Applications of Classical Mathematical Methods in Finance

Theory and Practice

Alexander Lipton

Bank of America Merrill Lynch & Imperial College

11th Winter School on Mathematical Finance, 23rd-24th January 2012

LECTURE I

Abstract, I

Recently, most fundamental and long-considered solved problems of financial engineering, such as construction of yield curves and calibration of implied volatility surfaces, have recently turned out to be more complex than previously thought. In particular, it has become apparent that one of the main challenges of options pricing and risk management is the sparseness of market data for model calibration, especially in severe conditions. Market quotes can be very sparse in both strike and maturity. As the spot price moves, options that were close to at-the-money at inception become illiquid, so that one has to find ways to interpolate and extrapolate the implied volatilities of liquid options to mark them to market. Moreover, for certain asset classes the concept of implied volatility surface is badly defined. For instance, for commodities it is not uncommon to have market prices of options for only a single maturity, while for foreign exchange it is customary to quote option prices with no more than five values of delta and very few maturities.

Abstract, II

The calibration of a model to sparse market data is needed not only for the consistent pricing of illiquid vanilla options, but also for the valuation of exotic options. The latter is particularly demanding since it requires the construction of implied and local volatility surfaces across a wide range of option strikes and maturities.
In this mini-course we shall discuss a universal volatility model (UVM) and discuss its applications to pricing of financial derivatives. First, we describe three sources of UVM, namely, local volatility model; stochastic volatility model; and jump-diffusion model. Second, we describe three component parts of UVM, namely, calibration of the model to the market; pricing of vanilla and first-generation exotic options; pricing of second-generation exotics. Third, we discuss main analytical, semi-analytical, and numerical techniques needed for efficient implementation of UVM from a practical standpoint with a particular emphasis on the Lewis-Lipton formula and its applications.

Acknowledgements

I am grateful to several current and former Bank of America Merrill Lynch colleagues, especially L. Andersen, S. Inglis, A. Rennie, A. Sepp, D. Shelton, and other members of the Global Quantitative Group. I am also grateful to my former Bankers Trust colleagues including T. Hyer, A. Langnau, M. Lipton, D. Pugachevsky, P. Romanelli, and my former Deutsche Bank colleague William McGhee.

The calibration problem for tiled volatility is discussed in a recent paper by A. Lipton & A. Sepp "Filling the Gaps", Risk, October 2011

The asymptotic of volatility for Lévy process driven underlyers is discussed in a recent working paper by L. Andersen & A. Lipton "Asymptotics for Exponential Levy Processes and their Volatility Smile: Survey and New Results", 2012.

Main textbook "Mathematical Methods for Foreign Exchange", A. Lipton, 2001

References

- L. Andersen, A. Lipton, "Asymptotics for Exponential Levy Processes and their Volatility Smile: Survey and New Results", Working Paper, 2012
- L. Andersen, R. Brotherton-Ratcliffe, "Asymptotic Expansions of European Option Prices for Separable Diffusions", Working Paper, 1999
- A. Lipton, "Mathematical Methods for Foreign Exchange", WSPC, Singapore, 2001
- A. Lipton, A. Rennie, "The Oxford Handbook of Credit Derivatives", OUP, Oxford, 2010
- A. Lipton, "The Vol Smile Problem", Risk, 2002, www.math.ku.dk/~rolf/Lipton_VolSmileProblem.pdf
- A. Lipton, W. McGhee, "Universal Barriers", Risk, 2002

The roots of Mathematical Finance

- The roots of Mathematical Finance are in currency exchange, insurance, gambling, and speculation
- The origin of Medici's wealth can be traced to creative usage of financial engineering, (more specifically, creative forex trading via *cambium sine litteris*)
- The origin of its disappearance can be traced to reckless usage of financial engineering (more specifically, lending large sums of money to sub-prime borrowers such as Charles the Bold)
- St. Petersburg paradox (Nicolas Bernoulli (1713) and Daniel Bernoulli (1738)). Utility theory. Log utility function
- Gambler's Ruin, Casino games, etc.
- Pari-mutuel betting (Joseph Oller (1865))
- Description of Paris Bourse by Louis Bachelier (Theory of Speculation, 1900)

ForEx Market, I

- Forex trading has been known since antiquity
- A well-known example is Arena Chapel in Padua paid for by the Scrovengi and painted by Giotto
- After WWII all major currencies were fixed against USD as part of the Bretton-Woods System (July 1944)
- In August 1971 this system was abandoned and exchange rates became floating
- According to BIS, FX market is the largest and the most liquid market in the world. Its turnover is $4 trillion (equity turnover is less than 400 billion).
- The market increased 5 times in 15 years
- This size is explained via the so-called hot potato effect

Giotto

ForEx Market, II

- FX market is predominantly OTC
- They operate 24 hours a day (from 20:15 GMT Sunday to 22:00 GMT Friday)
- Major centers: London (35%), NY (20%), Tokyo (10%)
- Participants include central banks, investment banks, corporations, hedge funds, speculators, tourists, etc.
- From time to time, the market is manipulated by central banks
- Top tier banks dominate the market place: DB 15.6%, BarCap 10.8%, UBS 10.6%, Citi 8.9%, JPM 6.4%, HSBC 6.3%, RBS 6.2%, CS 4.8%, GS 4.1%, MS 3.6%
- Principal transactions: spot transactions (1.49 T), forex swaps (1.77 T), outright swaps (475 B), options (207 B), cross-currency swaps (40 B)

ForEx Market, III

- FX quoting conventions are difficult for the novice to grasp and are fairly idiosyncratic (at best) especially with regards to options
- Consider a spot transaction to buy one unit of currency YYY for S units of currency XXX
- Dimension of S is $\frac{XXX}{YYY}$, so XXX is numerator, YYY denominator. The market terminology is different: YYY is base, XXX is quote
- Label for S is YYYXXX
- The most actively traded pairs are EURUSD, USDJPY, GBPUSD, USDCHF, and "crosses" EURJPY, EURGBP, EURCHF
- Turnover share USD 85%, EUR 40%, JPY 19%, GBP 13%, AUD 8%, CHF 6%, CAD 5%

ForEx Market, IV

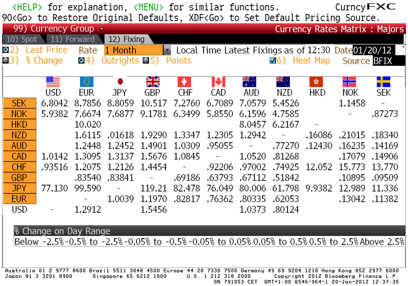
- Typically currencies are quoted with 5 significant digits, the 3rd one called "big figure", the 5th "pip"
- Dealers sell at ask(offer), buy at bid, they make profit on bid-ask spread
- FX triangulates

$$JPY\,/\,EUR = USD\,/\,EUR \times JPY\,/\,USD$$

$$EURJPY = EURUSD \times USDJPY$$

- To buy EUR (sell JPY) you buy EUR (sell USD) and buy USD (sell JPY)

Bloomberg FXC



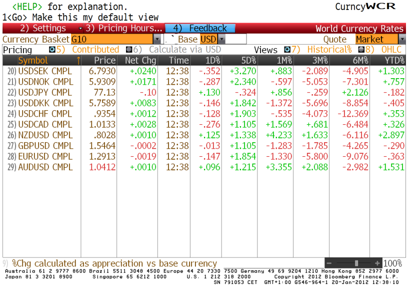
<HELP> for explanation, <MENU> for similar functions. CurncyFXC
90<Go> to Restore Original Defaults, XDF<Go> to Set Default Pricing Source.
99) Currency Group Currency Rates Matrix : Majors
10) Spot 11) Forward 12) Fixing
2) Last Price Rate 1 Month Local Time Latest Fixings as of 12:30 Date 01/20/12
3) % Change 4) Outrights 5) Points 6) Heat Map Source BFIX

	USD	EUR	JPY	GBP	CHF	CAD	AUD	NZD	HKD	NOK	SEK
SEK	6.8042	8.7856	8.8059	10.517	7.2760	6.7089	7.0579	5.4526		1.1458	-
NOK	5.9382	7.6674	7.6877	9.1781	6.3499	5.8550	6.1596	4.7585		-	.87273
HKD		10.020					8.0457	6.2167	-		
NZD		1.6115	.01618	1.9290	1.3347	1.2305	1.2942	-	.16086	.21015	.18340
AUD		1.2448	1.2452	1.4901	1.0309	.95055	-	.77270	.12430	.16235	.14169
CAD	1.0142	1.3095	1.3137	1.5676	1.0845	-	1.0520	.81268		.17079	.14906
CHF	.93516	1.2075	1.2126	1.4454	-	.92206	.97002	.74925	12.052	15.773	13.770
GBP		.83540	.83841	-	.69186	.63793	.67112	.51842		.10895	.09509
JPY	77.130	99.590	-	119.21	82.478	76.049	80.006	61.798	9.9382	12.989	11.336
EUR		-	1.0039	1.1970	.82817	.76362	.80335	.62053		.13042	.11382
USD	-	1.2912		1.5456			1.0373	.80124			

% Change on Day Range
Below -2.5% -0.5% to -2.5% -0.05% to -0.5% -0.05% to 0.05% 0.05% to 0.5% 0.5% to 2.5% Above 2.5%

Australia 61 2 9777 8600 Brazil 5511 3048 4500 Europe 44 20 7330 7500 Germany 49 69 9204 1210 Hong Kong 852 2977 6000
Japan 81 3 3201 8900 Singapore 65 6212 1000 U.S. 1 212 318 2000 Copyright 2012 Bloomberg Finance L.P.
SN 791053 CET GMT+1:00 G546-964-1 20-Jan-2012 12:37:35

Bloomberg G10



<HELP> for explanation. CurncyWCR

1<Go> Make this my default view

2) Settings 3) Pricing Hours... 4) Feedback World Currency Rates

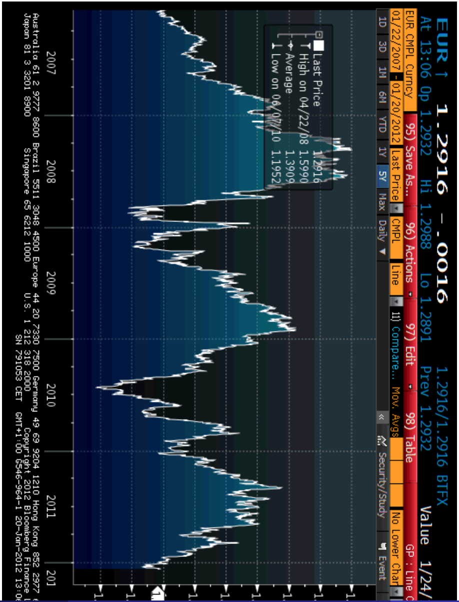
Currency Basket G10 Base USD Quote Market

Pricing 5) Contributed 6) Calculate via USD Views 7) Historical% 8) OHLC

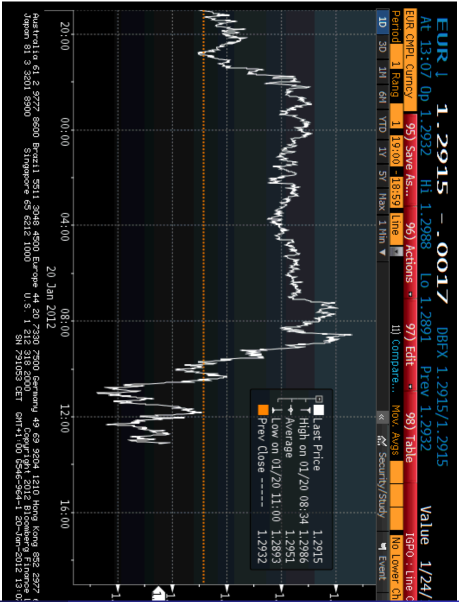
Symbol	Price	Net Chg	Time	1D%	5D%	1M%	3M%	6M%	YTD%
20) USDSEK CMPL	6.7930	+.0240	12:38	-.352	+3.270	+.883	-2.089	-4.905	+1.303
21) USDNOK CMPL	5.9309	+.0171	12:38	-.287	+2.340	-.597	-5.053	-7.301	+.757
22) USDJPY CMPL	77.13	-.10	12:38	+.130	-.324	+.856	-.259	+2.126	-.182
23) USDDKK CMPL	5.7589	+.0083	12:38	-.146	+1.842	-1.372	-5.696	-8.854	-.405
24) USDCHF CMPL	.9354	+.0012	12:38	-.128	+1.903	-.535	-4.073	-12.369	+.353
25) USDCAD CMPL	1.0133	+.0028	12:38	-.276	+1.105	+1.569	+.681	-6.484	+.326
26) NZDUSD CMPL	.8028	+.0010	12:38	+.125	+1.338	+4.233	+1.633	-6.116	+2.897
27) GBPUSD CMPL	1.5464	-.0002	12:38	-.013	+1.105	-1.283	-1.785	-4.265	-.290
28) EURUSD CMPL	1.2913	-.0019	12:38	-.147	+1.854	-1.330	-5.800	-9.076	-.363
29) AUDUSD CMPL	1.0412	+.0010	12:38	+.096	+1.215	+3.355	+2.088	-2.982	+1.531

9) %Chg calculated as appreciation vs base currency 100%

Bloomberg EURUSDGP



Bloomberg EURUSDGIP



ForEx Market , V

- We can define several FX rates: cash rate $X(t)$, spot rate $S(t)$, forward rate $F(t,T)$
- Cash rate is an artificial concept which is used in order to define $F(t,T)$ and $S(t)$ via the interest rate parity

$$F(t,T) = X(t)\frac{P_f(t,T)}{P_d(t,T)}$$

- This can be established by eliminating arbitrage. Carry trades exploit the fact spot tends to deviate from forward
- Now we can define $S(t)$

$$S(t) = F(t, t+\delta(t)),$$

where $\delta(t)$ is the delivery lag (typically 2 business days)

- The spread

$$F(t,T) - S(t) = F(t,T) - F(t, t+\delta(t)) \approx S(t)(r_d - r_f)(T - t -$$

is called "forward points"

ForEx Market VI

- FX forward is the most basic hedging instrument which is used to cover FX needs which occur in the future as well as for speculation
- A combination of two opposite FX forwards is known as FX swap (not to be confused with cross-currency swap)
- FX futures play the role similar to forwards but they trade electronically on exchanges such as CME, Euronext, Tokyo, etc.
- Since they are maked-to-market daily, they differ from forwards due to convexity effects. However, these effects are often ignored.

ForEx Options I

- The market for FX calls and puts is the largest options market in the world. The majority of these options are European. They all trade OTC.
- A typical European option gives the buyer the right (but not an obligation) to buy currency YYY (sell currency XXX) at an agreed rate K (expressed in terms of XXX/YYY) an an agreed date T. The price can be quoted in pips and in percentage and in both currencies!
- This is a YYY call (XXX put). It will be exercised if $S(T) > K$
- As usual, money is paid with delay $\delta(T)$
- Since market is global and options eventually become very short, one has to specify the place and the exact time for settling the option
- The cutoff time is as follows: Sydney 3pm, Tokyo 3pm, London 3pm, NY 10 am. It is clear that good systems for trading options have to interpret time in hours (but they often don't!)

Black-Scholes (BS) paradigm

BSM dynamics (1973) is the standard GBM:

$$\frac{dS_t}{S_t} = \left(r^d - r^f\right) dt + \sigma dW_t.$$

BS equation for the call price (per unit notional):

$$C_t^d + \left(r^d - r^f\right) S C_S^d + \frac{1}{2}\sigma^2 S^2 C_{SS}^d - r^d C^d = 0, \quad C^d\left(T, S\right) = \left(S - K\right)_+ .$$

BSM (Garman-Kolhagen) formula:

$$C^d\left(t, S_0, T, K; \sigma, r\right) = e^{-r^f \tau} S \Phi\left(d_+\right) - e^{-r^d \tau} K \Phi\left(d_-\right), \quad \Delta = e^{-r^f \tau} \Phi\left(d_+\right),$$

$$d_\pm = \frac{\ln\left(S_0/K\right) + \left(r^d - r^f\right)\tau}{\sigma\sqrt{\tau}} \pm \frac{\sigma\sqrt{\tau}}{2},$$

$\tau = T - t$. This is the price is domestic currency $\left(C^d\right)$. The price in foreign currency is $C^f = C^d/S$

Black-Scholes (BS) paradigm

Alternative representation of the BS formula (Lipton (2000)):

$$C^d\left(0, S_0, T, K; \sigma, r\right) = e^{-r^f T} S_0 - \frac{e^{-r^d T} K}{2\pi} \int\limits_{-\infty}^{\infty} \frac{e^{\left(iu + \frac{1}{2}\right)\left(\ln\left(S_0/K\right) + \left(r^d - r^f\right)T\right) - \frac{\sigma^2}{2}}}{\left(u^2 + \frac{1}{4}\right)}$$

This formula is derived by representing payoff of a call option in the form

$$\left(S - K\right)_+ = S - \min\left\{S, K\right\},$$

and dealing with the bounded component of the payout by changing the measure.

This expression has very useful generalization known as the Lewis-Lipton formula which has important implications.

