22 April 2005 - 17 December 2007).

Cumulative Emissions

We specify the process for the cumulative emissions in the framework of Carmona et al. by

$$q_{[0,t]} = \int_0^t Q_s ds$$

where the emission rate Q_t follows a Geometric Brownian motion. There is no closed-form density for $q_{[0,t]}$ available.

Approximation Approaches

- Linear approximation approach of Chesney and Taschini (2008)

$$q_{[t_1,t_2]} \approx \tilde{q}^{Lin}_{[t_1,t_2]} = Q_{t_2}(t_2 - t_1)$$

- Moment matching of Grüll and Kiesel (2009): Log-normal (moment matching)

$$q_{[t_1,t_2]} \approx \tilde{q}^{Log}_{[t_1,t_2]} = logN\left(\mu_L(t_1,t_2), \sigma_L^2(t_1,t_2)\right)$$

where the parameters $\mu_L(t_1,t_2)$, $\sigma_L(t_1,t_2)$ are chosen such that the first two moments of $\tilde{q}^{Log}_{[t_1,t_2]}$ and $\tilde{q}^{IG}_{[t_1,t_2]}$, respectively, match those of $q_{[t_1,t_2]}$.

Moment matching requires two steps

- Compute the first two moments m_k of a log-normal random variable and solve for the parameters.

  In the log-normal case we have that $m_k = e^{k\mu + k^2 \frac{\sigma^2}{2}}$ and

$$\sigma^2 = \ln\left(\frac{m_2}{m_1^2}\right) \qquad \mu = \ln(m_1) - \frac{1}{2}\sigma^2$$

- Compute the first two moments of the integral over a geometric Brownian motion

$$\begin{aligned} \mathbb{E}\left[q_{[t_1,t_2]}\right] &= Q_{t_1}\alpha_{t_2-t_1} \\ \mathbb{E}\left[\left(q_{[t_1,t_2]}\right)^2\right] &= 2Q_{t_1}^2\beta_{t_2-t_1} \end{aligned}$$

and plug those into the above equation.

Auxiliary functions for moments of integral over GBM

$$\alpha_{t_2-t_1} = \begin{cases} \frac{1}{\mu}\left(e^{\mu(t_2-t_1)} - 1\right) & \text{if } \mu \neq 0 \\ t_2 - t_1 & \text{if } \mu = 0 \end{cases} \tag{1}$$

$$\beta_{t_2-t_1} = \begin{cases} \frac{\mu e^{(2\mu+\sigma^2)(t_2-t_1)} + \mu + \sigma^2 - \left(2\mu+\sigma^2\right) e^{\mu(t_2-t_1)}}{\mu(\mu+\sigma^2)(2\mu+\sigma^2)} & \text{if } \mu \neq 0 \\ \frac{1}{\sigma^4}\left(e^{\sigma^2(t_2-t_1)} - 1\right) & \text{if } \mu = 0 \end{cases} \tag{2}$$

Permit price - linear approximation

The permit price at time t is given by

$$S_t^{Lin} = \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot \Phi\left(\frac{-\ln\left(\frac{1}{\tau}\left[\frac{N-q_{[0,t]}}{Q_t}\right]\right) + \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}} \right) & \text{if } q_{[0,t]} < N \end{cases}$$

where

$\tau = T - t$ is the time to compliance.

$\Phi(\cdot)$ denotes the c.d.f. of a standard normal random variable.

Permit price - log-normal moment matching

The permit price at time t is given by

$$S_t^{Log} = \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot \Phi\left(\frac{-\ln\left(\frac{N-q_{[0,t]}}{Q_t}\right) + 2\ln(\alpha_\tau) - \frac{1}{2}\ln(2\beta_\tau)}{\sqrt{\ln(2\beta_\tau) - 2\ln(\alpha_\tau)}} \right) & \text{if } q_{[0,t]} < N \end{cases}$$

where
$\tau = T - t$ is the time to compliance and
α_τ, β_τ are obtained by calculating the first and the second moment of the integral over a geometric Brownian motion.
$\Phi(\cdot)$ denotes the c.d.f. of a standard normal random variable.

