# An Introduction to Financial Networks and Systemic Risk (Part II)


Tom Hurd

Mathematics and Statistics
McMaster University


11th Winter School on Mathematical Finance



## Definitions and Notation

For any $N \geq 1$:

- The collection of directed graphs on $N$ nodes is denoted $\mathcal{G}(N)$.
- The set of nodes or vertices $\mathcal{N} = \{1, \dots, N\}$ is numbered by integers.
- The set of possible edges or links is $\mathcal{N} \times \mathcal{N}$.
- A graph $\mathcal{E} \in \mathcal{G}(N)$ is an arbitrary subset $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$.
- We write $v, w, v'$,etc for vertices, $\ell, \ell'$ etc for links.

We also write $\mathcal{G}_\infty = \cup_{N=1}^{\infty} \mathcal{G}(N)$ and $\mathcal{B}$ for its Borel sigma-algebra.

## Adjacency, Degrees and Types

- $\mathcal{E} \subset \mathcal{N} \times \mathcal{N}$ can be represented by its adjacency matrix $M(\mathcal{E})$

$$M_{vw}(\mathcal{E}) = \begin{cases} 1 & \text{if } (v,w) \in C\mathcal{E} \\ 0 & \text{if } (v,w) \in \mathcal{N} \times \mathcal{N} \setminus \mathcal{E} \end{cases}$$

- The in-degree $\deg^-(v)$ and out-degree $\deg^+(v)$ of a node $v$ are

$$\deg^-(v) = \sum_w M_{wv}(\mathcal{E}), \quad \deg^+(v) = \sum_w M_{vw}(\mathcal{E})$$

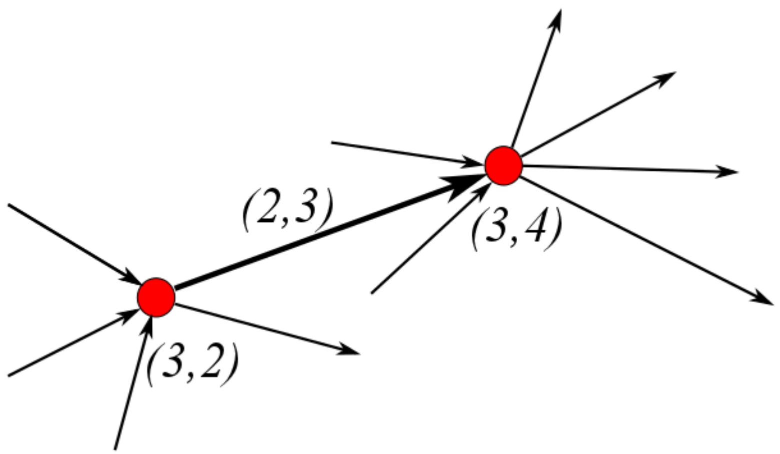
- $\deg^-(\ell) = j$ if $\ell$ is an in-edge of a node with in-degree $j$.
- $\deg^+(\ell) = k$ if $\ell$ is an out-edge of a node with out-degree $k$.
- $v \in \mathcal{N}$ has type $(j,k)$ if $\deg^-(v) = j$ and $\deg^+(v) = k$.
- $\ell \in g$ has type $(k,j)$ if $\deg^-(\ell) = j$ and $\deg^+(\ell) = k$.
- $j, j', j'', \dots$ refer to in-degrees;
- $k, k', k'', \dots$ refer to out-degrees.

## More Notation

For any graph $\mathcal{E}$:

- Decomposition by type: nodes $\mathcal{N} = \cup_{jk}\mathcal{N}_{jk}$ and edges $\mathcal{E} = \cup_{kj}\mathcal{E}_{kj}$ ;
- If $v \in \mathcal{N}_{jk}$ write $k_v = k, j_v = j$; if $\ell \in \mathcal{E}_{kj}$ write $k_\ell = k, j_\ell = j$.
- $\mathcal{E}_v^+$ is the set of out-edges of a given node $v$.
- Also $v_\ell^+$ is the node for which $\ell$ is an out-edge.
- Similarly for $\mathcal{E}_v^-$ and $v_\ell^-$.
- In-neighbourhood of $v$: $\mathcal{N}_v^- := \{w \in \mathcal{N} | M_{wv}(\mathcal{E}) = 1\}$.
- Out-neighbourhood of $v$: $\mathcal{N}_v^+ := \{w \in \mathcal{N} | M_{vw}(\mathcal{E}) = 1\}$.

## 2 Nodes and 1 Edge




## Systemic Interpretations

For any systemic graph $\mathcal{E}$:

- Nodes $v \in \mathcal{N}$ are "banks" or financial institutions;
- $v' \in \mathcal{N}_v^+$ means "$v'$ is exposed to $v$";
- $v' \in \mathcal{N}_v^-$ means "$v'$ owes to $v$";
- $v' \in \mathcal{N}_v^- \cap \mathcal{N}_v^+$ means $v, v'$ are mutually exposed and may want to "net" their exposures.