Implied volatility

"A wrong number which is substituted in a wrong formula to get the right price"

$$C^{(Mrkt)}(T,K) = C^{(BS)}\left(0, S_0, T, K; \sigma_{imp}(T,K), r^d, r^f\right).$$

Idealized equity market

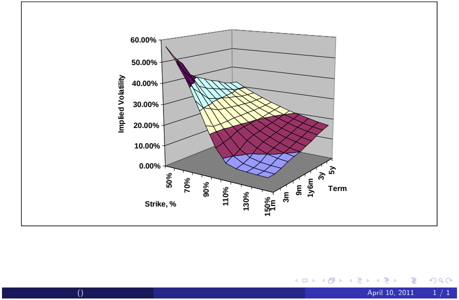


Figure:

Real equity market

k\t	0.025	0.101	0.197	0.274	0.523	0.772	1.769	2.267	2.784	3.781	4.778	5.774
51.31									33.66	32.91		
58.64									31.78	31.29	30.08	
65.97									30.19	29.76	29.75	
73.30									28.63	28.48	28.48	
76.97				32.62	30.79	30.01	28.43					
80.63				30.58	29.36	28.76	27.53	27.13	27.11	27.11	27.22	28.09
84.30				28.87	27.98	27.50	26.66					
86.13	33.65											
87.96	32.16	29.06	27.64	27.17	26.63	26.37	25.75	25.55	25.80	25.85	26.11	26.93
89.97	30.43	27.97	26.72									
91.63	28.80	26.90	25.78	25.57	25.31	25.19	24.97					
93.46	27.24	25.90	24.89									
95.29	25.86	24.88	24.05	24.07	24.04	24.11	24.18	24.10	24.48	24.69	25.01	25.84
97.12	24.66	23.90	23.29									
98.96	23.58	23.00	22.53	22.69	22.84	22.99	23.47					
100.79	22.47	22.13	21.84									
102.62	21.59	21.40	21.23	21.42	21.73	21.98	22.83	22.75	23.22	23.84	23.92	24.86
104.45	20.91	20.76	20.69									
106.29	20.56	20.24	20.25	20.39	20.74	21.04	22.13					
108.12	20.45	19.82	19.84									
109.95	20.25	19.59	19.44	19.62	19.88	20.22	21.51	21.61	22.19	22.69	23.05	23.99
111.78	19.33	19.29	19.20									
113.62				19.02	19.14	19.50	20.91					
117.28				18.85	18.54	18.88	20.39	20.58	21.22	21.86	22.23	23.21
120.95				18.67	18.11	18.39	19.90					
124.61				18.71	17.85	17.93	19.45		20.54	21.03	21.64	22.51
131.94									19.88	20.54	21.05	21.90
139.27									19.30	20.02	20.54	21.35
146.60									18.49	19.64	20.12	

Figure 1: Typical Vol Table (from Andreasen & Huge, 2011)

1

Forex Volatility Smile, I

- In equity market implied volatility exhibit a skew due to the fear of a crash
- In FX market crashes can be both sided, so we typically deal with a smile
- In equity market the concept of an option volatility is simple since prices are quoted per strike
- In FX market it is (very!) difficult since in the end of the day the prices are quoted as functions of delta

Forex Volatility Smile, II

- As we know
$$\Delta^{f}=e^{-r^{f}\tau}\Phi\left(d_{+}\right),$$
$$\Delta^{f}=e^{-r^{f}\left(T+\delta(T)-t-\delta(t)\right)}\Phi\left(d_{+}\right),$$
- It is more convenient to introduce forward delta
$$\Delta^{f,F}=\Phi\left(d_{+}\right),$$
representing the amount of foreign currency which need to be bought for forward delivery at time T for hedging purposes.
- Sadly, there are other complications such as premium-adjusted YYY delta and forward premium-adjusted delta
$$\Delta^{f,p}=\Delta^{f}-C^{f}$$
$$\Delta^{f,p,F}=e^{r^{f}\left(T+\delta(T)-t-\delta(t)\right)}\Delta^{f,p}$$
These deltas are not monotonic functions of strike.
- Market conventions are different for different currency pairs, say EURUSD, premium - USD, delta - regular, USDJPY - premium USD,

Forex Volatility Smile, III

- We start with ATM options. For such options the strike K is chosen in such a way that the sum of deltas for the corresponding call and put is equal to zero. (The corresponding deltas have to be premium-adjusted if needed).
- For non-premium-adjustded delta we have

$$d_+\left(K_{ATM}\right) = -d_+\left(K_{ATM}\right)$$

$$K_{ATM} = F\left(t, T\right) e^{\frac{1}{2}\sigma^2\tau}$$

- For premium-adjustded delta we have

$$K_{ATM} = F\left(t, T\right) e^{-\frac{1}{2}\sigma^2\tau}$$

Forex Volatility Smile, IV

- For non-ATM volatilities, the concept of risk reversals (RRs) and strangles or butterflies (STs or BFs) is needed.
- RR at a given level x is the difference of σ levels for the strikes of call and put with delta of x

$$RR\left(x\right)=\sigma\left(K_{RR}^{C}\left(x\right)\right)-\sigma\left(K_{RR}^{P}\left(x\right)\right)$$

- This definition is hard to deal with since it is sightly circular.
- In addition, if deltas are premium-adjusted, non-monototnicity comes into play.
- It is clear that additional information is needed.

Forex Volatility Smile, V

- This information is provided by strangles or butterflies (BFs), more specifically, market strangles
- These strangles are defined by the so-called "strangle volatility offset" ST
- We compute $\sigma_{ST} = \sigma_{ATM} + ST$; find K_{ST}^{C} and K_{ST}^{P} corresponding to $\Delta = \pm x$ with volatility σ_{ST}; compute the sum of the prices $C + P$ and require this sum to be equal to the sum of market prices of call and put with the given Δ.
- Thus, we need to know FOUR vols $\sigma\left(K_{RR}^{C}\right), \sigma\left(K_{RR}^{P}\right), \sigma\left(K_{ST}^{C}\right), \sigma\left(K_{ST}^{P}\right)$, in addition to $\sigma_{ATM}\left(K_{ATM}^{P}\right)$ for which we have TWO condition. It is clear that we have to choose some parametric form for $\sigma\left(K\right)$ to do the calibration.
- For simplicity, people deal with the so-called text-book strangles by assuming that $K_{RR}^{C} = K_{ST}^{C}, K_{RR}^{P} = K_{ST}^{P}$. This gives two conditions for two unknowns, but this is NOT what is done in practise.

Idealized forex market

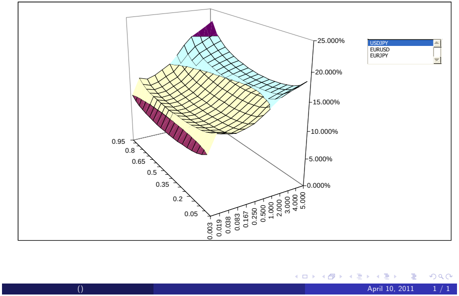
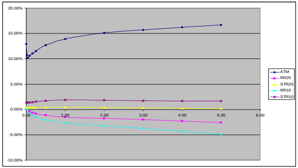


Figure:

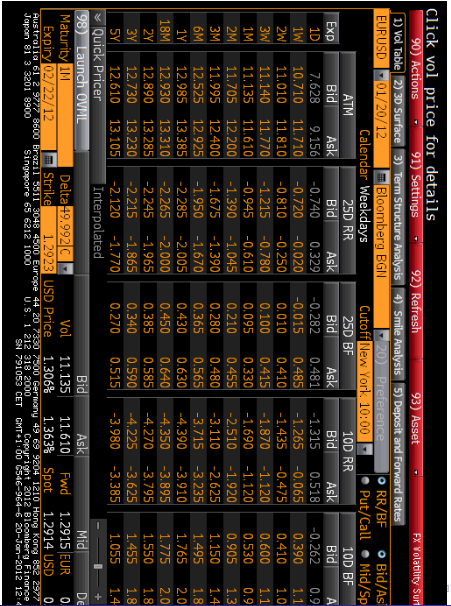
Real forex market

Maturity	USDJPY ATM	RR25	STR25	RR10	STR10
ON	12.9100%	0.0000%	0.3800%	0.0165%	1.3649%
1w	10.7100%	0.0000%	0.3800%	0.0364%	1.3647%
2w	10.2100%	0.0000%	0.3800%	0.0489%	1.3624%
1m	10.6000%	-0.4000%	0.3800%	-0.6947%	1.3864%
2m	11.0700%	-0.6500%	0.3900%	-1.1380%	1.4518%
3m	11.5500%	-0.8400%	0.4100%	-1.4937%	1.5574%
6m	12.7100%	-1.1500%	0.4300%	-2.0758%	1.7080%
1y	13.9000%	-1.5000%	0.4500%	-2.6154%	1.9199%
2y	15.1000%	-1.7500%	0.3900%	-3.1909%	1.8453%
3y	15.7000%	-2.0000%	0.3200%	-3.7547%	1.7464%
4y	16.2000%	-2.3000%	0.2400%	-4.3359%	1.6692%
5y	16.7000%	-2.6000%	0.1700%	-4.8962%	1.6445%

Real forex market



Bloomberg EURUSDData



Click vol price for details

90) Actions 91) Settings 92) Refresh 93) Asset FX Volatility Sur

1) Vol Table 2) 3D Surface 3) Term Structure Analysis 4) Smile Analysis 5) Deposit and Forward Rates

EURUSD 01/20/12 Bloomberg BGN 20) Preference RR/BF Bid/As

Calendar Weekdays Cutoff New York 10:00 Put/Call Mid/Sp

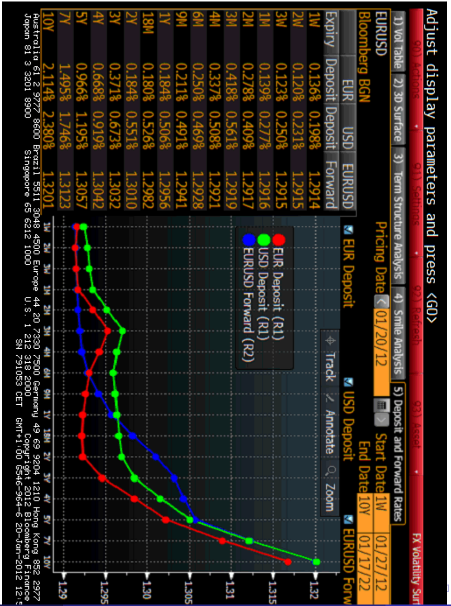
Exp	ATM Bid	ATM Ask	25D RR Bid	25D RR Ask	25D BF Bid	25D BF Ask	10D RR Bid	10D RR Ask	10D BF Bid	10D BF A
1D	7.628	9.156	-0.740	0.329	-0.282	0.481	-1.315	0.518	-0.262	0.9
1W	10.710	11.710	-0.720	-0.020	-0.015	0.485	-1.265	-0.065	0.390	1.1
2W	11.010	11.810	-0.810	-0.250	0.010	0.410	-1.435	-0.475	0.410	1.0
3W	11.140	11.770	-1.215	-0.780	0.100	0.415	-1.870	-1.120	0.600	1.1
1M	11.135	11.610	-0.945	-0.610	0.095	0.330	-1.690	-1.120	0.530	0.9
2M	11.705	12.200	-1.390	-1.045	0.210	0.455	-2.510	-1.920	0.905	1.3
3M	11.995	12.400	-1.675	-1.390	0.280	0.480	-3.110	-2.625	1.150	1.4
6M	12.525	12.925	-1.950	-1.670	0.365	0.565	-3.715	-3.235	1.495	1.8
1Y	12.985	13.385	-2.285	-2.005	0.430	0.630	-4.390	-3.910	1.765	2.0
18M	12.930	13.310	-2.265	-2.000	0.450	0.640	-4.350	-3.895	1.775	2.0
2Y	12.890	13.285	-2.245	-1.965	0.385	0.585	-4.270	-3.795	1.550	1.8
3Y	12.730	13.230	-2.215	-1.865	0.340	0.590	-4.225	-3.625	1.455	1.8
5Y	12.610	13.105	-2.120	-1.770	0.270	0.515	-3.980	-3.385	1.055	1.4

Quick Pricer Interpolated

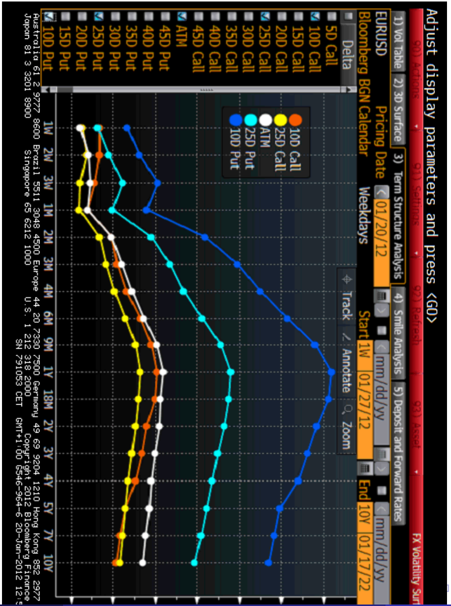
98) Launch OVML

			Bid	Ask	Mid	De
Maturity 1M	Delta 49.992 C	Vol	11.135	11.610	Fwd 1.2915 EUR	0.
Expiry 02/22/12	Strike 1.2923	USD Price	1.306%	1.363%	Spot 1.2914 USD	0.

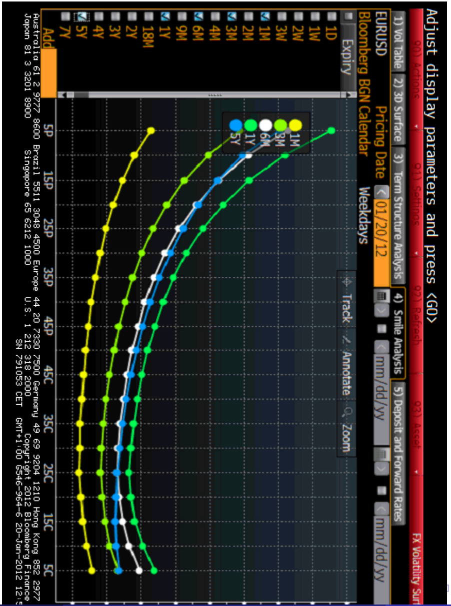
Bloomberg EURUSDForward



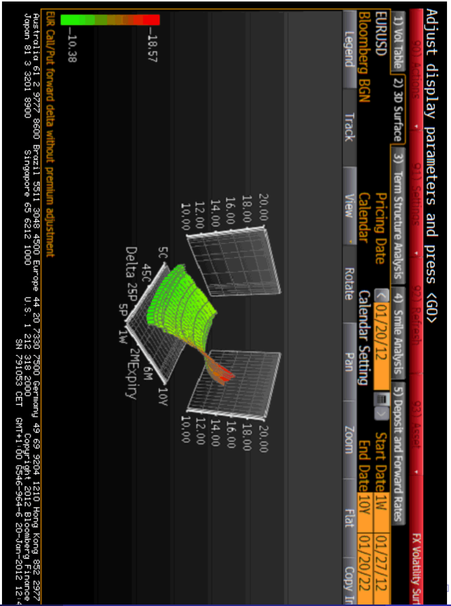
Bloomberg EURUSDTenor



Bloomberg EURUSDSmile



Bloomberg EURUSDSurface



Imagination versus reality

Implied volatility set or implied volatility surface.
Vol versus strike or vol versus delta? And if so, which delta?
How to preserve no arbitrage condition? And what should it be?
At the very least we have to have

$$C_T(T,K) \geq 0, \quad C_{KK}(T,K) \geq 0.$$

LECTURE III

Something needs to be done, but what?

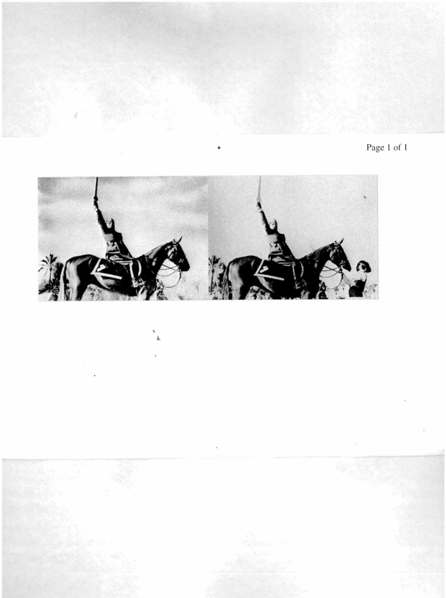
It is clear that we need to alter the basic premises of the BS theory.
Several possibilities present themselves:
(A) Parametric local volatility;
(B) Non-parametric local volatility;
(C) Stochastic volatility;
(D) Jumps;
(E) Regime switching;
(F) Various combinations of the above.

Why history of mathematical finance is difficult to study?

Mathematical finance is a scientific discipline dealing with unpredictable future. Sadly, as we shall see shortly, it also has unpredictable past.
Participants refuse to acknowledge earlier contributions. Typical excuses:
(A) Academics - "We do not subscribe to Risk";
(B) Practitioners - "We don't give a damn";
(C) Software providers - "We are just nuts and bots people, leave us alone".
All three camps play the "disappearing commissar" game thinking that they can get away with it.

Disappearing commissar I (Stalin)

Disappearing commissar II (Mussolini)



Replacing GBM with a different process was the earliest approach. In fact, it far predates BS approach. For instance, Bachelier (1900) postulated that stock price is governed by AMB:

$$\frac{dS_t}{S_t} = rdt + \frac{\sigma}{S_t} dW_t.$$

Later, several other possibilities have been considered, notably, CEV (Cox (1975), Cox & Ross (1976), Emanuel & MacBeth (1982)),

$$\frac{dS_t}{S_t} = rdt + \sigma S_t^{\beta-1} dW_t,$$

displaced diffusion (Rubinstein (1983)),

$$\frac{dS_t}{S_t} = rdt + \sigma \frac{(S_t + \beta)}{S_t} dW_t,$$

hyperbolic diffusion (Lipton (2000)),

$$\frac{dS_t}{S_t} = rdt + \sigma \left(\alpha S_t + \beta + \frac{\gamma}{S_t} \right) dW_t,$$

Non-parametric local volatility

In general, the so-called alternative stochastic processes do not match market prices exactly (although in many cases they come quite close). Accordingly, an idea to consider processes with unknown local volatility to be calibrated to the market somehow had been proposed by several researchers (Derman & Kani (1994), Dupire (1994), Rubinstein (1994)). The corresponding dynamics is

$$\frac{dS_t}{S_t} = rdt + \sigma_{loc}\left(t, S_t\right) dW_t.$$

The first and the third approaches were formulated via implied trees, while the second one in terms of PDEs. We discuss the actual calibration and the associated pitfalls shortly.

Stochastic volatility

Alternatively, several researchers suggested that volatility itself is stochastic. Several choices have been discussed in the literature:
(A) Hull & White (1988) model:

$$\frac{d\sigma_t}{\sigma_t} = \alpha dt + \gamma dW_t;$$

(B) Scott (1987) and Wiggins (1987) model:

$$\frac{d\sigma_t}{\sigma_t} = (\alpha - \beta\sigma_t)\, dt + \gamma dW_t;$$

(C) Stein & Stein (1991) model:

$$d\sigma_t = (\alpha - \beta\sigma_t)\, dt + \gamma dW_t;$$

Stochastic volatility

(D) Heston (1993) model:

$$d\sigma_t = \left(\frac{\alpha}{\sigma_t} - \beta\sigma_t\right) dt + \gamma dW_t;$$

(E) Lewis (2000) model:

$$\frac{d\sigma_t}{\sigma_t^2} = \left(\frac{\alpha}{\sigma_t} - \beta\sigma_t\right) dt + \gamma dW_t.$$

Occasionally, it is more convenient to deal with variance $v_t = \sigma_t^2$. The most popular assumption is that variance is driven by a square-root (Feller (1952)) process (Heston (1993)):

$$dv_t = \kappa\left(\theta - v_t\right) dt + \varepsilon\sqrt{v_t} dW_t.$$

My personal favorite model is Stein-Stein. The reasons are partly practical (easy to simulate) and partly sentimental (Kelvin wave analogy). Usual objections (negative vol) are irrelevant (vol is not a sign definite quantity).

Stochastic volatility

More recently Bergomi (2005) proposed to use HJM-style equations for stochastic volatility. Closer inspection suggests (to me?) that his model is more or less equivalent to Scott and Wiggins model.

Jump-diffusion based models

Meton (1976) proposed to add jumps to the standard BS dynamics:

$$\frac{dS_t}{S_t} = (r - \lambda m)\, dt + \sigma dW + \left(e^J - 1\right) dN,$$

where N is the Poisson process with intensity λ, and $m = \mathbb{E}\left\{e^J - 1\right\}$. Merton considered Gaussian distribution of jumps. Other distributions, such as exponential (Kou (2002) and others) and hyper-exponential (Lipton (2002)) have been popular as well. In reality though, it is exceedingly difficult to distinguish between different distributions, so that discrete one is perfectly adequate for many applications.