Permit price - reciprocal gamma moment matching

The permit price at time t is given by

$$S_t^{IG} = \begin{cases} Pe^{-r\tau} & \text{if } q_{[0,t]} \geq N \\ Pe^{-r\tau} \cdot G\left(\frac{Q_t}{N-q_{[0,t]}} \middle| \frac{4\beta_\tau - \alpha_\tau^2}{2\beta_\tau - \alpha_\tau^2}, \frac{2\beta_\tau - \alpha_\tau^2}{2\alpha_\tau \beta_\tau}\right) & \text{if } q_{[0,t]} < N \end{cases}$$

where

$\tau = T - t$ is the time to compliance and

α_τ, β_τ are obtained by calculating the first and the second moment of the integral over a geometric Brownian motion.

$G(x|a, b)$ denotes the c.d.f. of a gamma random variable with shape parameter a and scale parameter b.

Relating theoretical permit prices to allocation

We introduce the following two random variables that are very easy to interpret

Time needed to exhaust the remaining permits

$$x_t := \frac{N - q_{[0,t]}}{Q_t}$$

and

Over-/Underallocation in years

$$x_t - (T - t)$$

Numerical illustrations

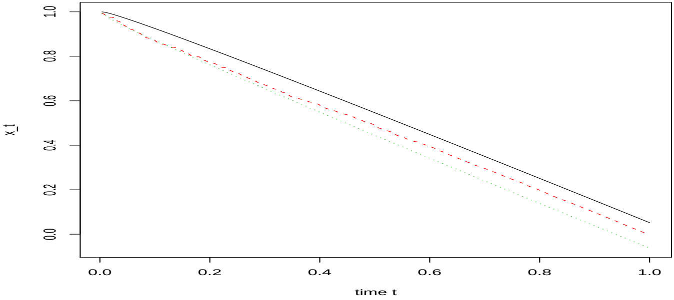

Figure: Trajectory of x_t for $t \in [0,1]$, $N = Q_0 = 100$, $\mu = 0.02$ and $\sigma = 0.05$.

Numerical illustrations

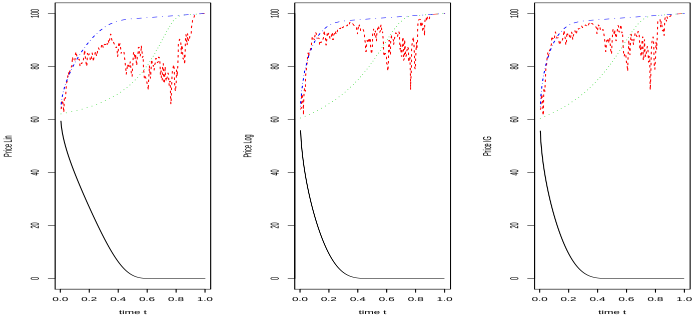

Figure: Trajectory of $S_t^{Lin}(x_t)$ (left), $S_t^{Log}(x_t)$ (middle) and $S_t^{IG}(x_t)$ (right) for $t \in [0, 1]$, $N = Q_0 = 100$, $\mu = 0.02$ and $\sigma = 0.05$.

Implied over-/underallocation during the first phase of the EU ETS

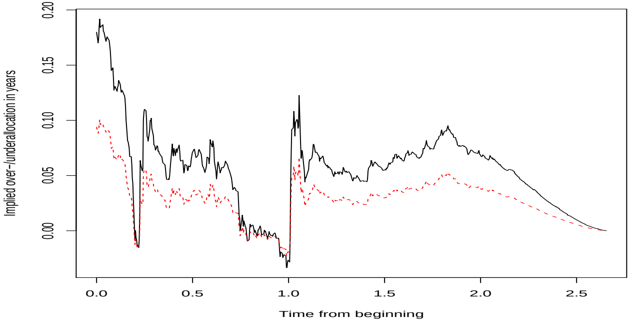

Figure: Implied $x_t - (T - t)$ for first phase for fixed $\mu = 0.02$ and $\sigma = 0.05$. Linear approximation approach (straight line), log-normal moment matching (dashed line). Positive values correspond to overallocation.