## Graph Functions

For any graph $\mathcal{E}$ of size $N$:

- Average degree: $z = \frac{1}{N}\sum_v \deg^-(v) = \frac{1}{N}\sum_v \deg^+(v)$
- Local clustering coefficient:

$$\kappa(v) = \frac{\#\{\ell \in \mathcal{E} | v_\ell^+ \in \mathcal{N}_v^+, v_\ell^- \in \mathcal{N}_v^-\}}{k_v j_v}$$

- Average clustering coefficient: $\bar{\kappa} = \frac{1}{N}\sum_v \kappa(v)$.

## Random Graphs

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

- A random graph of size $N$ is a random variable $\mathcal{E} : \Omega \to \mathcal{G}(N)$.
- We can also draw uniformly from the nodes and edges of the random graph $\mathcal{E}$.
- Node-type distribution: $P_{jk} = \mathbb{P}[v \in \mathcal{N}_{jk}]$.
- Edge-type distribution: $Q_{kj} = \mathbb{P}[\ell \in \mathcal{E}_{kj}]$.
- Node-degree distributions: $P_k^+ = \sum_j P_{jk}, P_j^- = \sum_k P_{jk}$.
- Edge-degree distributions: $Q_k^+ = \sum_j Q_{kj}, Q_j^- = \sum_k Q_{kj}$.
- Mean degree: $z = \sum_{jk} kP_{jk} = \sum_{jk} jP_{jk}$.
- Complete specification of the random graph ensemble involves choosing $\{P_{jk}, Q_{kj}\}$ and a full dependence structure.

## Erdös-Renyi Graphs $G_{N,p}$

- This classic model of random undirected graphs of size $N$ arises by adding each edge independently with probability $p$.
- Large $N$ asymptotics, taking $p = c/N$ with $c$ fixed:
  1. Poisson degrees $P_k \sim \frac{c^k e^{-c}}{k!}$.
  2. Average clustering goes to zero: $\bar{\kappa} \sim o(N)$.
  3. All average higher clustering coefficients go to zero.
- Results easily extend to the directed graph case.

## Configuration Graph Construction

Let $P = \{P_{jk}\}$ be any node-type distribution.

1. Draw $N$ independent samples $(j_n, k_n), n = 1, 2, \ldots, N$ from the $P$ distribution.
2. To "wire" the network with nodes $v_1, \ldots, v_N$ with these types, do the following loop for $n = 1, 2, \ldots, N$:
   - Take the $k_{v_n}$ out-stubs of $v_n$ one-at-a-time, and pair each with an in-stub chosen uniformly from the remaining un-paired in-stubs.

This gives a generalized Erdös-Renyi random graph in the large $N$ limit. The edge-type distribution $Q$ is "uncorrelated":

$$Q_{kj} = kjP_k^+ P_j^- / z^2 \ .$$

## Watts' Small World Graphs

Somehow "complex adapted networks" seem to evolve into the class of random graphs known as "Small World Networks". In some sense, the financial networks we see appear to have this structure. Are they random (directed) networks with the desired characteristics?

- Power law degree distribution "Scaling";
- Small average "path length" or degree of separation;
- High clustering coefficient.

## Barabasi-Albert Scale-free Graphs

- Power law degree distribution "Scaling";
- Small average "path length" or degree of separation;
- High clustering coefficient.

## Watts' 2002 Cascade Model

This is a model of a social network (network of "friendships"): the problem is to study if/how a newly introduced technology "percolates" through the network. The basic setup is as follows

- infinite (undirected) random graph $\mathcal{E}$;
- random threshold $\phi_v \in [0, 1]$ for each node $v \in \mathcal{E}$ drawn from distribution with CDF $F(\phi)$;
- Early adopters: initial states $\pi_v \in \{0, 1\}$, where $\pi_v = 1$ means $v$ has "adopted" the technology (or else, the bank has "defaulted").
- The cascade proceeds through a sequence of steps, in which each node $v$ "adopts" ($\pi_v$ changes from 0 to 1) if at least $k_v \phi_v$ of its neighbours were in state 1 in the previous step;

## Watts Main Result

Watts derives a "cascade condition" for global cascades (i.e ones containing a positive fraction of nodes) to occur with positive probability, starting with a randomly selected single early adopter.

## Watts Model: Details

- The random graph model is the undirected configuration model with degree distribution $P_k = \mathbb{P}[\mathsf{deg}(v) = k]$.
- Conditional degree distribution:

$$Q_k = \mathbb{P}[k_v = k | v \in \mathcal{N}_{v'} \text{ for some } v'] = \frac{kP_k}{z} \ .$$

- Recall the conditional probability formula: $\mathbb{P}[A|B] = \frac{\mathbb{P}[A \cap B]}{\mathbb{P}[B]}$.
- Initially adopting (defaulted) nodes: $\overline{\mathcal{M}} = \{v | \pi_v = 1\}$.
- Initial probabilities: for each possible degree $k$, $b_k := \mathbb{P}[v \in \overline{\mathcal{M}} | k_v = k]$.