Lévy process based models

Lévy process based models had been popularized by many researchers, for instance, Boyarchenko & Levendorsky (2000, 2002), Carr & Wu (2003, 2003), Cont and Tankov (2004), Eberlein (1995), and many others. These models assume that S_t is an exponential Lévy process of the jump-diffusion type

$$S_t = S_0 e^{X_t},$$

where

$$dX_t = \gamma dt + \sigma dW_t + \int_{\mathbb{R}} (e^x - 1) \left(\mu \left(dt, dx \right) - \nu \left(dx \right) dt \right), \quad X_0 = 0$$

Here the random measure $\mu(dt, dz)$ counting jumps in dz over the time-interval dt, must, from the properties of time-homogenous Lévy processes, have the form $dt \times \mu(dz)$, with expectation $dt \times v(dz)$.

Regime switching

Regime switching models have been less popular. However, some of them are quite good. For instance, ITO33 model which they modestly call nobody's model (ITO 33 (2004)) looks rather appealing.

Composite models

Each of the models considered above has its own attractions (as well as drawbacks). Hence several researchers tried to build combined models. The need for such models is particularly strong in forex market because in this markets several exotics are liquid and can be used for calibration purposes.
In particular, a class of the so-called LSV models was developed by Jex, Henderson, Wang (1999), Blacher (2001), Lipton (2002). The first model is tree-based, the other two are PDE based. The corresponding dynamics has the form

$$\begin{aligned}
\frac{dS_t}{S_t} &= rdt + \sqrt{v_t}\sigma\left(t, S_t\right) dW_t, \\
dv_t &= \kappa\left(\theta - v_t\right) dt + \varepsilon\sqrt{v_t} dZ_t, \\
dW_t dZ_t &= \rho dt.
\end{aligned}$$

Composite models

Blacher (2001) assumes that log-normal volatility is quadratic:

$$\sigma\left(t, S_t\right) = \left(a + bS_t + cS_t^2\right).$$

Lipton (2002) considers hyperbolic log-normal volatility:

$$\sigma\left(t, S_t\right) = \left(\frac{a}{S_t} + b + cS_t\right),$$

as well as purely non-parametric one.
Jäckel, Kahl (2010) consider other interesting possibilities.
Lipton's model is offered commercially by Murex (without proper acknowledgement).
We shall discuss its efficient implementation below.

SABR model

A very different composite model is proposed by Hagan *et al.* (2002) (the so-called SABR model). The corresponding dynamics is

$$\begin{aligned} \frac{dF_t}{F_t} &= \sigma_t F_t^{\beta-1} dW_t, \\ d\sigma_t &= \nu \sigma_t dZ_t, \\ dW_t dZ_t &= \rho dt. \end{aligned}$$

Although this model has several attractive features, including is scaling properties, it is clearly not dynamic in nature.

Universal model

In 2002 Lipton proposed a Universal Vol Model. This model incorporates most of the attractive features of the models considered so far. The corresponding dynamics has the form

$$
\begin{aligned}
\frac{dS_t}{S_t} &= (r - \lambda m)\, dt + \sqrt{v_t}\sigma_{loc}(t, S_t)\, dW_t + \left(e^J - 1\right) dN_t, \\
dv_t &= \kappa(\theta - v_t)\, dt + \varepsilon\sqrt{v_t} dZ_t \left(+\varphi dN_t\right).
\end{aligned}
$$

While this model is very attractive, it is very ambitious in its design and requires a lot of effort in order to be implemented properly.

Philosophical aside

Philosophical question: What kind of models are we looking for: plastic bags which assume the form of whatever goods are put into them, or cardboard boxes which can keep the form regardless and break if we put something too hard into them. Interesting ideas are developed by Aiyache (2004), and ITO 33 (2004).

Parametric local vol (such as CEV, quadratic, etc.) is a CB-style model. Might be difficult to match the market but can be good in other respects and provides a lot of insight.

Non-parametric local vol model is a PB-style model. It takes more or less arbitrary "market" prices (where it takes them from is entirely different question) and converts them into implied volatility.

Calibration of term structure model

We have to decide how to relate local vol and implied vol.
In the presence of term structure (but no skew) of the implied vol, there is a classical relation

$$\sigma_{loc}^2(T) = \frac{d\left(\sigma_{imp}^2(T)\,T\right)}{dT}$$

This formula is not as simple as it looks.

Calibration of LV models, Dupire solution

When interest rates are deterministic, Dupire (1994) shows how to combine forward Fokker-Planck equation for t.p.d. $P(t, S, T, K)$:

$$P_T - \frac{1}{2}\left(\sigma_{loc}^2 K^2 P\right)_{KK} + (rKP)_K + rP = 0,$$

with the famous Breeden & Litzenberger (1978) formula:

$$P = C_{KK},$$

in order to obtain an equation for the call prices

$$C_T(T,K) + rKC_K(T,K) - \frac{1}{2}\sigma_{loc}^2(T,K)K^2 C_{KK}(T,K) = 0,$$

$$C(0,K) = (S-K)_+ .$$

This equation is remarkable for its ruthless efficiency (it is much more economical than the BS equation).

Local vol via implied vol

Assuming that call prices $C(T, K)$ are known for all T, K (a very dubious assumption as we have seen), we can write

$$\sigma_{loc}^2(T, K) = 2\frac{C_T(T, K) + rKC_K(T, K)}{K^2 C_{KK}(T, K)}.$$

We emphasize that this approach breaks when interest rates are stochastic (the latter effect is particularly important for long-dated forex options). In this case one can use "the classic six" method of Lipton (1997) or simply solve the corresponding $2D$ or $3D$ equation which can be very time consuming.

Let us show how it can be done (in the simplest case).

Forward PDE, stochastic rates, I

Consider the following dynamics

$$\frac{dS_t}{S_t} = \left(r_{0t} + e^{-\kappa t} x_t\right) dt + \sigma_{loc}\left(t, S_t\right) dW_t, \quad dx_t = \eta e^{\kappa t} dZ_t, \quad dW_t dZ_t = \rho dt$$

The corresponding Fokker-Planck equation for $P\left(t, S, x, T, K, \xi\right)$ reads

$$\begin{aligned} P_T - \frac{1}{2}\left(\sigma_{loc}^2 K^2 P\right)_{KK} - \left(\rho\eta e^{\kappa T}\sigma_{loc} KP\right)_{K\xi} - \frac{1}{2}\left(\eta^2 e^{2\kappa T} P\right)_{\xi\xi} \\ + \left(\left(r_{0T} + e^{-\kappa T}\xi\right) KP\right)_K + \left(r_{0T} + e^{-\kappa T}\xi\right) P = 0. \end{aligned}$$

Introduce marginal distribution $Q\left(, S, x, T, K\right)$:

$$Q\left(, S, x, T, K\right) = \int_{-\infty}^{\infty} P\left(t, S, x, T, K, \xi\right) d\xi = C_{KK}\left(T, K\right).$$

Equation for Q is much simpler (but it is not closed!)

$$
\begin{aligned}
& Q_T - \frac{1}{2}\left(\sigma_{loc}^2 K^2 Q\right)_{KK} + r_{0T}\left((KQ)_K + Q\right) \\
= \quad & -e^{-\kappa T}\int_{-\infty}^{\infty}\xi\left((KP)_K + P\right)d\xi .
\end{aligned}
$$

Simple algebra yields:

$$
\begin{aligned}
& C_T(T,K) + rKC_K(T,K) - \frac{1}{2}\sigma_{loc}^2(T,K)K^2 C_{KK}(T,K) \\
= \quad & e^{-\kappa T}\int_0^{\infty}\int_{-\infty}^{\infty}\xi H\left(K' - K\right)PdK'd\xi .
\end{aligned}
$$

Forward PDE, stochastic rates, III

To find a closed equation, we perform expansion in powers of η and assume that

$$\begin{aligned}\sigma^2_{loc}\left(T,K\right) &= \sigma^2_{loc,0}\left(T,K\right)+\eta\sigma^2_{loc,1}\left(T,K\right),\\ P\left(T,K,\xi\right) &= P_0\left(T,K,\xi\right)+\eta P_1\left(T,K,\xi\right).\end{aligned}$$

It can be shown that

$$\begin{aligned}P_0\left(T,K,\xi\right) &= p_0\left(T,K\right)\delta\left(\xi\right),\\ P_1\left(T,K,\xi\right) &= p_{10}\left(T,K\right)\delta\left(\xi\right)+p_{11}\left(T,K\right)\delta'\left(\xi\right),\end{aligned}$$

and

$$\begin{aligned}\sigma^2_{loc,0}\left(T,K\right) &= \sigma^2_{loc,Dupire}\left(T,K\right),\\ \sigma^2_{loc,1}\left(T,K\right) &= -\frac{2e^{-\kappa T}\int_0^\infty H\left(K'-K\right)p_{11}\left(T,K'\right)dK'}{K^2C_{KK}\left(T,K\right)}.\end{aligned}$$

Application of Gyöngy (1986) theorem

In the most general case Dupire (1996) proved that

$$\mathbb{E}_0 \left\{ \sigma^2 \left(T, S_T, \sigma_T \right) \middle| \, S_T = K \right\} = \sigma^2_{loc} \left(T, K \right).$$

This result is useful in theory but relatively hard to use in practice directly

Local vol via implied vol

Alternatively, we can use the Dupire equation in order to find $\sigma_{loc}^2(T, K)$ such that certain given market prices are matched. This approach has been taken by Avellaneda *et al.* (1997), Coleman *et al.* (1999), and, more recently, by Andreasen & Huge (2011) among many others. Successful execution of this approach is much easier said than done.
We briefly discuss given a discrete set of calibration inputs.

For computational purposes, it is more convenient to deal with covered calls $\bar{C}(T,K) = S - C(T,K)$, which solve the following problem

$$\begin{array}{c} \bar{C}_T - \frac{1}{2}\sigma_{loc}^2(T,K)K^2\bar{C}_{KK} = 0, \\ \bar{C}(0,K) = S - (S-K)_+ . \end{array} \qquad (1)$$

We introduce a new independent variable X, $X = \ln(K/S)$, and a new dependent variable $B(T,X)$, $\bar{C}(T,X) = Se^{X/2}B(T,X)$:

$$\begin{array}{c} B_T(T,X) - \frac{1}{2}v(T,X)\left(B_{XX}(T,X) - \frac{1}{4}B(T,X)\right) = 0, \\ B(0,X) = e^{X/2}\mathbf{1}_{\{X\leq 0\}} + e^{-X/2}\mathbf{1}_{\{X>0\}}, \end{array} \qquad (2)$$

where $v(T,X) = \sigma_{loc}^2\left(T, Se^X\right)$.
Its solution can be represented as follows:

$$B(T,X) = \int_{-\infty}^{\infty} G(T,X,X')B(0,X')dX',$$

where $G(T,X,X')$ is the Green's function that solves equation (2) with initial condition given by delta function: $\delta(X-X')$.
Here X is a forward variable and X' is a backward variable.

Assume that v is a piece-wise constant function of time,

$$v\left(T,X\right)=v_i\left(X\right), \qquad T_{i-1}<T\leq T_i, \qquad 1\leq i\leq I,$$

so that equation (2) can be solved by induction.
On each time interval $T_{i-1}<T\leq T_i$, $1\leq i\leq I$, the corresponding problem is represented in the form

$$\begin{array}{c} B_{i,\tau}\left(\tau,X\right)-\frac{1}{2}v_i\left(X\right)\left(B_{i,XX}\left(\tau,X\right)-\frac{1}{4}B_i\left(\tau,X\right)\right)=0, \\ B_i\left(0,X\right)=B_{i-1}\left(X\right), \end{array} \tag{3}$$

$$B_i\left(\tau,X\right)=B\left(T,X\right), \qquad \tau=T-T_{i-1}, \qquad B_{i-1}\left(X\right)=B\left(T_{i-1},X\right).$$

Induction starts with

$$B_0\left(X\right)=e^{X/2}\mathbf{1}_{\{X\leq 0\}}+e^{-X/2}\mathbf{1}_{\{X>0\}}.$$

The solution of problem (3) can be written as

$$B_i(\tau,X)=\int_{-\infty}^{\infty}G_i(\tau,X,X')B_{i-1}\left(X'\right)dX', \tag{4}$$

where G_i is the corresponding Green's function for the corresponding time

Andreasen & Huge (2010) solution

As a crude approximation, the time-derivative $\partial/\partial\tau$ can be implicitly discretized and forward problem (3) can be cast in the form

$$B_i^{AH}(X) - \frac{1}{2}(T_i - T_{i-1})\, v_i(X)\left(B_{i,XX}^{AH}(X) - \frac{1}{4}B_i^{AH}(X)\right) = B_{i-1}^{AH}(X),$$

where $B_i^{AH}(X) \approx B(T_i, X)$.

This is the approach chosen by AH in the specific case of piecewise constant $v_i(X)$.

While intuitive and relatively simple to implement, this approach is not accurate, by its very nature, and its accuracy cannot be improved.

Moreover, for every τ, $0 < \tau \le T_i - T_{i-1}$, a separate equation single step equation from time T_{i-1} to $T_{i-1} + \tau$ has to be solved. These equations are solved in isolation and are not internally consistent.

Below an alternative approach is proposed. This approach is based on representation (4); by construction it is exact in nature.

Laplace Transform

It turns out that the problem (3) can be solved exactly, rather than approximately, via the direct and inverse Laplace transform (for applications of the Laplace transform in derivatives pricing see Lipton, 2001).
After performing the direct Carson-Laplace transform

$$\hat{B}_i(\lambda, X) = \lambda \mathfrak{L}\{B_i(\tau, X)\},$$

the following Sturm-Liouville problem is obtained:

$$\begin{array}{c} \hat{B}_i(\lambda, X) - \frac{1}{2}\frac{1}{\lambda} v_i(X)\left(\hat{B}_{i,XX}(\lambda, X) - \frac{1}{4}\hat{B}_i(\lambda, X)\right) = B(T_{i-1}, X), \\ \hat{B}_i(\lambda, X) \underset{X \to \pm\infty}{\to} 0. \end{array} \tag{5}$$

It is clear that

$$B_i^{AH}(X) = \hat{B}_i\left(\frac{1}{T_i - T_{i-1}}, X\right). \tag{6}$$

Sturm-Liouville equation

It is convenient to represent equation (5) in the standard Sturm-Liouville form

$$-\hat{B}_{i,XX}(\lambda,X)+q_i^2(\lambda,X)\hat{B}_i(\lambda,X)=\left(q_i^2(\lambda,X)-\tfrac{1}{4}\right)B(T_{i-1},X),$$
$$\hat{B}_i(\lambda,X)\underset{X\to\pm\infty}{\to}0,$$

where

$$q_i^2(\lambda,X)=\frac{2\lambda}{v_i(X)}+\frac{1}{4}.$$

The corresponding Green's function $\hat{G}_i(\lambda,X,X')$ solves the following adjoint Sturm-Liouville problems

$$-\hat{G}_{i,XX}(\lambda,X,X')+q_i^2(\lambda,X)\hat{G}_i(\lambda,X,X')=\delta(X-X'),$$
$$\hat{G}_i(\lambda,X,X')\underset{X\to\pm\infty}{\to}0,$$

$$-\hat{G}_{i,X'X'}(\lambda,X,X')+q_i^2(\lambda,X')\hat{G}_i(\lambda,X,X')=\delta(X-X'),$$
$$\hat{G}_i(\lambda,X,X')\underset{X'\to\pm\infty}{\to}0. \tag{7}$$

ODE solution

To be concrete, backward problem (7) is considered and its fundamental solutions are denoted by $\hat{g}_i^{\pm}(\lambda, X')$:

$$-\hat{g}_{i,X'X'}^{\pm}(\lambda, X') + q_i^2(\lambda, X')\,\hat{g}_i^{\pm}(\lambda, X') = 0, \qquad \hat{g}_i^{\pm}(\lambda, X') \underset{X' \to \pm\infty}{\to} 0.$$

These solutions are unique (up to a constant). It is well-known (see, e.g., Lipton (2001)) that

$$\hat{G}_i(\lambda, X, X') = \frac{1}{W(\lambda)} \begin{cases} \hat{g}_i^{+}(\lambda, X)\,\hat{g}_i^{-}(\lambda, X'), & X' \leq X, \\ \hat{g}_i^{-}(\lambda, X)\,\hat{g}_i^{+}(\lambda, X'), & X' > X, \end{cases}$$

where $W(\lambda)$ is the so-called Wronskian

$$W(\lambda) = \hat{g}_i^{-}(\lambda, X)\,\hat{g}_{i,X'}^{+}(\lambda, X') - \hat{g}_i^{+}(\lambda, X)\,\hat{g}_{i,X'}^{-}(\lambda, X').$$

Summary

Once the Green's function is found, the solution of equation (5) can be represented in the form

$$\hat{B}_i(\lambda, X) = \int_{-\infty}^{\infty} \hat{G}_i(\lambda, X, X') \left(q_i^2(\lambda, X') - \frac{1}{4} \right) B_{i-1}(X')\, dX'. \qquad (8)$$

This is a generic formula in models where the Green's function $\hat{G}(\lambda, X, X')$ is known in the closed form.

The inverse Carson-Laplace transform yields $B(T_{i-1} + \tau, X)$ for $0 < \tau \leq T_i - T_{i-1}$, including $B_i(X)$.

In order to compute the integral in equation (8) it is assumed that X and X' are defined on the same grid $X_{\min} < X < X_{\max}$ and the trapezoidal rule is applied.

Once $B_i(X)$ is computed for a given $v_i(X)$, the latter function is changed until market prices are reproduced. The latter operation is non-linear in nature and might or might not be feasible. This depends on whether or not market prices are internally consistent.

Calibration problem for a tiled local volatility case

We apply the generic calibration method to the case of tiled local volatility considered by AH.
Given a *discrete* set of market call prices $C_{mrkt}\left(T_i, K_j\right)$, $0 \leq i \leq I$, $0 \leq j \leq J_i$, we consider a tiled local volatility $\sigma_{loc}\left(T, K\right)$

$$\sigma_{loc}\left(T, K\right) = \sigma_{ij}, \quad T_{i-1} < T \leq T_i, \quad \bar{K}_{j-1} < K \leq \bar{K}_j, \quad 1 \leq i \leq I, \quad 0 \leq$$
$$\bar{K}_{-1} = 0, \quad \bar{K}_j = \tfrac{1}{2}\left(K_j + K_{j+1}\right), \quad 0 \leq j \leq J_i - 1, \quad \bar{K}_{J_i} = \infty,$$

Clearly, non-median break points can be chosen if needed. Equivalently, $\sigma_{loc}\left(T, X\right)$ has the form

$$\sigma_{loc}\left(T, X\right) = v_{ij}, \quad T_{i-1} < T \leq T_i, \quad \bar{X}_{j-1} < X \leq \bar{X}_j, \quad \bar{X}_j = \ln\left(\bar{K}_j / S\right).$$

By construction, for every T_i, $\sigma_{loc}\left(T_i, K\right)$ depends on as many parameters as there are market quotes.
On every step of the calibration procedure these parameters are adjusted in such way that the corresponding model prices $C_{mdl}\left(T_i, K_j\right)$ and market prices $C_{mrkt}\left(T_i, K_j\right)$ coincide within prescribed accuracy.