Permit price Delta

For $t \in [0, T)$ and $q_{[0,t]} < N$

- $$\frac{dS_t^{Lin}}{dx_t}(x_t) := -\frac{Pe^{-r\tau}}{\sigma\sqrt{\tau}} \cdot \frac{1}{x_t}\phi\left(\frac{-\ln\left(\frac{1}{\tau}x_t\right) + \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right) < 0$$

- $$\frac{S_t^{Lin}((1+h)x_t) - S_t^{Lin}(x_t)}{S_t^{Lin}(x_t)} = -\frac{\phi\left(\frac{-\ln\left(\frac{1}{\tau}x_t\right)+\left(\mu-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)}{\Phi\left(\frac{-\ln\left(\frac{1}{\tau}x_t\right)+\left(\mu-\frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right)} \cdot \frac{h}{\sigma\sqrt{\tau}}$$

Price slumps and allocation

We show that a price slump of more than 50% can be related to an implicit change in x_t of less than 5%.

We introduce the following notation

- $t - \Delta$ is the date before the publication of verified emissions that affected the permit price (28 April 2006)
- t is the date of the announcement of cumulative emissions (15 May 2006)

Price slumps and allocation

Using

- the cumulative emissions until t denoted by $q_{[0,t]}$
- the futures permit price at and before publication of emission data denoted by $F(t,T)$ and $F(t-\Delta,T)$, respectively

the implicit time needed to exhaust the remaining permits before the announcement was $h(\sigma)$ per cent larger than the previous estimate $\bar{x}_t$ where

$$h(\sigma) = \frac{F(t,T) - F(t-\Delta,T)}{P\phi\left(\Phi^{-1}\left(\frac{F(t,T)}{P}\right)\right)} \cdot f^{approx}(\sigma, t, \bar{x}_t)$$

Price slumps and allocation

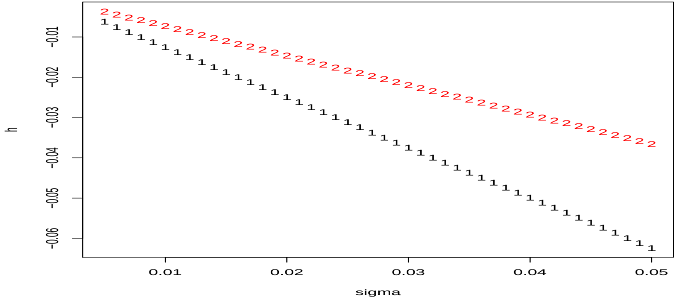

Figure: Linear approximation ("1"), log-normal moment matching ("2").

Price Floor Using a Subsidy

- The severe permit price drop, followed by a price hovering above zero for more than five months during the first phase of the EU ETS, persuaded several policy makers that new cap-and-trade schemes would need additional safety-valve features.
- In particular, policy makers have been concerned about permit prices that are either too low or too high.
- Thus setting a price floor and/or ceiling has been proposed.

Price Floor Using a Subsidy – Regulation

- A company with a permit shortage at compliance date faces a penalty P.
- If a company ends up with an excess of permits, it receives a subsidy S per unit of permit.
- Let $0 < S \leq P$ and let N be the initial amount of permits allocated to relevant companies.