## Watts Model: the Cascade

- We want to construct probabilities for the increasing sequence of node-sets :

$$\overline{\mathcal{M}} \to \mathcal{M}_1 \cup \overline{\mathcal{M}} \to \cdots \to \mathcal{M}_n \cup \overline{\mathcal{M}} \to \cdots$$

- Let $a_k^n = \mathbb{P}[v \in \mathcal{D}_n | k_v = k]$ for $n = 1, 2, \ldots$, and all $k$.
- Note: we assume disjointness $\mathcal{M}_n \cap \overline{\mathcal{M}} = \emptyset$ for all $n \geq 1$.
- In the limit $n \to \infty$ we obtain the collection $a_k^\infty$, and the probabilities

$$\mathbb{P}[v \text{ eventually adopts } | \ k_v = k] = b_k + a_k^\infty$$

## Watts Model: Cascade Analysis

- Define "property $P$ holds for $v$ without regard to (WORT) $v' \in \mathcal{N}_v$" means $P$ holds in the equivalent graph with $v'$ removed.
- Now define

$$c_n := \mathbb{P}[v \in \overline{\mathcal{M}} \cup \mathcal{M}_n \text{ WORT } v' | v' \in \mathcal{N}_v], \quad n = 0, 1, \ldots$$

- Note that $c_0 = \sum_k b_k Q_k$.
- "$v \in \overline{\mathcal{M}} \cup \mathcal{M}_n$ WORT $v'$" means either $v \in \overline{\mathcal{M}}$ or a sufficient number of the remaining $k-1$ neighbours of $v$ have adopted at level $n-1$. By an IID condition, this number is Binomial$(k-1, p)$ with $p = c_{n-1}$, and using intermediate conditioning on the RV $\phi_v$ we obtain:

$$c_n = \sum_k \left[ b_k + (1 - b_k) \sum_{j=1}^{k-1} \binom{k-1}{j} c_{n-1}^j (1 - c_{n-1})^{k-1-j} F(j/k) \right] Q_k$$

## Basic Theorem on Watts' Cascade

**Theorem**
*For any initial "seed probabilities" $b = \{b_k \in [0,1]\}_{k=0,1,\ldots}$*

1. *The probability $c = \lim_{n\to\infty} c_n$ exists and solves the fixed point equation $c = G(c; b)$*

$$G(x; b) = \sum_k \left[ b_k + (1 - b_k) \sum_{j \leq k-1} \binom{k-1}{j} x^j (1 - x)^{k-1-j} F(j/k) \right] Q_k$$

2. *The fraction of adopting nodes at the completion of the Watts cascade is*

$$\begin{aligned} a_k &:= \mathbb{P}[v \text{ eventually adopts } | \ k_v = k] \\ &= b_k + (1 - b_k) \sum_{j \leq k} \binom{k}{j} c^j (1 - c)^{k-j} F(j/k) \end{aligned}$$

## Watts' 2002 Cascade Condition

**Theorem**
*In the infinite Watts network:*

1. *The probability for a single random seed (an "infinitesimal seed") to grow to a finite fraction of the network is positive provided $G'(0; 0) > 1$.*
2. *This probability is zero if $G'(0; 0) < 1$.*
3. *Here $G'(0; 0) = \sum_k k(k-1) F(1/k) P_k / z$.*

## Percolation

- $F(1/k) = \mathbb{P}[k_v \phi_v \leq 1 | k_v = k]$ is the probability that a node of degree $k$ will adopt if only one neighbour adopts: we call such a node "vulnerable".
- Let $\mathcal{V} = \cup_k \mathcal{V}_k$ be the set of vulnerable nodes and note that $\mathbb{P}[v \in \mathcal{V}] = \sum_k F(1/k) P_k$.
- Let $\mathcal{S}$ be the "giant vulnerable cluster", i.e. the largest connected component of $\mathcal{V}$.
- Let $\pi := \mathbb{P}[v \notin \mathcal{S} \text{ WORT } v' | v' \in \mathcal{N}_v]]$.
- Note that $v \notin \mathcal{S}$ WORT $v'$ iff either $v \notin \mathcal{V}$ or none of its remaining $k-1$ neighbours are in $\mathcal{S}$.
- Then we deduce $\pi$ is a fixed point of

$$H(x) = \sum_k \left[ (1 - F(1/k)) + F(1/k) x^{k-1} \right] Q_k$$

## Watts' 2002 Percolation Condition

**Theorem**
*In the infinite Watts network:*

1. *The fixed point $\pi = 1$ is stable and the giant vulnerable cluster is finite almost surely if $H'(1) < 1$*
2. *If $H'(1) > 1$ there is a stable fixed point $\pi^* < 1$ , and the fractional size of the giant vulnerable cluster is $1 - \pi^*$. Moreover, the frequency that a random single seed will trigger a global cascade is*

$$f* = 1 - \sum_k (\pi^*)^k P_k$$

3. *Here $H'(1) = G'(0;0) = \sum_k k(k-1) F(1/k) P_k / z$.*

## Gai-Kapadia 2010: "Contagion in Financial Networks"

This is a very basic model of contagion through shocks to the asset side of the balance sheet. It is designed to mimic the general features of the Watts 2002 Cascade Model. The specification will consist of three levels.