Calibration problem (7) for a tiled local volatility case

For calibration it is sufficient to consider $X = X_j$, where $X_j = \ln(S_j/K)$; however, to propagate the solution forward from T_{i-1} to T_i, it is necessary to consider all X.

A new set of ordered points is introduced

$$\{Y_k\} = \{\bar{X}_j\} \cup X, \quad -1 \leq k \leq J_1 + 1, \quad Y_{-1} = -\infty, \quad Y_{J_1+1} = \infty,$$

and it is assumed that $X = Y_{k^*}$.

On each interval

$$\mathcal{J}_k = \left\{ X' \middle| \, Y_{k-1} \leq X' \leq Y_k \right\},$$

except for the first and the last one, the general solution of equation (7) has the form

$$g_k\left(X'\right) = \alpha_{k,+} e^{q_k\left(X' - Y_{k^*}\right)} + \alpha_{k,-} e^{-q_k\left(X' - Y_{k^*}\right)},$$

while on the first and last intervals it has the form

$$g_0\left(X'\right) = \alpha_{0,+} e^{q_0\left(X' - Y_{k^*}\right)}, \quad g_{J_1+1}\left(X'\right) = \alpha_{J_1+1,-} e^{-q_{J_1+1}\left(X' - Y_{k^*}\right)}$$

so that the corresponding Green's function decays at infinity. Here q_k are constant values of the potential on the corresponding interval.

Calibration problem for a tiled local volatility case

For $k \neq k^*$ both $\hat{G}$ and $\hat{G}_X$ have to be continuous, while for $k = k^*$ only $\hat{G}$ is continuous, while $\hat{G}_X$ has a jump of size -1.
Thus, the following system of $2(J_1 + 1)$ linear equations can be obtained:

$$\begin{pmatrix} -E^+_{kk} & -E^-_{kk} & E^+_{k+1k} & E^-_{k+1k} \\ -q_k E^+_{kk} & q_k E^-_{kk} & q_{k+1} E^+_{k+1k} & -q_{k+1} E^-_{k+1k} \end{pmatrix} \begin{pmatrix} \alpha_{k,+} \\ \alpha_{k,-} \\ \alpha_{k+1,+} \\ \alpha_{k+1,-} \end{pmatrix} = \begin{pmatrix} 0 \\ -\delta_{kk^*} \end{pmatrix}$$

Here

$$E^{\pm}_{kl} = e^{\pm q_k (Y_l - Y_{k^*})},$$

and δ_{kk^*} is the Kronecker symbol.

Calibration problem for a tiled local volatility case

In matrix form these equations can be written as

$$\mathcal{R}\vec{A} = \vec{B}_{k^*}, \tag{9}$$

For instance, for $J = 3$, $k^* = 1$, we have

$$\mathcal{R} = \begin{pmatrix} -E_{00}^{+} & E_{10}^{+} & E_{10}^{-} & & & & & \\ -q_0 E_{00}^{+} & q_1 E_{10}^{+} & -q_1 E_{10}^{-} & & & & & \\ & -1 & -1 & 1 & 1 & & & \\ & -q_1 & q_1 & q_2 & -q_2 & & & \\ & & & -E_{22}^{+} & -E_{22}^{-} & E_{32}^{+} & E_{32}^{-} & \\ & & & -q_2 E_{22}^{+} & q_2 E_{22}^{-} & q_3 E_{32}^{+} & -q_3 E_{32}^{-} & \\ & & & & & -E_{33}^{+} & -E_{33}^{-} & \\ & & & & & -q_3 E_{33}^{+} & q_3 E_{33}^{-} & \end{pmatrix}$$

$$\vec{A} = \begin{pmatrix} \alpha_{0,+} & \alpha_{1,+} & \alpha_{1,-} & \alpha_{2,+} & \alpha_{2,-} & \alpha_{3,+} & \alpha_{3,-} & \alpha_{4,-} \end{pmatrix}^{T},$$

$$\vec{B}_{k^*} = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}^{T},$$

Calibration problem for a tiled local volatility case

Although matrix equation (9) is five-diagonal (rather than tri-diagonal), it can still be solved very efficiently via forward elimination and backward substitution.

First, $(\alpha_{1,+}, \alpha_{1,-})^T$ is eliminated in favor of $\alpha_{0,+}$ and $(\alpha_{J,+}, \alpha_{J,-})^T$ is eliminated in favor of $\alpha_{J+1,-}$:

$$\begin{pmatrix} \alpha_{1,+} \\ \alpha_{1,-} \end{pmatrix} = \frac{1}{2q_1} \begin{pmatrix} (q_1 + q_0) E_{10}^- E_{00}^+ \\ (q_1 - q_0) E_{10}^+ E_{00}^+ \end{pmatrix} \alpha_{0,+} \equiv \vec{C}_1 \alpha_{0,+},$$

$$\begin{pmatrix} \alpha_{J,+} \\ \alpha_{J,-} \end{pmatrix} = \frac{1}{2q_J} \begin{pmatrix} (q_J - q_{J+1}) E_{JJ}^- E_{J+1J}^- \\ (q_J + q_{J+1}) E_{JJ}^+ E_{J+1J}^- \end{pmatrix} \alpha_{J+1,-} \equiv \vec{D}_J \alpha_{J+1,-},$$

Next, $(\alpha_{k,+}, \alpha_{k,-})^T$ is eliminated in favor of $(\alpha_{k-1,+}, \alpha_{k-1,-})^T$, $2 \le k \le k^*$ and $(\alpha_{k,+}, \alpha_{k,-})^T$ is eliminated in favor of $(\alpha_{k+1,+}, \alpha_{k+1,-})^T$, $k^* + 1 \le k \le J - 1$:

$$\begin{pmatrix} \alpha_{k,+} \\ \alpha_{k,-} \end{pmatrix} = \frac{1}{2q_k} \begin{pmatrix} (q_k + q_{k-1}) E_{kk-1}^- E_{k-1k-1}^+ & (q_k - q_{k-1}) E_{kk-1}^- E_{k-1k-1}^- \\ (q_k - q_{k-1}) E_{kk-1}^+ E_{k-1k-1}^+ & (q_k + q_{k-1}) E_{kk-1}^+ E_{k-1k-1}^- \end{pmatrix}$$

$$\begin{pmatrix} \alpha_{k,+} \\ \alpha_{k,-} \end{pmatrix} = \frac{1}{2q_k} \begin{pmatrix} (q_k + q_{k+1}) E^{-}_{kk} E^{+}_{k+1k} & (q_k - q_{k+1}) E^{-}_{kk} E^{-}_{k+1k} \\ (q_k - q_{k+1}) E^{+}_{kk} E^{+}_{k+1k} & (q_k + q_{k+1}) E^{+}_{kk} E^{-}_{k+1k} \end{pmatrix} \begin{pmatrix} \alpha_{k+} \\ \alpha_{k+} \end{pmatrix}$$

and a recursive set of vectors is computed

$$\vec{C}_k = \mathcal{S}_k \vec{C}_{k-1}, \quad 2 \leq k \leq k^*, \qquad \vec{D}_k = \mathcal{T}_k \vec{D}_{k+1}, \quad k^* + 1 \leq k \leq J - 1.$$

Finally, a system of 2×2 equations for $\alpha_{0,+}, \alpha_{J+1,-}$ is obtained and solved

$$\begin{pmatrix} -1 & -1 \\ -q_{k^*} & q_{k^*} \end{pmatrix} \vec{C}_{k^*} \alpha_{0,+} + \begin{pmatrix} 1 & 1 \\ q_{k^*+1} & -q_{k^*+1} \end{pmatrix} \vec{D}_{k^*+1} \alpha_{J+1,-} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \tag{10}$$

Once $\alpha_{0,+}, \alpha_{J+1,-}$ are determined, $(\alpha_{k,+}, \alpha_{k,-})^T$ are calculated by using vectors $\vec{C}_k$ or $\vec{D}_k$.

This procedure is just icing on the cake since the size of the corresponding system (determined by the number of market quotes) is quite small and is not related to the size of the interpolation grid.

Calibration problem for a tiled local volatility case

Once the coefficient vector $\vec{A}$ is known, $\hat{B}_i(\lambda, X)$ can be computed semi-analytically via equation (8):

$$\begin{aligned}\hat{B}_i(\lambda, X) = & \left(q_0^2 - \tfrac{1}{4}\right) \alpha_{0,+} e^{-q_0 Y_{k^*}} \int_{-\infty}^{Y_0} e^{q_0 X'} B_{i-1}(X')\, dX' \\ & + \textstyle\sum_{k=1}^{J_i} \left(q_k^2 - \tfrac{1}{4}\right) \left(\alpha_{k,+} e^{-q_k Y_{k^*}} \int_{Y_{k-1}}^{Y_k} e^{q_k X'} B_{i-1}(X')\, dX' \right. \\ & \left. + \alpha_{k,-} e^{q_k Y_{k^*}} \int_{Y_{k-1}}^{Y_k} e^{-q_k (X' - Y_{k^*})} B_{i-1}(X')\, dX' \right) \\ & + \left(q_{J_i+1}^2 - \tfrac{1}{4}\right) \alpha_{J_i+1,-} e^{q_{J_{i+1}} Y_{k^*}} \int_{Y_{J_1}}^{\infty} e^{-q_{J_i+1} X'} B_{i-1}(X')\, dX'.\end{aligned} \tag{11}$$

i) The corresponding integrals are computed via the trapezoidal rule.
ii) Variables X and X' are defined on the same dense spatial grid $X_{\min} < X, X' < X_{\max}$; this grid is similar to the one used in a conventional finite-difference solver.
iii) For $X' > X_{\max}$ or $X' < X_{\min}$ it is assumed that $B_{i-1}(X') = e^{-|X|/2}$.
iv) Since these integrals are independent on Y_{k^*}, they can be pre-computed for all X; thus the corresponding calculation has complexity linear in J_i.
v) For $i = 1$, the corresponding integrals can be computed analytically.

Calibration problem for a tiled local volatility case

The inverse Carson-Laplace transform generates $B(T,X)$:

$$B(T_{i-1}+\tau,X)=\mathfrak{L}_\tau^{-1}\left\{\frac{\hat{B}(\lambda,X)}{\lambda}\right\}. \tag{12}$$

This transform can be performed efficiently via the Stehfest algorithm:

$$B(T_{i-1}+\tau,X)=\sum_{k=1}^{N}\frac{St_k^N}{k}\hat{B}(k\Lambda,X), \quad \Lambda=\frac{\ln 2}{\tau}. \tag{13}$$

Choosing $N=12$ is sufficient. Coefficients St_k^{12} are given below

1	2	3	4	5	6
$-0.01(6)$	$16.01(6)$	-1247	$27554.(3)$	$-263280.8(3)$	132
7	8	9	10	11	12
$-3891705.5(3)$	$7053286.(3)$	-8005336.5	5552830.5	-2155507.2	35

It is obvious that these coefficients are very stiff.

The above procedure allows one to calculate $B(T_i,X_j)$ *for given* v_{ij}.

To calibrate the model to the market, v_{ij} are changed until model and market prices agree. It is worth noting that, as always, vectorizing

Calibration problem for a tiled local volatility case

The calibration algorithm is summarized as follows:
(A) At initialization, $B_0(X)$ given by equation (2) is computed on the spatial grid;
(B) At time T_{i+1}, given $B_i(X)$, equations (11) and (12) are used to compute $B(\lambda, X_j)$ and $B\left(T_i, X_j\right)$ at specified market strikes only; $\{v_{ij}\}$ are adjusted until model prices match market prices;
(C) After calibration at time T_{i+1} is complete, $B\left(T_i, X\right)$ is computed on the entire spatial grid using new model parameters at time T_{i+1};
(D) The algorithm is repeated for the next time slice.
If so desired, $B\left(T, X\right)$ can be calculated on the entire temporal-spatial grid with very limited additional effort.

Table: calibrated SPX index

K/T	0.101		0.197		0.274		0.523		0.772		1.769		2.267		2.784		3.781		4.778		5.774	
	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt	mdl	mkt
51.31			34.40		40.66		38.42		36.88		33.62		32.44		33.66	33.66	32.91	32.91	30.92		32.27	
58.64	35.80		34.29		39.25		36.60		35.21		32.30		31.25		31.78	31.78	31.29	31.29	30.18	30.08	31.30	
65.97	34.65		33.37		37.20		34.52		33.34		30.86		29.98		30.19	30.19	29.76	29.76	29.56	29.75	30.28	
73.30	33.80		32.10		34.39		32.17		31.22		29.30		28.63		28.63	28.63	28.49	28.48	28.54	28.48	29.22	
76.97	33.05		31.25		32.59	32.62	30.79	30.79	30.00	30.01	28.42	28.43	27.89		27.83		27.78		27.88		28.66	
80.63	32.06		30.27		30.66	30.58	29.37	29.36	28.75	28.76	27.53	27.53	27.13	27.13	27.11	27.11	27.11	27.11	27.25	27.22	28.09	28.09
84.30	30.76		29.09		28.82	28.87	27.96	27.98	27.51	27.50	26.63	26.66	26.33		26.44		26.46		26.66		27.51	
86.13	29.95		28.38		27.94		27.27		26.92		26.19		25.92		26.11		26.14		26.36		27.20	
87.96	29.10	29.06	27.65	27.64	27.16	27.17	26.64	26.63	26.38	26.37	25.79	25.75	25.55	25.55	25.80	25.80	25.85	25.85	26.10	26.11	26.93	26.93
89.97	27.90	27.97	26.66	26.72	26.26		25.88		25.71		25.30		25.11		25.42		25.49		25.77		26.60	
91.63	26.92	26.90	25.82	25.78	25.57	25.57	25.30	25.31	25.20	25.19	24.93	24.97	24.79		25.12		25.23		25.53		26.36	
93.46	25.86	25.90	24.86	24.89	24.76		24.64		24.63		24.54		24.43		24.78		24.94		25.26		26.08	
95.29	24.98	24.88	24.08	24.05	24.04	24.07	24.06	24.04	24.11	24.11	24.19	24.18	24.10	24.10	24.48	24.48	24.71	24.69	25.00	25.01	25.84	25.84
97.12	23.86	23.90	23.24	23.29	23.31		23.41		23.53		23.81		23.73		24.14		24.46		24.71		25.57	
98.96	22.98	23.00	22.58	22.53	22.69	22.69	22.83	22.84	23.00	22.99	23.47	23.47	23.40		23.82		24.26		24.45		25.33	
100.79	22.14	22.13	21.78	21.84	21.97		22.20		22.42		23.11		23.03		23.48		24.03		24.15		25.07	
102.62	21.38	21.40	21.24	21.23	21.43	21.42	21.72	21.73	21.98	21.98	22.81	22.83	22.75	22.75	23.22	23.22	23.81	23.84	23.93	23.92	24.86	24.86
104.45	20.78	20.76	20.70	20.69	20.89		21.22		21.50		22.48		22.45		22.95		23.55		23.69		24.63	
106.29	20.24	20.24	20.23	20.25	20.40	20.39	20.74	20.74	21.04	21.04	22.15	22.13	22.16		22.68		23.25		23.47		24.41	
108.12	19.84	19.82	19.86	19.84	19.95		20.28		20.59		21.81		21.87		22.43		22.95		23.24		24.19	
109.95	19.57	19.59	19.45	19.44	19.58	19.62	19.87	19.88	20.20	20.22	21.50	21.51	21.61	21.61	22.19	22.19	22.71	22.69	23.04	23.05	23.99	23.99
111.78	19.28	19.29	19.18	19.20	19.26		19.49		19.84		21.20		21.35		21.94		22.48		22.84		23.79	
113.62	19.04		18.98		19.01	19.02	19.13	19.14	19.50	19.50	20.91	20.91	21.08		21.69		22.27		22.63		23.60	
117.28	18.73		18.68		18.84	18.85	18.55	18.54	18.90	18.88	20.39	20.39	20.58	20.58	21.22	21.22	21.85	21.86	22.24	22.23	23.21	23.21
120.95	18.65		18.48		18.66	18.67	18.11	18.11	18.38	18.39	19.89	19.90	20.14		20.86		21.41		21.91		22.84	
124.61	18.92		18.34		18.71	18.71	17.85	17.85	17.92	17.93	19.45	19.45	19.79		20.54	20.54	21.03	21.03	21.62	21.64	22.51	22.51
131.94	19.85		18.19		18.72		17.59		17.32		18.77		19.22		19.88	19.88	20.54	20.54	21.06	21.05	21.90	21.90
139.27			18.12		18.61		17.47		17.00		18.29		18.79		19.30	19.30	20.02	20.02	20.53	20.54	21.35	21.35
146.60			17.31		18.63		17.43		16.83		17.92		18.44		18.49	18.49	19.64	19.64	20.12	20.12	20.90	

Figure: Market and model implied volatilities

Illustrations

For illustrative purposes a tiled local volatility model is calibrated to SX5E equity volatility data as of 01/03/2010. This data is taken from AH. Depending on maturity, one needs up to 13 tiles to be able to calibrate the model to the market.

In Figure 1 model and market implied volatilities for the index are shown graphically, while in Table 1 the same volatilities are presented numerically. In Figure 2 the calibrated tiled local volatility is shown. In Figure 3, the Laplace transforms of the Green's functions $\hat{G}_1(\lambda, X_j, X')$ are shown as functions of X' for fixed λ. In Figure 4, the Laplace transforms of option prices $\hat{B}(\lambda, X_j)$ are shown as functions of λ. In Figure 5, functions $B(T_i, X)$ are shown as functions of X. In all these Figures model parameters calibrated to the SX5E volatility surface are used.

Market and model implied vol

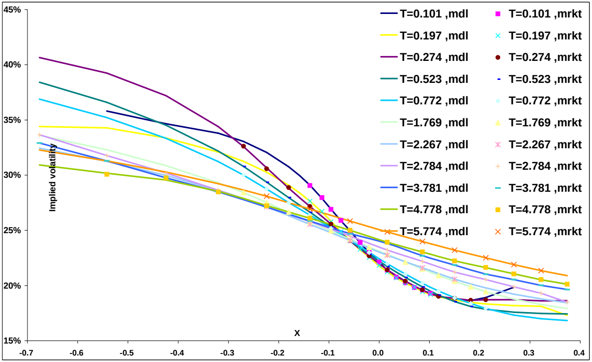


Figure: Market and model SX5E implied volatility quotes for March 1, 2010.

Calibrated local vol

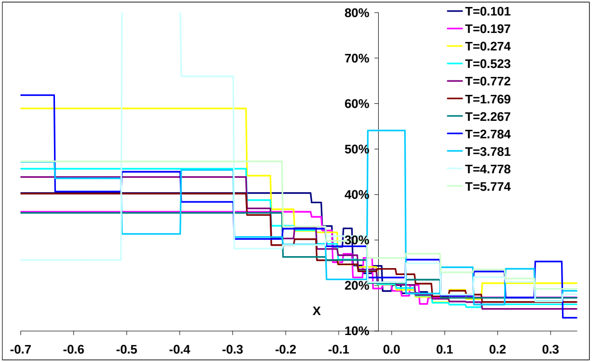


Figure: Calibrated local volatility for March 1, 2010.