Permit Price in hybrid system

Denote the futures permit price by $\tilde{F}(t, T)$:

$$\begin{aligned}
\tilde{F}(t, T) &= P \cdot \mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right) + S \cdot \mathbb{P}\left(q_{[0,T]} \leq N \mid \mathcal{F}_t\right) \\
&= P \cdot \mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right) + S \cdot \left(1 - \mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right)\right) \\
&= S + \frac{P - S}{P} \cdot P \cdot \mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right) \\
&= S + \frac{P - S}{P} \cdot F(t, T),
\end{aligned}$$

where $F(t, T) = P \cdot \mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right)$ is the futures permit price in an ordinary system.

Decomposition of permit price in hybrid system

Computing the value of a put with strike S shows that the price in the hybrid scheme is the price in the ordinary scheme plus the value of a put option on the price in the ordinary scheme with strike S and maturity T:

$$
\begin{aligned}
& \mathbb{E}\left[(S - F(T,T))^+ \mid \mathcal{F}_t\right] \\
= \quad & \mathbb{E}\left[\left(S - P\mathbf{1}_{\{q_{[0,T]}>N\}}\right)^+ \mid \mathcal{F}_t\right] \\
= \quad & (S-P)^+\,\mathbb{P}\left(q_{[0,T]} > N \mid \mathcal{F}_t\right) + (S-0)^+\,\mathbb{P}\left(q_{[0,T]} \leq N \mid \mathcal{F}_t\right) \\
\overset{S \leq P}{=} \quad & S \cdot \mathbb{P}\left(q_{[0,T]} \leq N \mid \mathcal{F}_t\right).
\end{aligned}
$$

Expected enforcement costs for regulated companies

Let f_q be the probability density function of the cumulative emissions $q_{[0,T]}$ in the entire regulated period. The expected enforcement costs for relevant companies in an ordinary system are

$$EEC = P\int_N^\infty (x-N)f_q(x)dx \geq 0.$$

Similarly, the expected enforcement costs for regulated companies in this hybrid system are

$$EEC^{PF} = P\int_N^\infty (x-N)f_q(x)dx - S\int_0^N (N-x)f_q(x)dx.$$

So, the total expected enforcement costs for regulated companies under this hybrid system are lower than under an ordinary system.

$$EEC - EEC^{PF} = S\int_0^N (N-x)f_q(x)dx \geq 0.$$

Enforcement costs for regulator

- A price floor ensured by the presence of a subsidy is relatively easy to implement and has the further advantage of lowering the expected enforcement costs for regulated companies.
- The presence of the subsidy could induce a higher stimulus in technology and abatement investments, favoring the achievement of emission reduction targets.
- However, the implementation of such a hybrid system might result in a significant financial burden for the environmental policy regulator. The current magnitude of this burden can be obtained by calculating the price of the put option.

Hybrid systems

Scheme	Price bound	Prices can exceed bounds	Link with offsets market	Description of the mechanism
Existing cap-and-trade scheme				
Offset safety-valve	Upper	Yes	Yes	Flexible limit on the use of offsets
Proposed safety-valve mechanisms for cap-and-trade schemes				
Subsidy price floor	Lower	No	No	Subsidy
Price collar	Upper & Lower	No	No	Regulator sells unlimited amount of permits at the price ceiling and buys unlimited amount of permits at the price floor
Allowance reserve	Upper & Lower	Yes	No	Regulator sells limited amount of permits at the price ceiling and buys limited amount permits at price floor
Regulator offers options	Upper & Lower	No (for owner of options	No	Regulator sells options at a market price