- The random directed graph model for the "skeleton" of the network;
- a specification of balance sheet values for all nodes and edges;
- a specification of the type of initial shocks that will be considered.

## Extended GK Skeleton Graph

In Hurd-Gleeson 2011, we developed the following extended framework.

- Directed Assortative Configuration Graph $\mathcal{E}$ on $N \leq \infty$ nodes;
- Type and degree distributions:
  - $P_{jk} = \mathbb{P}[v \in \mathcal{N}_{jk}]$;
  - $Q_{kj} = \mathbb{P}[\ell \in \mathcal{E}_{kj}]$;
  - $P^+_k = \mathbb{P}[k_v = k]$; $P^-_j = \mathbb{P}[j_v = j]$;
- Consistency Condition:

$$Q^+_k = \mathbb{P}[k_\ell = k] = \frac{k P^+_k}{z}; \quad Q^-_j = \mathbb{P}[j_\ell = j] = \frac{j P^-_j}{z}.$$

- GK 2010 assumed Independent edge condition $Q_{kj} = Q^+_k Q^-_j$.

## Balance Sheets

- For each node $v$:
  - the external assets $Y_v$;
  - and external liabilities $Z_v$.
- For each edge $\ell$ of the network, an exposure size or weight $w_\ell$.
- The *net worth* or *buffer* of a node $v$ is

$$\gamma_v = Y_v + \sum_{\ell \in \mathcal{N}_v^-} w_\ell - Z_v - \sum_{\ell \in \mathcal{N}_v^+} w_\ell \,.$$

- The GK 2010 framework only depends on partial information:
$$\{\gamma_v, v \in \mathcal{N}\} \cup \{w_\ell, \ell \in \mathcal{E}\}.$$
- $\gamma_v$ may depend on the node type $(j,k)$ (GK 2010: $\gamma_v = 0.035$);
- $w_\ell$ depend only on $\deg^-(\ell)$ (GK 2010: $w_\ell = 1/(5j_\ell)$)

## Cascade Equilibrium

- Initially, all banks are solvent;
- meaning $\gamma_v > 0$ at every node $v$.
- $\gamma_v$ is a buffer against balance sheet shocks.
- Equilibrium system is hit by an external shock that "removes" a number of nodes.
- Suppose an initial set $\overline{\mathcal{M}} \subset \mathcal{N}$ of nodes become insolvent;
- $\overline{\mathcal{M}}$ is drawn randomly, with the fraction of type $(j,k)$ nodes that are defaulted denoted by

$$\rho^0_{jk} := \mathbb{P}[v \in \overline{\mathcal{M}} | v \in \mathcal{N}_{jk}] \,.$$

## The Solvency Condition

- GK "zero recovery" assumption: an insolvent bank can pay none of its interbank credit obligations, and each insolvent node $v \in \overline{\mathcal{M}}$ triggers all its out-edges $\ell \in \mathcal{E}_v^+$ to have zero value.
- Write $\overline{\mathcal{D}} = \cup_{v \in \overline{\mathcal{M}}} \mathcal{E}_v^+$;
- Each defaulted out-edge $\ell \in \overline{\mathcal{D}}$ supplies an asset shock to the creditor bank $v_\ell^-$.
- The new solvency condition on a bank $v$ is now:

$$\gamma_v > \sum_{\ell \in \mathcal{E}_v^-} \mathbf{1}_{\{\ell \in \mathcal{D}\}} w_\ell \,.$$

## Default Cascade Steps

1. We analyze the sequence of "updates":

$$\begin{pmatrix} \overline{\mathcal{M}} \\ \overline{\mathcal{D}} \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{M}} \cup \mathcal{M}_1 \\ \overline{\mathcal{D}} \cup \mathcal{D}_1 \end{pmatrix} \to \begin{pmatrix} \overline{\mathcal{M}} \cup \mathcal{M}_2 \\ \overline{\mathcal{D}} \cup \mathcal{D}_2 \end{pmatrix} \cdots \to \begin{pmatrix} \overline{\mathcal{M}} \cup \mathcal{M}_n \\ \overline{\mathcal{D}} \cup \mathcal{D}_n \end{pmatrix} \to \cdots$$

   where set unions are assumed to be disjoint.
2. Increasing sequences of sets:

$\mathcal{M}_n :=$ defaulted nodes not in $\overline{\mathcal{M}}$ "triggered" by edges in $\overline{\mathcal{D}} \cup \mathcal{D}_{n-1}$
$\mathcal{D}_n :=$ defaulted edges not in $\overline{\mathcal{D}}$ "triggered" by nodes in $\overline{\mathcal{M}} \cup \mathcal{M}_n$.