Laplace Transforms as functions of

X

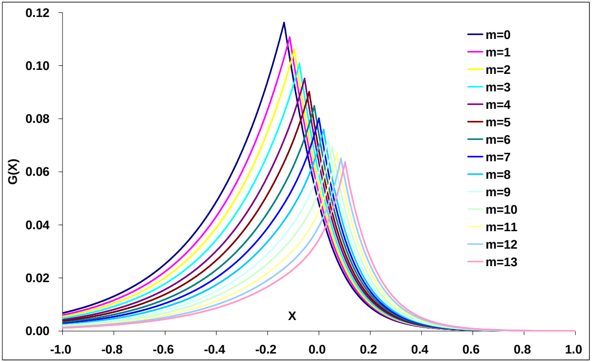


Figure: The Laplace transforms of the Green's functions $\hat{G}_1 \left(\lambda, X_j, X'\right)$ as functions of X', where $\lambda = 1$ and $X = X_j$, $0 < j < 13$, are given in Table 1.

Laplace Transforms as functions of λ

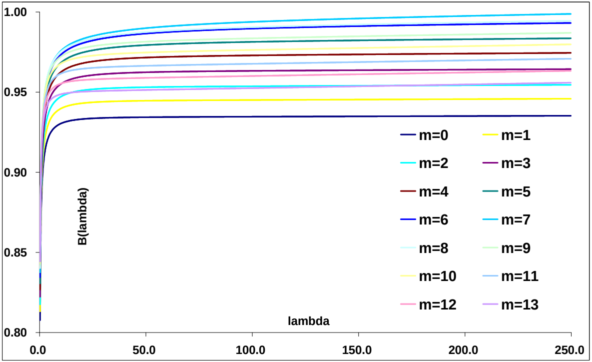


Figure: Option prices $\hat{B}(\lambda, X_j)$ as functions of λ, where X_j are given in Table 1.

Option prices

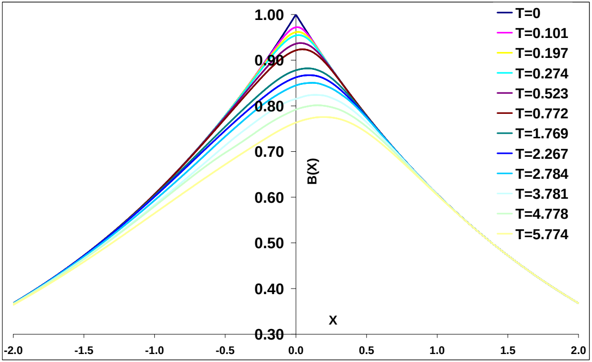


Figure: Option prices $B(T_i, X)$ as functions of X, where T_i are given in Table 1.

LECTURE III

Consider the one-tile case with $\sigma = \sigma_0$, which is the classical Black-Scholes case. In this case matrix equation (10) is trivial

$$\begin{pmatrix} -1 \\ -q_0 \end{pmatrix} \alpha_{0,+} + \begin{pmatrix} 1 \\ -q_0 \end{pmatrix} \alpha_{1,-} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad q_0 = \sqrt{\frac{2\lambda}{\sigma_0^2} + \frac{1}{4}}.$$

Accordingly, $\alpha_{0,+} = \alpha_{1,-} = 1/2q_0$.

The Green's function $\hat{G}(\lambda)$ and the corresponding option price $\hat{B}(\lambda)$ have the form

$$\hat{G}(\lambda, X, X') = \frac{e^{-q_0|X-X'|}}{q_0}, \quad \hat{B}(\lambda, X) = e^{-\frac{|X|}{2}} - \frac{e^{-q_0|X|}}{2q_0}.$$

The inverse Carson-Laplace transform of $\hat{B}(\lambda)$ yields $B(T)$:

$$B(T, X) = e^{-\frac{|X|}{2}} \Phi\left(\frac{|X| - \frac{\sigma_0^2 T}{2}}{\sqrt{\sigma_0^2 T}}\right) + e^{\frac{|X|}{2}} \Phi\left(-\frac{|X| + \frac{\sigma_0^2 T}{2}}{\sqrt{\sigma_0^2 T}}\right).$$

Here $\Phi(\xi)$, $\phi(\xi)$ are the cumulative density and density of the standard Gaussian variable.

While the above transforms can be computed in a closed form, in multi-tile case it is not possible.
Accordingly, an approximation valid for $\lambda \to \infty$ is useful:

$$q_0 \approx \frac{\zeta}{\sigma_0} + \frac{\sigma_0}{8\zeta}, \qquad e^{-q_0|X|} \approx e^{-\frac{\zeta}{\sigma_0}|X|}\left(1 - \frac{\sigma_0 |X|}{8\zeta}\right),$$

$$\hat{B}_a(\lambda, X) \approx e^{-\frac{|X|}{2}} - \frac{\sigma_0 e^{-\frac{\zeta}{\sigma_0}|X|}}{2\zeta} + \frac{\sigma_0^2 |X| e^{-\frac{\zeta}{\sigma_0}|X|}}{16\zeta^2},$$

where $\zeta = \sqrt{2\lambda}$. It is well-known that for $z < 0$

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{e^{\zeta z}}{\zeta^3}\right) &= \sqrt{T}\Psi_3\left(\frac{z}{\sqrt{T}}\right), \\ \mathcal{L}^{-1}\left(\frac{e^{\zeta z}}{\zeta^4}\right) &= T\Psi_4\left(\frac{z}{\sqrt{T}}\right), \end{aligned} \tag{14}$$

where

$$\begin{aligned} \Psi_3(\xi) &= \xi\Phi(\xi) + \phi(\xi), \\ \Psi_4(\xi) &= \tfrac{1}{2}\left(\left(\xi^2+1\right)\Phi(\xi) + \xi\phi(\xi)\right). \end{aligned}$$

Equation (14) shows that the inverse Carson-Laplace transform of $\hat{B}_a(\lambda)$ yields an approximate option price:

$$B_a(T,X) = e^{-\frac{|X|}{2}} - \sigma_0\sqrt{T}\Psi_3\left(-\frac{|X|}{\sigma_0\sqrt{T}}\right) + \frac{\sigma_0^2 T|X|}{8}\Psi_4\left(-\frac{|X|}{\sigma_0\sqrt{T}}\right). \quad (15)$$

Exact and approximate implied volatilities for several representative maturities are shown in Figure 6.

It is clear that exact implied volatility is equal to σ_0.

This Figure shows that the above approximation is reasonably accurate provided that $\sigma_0^2 T$ is sufficiently small.

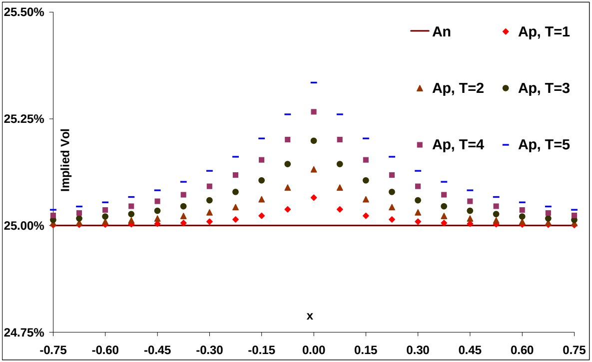


Figure: Implied volatility given by equation (15) with $\sigma_0 = 25\%$ vs. exact implied volatility σ_0.

Two-tile case

Consider two-tiled case:

$$\sigma(X) = \begin{cases} \sigma_0, & X \leq \bar{X}, \\ \sigma_1 & X > \bar{X}. \end{cases}$$

For concreteness, the case when $X < \bar{X}_0$ is considered. In the case in question equation (9) has the form

$$\begin{pmatrix} -1 & 1 & 1 & 0 \\ -q_0 & q_0 & -q_0 & 0 \\ 0 & -e^{q_0(\bar{X}_0 - X)} & -e^{-q_0(\bar{X}_0 - X)} & e^{-q_1(\bar{X}_0 - X)} \\ 0 & -q_0 e^{q_0(\bar{X}_0 - X)} & +q_0 e^{-q_0(\bar{X}_0 - X)} & -q_1 e^{-q_1(\bar{X}_0 - X)} \end{pmatrix} \begin{pmatrix} \alpha_{0,+} \\ \alpha_{1,+} \\ \alpha_{1,-} \\ \alpha_{2,-} \end{pmatrix} = ($$

so that

$$\begin{pmatrix} \alpha_{0,+} & \alpha_{1,+} & \alpha_{1,-} & \alpha_{2,-} \end{pmatrix}$$

$$= \left(\frac{1}{2q_0} + \frac{(q_0 - q_1)}{2q_0(q_0 + q_1)} e^{-2q_0(\bar{X}_0 - X)} \quad \frac{(q_0 - q_1)}{2q_0(q_0 + q_1)} e^{-2q_0(\bar{X}_0 - X)} \quad \frac{1}{2q_0} \quad \frac{1}{(q_0 + q_1)} e^{(q_1 -} \right.$$

Two-tile case

Accordingly, when $X > \bar{X}_0$, $\hat{G}(\lambda, X, X')$ can be written as

$$\hat{G}(\lambda, X, X') = \begin{cases} \frac{1}{2q_0} e^{-q_0|X'-X|} + \frac{(q_0-q_1)}{2q_0(q_0+q_1)} e^{q_0(X'+X-2\bar{X}_0)}, & X' \leq \bar{X}_0, \\ \frac{1}{(q_0+q_1)} e^{-q_1(X'-\bar{X}_0)-q_0(\bar{X}_0-X)}, & \bar{X}_0 < X'. \end{cases}$$

By the same token, when $X > \bar{X}_0$, it can be written as

$$\hat{G}(\lambda, X, X') = \begin{cases} \frac{1}{(q_0+q_1)} e^{q_1(\bar{X}_0-X)+q_0(X'-\bar{X}_0)}, & X' \leq \bar{X}_0, \\ \frac{1}{2q_1} e^{-q_1|X'-X|} + \frac{(q_1-q_0)}{2q_1(q_1+q_0)} e^{-q_1(X'+X-2\bar{X}_0)}, & \bar{X}_0 < X'. \end{cases}$$

As expected, $\hat{G}(\lambda, X, X') = \hat{G}(\lambda, X', X)$
It can be shown that

$$\hat{B}(\lambda, X) = e^{-\frac{|X|}{2}} - \frac{(q(X)+q(0)-(q_0+q_1))e^{-q(X)|X-\bar{X}_0|-q(0)|\bar{X}_0|}}{2q(0)(q_0+q_1)} - \frac{e^{-q(X)(|X|-|\bar{X}_0|)-q(0)|\bar{X}_0}}{(q(X)+q(0))} \tag{16}$$

$$q(X) = \sqrt{\frac{2\lambda}{\sigma^2(X)} + \frac{1}{4}} = \begin{cases} \sqrt{\frac{2\lambda}{\sigma_0^2} + \frac{1}{4}}, & X \leq \bar{X}_0, \\ \sqrt{\frac{2\lambda}{\sigma_1^2} + \frac{1}{4}}, & X > \bar{X}_0. \end{cases}$$

Two-tile case

When $\lambda \to \infty$, $\hat{B}(\lambda, X)$ can be expanded as follows

$$\hat{B}(\lambda, X) = e^{-\frac{|X|}{2}}$$
$$-\frac{\left(\left(\frac{1}{\sigma(0)}+\frac{1}{\sigma(X)}\right)-\left(\frac{1}{\sigma_0}+\frac{1}{\sigma_1}\right)\right)}{\frac{2}{\sigma(0)}\left(\frac{1}{\sigma_0}+\frac{1}{\sigma_1}\right)\zeta} e^{-\left(\frac{|X-\bar{X}_0|}{\sigma(X)}+\frac{|\bar{X}_0|}{\sigma(0)}\right)\zeta}\left(1-\frac{\sigma(X)|X-\bar{X}_0|+\sigma(0)|\bar{X}_0|}{8\zeta}\right)$$
$$-\frac{1}{\left(\frac{1}{\sigma(0)}+\frac{1}{\sigma(X)}\right)\zeta} e^{-\left(\frac{|X|-|\bar{X}_0|}{\sigma(X)}+\frac{|\bar{X}_0|}{\sigma(0)}\right)\zeta}\left(1-\frac{\sigma(X)(|X|-|\bar{X}_0|)+\sigma(0)|\bar{X}_0|}{8\zeta}\right).$$

$$\hat{B}(\lambda, X) = e^{-\frac{|X|}{2}}$$
$$-\frac{\sigma(0)\left(\frac{\sigma_0\sigma_1(\sigma(0)+\sigma(X))}{\sigma(0)\sigma(X)}-(\sigma_0+\sigma_1)\right)}{2(\sigma_0+\sigma_1)\zeta} e^{-\left(\frac{|X-\bar{X}_0|}{\sigma(X)}+\frac{|\bar{X}_0|}{\sigma(0)}\right)\zeta}\left(1-\frac{\sigma(X)|X-\bar{X}_0|+\sigma(0)|\bar{X}_0|}{8\zeta}\right)$$
$$-\frac{\sigma(0)\sigma(X)}{(\sigma(0)+\sigma(X))\zeta} e^{-\left(\frac{|X|-|\bar{X}_0|}{\sigma(X)}+\frac{|\bar{X}_0|}{\sigma(0)}\right)\zeta}\left(1-\frac{\sigma(X)(|X|-|\bar{X}_0|)+\sigma(0)|\bar{X}_0|}{8\zeta}\right).$$

The inverse Carson-Laplace transform using on equation (14) yields

$$
\begin{aligned}
B\left(T,X\right) = e^{-\frac{|X|}{2}} - \sigma\left(0\right)\sqrt{T} \\
\left(\frac{(\sigma_0+\sigma_1-2\sigma(0))}{(\sigma_0+\sigma_1)}\Psi_3\left(-\left(\frac{|X-\bar{X}_0|}{\sigma(X)\sqrt{T}}+\frac{|\bar{X}_0|}{\sigma(0)\sqrt{T}}\right)\right)\right. \\
\left.+\Psi_3\left(-\left(\frac{|X|-|\bar{X}_0|}{\sigma(X)\sqrt{T}}+\frac{|\bar{X}_0|}{\sigma(0)\sqrt{T}}\right)\right)\right) \\
+\frac{\sigma(0)\sigma(X)T}{8}\left(\frac{\left(\frac{\sigma_0\sigma_1(\sigma(0)+\sigma(X))}{\sigma(0)\sigma(X)}-(\sigma_0+\sigma_1)\right)(\sigma(X)|X-\bar{X}_0|+\sigma(0)|\bar{X}_0|)}{(\sigma_0+\sigma_1)\sigma(X)}\right. \\
\Psi_4\left(-\left(\frac{|X-\bar{X}_0|}{\sigma(X)\sqrt{T}}+\frac{|\bar{X}_0|}{\sigma(0)\sqrt{T}}\right)\right) \\
\left.+\frac{2(\sigma(X)(|X|-|\bar{X}_0|)+\sigma(0)|\bar{X}_0|)}{(\sigma(0)+\sigma(X))}\Psi_4\left(-\left(\frac{|X|-|\bar{X}_0|}{\sigma(X)\sqrt{T}}+\frac{|\bar{X}_0|}{\sigma(0)\sqrt{T}}\right)\right)\right).
\end{aligned}
\tag{17}
$$

Figure 7 shows the implied volatility computed using approximate formula (17) and the implied volatility computed by the exact algorithm. This formula provides adequate solution to the original problem. Note that the classical short-time approximation

$$
\sigma_{imp}\left(X\right) = \frac{X}{\int_0^X \frac{d\xi}{\sigma_{loc}(\xi)}} = \frac{|X|}{\frac{|X|-|\bar{X}_0|}{\sigma(X)}+\frac{|\bar{X}_0|}{\sigma(0)}}, \tag{18}
$$

Two-tile vol approximation

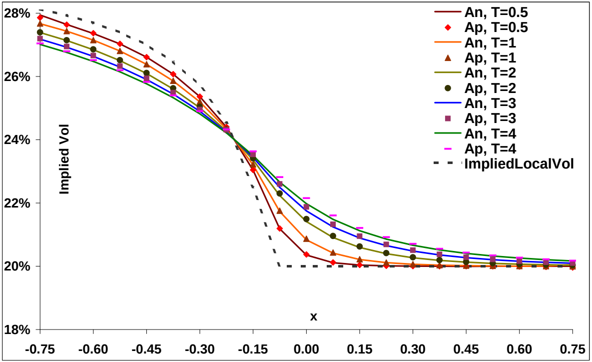


Figure: Two-tile case with $\sigma_0 = 30\%$, $\sigma_1 = 20\%$, $\bar{X}_0 = -0.1$. Implied volatility computed using equation (17) vs. implied volatility computed via the exact Carson-Laplace using equation (16); for five representative maturities $T = 1, \ldots, 5$.

Short time asymptotics for local vol, I

Following Andersen & Brotherton-Ratcliffe (1999) we consider Dupire equation written in terms of $\Sigma = \sigma_L$ and $\sigma = \sigma_I$:

$$\Sigma^2 = \frac{2T\sigma\sigma_T + \sigma^2}{\left(1 - \frac{k\sigma_k}{\sigma}\right)^2 + T\sigma\sigma_{kk} - \frac{1}{4}T^2\sigma^2\sigma_k^2},$$

or, equivalently,

$$\sigma^2 - \Sigma^2 \left(1 - \frac{k\sigma_k}{\sigma}\right)^2 + 2T\sigma\sigma_T - T\Sigma^2\sigma\sigma_{kk} + \frac{1}{4}T^2\Sigma^2\sigma^2\sigma_k^2 = 0.$$

Expansion

$$\sigma = A\left(1 + TB + ...\right).$$

The zero-order equation

$$A^2 - \Sigma^2 \left(1 - \frac{kA_k}{A}\right)^2 = 0.$$

Short time asymptotics for local vol, II

Ansatz

$$A(k) = \frac{k}{\left(\int_0^k \frac{d\xi}{\Sigma(\xi)}\right)},$$

yields

$$\frac{kA_k}{A} = \left(1 - \frac{A}{\Sigma}\right), \quad 1 - \frac{kA_k}{A} = \frac{A}{\Sigma},$$

and

$$A^2 - \Sigma^2 \left(\frac{A}{\Sigma}\right)^2 = 0,$$

as desired.