Comparison of schemes

Mechanism	Advantages	Disadvantages
Offset safety valve	(a) Relatively simple to implement (b) Lower expected enforcement costs for regulated companies than in an ordinary cap-and-trade system (c) Regulator faces no financial burden	(a) Price ceiling is not guaranteed under all circumstances (b) Creates uncertainty on the projects for active emission reduction (c) Weakens the pressure for actions within the system, i.e. environmental targets are not ensured
Subsidy	(a) Relatively simple to implement (b) Reduces investment uncertainty under all circumstance (c) Stimulates reduction efforts in the system	(a) Regulator might face a significant financial burden whose size is hardly quantifiable a priori
Price collar	(a) Price collar is guaranteed under all circumstances (b) Lower expected enforcement costs for regulated companies than in an ordinary cap-and-trade system	(a) Permit prices do not reflect real expectations on the level of cumulative emissions after market intervention. The permit price volatility is not necessarily reduced (b) Regulator might face a significant financial burden when the price floor is reached (c) Regulator cannot plan the size of the financial burden and when the cash outflows will occur (d) Environmental targets are loosened when the price ceiling is reached.
Allowance reserve	(a) Compared to price collar, environmental target is only weakened up to a certain level	(a) Price bounds cannot be guaranteed under all circumstances (b) Drawbacks of price collar (see above)
Regulator offers options	(a) Regulator faces no financial burden (b) Price bounds are guaranteed for those companies willing to pay for these options (c) Environmental targets are not affected	(a) Policy regulator bears the price risk of the options written

Table 2: Advantages and disadvantages of the different schemes under investigation.

Agenda

1 Permit Price Analysis

2 Dynamics of CO2 permit prices
- An Equilibrium Model
- Central Planner and Equilibrium

3 Reduced Form Models

Basic Model

- Risk-neutral companies with total initial endowment e_0
- Total emission dynamics are

$$dy_t = \mu(t, y_t)dt + \sigma(t, y_t)dW_t \tag{3}$$

  with deterministic drift and volatility.
- Central planner who minimizes total expected cost over a trading period $[0, T]$ by deciding at any time instant whether to costly abate some of the CO2 emissions or not.
- At the end of the period actual accumulated emissions and penalty costs are determined.

Basic Model II

- x_t are the total expected emissions over the trading period
- Then

$$x_t = -\int_0^t u_s ds + \mathbb{E}_t \left[\int_0^T y_s ds \right] \tag{4}$$

- u_t is the optimal rate of abatement which is actively chosen by the central planner.
- So x_t is a controlled stochastic process.

Total Emissions

- x_T are the realized emissions that relate to a potential penalty function
- Without abatement total expected emission are

$$x_0 = \mathbb{E}\left[\int_0^T y_s ds\right]$$

- The dynamics of the total expected emissions are

$$dx_t = -u_t dt + G(t)dW_t \tag{5}$$

- $G(t)$ is the volatility of the uncontrolled part of x_t and depends both on the drift $\mu(t, y_t)$ and the volatility $\sigma(t, y_t)$ of the emission rate.

Optimisation problem of the central planner I

$$\max_{u_t} \mathbb{E}_0 \left[\int_0^T e^{-rt} C(t, u_t) dt + e^{-rT} P(x_T) \right] \tag{6}$$

with

$$\begin{aligned} C(t, u_t) &= -\tfrac{1}{2} c u_t^2 \\ P(x_T) &= \min[0, p(e_0 - x_T)] \end{aligned}$$

Optimisation problem of the central planner II

- $C(t, u_t)$ are the abatement costs per unit of time. c constant implies no change in technology occurs. The quadratic form implies linearly increasing marginal abatement costs.
- $P(x_T)$ is the penalty function, with p the penalty including all costs.
- r is the constant interest rate.

Solution of the control problem

Let $V(t, x_t)$ be the expected value of the optimal policy given x_t. By a standard Hamilton-Jacobi-Bellman argument we arrive at