3. We keep track of the following sets of probabilities:

$$\begin{aligned} \rho^n_{jk} &:= \mathbb{P}[v \in \mathcal{M}_n | v \in \mathcal{N}_{jk}] \\ \sigma^n_{kj} &:= \mathbb{P}[\ell \in \mathcal{D}_n | \ell \in \mathcal{E}_{kj}] \\ a_j^{(n)} &:= \mathbb{P}[\ell \in \mathcal{D}_n | j_\ell = j] \end{aligned}$$

## First Result on Default Cascades

### Proposition

*In the infinite $N$ extended GK model specified by $(P, Q, \gamma, w, \rho)$ the quantities $\rho^n, \sigma^n, a^{(n)}$ obey the recursion formulas:*

$$\begin{aligned}
\rho^n_{jk} &= (1-\rho^0_{jk}) \sum_{m=\lceil \gamma_{jk}/w_j \rceil}^{j} \binom{j}{m} (a_j^{(n-1)})^m (1-a_j^{(n-1)})^{j-m} \\
\sigma^n_{kj} &= \frac{\sum_{j'} (\rho^n_{j'k} P_{j'k})}{P^+_k} \\
a^{(n)}_j &= \frac{\sum_k (Q_{kj}(\sigma^0_{kj} + \sigma^n_{kj}))}{Q^-_j} .
\end{aligned}$$

## Inductive Proof

To derive $\rho^n$ given the set $\mathbf{a}^{(n-1)} = \{a_k^{(n-1)}\}_{k=0,1,\ldots}$:

- Note

$$\mathbb{P}[v \in \mathcal{M}_n | v \in \mathcal{N}_{jk}] = (1-\rho^0_{jk}) \mathbb{P}[v \in \mathcal{M}_n | v \in \mathcal{N}_{jk} \setminus \overline{\mathcal{M}}] , \tag{1}$$

- Also that a node $v \in \mathcal{N}_{jk} \setminus \overline{\mathcal{M}}$ will be in default if and only if at least $M_{jk}$ in-edges to $v$ are in $\mathcal{D}_{n-1}$.
- By an IID property the number $\#\{\mathcal{E}_v^- \cup \mathcal{D}_{n-1}\}$ is $Bin(j, a_j^{(n-1)})$.
- Putting these facts together gives

$$\mathbb{P}[v \in \mathcal{M}_n | v \in \mathcal{N}_{jk} \setminus \overline{\mathcal{M}}] = \sum_{m=M_{jk}}^{j} \binom{j}{m} (a_j^{(n-1)})^m (1-a_j^{(n-1)})^{j-m}$$

Showing the steps $\rho \to \sigma$ and $\sigma \to a$ are easier.

## Default Cascade Mapping

1. The sequence $\{\mathbf{a}^{(n)}\}_{n=0,1,\ldots}$ satisfies a recursion

$$\mathbf{a}^{(n+1)} = G(\mathbf{a}^{(n)}), \quad n = 0, 1, \ldots$$

2. Here mapping $G : (\mathbb{R}_+)^{\mathbb{Z}_+} \to (\mathbb{R}_+)^{\mathbb{Z}_+}$ depends explicitly on the structure of the network and the initial shock distribution.
3. The sequence converges to a fixed point $\mathbf{a}^{(\infty)}$.

## GK Cascade Condition

### Proposition

*In the infinite GK network, let the Jacobian matrix of the mapping $G$ be*

$$\begin{aligned}
D_{jj'} &= \partial G_j / \partial a_{j'} |_{\vec{a}=\mathbf{0}, \rho^0=0} \\
&= \sum_k \frac{j' Q_{kj} P_{j'k} \mathbf{1}_{\{\gamma_{j'k} \le w_{j'}\}}}{Q^-_j P^+_k} .
\end{aligned}$$

*and let $\|D\|$ denote its spectral radius.*

1. *The probability that a single random seed will grow to a finite cascade is positive provided $\|D\| > 1$*
2. *If $\|D\| < 1$, then the network will not exhibit large scale cascades for almost all single random seeds.*

*We say the network satisfies the cascade condition if $\|D\| > 1$.*

## Percolation on Directed Graphs

it is well-known that the frequency of global cascades in infinite random graphs is given by the fractional size of the so-called in-component associated to the giant vulnerable cluster.
Define:

- A node $v_{jk}$ is vulnerable if $\gamma_{jk} \le w_j$;
- $\Gamma_{jk} = \mathbf{1}_{\{\gamma_{jk} \le w_j\}}$.
- $\mathcal{N}_{(v)}$ is the set of vulnerable nodes, .
- $\mathcal{S}_s$ is the giant strongly connected set of vulnerable nodes (the "giant vulnerable cluster");
- $\mathcal{S}_i$ is the set of (possibly not vulnerable) nodes that are forward connected to $\mathcal{S}_s$ by a path of vulnerable nodes (the "in-component" of the giant vulnerable cluster);