Equation for B:

$$2A^2 B + 2\Sigma^2 \frac{A}{\Sigma} k B_k + 2A^2 B - \Sigma^2 A A_{kk} = 0,$$

or

$$k B_k + 2\frac{A}{\Sigma} B - \frac{1}{2} \Sigma A_{kk} = 0,$$

It is easy to check that

$$\Sigma A_{kk} = -\frac{2A^2}{k} \left(\frac{A_k}{A} - \frac{\Sigma_k}{2\Sigma} \right),$$

so that we can rewrite the equation as follows

$$k B_k + 2\frac{A}{\Sigma} B + \frac{A^2}{k} \left(\frac{A_k}{A} - \frac{\Sigma_k}{2\Sigma} \right) = 0$$

Short time asymptotics for local vol, IV

Ansatz

$$B(k) = \frac{A^2(k)}{k^2} \ln\left(\frac{\sqrt{\Sigma(0)\Sigma(k)}}{A(k)}\right),$$

yields

$$lhs = \frac{2A^2}{k^2}\left(\frac{kA_k}{A} - 1\right)\ln\left(\frac{\sqrt{\Sigma(0)\Sigma(k)}}{A(k)}\right) + \frac{A^2}{k}\left(\frac{\Sigma_k}{2\Sigma} - \frac{A_k}{A}\right)$$

$$+\frac{2A^3(k)}{k^2\Sigma}\ln\left(\frac{\sqrt{\Sigma(0)\Sigma(k)}}{A(k)}\right) + \frac{A^2}{k}\left(\frac{A_k}{A} - \frac{\Sigma_k}{2\Sigma}\right) = 0,$$

as needed.

Short time asymptotics for local vol, V

It is useful to compute $\sigma(0), \sigma_k(0), \sigma_{kk}(0), \ldots$. It is clear that $\sigma(0) = \Sigma(0)$. We have

$$\frac{k}{\sigma(k)} = \int_0^k \frac{d\xi}{\Sigma(\xi)},$$

$$\frac{1}{\sigma(k)} - \frac{k\sigma_k(k)}{\sigma^2(k)} = \frac{1}{\Sigma(k)},$$

$$-\frac{2\sigma_k(k)}{\sigma^2(k)} - \frac{k\sigma_{kk}(k)}{\sigma^2(k)} + \frac{2k\sigma_k^2(k)}{\sigma^3(k)} = -\frac{\Sigma_k(k)}{\Sigma^2(k)},$$

$$-\frac{3\sigma_{kk}(k)}{\sigma^2(k)} + \frac{6\sigma_k^2(k)}{\sigma^3(k)} + \ldots = -\frac{\Sigma_{kk}(k)}{\Sigma^2(k)} + \frac{2\Sigma_k^2(k)}{\Sigma^3(k)}$$

so that

$$\sigma(0) = \Sigma(0), \quad \sigma_k(0) = \frac{\Sigma_k(0)}{2}, \quad \sigma_{kk}(0) = \frac{2\Sigma(0)\Sigma_{kk}(0) - \Sigma_k^2(0)}{6\Sigma(0)}.$$

LECTURE IV

Analytical solution for local vol, I

We consider options on the underlying driven by Brownian motion with local volatility. The corresponding backward pricing problem has the form

$$V_t + \frac{1}{2}\left(\sigma_{loc}^N(F)\right)^2 V_{FF} = 0,$$

$$V(T,F) = v(F).$$

We can simplify this problem via the Liouville transform

$$\frac{dF}{\sigma_{loc}^N(F)} = dY, \qquad Y(F_0) = 0, \qquad \frac{V(t,F)}{\sqrt{\sigma_{loc}^N(F)}} = U(t,Y),$$

which results in the following pricing problem

$$\begin{aligned} U_t + \frac{1}{2}U_{YY} + \tfrac{1}{8}\Theta^N(Y)\,U &= 0, \\ U(T,Y) &= u(Y), \end{aligned} \tag{19}$$

where $Y_0 \leq Y \leq Y_\infty$,

Analytical solution for local vol, II

$$u(Y) = \frac{v(F(Y))}{\sqrt{\sigma_{loc}^N(F(Y))}},$$

$$\Theta^N(Y) = 4\left(\sigma_{loc}^N\right)^{3/2}\left(\left(\sigma_{loc}^N\right)^{1/2}\right)_{FF} = 2\sigma_{loc}^N\sigma_{loc,FF}^N - \left(\sigma_{loc,F}^N\right)^2 = 2\left(\ln\left(\right.\right.$$

$$Y_0 = -\int_0^{F_0}\frac{dF}{\sigma_{loc}^N(F)}, \qquad Y_\infty = \int_{F_0}^{\infty}\frac{dF}{\sigma_{loc}^N(F)}.$$

The corresponding interval can be bounded or unbounded depending on $\sigma_{loc}^N(F)$. It is clear that, if so desired, we can express σ_{loc}^N in terms of Θ via the standard Riccati transform. A simple calculation yields $\sigma_{loc}^N = \zeta^{-2}$, where ζ is a solution of the following equation

$$\frac{1}{2}\zeta_{YY} + \tfrac{1}{8}\Theta^N(Y)\zeta = 0.$$

Thus, we can expect to find a (semi)-analytical solution of the pricing equation provided that we know the solution of the spectral problem

$$-\frac{1}{2}\Upsilon_{YY}(Y,\lambda)-\frac{1}{8}\Theta^{N}(Y)\Upsilon(Y,\lambda)=\lambda\Upsilon(Y,\lambda), \tag{20}$$

supplied with appropriate boundary conditions at Y_0, Y_∞. In this case we call the potential $\Theta^{N}(Y)$ solvable. Choosing $\Theta=\mathrm{const}$ is an obvious possibility, others include quadratic potential Θ, etc.

The corresponding spectrum can be both continuous and discrete.

Assuming that the spectrum is parametrized in the most convenient way, so that $\lambda=\lambda(k)$, $\Upsilon(Y,\lambda)=\Upsilon(Y,k)$ we can symbolically represent the solution of the problem (19) in the form

$$U(\tau,Y)=\int\limits_{\mathbb{K}} e^{-\lambda(k)\tau}\Upsilon(Y,k)\psi(k)\,dk, \tag{21}$$

where $\tau=T-t$, and the coefficients $\psi(k)$ are uniquely determined by the expansion of the terminal condition

Quadratic vol, I

In particular, when volatility is quadratic, the corresponding transformed PDE also has constant coefficients:

$$U_t + \frac{1}{2} U_{YY} - \frac{1}{8} \omega U = 0,$$
$$\omega = \beta^2 - 4\alpha\gamma.$$

Assuming that

$$\sigma_{loc}^N (F) = \alpha (F - p)(F - q), \qquad p < q < 0,$$
$$\omega = \alpha^2 (q - p)^2 > 0.$$

we have

$$Y = \frac{1}{\sqrt{\omega}} \ln \left(\frac{(F - q)(F_0 - p)}{(F - p)(F_0 - q)} \right), \qquad F = \frac{p (F_0 - q) e^{\sqrt{\omega} Y} - q (F_0 - p)}{(F_0 - q) e^{\sqrt{\omega} Y} - (F_0 - p)}.$$

The positive semi-axis is compactified and mapped into a finite interval:

$$I = \frac{1}{\sqrt{\omega}} \left[\ln \left(\frac{q (F_0 - p)}{p (F_0 - q)} \right), \ln \left(\frac{F_0 - p}{F_0 - q} \right) \right] = [Y_0, Y_\infty], \qquad |I| = \frac{\ln \left(\frac{p}{q} \right)}{\sqrt{\omega}}.$$

Quadratic vol, II

The corresponding call spread payoff for u has the form

$$u(Y) = \frac{\sqrt{\alpha}}{\sqrt{\omega}} \begin{cases} -p\sqrt{\frac{F_0-q}{F_0-p}} e^{\sqrt{\omega}Y/2} + q\sqrt{\frac{F_0-p}{F_0-q}} e^{-\sqrt{\omega}Y/2}, & Y \in [Y_0, Y_K], \\ K\left(-\sqrt{\frac{F_0-q}{F_0-p}} e^{\sqrt{\omega}Y/2} + \sqrt{\frac{F_0-p}{F_0-q}} e^{-\sqrt{\omega}Y/2}\right), & Y \in [Y_K, Y_\infty] \end{cases}$$

where

$$Y_K = \frac{1}{\sqrt{\omega}} \ln\left(\frac{(K-q)(F_0-p)}{(K-p)(F_0-q)}\right).$$

Quadratic vol, III

Solutions of the corresponding problems can be found either via the eigenfunction expansion method or the method of images. When the eigenfunction expansion method is used, solutions are represented in the form

$$U(\tau, Y) = \sum_{l=1}^{\infty} e^{-\frac{1}{2}\left(k_l^2+\frac{1}{4}\right)\omega\tau} \phi_l \sin\left(\sqrt{\omega} k_l (Y - Y_0)\right),$$

where

$$k_l = \frac{\pi l}{\sqrt{\omega}\,|I|} = \frac{\pi l}{\ln\left(\frac{p}{q}\right)} > 0,$$

and ϕ_l are the corresponding Fourier coefficients. The calculation of ϕ_l is tedious but straightforward and gives the following answer

$$\phi_l = \frac{2(-1)^{l+1}\sqrt{\sigma_{loc}^N(K)}}{|I|\,\omega\left(k_l^2+\frac{1}{4}\right)} \sin\left(k_l \ln\left(\frac{K-p}{K-q}\right)\right),$$

We now solve the problem via the method of images. We define

$$\begin{aligned} Y_l &= \frac{1}{\sqrt{\omega}} \ln\left(\frac{p^{2l}}{q^{2l}}\right), \quad Y_0 = \frac{1}{\sqrt{\omega}} \ln\left(\frac{q\left(F_0 - p\right)}{p\left(F_0 - q\right)}\right), \\ Y_\infty &= \frac{1}{\sqrt{\omega}} \ln\left(\frac{F_0 - p}{F_0 - q}\right), \quad Y_K = \frac{1}{\sqrt{\omega}} \ln\left(\frac{\left(F_0 - p\right)\left(K - q\right)}{\left(F_0 - q\right)\left(K - p\right)}\right). \end{aligned}$$

The Green's function has the form

$$\begin{aligned} G\left(\tau, Y, Y'\right) &= \sum_{l=-\infty}^{\infty} \left(g\left(\tau, Y - Y' - Y_l\right) - g\left(\tau, Y + Y' + Y_l - 2Y_0\right)\right) \\ &= \sum_{l=-\infty}^{\infty} \left(g\left(\tau, Y' + Y_l - Y\right) - g\left(\tau, Y' + Y_l + Y - 2Y_0\right)\right). \end{aligned}$$

Quadratic vol, V

We are interested in $Y = 0$, so that

$$G\left(\tau, Y'\right) \equiv G\left(\tau, 0, Y'\right) = \sum_{l=-\infty}^{\infty} \left(g\left(\tau, Y' + Y_l\right) - g\left(\tau, Y' + Y_l - 2Y_0\right)\right).$$

Generic integral

$$m\left(x; a, b, c\right) = \int e^{cx}\phi\left(ax + b\right)dx = \frac{1}{a}e^{-\frac{bc}{a}+\frac{c^2}{2a^2}}\Phi\left(ax + b - \frac{c}{a}\right),$$

so that

$$I_l^{\pm}\left(\tau, Y'\right) = \int e^{\pm\frac{\sqrt{\omega}Y'}{2}}g\left(\tau, Y' + Y_l\right)dY' = e^{\mp\frac{\sqrt{\omega}Y_l}{2}}\Phi\left(\frac{Y' + Y_l}{\sqrt{\tau}} \mp \frac{\sqrt{\omega\tau}}{2}\right),$$

$$\begin{aligned} J_l^{\pm}\left(\tau, Y'\right) &= \int e^{\pm\frac{\sqrt{\omega}Y'}{2}}g\left(\tau, Y' + Y_l - 2Y_0\right)dY' \\ &= e^{\mp\frac{\sqrt{\omega}\left(Y_l - 2Y_0\right)}{2}}\Phi\left(\frac{Y' + Y_l - 2Y_0}{\sqrt{\tau}} \mp \frac{\sqrt{\omega\tau}}{2}\right). \end{aligned}$$

Quadratic vol, VI

In the case when volatility is quadratic with negative roots, we can represent the price of a call option in two complementary ways. The eigenfunction expansion representation has the form

$$C(\tau, F_0, K) = F_0 - \frac{2}{\ln(p/q)} \sqrt{\frac{\sigma_{loc}^N(F_0)\,\sigma_{loc}^N(K)}{\omega}}$$

$$\times \sum_{l=1}^{\infty} \frac{e^{-\frac{1}{2}\omega\tau Q(k_l)}}{Q(k_l)} \sin(k_l Z_\infty) \sin(k_l (Z_\infty - Z_K)),$$

where $k_l = \pi l / \ln(p/q)$, and

$$Z_\infty = \ln\left(\frac{F_0 - p}{F_0 - q}\right), \quad Z_K = \ln\left(\frac{(F_0 - p)(K - q)}{(F_0 - q)(K - p)}\right).$$

It can be shown that for $p \to -\infty$, $q \to 0$, $\alpha p \to \mathrm{const}$, this formula becomes the standard LL formula.

The method of images provides a viable alternative. The corresponding representation has the form

$$
\begin{aligned}
& C(\tau, F_0, K) \\
= \quad & F_0 - \sqrt{\frac{\sigma^N_{loc}(F_0)}{\omega}} \left(\sqrt{\sigma^N_{loc}(0)} \sum_{l=-\infty}^{\infty} \left(\Upsilon(\omega\tau, Z_l + Z_0) - \Upsilon(\omega\tau, Z_l - Z_0 \right) \right. \\
& + \sqrt{\alpha K^2} \sum_{l=-\infty}^{\infty} \left(\Upsilon(\omega\tau, Z_l - Z_\infty + 2Z_0) - \Upsilon(\omega\tau, Z_l - Z_\infty) \right) \\
& \left. + \sqrt{\sigma^N_{loc}(K)} \sum_{l=-\infty}^{\infty} \left(\Upsilon(\omega\tau, Z_l - Z_K) - \Upsilon(\omega\tau, Z_l - Z_K + 2Z_0) \right) \right),
\end{aligned}
$$

$$
\Upsilon(v, k) = e^{-\frac{k}{2}} CS(v, k) = e^{-\frac{k}{2}} \Phi\left(\frac{k}{\sqrt{v}} - \frac{\sqrt{v}}{2} \right) + e^{\frac{k}{2}} \Phi\left(-\frac{k}{\sqrt{v}} - \frac{\sqrt{v}}{2} \right),
$$

$$
Z_l = \ln\left(\frac{p^{2l}}{q^{2l}} \right), \quad Z_0 = \ln\left(\frac{q(F_0 - p)}{p(F_0 - q)} \right).
$$

It can be checked that for $p \to -\infty$, $q \to 0$, $\alpha p \to$ const, this formula

It is worth noting that processes with quadratic volatility are not matringales, but rather supermatingales, see., e.g., Andersen (2008). As a consequence, the solution of the corresponding pricing problem is not unique and a proper one has to be chosen. This choice is implicitly done above. Non-uniqueness also means that there are (many) non-zero solutions of the pricing problem with zero initial and boundary conditions. Assuming for brevity that $p = q = 0$, so that

$$\sigma_{loc}^{N}\left(F\right) = \alpha F^{2}, \qquad \alpha > 0,$$

we can represent the corresponding homogeneous pricing problem as follows:

$$V_{\tau} - \frac{1}{2}\alpha^{2} F^{4} V_{FF} = 0, \quad V\left(0, F\right) = 0, \quad V\left(\tau, 0\right) = 0.$$

Quadratic vol, IX

It can be shown that the generic non-trivial solution of the above problem has the form

$$V(\tau, F; h) = \frac{1}{\alpha} \int_0^\tau \frac{e^{-\left(2\alpha^2 F^2 (\tau - \tau')\right)^{-1}}}{\sqrt{2\pi (\tau - \tau')^3}} h(\tau') \, d\tau'.$$

In particular, when $h(\tau) = 1$, we have

$$V(\tau, F; 1) = 2F\Phi\left(-\frac{1}{\alpha F \sqrt{\tau}}\right).$$

Poisson summation formula

The famous Poisson summation formula states that under appropriate regularity conditions we have

$$\sum_{k=-\infty}^{\infty} g\left(x+2\pi k\right) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} \hat{g}\left(n\right) e^{inx},$$

$$\hat{g}\left(n\right) = \int_{-\infty}^{\infty} g\left(x\right) e^{-inx} dx.$$

Consider $h\left(x\right) \equiv \sum_{k=-\infty}^{\infty} g\left(x+2\pi k\right)$. It is 2π periodic and

$$h\left(n\right) = \frac{1}{2\pi}\int_{0}^{2\pi} h\left(x\right) e^{-inx} dx = \frac{1}{2\pi}\int_{-\infty}^{\infty} g\left(x\right) e^{-inx} dx = \frac{1}{2\pi}\hat{g}\left(n\right).$$

Green's function for correlated Brownian motions, I

Without jumps, we need to find Green's function for the correlated heat equation in the quarter-plane, i.e., to solve the following problem

$$G_t - \frac{1}{2}\left(\sigma_1^2 G_{x_1 x_1} + 2\rho\sigma_1\sigma_2 G_{x_1 x_2} + \sigma_2^2 G_{x_2 x_2}\right) + \mu_1 G_{x_1} + \mu_2 G_{x_2} = 0$$

$$G\left(t, x_1, 0\right) = 0, \qquad G\left(t, 0, x_2\right) = 0$$

$$G\left(t, x_1, x_2\right) \to \delta\left(x_1 - x_1'\right)\delta\left(x_2 - x_2'\right)$$

Standard transform

$$G = \exp\left(\alpha t + \beta_1\left(x_1 - x_1'\right) + \beta_2\left(x_2 - x_2'\right)\right) u$$

removes drift terms provided that

$$\Sigma\beta = \mu, \qquad \alpha = -\frac{1}{2}\Sigma^{-1}\mu\cdot\mu$$

Green's function for correlated Brownian motions, II

$$\mathcal{L}u = u_t - \frac{1}{2}\left(\sigma_1^2 u_{x_1x_1} + 2\rho\sigma_1\sigma_2 u_{x_1x_2} + \sigma_2^2 u_{x_2x_2}\right) = 0$$

We rescale x_1, x_2, u and introduce ξ_1, ξ_2, v such that

$$x_i = \sigma_i \xi_i, \qquad x_i' = \sigma_i \xi_i', \qquad u = \frac{v}{\sigma_1\sigma_2},$$

In these variables the pricing problem becomes

$$\mathcal{M}v = v_t - \frac{1}{2}\left(v_{\xi_1\xi_1} + 2\rho v_{\xi_1\xi_2} + v_{\xi_2\xi_2}\right) = 0$$

$$v\left(t, \xi_1, 0\right) = 0, \qquad v\left(t, 0, \xi_2\right) = 0$$

$$v\left(t, \xi_1, \xi_2\right) \to \delta\left(\xi_1 - \xi_1'\right)\delta\left(\xi_2 - \xi_2'\right).$$

Green's function for correlated Brownian motions, III

Next, we make a further linear transform

$$(\xi_1, \xi_2) = (\bar{\rho} p_1 + \rho p_2, p_2), \qquad (\xi_1', \xi_2') = (\bar{\rho} p_1' + \rho p_2', p_2'), \qquad v = \frac{w}{\bar{\rho}}$$

where $\bar{\rho} = \sqrt{1 - \rho^2}$, and rewrite the pricing problem as follows

$$\mathcal{N} w = w_t - \frac{1}{2} \left(w_{p_1 p_1} + w_{p_2 p_2} \right) = 0$$

$$w(t, p_1, 0) = 0, \qquad w(t, -\rho p_2 / \bar{\rho}, p_2) = 0$$

$$w(t, p_1, p_2) \to \delta(p_1 - p_1') \, \delta(p_2 - p_2')$$

where $(p_1', p_2') = \left(\left(\xi_1' - \rho \xi_2' \right) / \bar{\rho}, \xi_2' \right)$. We now have the standard heat equation in an angle and can use our previous results. This angle is formed by the horizontal axis $p_2 = 0$ and a sloping line $p_1 = -\rho p_2 / \bar{\rho}$. It is acute when $\rho < 0$ and blunt otherwise. The size of this angle is $\alpha = \text{atan}(-\bar{\rho}/\rho)$.