$$V_t = -\frac{1}{2}(G(t))^2 V_{xx} - \frac{1}{2c}e^{rt}(V_x)^2 \qquad (7)$$

with boundary condition

$$V(T, x_T) = e^{-rT}P(x_T)$$

and optimal control

$$u_t = -\frac{1}{c}e^{rt}V_x$$

Permit Prices

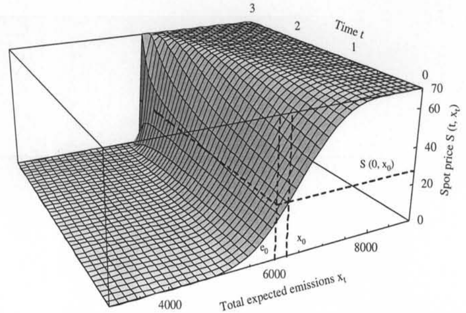

Fig. 1. Equilibrium spot price $S(t, x_t)$—risk neutral special case. The figure shows the equilibrium spot price of emission certificates dependent on time t and total expected emissions x_t within a trading period, where the emission rate y_t follows a white noise process and interest $r = 0$. Initial endowment $e_0 = 6000$, initial total expected emissions $x_0 = 6240$, and expected spot price level $S(0, x_0) = 27.46$ are indicated by dashed lines. Upper price bound is $p = 70$.

Permit Price Dynamics

- Recall that the permit price must equal the marginal abatement costs, so

$$S(t, x_t) = cu_t = -e^{rt} V_x(t, x_t) \qquad (8)$$

- Using Itô's formula and the HJB-PDE we find that the discounted permit price is a martingale.
- Its dynamics are

$$dS(t, x_t) = G(t) S_x(t, x_t) dW_t \qquad (9)$$

Implied Permit Price Volatility

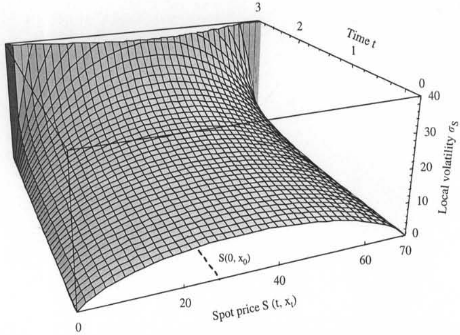

Fig. 2. Local volatility σ_S—risk neutral special case. This plot presents the local volatility σ_S for the resulting spot price process $S(t, x_t)$ dependent on time and spot price level, where the emission rate y_t follows a white noise process. The expected spot price level $S(0, x_0) = 27.46$ for $x_0 = 6240$ is indicated by the dashed line. The upper price bound is $p = 70$. The plot is cut at $\sigma_S = 40$ because σ_S reaches infinity at $t = T$.

Individual Company Models

- Each individual company has an endowment e_{i0}
- Individual emission dynamics are

$$dy_{it} = \mu(t, y_{it})dt + \sigma(t, y_{it})dW_{it} \tag{10}$$

with deterministic drift and volatility.

Individual Emissions

- x_{it} are the total expected emissions of company i over the trading period
- Then

$$x_{it} = -\int_0^t u_{is} ds - \int_0^t z_{is} ds + \mathbb{E}_t \left[\int_0^T y_{is} ds \right] \quad (11)$$

- u_{it} is the individual rate of abatement
- and z_{it} is the instantaneous amount of permits bought or sold.

Individual Emissions Dynamics

- The dynamics of the total expected emissions are

$$dx_{it} = -[u_{it} - z_{it}]dt + G_i(t)dW_{it} \tag{12}$$

- $G_i(t)$ is the volatility of the uncontrolled part of x_{it} and depends both on the drift $\mu_i(t, y_{it})$ and the volatility $\sigma_i(t, y_{it})$ of the emission rate.