## Percolation Analysis

Let $\pi = \{\pi_k\}$ denote the sequence of probabilities $\mathbb{P}[v \in \mathcal{S}_i^c | k_v = k]$

- $v \in \mathcal{S}_i^c$ and $k_v = k \iff v' \in \mathcal{V}^c \cup (\mathcal{S}^c \cap \mathcal{V})$ and $k_v = k$ for all $v' \in \mathcal{N}_v^+$;
- Thus $\pi_k = (\tilde{\pi}_k)^k$ where

$$\begin{aligned}\tilde{\pi}_k &= \mathbb{P}[v' \in \mathcal{V}^c \cup (\mathcal{S}^c \cap \mathcal{V}) | k_v = k, v' \in \mathcal{N}_v^+] \\ &= \sum_{j',k'} \left(\Gamma_{j'k'}\pi_{k'} + (1 - \Gamma_{j'k'})\right) \mathbb{P}[v' \in \mathcal{N}_{j'k'} | k_v = k, v' \in \mathcal{N}_v^+] \\ &= \sum_{j',k'} \left(\Gamma_{j'k'}\pi_{k'} + (1 - \Gamma_{j'k'})\right) \frac{P_{j'k'}Q_{kj'}}{P_{j'}^- Q_k^+}\end{aligned}$$

- i.e. $\tilde{\pi}$ is the fixed point of the sequence-valued function $H : [0,1]^\infty \to [0,1]^\infty$ where

$$H_k(x) = \sum_{j',k'} \left(\Gamma_{j'k'}(x_{k'})^{k'} + (1 - \Gamma_{j'k'})\right) \frac{P_{j'k'}Q_{kj'}}{P_{j'}^- Q_k^+}$$

## GK: Theorem on Percolation

**Proposition**

*Frequency of global cascades $f$ is determined by the size of the in-component of the giant strongly connected vulnerable cluster.*

1. *$f$ will be positive if and only if there is a nontrivial solution $\tilde{\pi} = \{\tilde{\pi}_k\}$ of the fixed point equation $x = H(x)$.*
2. *This occurs if and only if the cascade condition holds.*
3. *The frequency is given by $f = \sum_k (1 - (\tilde{\pi}_k)^k) P_k^+$.*

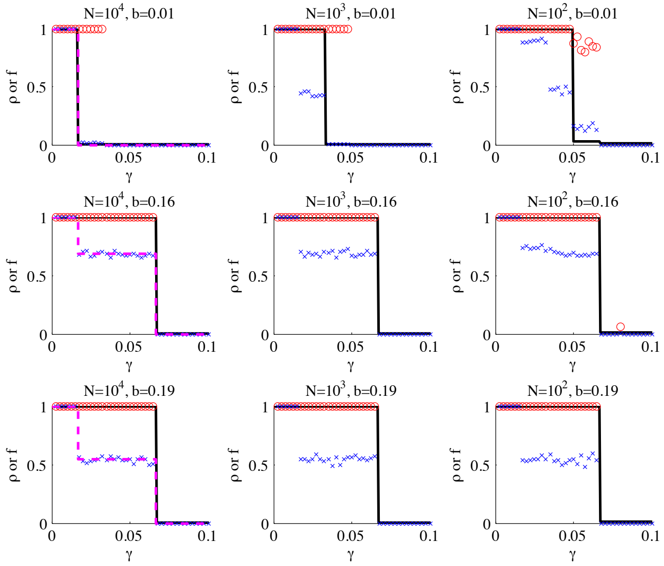
## A Simple Random Network Model

1. We simulated networks with nodes of types $(3,3), (3,12), (12,3), (12,12)$ and edges of the same types.
2. For parameters $a \in [0, 1/2]$ and $b \in [0, 1/5]$ the following $P$ and $Q$ probabilities are consistent:

$$\begin{pmatrix} P_{3,3} & P_{3,12} \\ P_{12,3} & P_{12,12} \end{pmatrix} = \begin{pmatrix} 1/2 - a & a \\ a & 1/2 - a \end{pmatrix};$$
$$\begin{pmatrix} Q_{3,3} & Q_{3,12} \\ Q_{12,3} & Q_{12,12} \end{pmatrix} = \begin{pmatrix} 1/5 - b & b \\ b & 4/5 - b \end{pmatrix}.$$