The solution of the above problem via the method of images was independently introduced in finance by He *et al.*, Zhou and Lipton. We want to find the Green's function for the following parabolic equation

$$w_t - \frac{1}{2}\left(w_{rr} + \frac{1}{r}w_r + \frac{1}{r^2}w_{\phi\phi}\right)$$

supplied with the boundary conditions of the form

$$w\left(t,r,\phi\right) \underset{r\to 0}{\to} C < \infty, \quad w\left(t,r,\phi\right) \underset{r\to\infty}{\to} 0, \quad w\left(t,r,0\right) = 0, \quad w\left(t,r,\alpha\right) =$$

and the initial condition

$$w\left(t,r,\phi\right) \underset{t\to 0}{\to} \frac{\delta\left(r-r'\right)\delta\left(\phi-\phi'\right)}{r'}$$

The fundamental solution has the form

$$\Psi_{\alpha}\left(t,r,\phi|0,r',\phi'\right) = \frac{2e^{-\left(r^2+r'^2\right)/2t}}{\alpha t}\sum_{n=1}^{\infty} I_{n\pi/\alpha}\left(\frac{rr'}{t}\right)\sin\left(\frac{n\pi\phi'}{\alpha}\right)\sin\left(\frac{n\pi\phi}{\alpha}\right.$$

Method of images, I

We introduce the following function $f(p,q)$ with $p \geq 0$, $-\infty < q < \infty$, (which is a close relation of α-stable Levy distributions):

$$f(p,q) = 1 - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-p(\cosh(2q\zeta)-\cos(q))}}{\zeta^2 + \frac{1}{4}} d\zeta$$

We also define the following auxiliary function $h(p,\phi)$:

$$h(p,q) = \frac{1}{2}\left(s_+ f(p,\pi+q) + s_- f(p,\pi-q)\right)$$

where

$$s_\pm = \mathrm{sign}(\pi \pm q)$$

Method of images, II

Then we can represent unbounded $\Psi\left(t,r,\phi|0,r',\phi'\right)$ in the following remarkable form, which can be viewed as a direct generalization of the planar 2D case:

$$\Psi\left(t,r,\phi|0,r',\phi'\right)=\frac{1}{\sqrt{2\pi}t}g\left(\frac{d\left(r,r',\phi-\phi'\right)}{\sqrt{t}}\right)h\left(\frac{rr'}{t},\phi-\phi'\right)$$

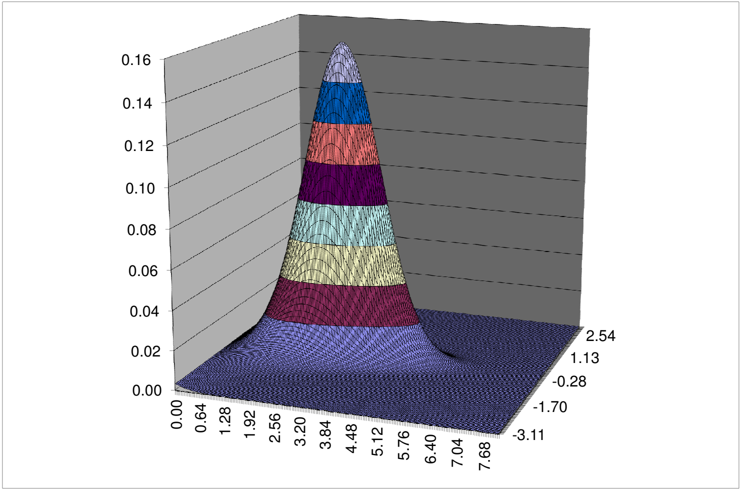
where

$$d\left(r,r',\phi-\phi'\right)=\sqrt{r^2+r'^2-2\cos\left(\phi-\phi'\right)rr'}$$

The corresponding Ψ_α can be written as follows:

$$\Psi_\alpha\left(*|0,r',\phi'\right)=\sum_{n=-\infty}^{\infty}\left[\Psi\left(*|0,r',\phi'+2n\alpha\right)-\Psi\left(*|0,r',-\phi'+2n\alpha\right)\right]$$

Non-periodic Green's function



Important remark

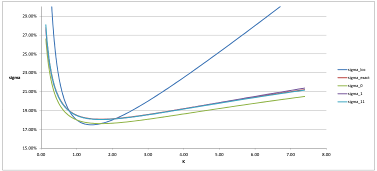
For short times, implied volatility generated by diffusion processes is very flat (viewed as function of Δ). Hence these processes can not be used to explain forex smile.
Indeed, for the quadratic volatility model with negative real roots, say, expressions for $\sigma_{imp}^{(0)}, \sigma_{imp}^{(1)}$become especially simple

$$\sigma_{imp}^{(0)}(k) = \frac{\sqrt{\omega}k}{\ln\left(\frac{(Fe^k - q)(F-p)}{(Fe^k - p)(F-q)}\right)} = \frac{\sqrt{\omega}\ln\left(\frac{K}{F}\right)}{\ln\left(\frac{(K-q)(F-p)}{(K-p)(F-q)}\right)},$$

$$\begin{aligned}\sigma_{imp}^{(1)}(\tau, k) &= \sigma_{imp}^{(0)}(k)\left(1 + \frac{\tau}{24}\left(\left(\sigma_{loc}^{LN}(\bar{k})\right)^2 - \omega\right)\right) \\ &= \sigma_{imp}^{(0)}(\tau, k)\left(1 + \frac{\tau}{24}\left(\left(\alpha\bar{K} + \beta + \frac{\gamma}{\bar{K}}\right)^2 - \omega\right)\right).\end{aligned}$$

This observation is illustrated in the following Figures

Short-time asymptotics for QVP



Implied vol for quad vol model for fixed delta as function of time

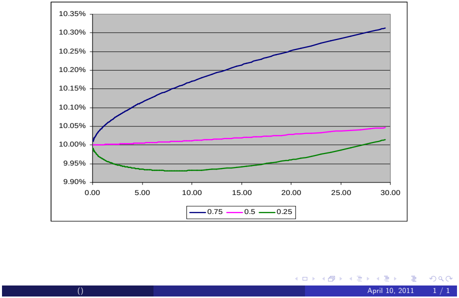


Figure:

Implied vol for quad vol model for fixed time as function of delta

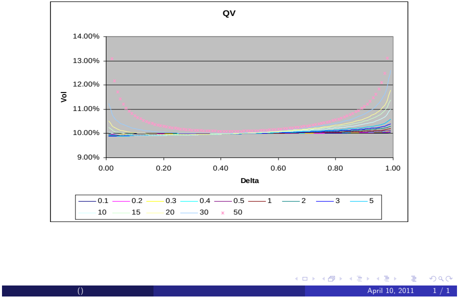


Figure:

Calibration of Heston model, I

There are several approaches one can use.
First, when $\rho = 0$ one can use simple scenario analysis of Hull & White (1988) and write

$$C^{(SV)}(0, S, T, K) = \int_0^\infty C^{(BS)}(0, S, T, K; \hat{\sigma}) f(\hat{\sigma}) d\hat{\sigma}, \qquad \hat{\sigma}^2 = \frac{\int_0^T \sigma_t^2 dt}{T}.$$

When $\rho \neq 0$ this formula can be extended (as was done by Willard (1997)):

$$\begin{aligned} C^{(SV)}(0, S, T, K) &= \int_0^\infty C^{(BS)}\left(0, e^{-\frac{1}{2}\rho^2\hat{\sigma}^2 T + \rho J} S, T, K; \sqrt{1-\rho^2}\hat{\sigma}\right) g(\hat{\sigma} \\ J &= \int_0^T \sigma_t dW_t. \end{aligned}$$

These formulas reduce dimensionality of the problem to $1D$.

Willard formula gives the following simple expression for the "equivalent" local vol (see Lee (2001), Henry-Labordére (2009)):

$$\sigma_{loc}^2 = \frac{\mathbb{E}\left\{\sigma_T^2 \frac{e^{-\frac{X^2}{2(1-\rho^2)T\hat{\sigma}^2}}}{T\hat{\sigma}^2}\right\}}{\mathbb{E}\left\{\frac{e^{-\frac{X^2}{2(1-\rho^2)T\hat{\sigma}^2}}}{T\hat{\sigma}^2}\right\}}, \quad X = \ln\left(\frac{K}{S}\right) + \frac{1}{2}\rho^2\hat{\sigma}^2 T - \rho J.$$

Second, Assuming that vol-of-vol ε is a small parameter, we can write Heston price as a series in powers of ε. Inspired by earlier work of Hull & White (1987), Lipton (1997, 2001) and Lewis (2000) obtained the following expansions:

$$
\begin{aligned}
C^{(H)} &= C^{BS}\left(\sqrt{\gamma}\right) \\
&+ \left(\frac{\varepsilon \rho f_1^{(1)} d_-}{\gamma} + \varepsilon^2 \left(-\frac{f_1^{(2)}\left(1 - d_+ d_-\right)}{\gamma\sqrt{\gamma}} + \frac{\rho^2 f_2^{(2)}\left(1 - d_-^2\right)}{\gamma\sqrt{\gamma}} \right.\right. \\
&\left.\left. + \frac{\rho^2 f_3^{(2)}\left(3 - 3d_+ d_- - 3d_-^2 + d_+ d_-^3\right)}{\gamma^2\sqrt{\gamma}}\right)\right) \Phi\left(d_-\right)
\end{aligned}
$$

$$
\begin{aligned}
\gamma &= \theta\tau + \frac{(v-\theta)}{\kappa}\left(1 - e^{-\chi}\right), \quad \chi = \kappa\tau, \\
f_1^{(1)} &= -\frac{1}{2\kappa^2}\left(v\left(1 - (1+\chi)e^{-\chi}\right) - \theta\left(2 - \chi - (2+\chi)e^{-\chi}\right)\right).
\end{aligned}
$$

etc.

Calibration of Heston model, IV

These formulas show that

$$\sigma_{imp,0}(l) = \sqrt{\varpi}\left(1 + \frac{\rho l}{4} + \frac{\left(1 - \frac{5}{2}\rho^2\right) l^2}{24}\right),$$

where $l = \varepsilon k/\varpi$.

Calibration of Heston model, V

Third, we can use the stoch vol version of the general Lewis (2000, 2001) and Lipton (2000, 2001, 2002) model and represent the price in the form (for brevity we put $r = 0$)

$$C^{(SV)}(T,K) = S - \frac{K}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\left(-iu+\frac{1}{2}\right)\ln(S/K)+\alpha^{(SV)}(T,u)-\beta^{(SV)}(T,u)v\left(u^2+\frac{1}{4}\right)}}{\left(u^2+\frac{1}{4}\right)} du,$$

$$\alpha^{(SV)} = -\frac{\kappa\theta}{\varepsilon^2}\left(\psi_+ T + 2\ln\left(\frac{\psi_- + \psi_+ e^{-\zeta T}}{2\zeta}\right)\right), \quad \beta^{(SV)} = \frac{1-e^{-\zeta T}}{\psi_- + \psi_+ e^{-\zeta T}},$$

$$\psi_\pm = \mp(iu\varepsilon\rho + \hat{\kappa}) + \zeta, \quad \hat{\kappa} = \kappa - \frac{\varepsilon\rho}{2}$$

$$\zeta = \sqrt{u^2\varepsilon^2\left(1-\rho^2\right) + 2iu\varepsilon\rho\hat{\kappa} + \hat{\kappa}^2 + \frac{\varepsilon^2}{4}}.$$

This formula is more efficient than the original Heston (1993) formula.

General LL formula

When jumps are present (but local vol is flat) the LL formula can easily be generalized

$$C^{(SVJD)}(T,K) = S - \frac{K}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\left(-iu+\frac{1}{2}\right)\ln(S/K)+\alpha^{(JD)}(T,u)+\alpha^{(SV)}(T,u)-\beta^{(SV)}(T,u)v}}{\left(u^2+\frac{1}{4}\right)}$$

This formula is very efficient and can be used in order to analyze both the qualitative and asymptotic behavior of the implied vol surface. In particular, one can derive all known (and many unknown) results regarding short-time ($T \to 0$), long-time ($T \to \infty$), wing ($\ln(S/K) \to \pm\infty$), and weak perturbation ($\varepsilon \to 0, \lambda \to 0$) asymptotics in a very straightforward fashion.

For that reason (?), it has been a subject of the disappearing commissar treatment.

In the most general case, we still can follow Dupire recipe (modified as appropriate) and specify his (and Gyöngy's) formula for the problem at hand. For brevity we put $r = 0$. We follow Lipton (2002).
The corresponding Fokker-Planck equation for the t.p.d.f. $P(T, K, w)$ reads

$$P_T - \frac{1}{2}\left(w\sigma_{loc}^2(T,K)K^2P\right)_{KK} - \left(\rho\varepsilon w\sigma_{loc}(T,K)KP\right)_{Kw} - \frac{1}{2}\left(\varepsilon^2 P\right)_{ww}$$

$$-\left((-\lambda m)KP\right)_K + \left(\kappa(\theta - w)P\right)_w - \lambda\int P\left(e^{-j}K\right)e^{-j}\phi(j)\,dj + \lambda P = 0,$$

$$P(0.K, w) = \delta(S - K)\,\delta(v - w).$$

Calibration of the Universal vol model, II

For our purposes we mostly interested in marginal t.p.d.f. $Q(T,K)$ given by

$$Q(T,K) = \int_0^\infty P(T,K,w)\,dw.$$

Integration of the FP equation yields

$$\begin{aligned} Q_T - \frac{1}{2}\left(V(T,K)\,\sigma_{loc}^2(T,K)\,K^2 Q\right)_{KK} - ((-\lambda m)\,KQ)_K \\ -\lambda \int Q\left(e^{-j}K\right)e^{-j}\phi(j)\,dj + \lambda Q = 0, \end{aligned}$$

where

$$V(T,K) = \frac{\int_0^\infty wP(T,K,w)\,dw}{Q(T,K)}.$$

Finally, application of the Bredeen-Litzenberger yields

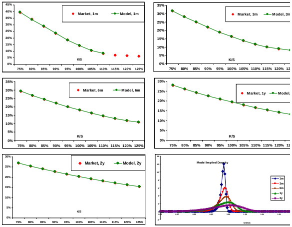
$$
\begin{gathered}
C_T - \frac{1}{2} V\left(T, K\right) \sigma_{loc}^2 \left(T, K\right) K^2 C_{KK} - \lambda m K C_K \\
-\lambda \int C\left(e^{-j} K\right) e^{-j} \phi\left(j\right) dj + \lambda\left(m+1\right) C = 0,
\end{gathered}
$$

$$
C\left(0, K\right) = \left(S - K\right)_+ .
$$

Various special cases had been studied in the literature earlier, notably, by Andersen & Andreasen (2000), but the general formula was not known. This calibration procedure works very well, as is shown in the following figure.

It has been rediscovered several times.

LSV Calibration Example (thanks to A Sepp)



1

Extension of Heston model

This formula can be extended to the case when vol is composite (provided that $\rho = 0$) by replacing E_n as follows

$$E_n = e^{\alpha^{(SV)}(T,u_n) - \beta^{(SV)}(T,u)v\left(u_n^2 + \frac{1}{4}\right)}.$$

Using the corresponding expression as zeroth-order approximation, we can perform expansion in powers of ρ. This expansion works very well. Alternatively, we can perform expansion in powers of ε which results in the so-called "classic six" set of equations (Lipton (1997)) which works very well and has been recently used by Andreasen and Huge (2010) with great success.

LECTURE V

Asymptotics I

Consider

$$F(t) = F(0)e^{X(t)}, \tag{22}$$

where $X(t)$ is a Lévy process. In this setup, particular emphasis has been put on the small-time asymptotic of $C(t;\tau,K)$ for $\tau \to 0$ and K fixed at some level different from $F(t)$. A typical result in this area of research is that

$$C(t;\tau,K) \sim \tau, \quad K \neq F(t),$$

where $\sim$ indicates the leading order term as $\tau \to 0$. Small-time results for at-the-money (ATM) options where $K = F(t)$ are scarcer, but Carr & Wu (2003) use Tanaka's formula to demonstrate that

$$C(t;\tau,F(t)) \sim \begin{cases} \tau, & \sigma = 0, \\ \sqrt{\tau}, & \sigma \neq 0, \end{cases}$$

Asymptotics II

Tankov and many others further elaborate on this result in implied volatility space, and demonstrates that for a finite variation Lévy process (necessarily without a diffusion component)

$$\sigma_{imp}\left(t;\tau,F\left(t\right)\right)\sim\sqrt{2\pi\tau}\max\left\{\int_{\mathbb{R}}\left(e^{x}-1\right)^{+}\nu\left(dx\right),\int_{\mathbb{R}}\left(1-e^{x}\right)^{+}\nu\left(dx\right)\right\},\tag{23}$$

where $\nu(\cdot)$ is the Lévy measure of X. In the presence of a diffusion component with constant volatility σ, Tankov (2010) shows that if

$$\int_{\mathbb{R}}x^{2}\nu(dx)<\infty,$$

then

$$\sigma_{imp}\left(t;\tau,F\left(t\right)\right)\sim\sigma$$

for $\tau\downarrow 0$.

Let $X(t)$ be a Lévy process, i.e. a cadlag process with stationary and independent increments, satisfying $X(0) = 0$. It is known that every Lévy process is characterized by a triplet $(\bar{\gamma}, \sigma, \nu)$, where $\bar{\gamma}$ and σ are constants, and ν is a (possibly infinite) Radon measure, known as the *Lévy measure*. The Lévy measure must always satisfy

$$\int_{\mathbb{R}} \min\left(x^2, 1\right) v(dx) < \infty. \tag{24}$$

To characterize the infinitesimal generator of a Lévy process, let E be the expectation operator, and define

$$V(t, x) = \mathrm{E}\left(f(X(T)) | X(t) = x\right)$$

for some suitably regular function $f(\cdot)$. It can be shown that V solves a partial integro-differential equation (PIDE) of the form

$$V_t + \gamma V_x + \frac{1}{2}\sigma^2\left(V_{xx} - V_x\right) + \int_{\mathbb{R}} \left[V(t, x+y) - V(t, x) - y 1_{|y|\leq 1} V_x(t, x)\right] \tag{25}$$

where $\gamma = \bar{\gamma} + \sigma^2/2$, subject to the terminal condition $V(T, x) = f(x)$.