Optimisation Problem for the individual Company

$$\max_{u_{it},z_{it}} \mathbb{E}\left[\int_0^T e^{-rt} C_i(t,u_{it})dt - \int_0^T e^{-rt} S(t) z_{it} dt + e^{-rT} P_i(x_{iT})\right] \tag{13}$$

with $S(t)$ the permit price and

$$\begin{aligned} C_i(t,u_{it}) &= -\tfrac{1}{2} c_i u_{it}^2 \\ P_i(x_{iT}) &= \min[0, p(e_{i0}) - x_{iT})] \end{aligned}$$

Solution of the control problem

Let $V^i(t, x_{it})$ be the expected value of the optimal policy for company i. By a standard Hamilton-Jacobi-Bellman argument we arrive at

$$\begin{aligned} 0 &= \max_{u_{it}, z_{it}} \quad \Big[e^{-rt} (C_i(t, u_{it}) - S(t) z_{it}) \\ &\qquad + V_t^i - V_x^i (u_{it} + z_{it}) + \frac{1}{2} (G_i(t))^2 V_{xx}^i \Big] \end{aligned}$$

with boundary condition

$$V^i(T, x_{iT}) = e^{-rT} P_i(x_{iT}).$$

Equilibrium Solution

- We solve the HJB for N companies and use the market clearing condition

$$\sum_{i=1}^{N} z_{it}^{*} = 0$$

- The first-order conditions give

$$\begin{aligned} u_{it}^{*} &= \; = -\frac{1}{c_i} e^{rt} V_x^i \quad i = 1, \ldots N \\ S(t) &= -e^{rt} V_x^i \qquad i = 1, \ldots N \end{aligned}$$

- So again

$$S(t) = c_i u_{it}^{*}, \; i = 1, \ldots N.$$

Joint Cost Problem I

Again we image a central planner who has to solve

$$\max_{u_{it}} \mathbb{E}\left[\int_0^T e^{-rt}\sum_{i=1}^N C_i(t,u_{it})dt + e^{-rT}\sum_{i=1}^N P_i(x_{iT})\right] \qquad (14)$$

with C_i and P_i as before.
We assume only one source of randomness, i.e. $W_{it} = W_t$, then we have the joint value function as

$$V(t, x_{1t}, \dots, x_{Nt}) = \sum_{i=1}^N V_i(t, x_{it}).$$

Again we can show that the individual firm solution and the joint planer solution are equivalent.

Agenda

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3 Reduced Form Models
- Motivation
- Dynamics of the permit process in the reduced form model
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Permit Prices

- Recall the emission rate

$$dQ_t = Q_t(\mu dt + \sigma dW_t)$$

- The cumulative emission are

$$q_{[0,t]} = \int_0^t Q_s ds$$

- The futures permit price is given as

$$F(t,T) = P\mathbb{P}\left[q_{[0,T]} > N|\mathcal{F}_t\right]$$

Approximative Pricing

- Linear approximation approach

$$q_{[t_1,t_2]} \approx \tilde{q}^{Lin}_{[t_1,t_2]} = Q_{t_2}(t_2 - t_1)$$
$$= Q_{t_1} e^{\left\{\log(t_2-t_1)+\left(\mu-\frac{\sigma^2}{2}\right)(t_2-t_1)+\int_{t_1}^{t_2}\sigma dW_t\right\}}$$

- Moment matching

$$q_{[t_1,t_2]} \approx \tilde{q}^{Log}_{[t_1,t_2]}$$
$$= Q_{t_1} \exp\left\{\int_{t_1}^{t_2}\mu_t dt + \int_{t_1}^{t_2}\sigma_t dW_t\right\}$$

where the functions μ_t and σ_t are defined by the functions α_t, β_t from the moment matching.

Carmona-Hinz Approach

- Use a lognormal process

$$\Gamma_T = \Gamma_0 \exp\left\{ \int_0^T \sigma_t dW_t - \frac{1}{2} \int_0^T \sigma_t^2 dt \right\}$$

with $\Gamma_0 > 0$ and $\sigma(.)$ a deterministic square-integrable function.