3. We compared simulations with 500 runs to theory.

# Model 1




# Assortativity: Is it Important?

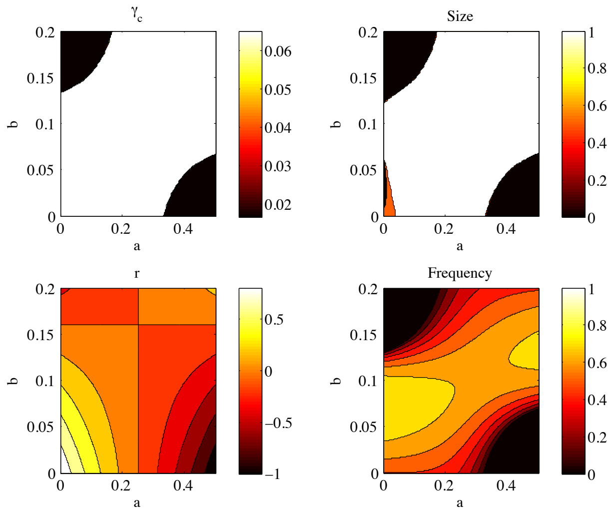
Haldane observes that "diversity" should be positively related to systemic stability. Other people claim networks with homogeneous nodes are more stable. Can we test who is right?

- Edge-assortativity: Pearson correlation of matrix $Q_{kj}$;
- Node-assortativity: Pearson correlation of matrix $P_{jk}$;
- Graph-assortativity $r$: Pearson correlation of matrix

$$B_{jj'} = \sum_k \frac{P_{jk} Q_{kj'}}{P_k^+} = \mathbb{P}[j_v = j, j_{v'} = j' | v, v' \text{are connected}]$$

- How do these skeleton graph quantities relate to systemic risk?

# Model 1: More Results



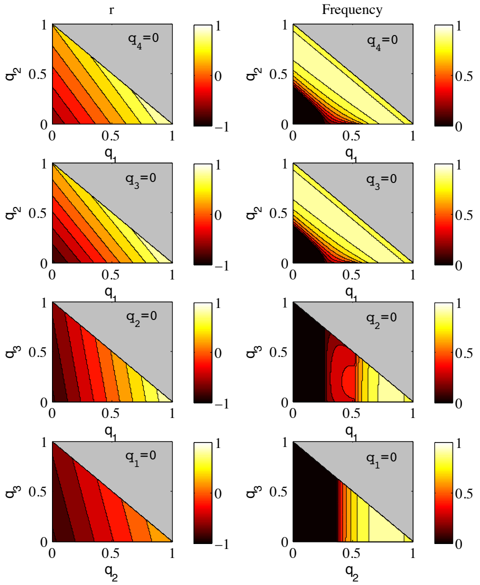

# Assortativity: Test on Another Family of Networks

Networks with node types $(2,2), (4,4), (8,8), (16,16)$ and various $Q$ matrices:

$$P = \frac{1}{15}\begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, Q(q_1, q_2, q_3, q_4) = \frac{1}{4}\begin{pmatrix} q_1 & q_2 & q_3 & q_4 \\ q_2 & q_1 & q_4 & q_3 \\ q_3 & q_4 & q_1 & q_2 \\ q_4 & q_3 & q_2 & q_1 \end{pmatrix},$$

restricted to $q_1 + q_2 + q_3 + q_4 = 1$.

## $r$ (left) and $f$ frequency (right)




## Gai-Haldane-Kapadia 2011: "Complexity, Concentration and Contagion"

This adapts the GK 2010 framework to a new cascade problem: the hoarding of assets undertaken by stressed banks, and the transmission of liquidity shocks to the liabilities of its counterparties.
The basic setup is:

- A skeleton network of banks as in GK 2010;
- A refined breakdown of the banks' balance sheets;
- Initial shocks.

## Interbank Links: Repos

Overnight "collateralized" loans:

- Fully liquid assets can always be used as repo collateral without a haircut;
- Certain other assets can be collateral with a haircut $h \in (0, 1)$;
- "Fixed assets" and unsecured interbank assets can never be used as repo collateral.
- "Reverse repo" assets, may be "rehypothecated" to create further repos with the same haircut.
- Repos command interest at the "repo rate";
- Haircuts are just high enough that there is negligible counterparty risk.

## Interbank Links: Unsecured Loans

Overnight unsecured loans:

- Draw a higher interest rate (the LIBOR rate);
- Have full counterparty risk.

## GHK 2011: Bank Assets

Five asset classes are considered:

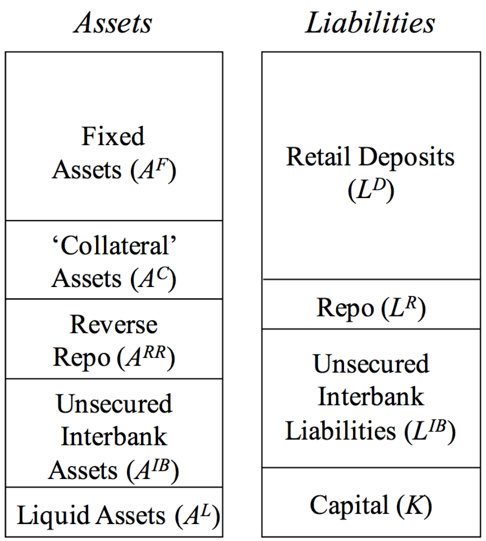
- Fixed assets $A^F$: commercial loans and mortgages;
- Collateral assets $A^C$: liquid market securities;
- Reverse repo assets $A^{RR}$;
- Unsecured interbank assets $A^{IB}$;
- Liquid assets $A^L$: cash or treasuries.

## GHK 2011: Bank Liabilities

Four liability classes are considered:

- Retail deposits $L^D$;
- Repos $L^R$;
- Unsecured interbank liabilities $L^{IB}$;
- Capital or net worth or buffer $\gamma$.

## Balance Sheet (from GHK 2011)




## Gai-Haldane-Kapadia 2011

This paper introduces the idea that a stressed bank will begin hoarding liquidity, thereby transmitted a liability shock to its debtor banks. This mechanism can create liability cascades.
A liquid or unstressed bank is such that:

1. total collateral available exceeds existing repo funding...
2. plus any possible IB shock...
3. plus any balance sheet shock $\epsilon$.

The "liquid" condition is thus

$$K := A^L + (1-h)[A^C + A^{RR}/(1-h)] - L^R - \lambda L^{IB} - \epsilon > 0. \quad (2)$$

## Liquidity Hoarding

A stressed bank is one with $K \le 0$. Such a bank reacts defensively to restore liquidity in order as follows:

1. it withdraws a fraction $\lambda \in (0, 1]$ of its IB lending;
2. it will sell fixed assets as needed.

$\lambda$ (taken as a constant) is a key amplification parameter. Pessimistically, one can take $\lambda = 1$.

## GHK 2011: Network Specifications

- A directed network of $N = 250$ banks (statistically homogeneous);
- Degree distribution $P$: either Poisson or Geometric with variable mean $z$;
- $\lambda = 1$;

## GHK 2011: Balance Sheet Structure

Liabilities:

- $\gamma = 0.04A$ (where $A = L$ is the total balance sheet);
- $L^{IB} = 0.15A$ and IB lending links are equidistributed, hence $w_\ell = 0.15A/k_\ell$;
- $L^R = 0.2A = (1-h)[A^C + A^{RR}/(1-h)]$ (i.e. full repo funding);
- $L^D = 0.61A$.

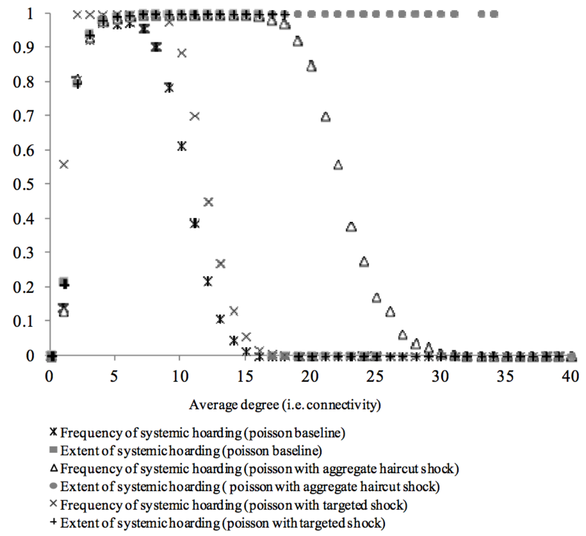
Assets:

- $A^L = 0.02A$;
- $A^{RR} = 0.11A$;
- $A^C = 0.10A$ (hence $h = 0.1$);
- $A^{IB} = \sum_{\ell \in \mathcal{E}^+} w_\ell$ and $A^F$ are endogenous.

## GHK 2011: Computer Simulations

- Generate 1000 MC simulations of the network
- Generate several types of shocks, eg. a single random bank gets stressed;
- Evaluate the liquidity cascade.
- Reported results: a “systemic event” leads to $\ge 25$ stressed banks.
- GHK plot frequency and impact conditioned on a systemic event having occurred.

## GHK 2011: Typical Results




## GHK 2011: Conclusions

- Their simulations reproduce several systemic effects that were observed during the 07/08 crisis.
- They draw many conclusions about policy and regulation based on their model.
- Although GHK 2011 is mathematically identical to GK 2010, they present no analytical results. (Why not?)

## Open Problems: Mathematical

This graph theoretical approach has just begun. Many accessible mathematical problems remain to be addressed:

- Stochastic Balance Sheets;
- Mixed liquidity/default cascades (i.e. combine GK + GHK);
- Non-zero recovery at default;
- The systemic effect of asset "firesales";
- The impact of derivatives, counterparty risk, and CDS;
- Graph theoretical measures of systemic risk.

## Open Problems: Financial

The financial implications also need development:

- What systemic databases are available or will become available?
- How can one use such data?
- How can one identify "systemically important" banks?
- What are some policy and regulatory implications of such models?

## Overall Summary

- We are beginning to understand something about how systemic stability is related to the structure of the network;
- We might be able to learn a lot from "Deliberately Simplified Models".
-
- Open problems abound.

### Proposition

*Thanks for Attending!*