- Define the futures price under a risk-neutral measure $\mathbb{Q}$ as

$$F(t, T) = P\mathbb{Q}\left[\Gamma_T > 1 | \mathcal{F}_t\right]$$

Reduced-Form Dynamics

The martingale

$$a_t = \mathbb{E}^{\mathbb{Q}} \left[\mathbf{1}_{\{\Gamma_T > 1\}} | \mathcal{F}_t \right]$$

is given by

$$a_t = \Phi \left[\frac{\Phi^{-1}(a_0)\sqrt{\int_0^T \sigma_s^2 ds} + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}} \right]$$

and solves the stochastic differential equation

$$da_t = \Phi' \left(\Phi^{-1}(a_t) \right) \sqrt{z_t} dW_t$$

with

$$z_t = \frac{\sigma_t^2}{\int_t^T \sigma_u^2 du}$$

Reduced-Form Dynamics – Proof

- a_t formula is straightforward calculation
- For dynamics use that

$$a_t = \Phi(\xi_t)$$

with

$$\xi_t = \frac{\xi_{0,T} + \int_0^t \sigma_s dW_s}{\sqrt{\int_t^T \sigma_s^2 ds}} \text{ and } \xi_{0,T} = \log \Gamma_0 - \frac{1}{2}\int_0^T \sigma_s^2 ds.$$

Starting with the dynamics of ξ_t an application of Itô's formula gives the result.

Model Parametrization

- For constant σ we find $z_t = (T-t)^{-1}$, so a richer specification is needed.
- A standard model is

$$da_t = \Phi'\left(\Phi^{-1}(a_t)\right)\sqrt{\beta(T-t)^{-\alpha}}dW_t$$

  which specifies a family $\sigma_s(\alpha,\beta)$.
- So $z_t(\alpha,\beta) = \beta(T-t)^{-\alpha}$ and

$$\begin{aligned}\sigma_t^2(\alpha,\beta) &= z_t(\alpha,\beta)\exp\left\{-\int_0^t z_s(\alpha,\beta)ds\right\}\\ &= \begin{cases}\beta(T-t)^{-\alpha}e^{-\frac{\beta}{1-\alpha}[T^{1-\alpha}-(T-1)^{1-\alpha}]} & \alpha\neq 1\\ \beta(T-t)^{\beta-1}T^{-\beta} & \alpha = 1.\end{cases}\end{aligned}$$

Objective Measure

- We do a historical calibration and change measure to the objective measure.
- The standard change of measure gives

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left\{\int_0^T H_s dW_s - \frac{1}{2}\int_0^T H_s^2 ds\right\}).$$

- Under constant market price of risk $H_t \equiv h$ and by Girsanov's theorem

$$\tilde{W}_t = W_t - ht$$

is a $\mathbb{P}$ Brownian motion.

Objective Measure

- Under $\mathbb{P}$

$$d\xi_t = \left(\frac{1}{2}z_t\xi_t + h\sqrt{z_t}\right)dt + \sqrt{z_t}d\tilde{W}_t,$$

so ξ_τ given ξ_t is Gaussian.

- So we can invert permit prices to obtain ξ values and calculate the log-likelihood to obtain estimates for α and β.

Pricing Formula

For a European call with strike K and maturity τ the option price is

$$C_t = e^{-\int_t^\tau r_s ds} \int_{-\infty}^{\infty} (P\Phi(x) - K)^+ \Phi_{\mu_{t,\tau},\sigma_{t,\tau}}(dx)$$

with

$$\mu_{t,\tau} = \begin{cases} \xi_t \left(\frac{T-t}{T-\tau}\right)^{\frac{\beta}{2}} & \alpha = 1 \\ \xi_t \exp\left\{\frac{\beta}{2(1-\alpha)}[(T-t)^{1-\alpha} - (T-\tau)^{1-\alpha}]\right\} & \alpha \neq 1. \end{cases}$$

and

$$\sigma^2_{t,\tau} = \begin{cases} \left(\frac{T-t}{T-\tau}\right)^{\beta} - 1 & \alpha = 1 \\ \exp\left\{\frac{\beta}{1-\alpha}[(T-t)^{1-\alpha} - (T-\tau)^{1-\alpha}]\right\} - 1 & \alpha \neq 1. \end{cases}$$