By choosing $f(x) = \exp(iux)$ and solving (25) through an affine ansatz

$$V(t,x,T) = \exp(\psi(u)(T-t) + iux),$$

one arrives at the famous *Lévy-Khinchine formula,*

$$\phi(t,u) \triangleq \mathrm{E}\left(e^{iuX(t)}\right) = \exp\left(\psi(u)t\right),$$

$$\psi(u) = \gamma iu - \frac{1}{2}\sigma^2 u\left(u+i\right) + \int_{\mathbb{R}} \left[e^{iux} - 1 - iux1_{|x|\leq 1}\right] \nu(dx), \quad u \in \mathbb{R}, \tag{26}$$

where ψ is the so-called *Lévy exponent*. In practice, a Lévy process may be specified either by its exponent ψ or by its Lévy measure ν.

Levy Processes, III

We wish for $F(t)$ to be a martingale in some pricing measure. For this, we impose that, for any $t > 0$,

$$\mathrm{E}\left(e^{X(t)}\right) = 1, \tag{27}$$

which requires that the first exponential moment of $X(t)$ exists in the first place, i.e. that large positive jumps be suitably bounded

$$\int_{|x|>1} e^x \nu(dx) < \infty. \tag{28}$$

Equivalently, we require that $\phi(t, z)$ exists in the complex plane strip

$$\mathcal{S} = \{z \in \mathbb{C} | \mathrm{Im}(z) \in [-1, 0]\},$$

and that

$$\phi(t, -i) = \exp\left(\gamma t + t \int_{\mathbb{R}} \left[e^x - 1 - x 1_{|x|\leq 1}\right] \nu(dx)\right) = 1,$$

in order to satisfy condition (27). This implies the constraint

Using (29) to eliminate γ from (26), the martingale restriction on F allows us to write the Lévy exponent in the form

$$\psi(u) = -\frac{1}{2}\sigma^2 u(u+i) + \int_{\mathbb{R}} \left[e^{iux} - 1 - iu\left(e^x - 1\right)\right] \nu(dx). \qquad (30)$$

Similarly, we may write the PIDE (25) as

$$V_t + \frac{1}{2}\sigma^2\left(V_{xx} - V_x\right) + \int_{\mathbb{R}} \left[V(t, x+y) - V\left(t, x\right) - \left(e^y - 1\right) V_x\left(t, x\right)\right] \nu(dy \qquad (31)$$

Whenever possible, it is more convenient to use the following forms of the PIDE (25), namely,

$$V_t + \gamma' V_x + \frac{1}{2}\sigma^2\left(V_{xx} - V_x\right) + \int_{-\infty}^{\infty} \left[V\left(t, x+y\right) - V\left(t, x\right)\right] \nu(dy) = 0, \qquad (32)$$

$$V_t + \gamma'' V_x + \frac{1}{2}\sigma^2\left(V_{xx} - V_x\right) + \int_{-\infty}^{\infty} \left[V\left(t, x+y\right) - V\left(t, x\right) - yV_x\left(t, x\right)\right] \nu(c \qquad (33)$$

Characteristic function, II

First, if the Lévy measure is finite (i.e., the jump component of the process has *finite activity*), the resulting process for $X(t)$ is a combination of a Brownian motion and an ordinary compound Poisson jump-process. We may then replace $\nu(dx)$ with

$$\nu(dy) = \lambda j(dy), \quad \lambda \triangleq \int_{\mathbb{R}} \nu(dx) < \infty, \tag{34}$$

where $j(dx) = \nu(dx)/\lambda$ is now a properly normed probability measure for the distribution of jump sizes in X, and λ is the (Poisson) arrival intensity of jumps. We may now write

$$\psi(u) = \gamma' iu - \frac{1}{2}\sigma^2 u(u+i) + \lambda \left(\int_{\mathbb{R}} e^{iux} j(\,dx) - 1 \right),$$

where the martingale restriction requires that γ' satisfies the martingale condition. Notice that if we define a random variable J with density $j(dx)$, then we can, in the finite activity case, interpret

$$\psi(u) = \gamma' iu - \frac{1}{2}\sigma^2 u(u+i) + \lambda \left[\phi_J(u) - 1\right], \tag{35}$$

The jump component of a Lévy process is said to have *finite variation* if

$$\int_{|x|\leq 1} |x| \nu(dx) < \infty. \tag{36}$$

Under this condition, truncation of the Lévy exponent ψ around the origin is not necessary, and we may write

$$\psi(u) = \gamma' iu - \frac{1}{2}\sigma^2 u\,(u+i) + \int_{\mathbb{R}} \left[e^{iux} - 1\right] \nu(\,dx), \quad \gamma' = \gamma - \int_{|x|\leq 1} x\nu(dx), \tag{37}$$

For the PIDE (25), we then get the simpler form (32).
Finally, for the case where the first moment exists,

$$\int_{\mathbb{R}} |x| 1_{|x|>1} \nu(\,dx) < \infty, \tag{38}$$

we may write the Lévy exponent in purely compensated form:

$$\psi(u) = \gamma'' iu - \frac{1}{2}\sigma^2 u\,(u+i) + \int_{\mathbb{R}} \left[e^{iux} - 1 - iux\right] \nu(\,dx), \quad \gamma'' = \gamma + \int_{|x|\leq 1} x$$

The corresponding PIDE can be written in the simpler form (33).

Classification

Establishing short-time ATM volatility smile asymptotics for the completely generic class of exponential Lévy processes appears to be a difficult problem, so we narrow our focus to classes of processes important in applications. Of primary importance to us are processes characterized by Lévy measures of the form

$$\nu(dx) = \left(\frac{c_+}{x^{\alpha+1}} e^{-\kappa_+ x} 1_{x>0} + \frac{c_-}{|x|^{\alpha+1}} e^{-\kappa_- |x|} 1_{x<0} \right) dx, \qquad (39)$$

where we require that $\kappa_+ \geq 1$, $\kappa_- \geq 0$, $c_+ \geq 0$, $c_- \geq 0$, and $\alpha < 2$. The resulting class of processes is known as *tempered α-stable (TS) Lévy processes*.

The overall behavior of the TS class is closely tied to the selection of the power α, as demonstrated in Table I.

α	Activity	Variation
< 0	Finite	Finite
$(0, 1)$	Infinite	Finite
$[1, 2)$	Infinite	Infinite

Special cases

Some important special cases of the TS class include:

- The KoBoL (CGMY) model, where $c_+ = c_-$;
- The exponential jump model, where $\alpha = -1$;
- The Gamma process, where $\alpha = 0$ and either $c_- = 0$ or $c_+ = 0$;
- The Variance Gamma model, where $\alpha = 0$;
- The Inverse Gaussian process, where $\alpha = 1/2$ and either $c_- = 0$ or $c_+ = 0$.

For some of the special cases above, explicit formulas exists for the density of $X(t)$ and for European call options.

Characteristic function, IV

When $\alpha \neq 0$ and $\alpha \neq 1$, the characteristic function for the TS Lévy process can easily be shown to be

$$\psi(u) = \sum_{s=\pm} a_s \left(\kappa_s - siu\right)^{\alpha} + \gamma iu + \delta,$$

where

$$\begin{aligned}
a_s &= \Gamma\left(-\alpha\right) c_s, \quad \zeta_s = s\left(\kappa_s^{\alpha} - \left(\kappa_s - s\right)^{\alpha}\right), \quad \eta_s = -\kappa_s^{\alpha}, \\
\gamma &= a_+\left(\kappa_+^{\alpha} - \left(\kappa_+ - 1\right)^{\alpha}\right) + a_-\left(\kappa_-^{\alpha} - \left(\kappa_- + 1\right)^{\alpha}\right) = a_+\zeta_+ - a_-\zeta_-, \\
\delta &= -\left(a_+\kappa_+^{\alpha} + a_-\kappa_-^{\alpha}\right).
\end{aligned}$$

Consider now the TS Lévy class with an added Brownian motion with volatility σ. For $\alpha \in (0,1)$ the PIDE (32) applies, and has the form

$$V_t + \gamma V_x + \frac{1}{2}\sigma^2 \left(V_{xx} - V_x\right) + \sum_{s=\pm} c_s \int_0^\infty \left(V\left(x+sy\right) - V\left(x\right)\right) \frac{e^{-\kappa_s y} dy}{y^{1+\alpha}} = 0, \tag{40}$$

for $\alpha \in (1,2)$ the PIDE (33) can be used,

$$V_t + \gamma V_x + \frac{1}{2}\sigma^2 \left(V_{xx} - V_x\right) + \sum_{s=\pm} c_s \int_0^\infty \left(V\left(x+sy\right) - V\left(x\right) - syV_x\left(x\right)\right) \frac{e^{-}}{} \tag{41}$$

Interestingly, it is possible to rewrite both (40) and (41) in terms of so-called *fractional derivatives*.

For non-integer values of α, consider first $\alpha \in (0,1)$, and define left $(s=-1)$ and right $(s=1)$ fractional derivatives of order α as follows

$$\mathfrak{D}_s^{\alpha} V(x) = \frac{(-s)^{\alpha}}{\Gamma(-\alpha)} \int_0^{\infty} \left(V(x+sy) - V(x) \right) \frac{dy}{y^{1+\alpha}}, \qquad s = \pm.$$

We emphasize that with this definition, irrespective of s,

$$\mathfrak{D}_s^{\alpha} e^{iux} = (iu)^{\alpha} e^{iux},$$

consistent with what one would expect from a generalization of a regular derivative.

Fractional derivatives, III

Under some mild regularity assumptions we can write

$$\mathfrak{D}_s^{\alpha} V(x) = \frac{(-s)^{\alpha-1}}{\Gamma(1-\alpha)} \int_0^{\infty} V_x(x+sy) \frac{dy}{y^{\alpha}} = \frac{(-s)^{\alpha-1}}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^{\infty} V(x+sy) \frac{dy}{y^{\alpha}}$$

For all non-integer values of $\alpha \in (1,\infty)$ we may then define

$$\mathfrak{D}_s^{\alpha} V(x) = (-s)^{\lfloor\alpha\rfloor} \mathfrak{D}_s^{\alpha-\lfloor\alpha\rfloor} \left(\mathfrak{D}^{\lfloor\alpha\rfloor} V(x) \right),$$

where $\lfloor\alpha\rfloor$ is the floor function, i.e., the largest integer such that $\lfloor\alpha\rfloor < \alpha$, so that

$$\mathfrak{D}_s^{\alpha} V(x) = \frac{(-s)^{\alpha-1}}{\Gamma(1-\alpha+\lfloor\alpha\rfloor)} \frac{d^{\lfloor\alpha\rfloor+1}}{dx^{\lfloor\alpha\rfloor+1}} \int_0^{\infty} V(x+sy) \frac{dy}{y^{\alpha-\lfloor\alpha\rfloor}}.$$

Notice that in general $\mathfrak{D}_+^{\alpha} V(x)$ is complex-valued even when $V(x)$ is real-valued.

For $\alpha \in (0,1)$, the PIDE (40) may be written

$$V_t + \gamma V_x + \frac{1}{2}\sigma^2 \left(V_{xx} - V_x\right) + \sum_{s=\pm} (-s)^\alpha a_s e^{s\kappa_s x} \mathfrak{D}_s^\alpha \left(e^{-s\kappa_s x} V\right) + \delta V = 0. \tag{42}$$

For $\alpha \in (1,2)$, the PIDE (41) can be written as

$$V_t + \left(\gamma + \sum_{s=\pm} s a_s \alpha \kappa_s^{\alpha-1}\right) V_x + \frac{1}{2}\sigma^2 \left(V_{xx} - V_x\right) + \sum_{s=\pm} (-s)^\alpha a_s e^{s\kappa_s x} \mathfrak{D}_s^\alpha \left(e^{-s\right. \tag{43}$$

In particular, for a regular stable process with $\kappa_\pm = 0$, the corresponding PIDE has the form

$$V_t + \gamma V_x + \frac{1}{2}\sigma^2 \left(V_{xx} - V_x\right) + \sum_{s=\pm} (-s)^\alpha a_s \mathfrak{D}_s^\alpha V = 0.$$

Fractional derivatives, V

We introduce the upper and lower computational boundaries $B_{\pm}$ to represent the infinite computational domain, and then construct the finite-difference grid $x_n = B_- + nh$, $h = (B_+ - B_-)/N$, and discretize $\mathfrak{D}_s^{\alpha} f(x_n)$ at a representative grid point as follows

$$\begin{aligned} \mathfrak{D}_s^{\alpha} f(x_n) &= \frac{(-s)^{\alpha}}{\Gamma(-\alpha) h^{\alpha}} \sum_{k=0}^{K_{n,s}} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)} f(x_n + s(k-1)h), \\ K_{n,-} &= n-1, \quad K_{n,+} = N - (n-1). \end{aligned}$$

For $s = +$ and $s = -$ this discretization is based on the values of $f(\cdot)$ computed at all the grid points lying to the right (left) of the point x_n, the point x_n itself, and an additional point immediately to the left (right) of it. This discretization guarantees that the resulting finite difference scheme is stable.

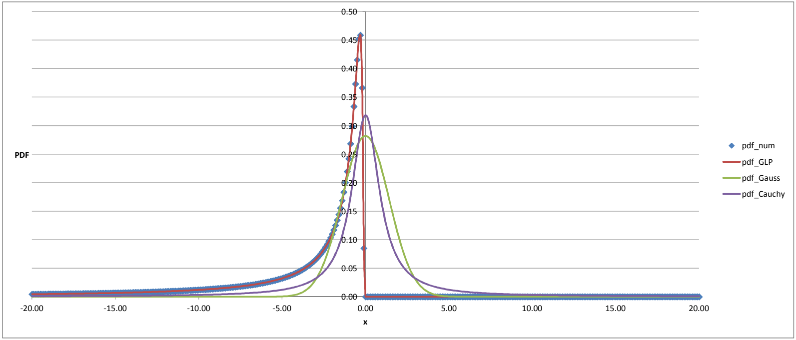
As a result of this discretization $\mathfrak{D}_s^\alpha$ turns into a Hessenberg matrix H_s^α. In order to take full advantage of this fact we solve a generic evolution equation of the form (42), (43) by splitting a typical time step into two:

$$\frac{V_n^{m+1} - V_n^*}{\Delta t} + \frac{1}{4}\left(MV^* + MV^{m+1}\right)_n + \frac{1}{2}(-1)^\alpha a_+\left(H_+^\alpha V^* + H_+^\alpha V^{m+1}\right)_n =$$

$$\frac{V_n^* - V_n^m}{\Delta t} + \frac{1}{4}\left(MV^* + MV^m\right)_n + \frac{1}{2}a_-\left(H_-^\alpha V^* + H_+^\alpha V^m\right)_n =$$

where M is the standard tri-diagonal matrix representing advection and diffusion terms. It is clear that at each half-step we have to solve a Hessenberg system of equations, so that our scheme is computationally efficient, of order $O(N)$. We illustrate the efficiency of the scheme by numerically constructing the known PDF for the Inverse Gaussian process; see Figure 31.

Numerics vs. analytics



Lewis-Lipton formula for Levy Processes, I

The key formula allowing one to analyze option prices for Lévy processes has been independently proposed by Lewis (2000) and Lipton (2000). The normalized price of a call written on an inderlying driven by an exponential Lévy process has the form

$$C(\tau, k) = 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{E(\tau, u)}{Q(u)} e^{-k\left(iu - \frac{1}{2}\right)} du, \qquad (44)$$

where

$$\begin{aligned} E(\tau, u) &= \exp\left(\tau v(u) - \frac{1}{2}\sigma^2 \tau Q(u)\right), \\ v(u) &= \psi_0\left(u - \frac{i}{2}\right), \\ Q(u) &= u^2 + \frac{1}{4}. \end{aligned}$$

The corresponding derivatives C_k and C_{kk} can be written as

$$C_k(\tau, k) = u - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E(\tau, u)}{Q(u)} e^{-k\left(iu - \frac{1}{2}\right)} \left(-iu + \frac{1}{2}\right) du,$$

$$C_{kk}(\tau, k) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E(\tau, u)}{Q(u)} e^{-k\left(iu - \frac{1}{2}\right)} \left(-iu + \frac{1}{2}\right)^2 du.$$

For TSP

$$E_{TSP}(\tau, u) = \exp\left(\tau\left(-\frac{1}{2}\sigma^2 Q(u) + \sum_{s=\pm} a_s \left(\kappa_s - s\left(iu + \frac{1}{2}\right)\right)^{\alpha} + \gamma\left(iu -\right.\right.\right.$$

Lewis-Lipton formula for Levy Processes, III

In the case of a standard diffusion process, Eq. (44) yields

$$BS\left(v,k\right) = 1 - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}vQ\left(u\right)-k\left(iu-\frac{1}{2}\right)}}{Q\left(u\right)}du,$$

so that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2}vQ\left(u\right)-k\left(iu-\frac{1}{2}\right)}}{Q\left(u\right)}du = \Phi\left(\frac{k-\frac{1}{2}v}{\sqrt{v}}\right) + e^{k}\Phi\left(\frac{-k-\frac{1}{2}v}{\sqrt{v}}\right).$$

In the limiting case $v = 0$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{-k\left(iu-\frac{1}{2}\right)}}{Q\left(u\right)}du = e^{k_{-}}.$$

These formulas are used below for studying properties of MPs and other purposes.

Lewis-Lipton formula for Levy Processes, IV

When jump component of a Lévy process is small compared to its diffusion component, the call price can be written in the form

$$
\begin{aligned}
C(\tau, k) &= 1 - \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\sigma^2 \tau Q(u) - k\left(iu - \frac{1}{2}\right)}}{Q(u)} du \\
& \quad - \tau \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\sigma^2 \tau Q(u) - k\left(iu - \frac{1}{2}\right)}}{Q(u)} \upsilon(u)\, du + \ldots,
\end{aligned}
$$

provided that the second integral converges. In particular

$$
\sigma_{imp}(\tau, k) = \sigma + \sigma_1(\tau, k) + \ldots,
$$

where σ_1 is of the same order of magnitude as υ, and is given by the following expression

$$
\begin{aligned}
\sigma_1(\tau, k) &= \tau^{1/2} \frac{e^{k^2/2\sigma^2\tau}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\sigma^2 \tau u^2 - iku}}{Q(u)} \upsilon(u)\, du \\
&= \tau^{1/2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-\frac{1}{2}\left(\sigma\tau^{1/2} u + \frac{ik}{\sigma\tau^{1/2}}\right)^2}}{Q(u)} \upsilon(u)\, du.
\end{aligned}
$$

Lewis-Lipton formula for Heston processes, I

The normalized price of a call written on an underlying driven by a square-root stochastic volatility process has the form

$$C(\tau, k) = 1 - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{E(\tau, u)}{Q(u)} e^{-k\left(iu-\frac{1}{2}\right)} du, \qquad (45)$$

where

$$\begin{aligned}
E(\tau, u) &= e^{\alpha(\tau,u)-\beta(\tau,u)\varpi Q(u)} \equiv e^{\gamma(\tau,u)}, \qquad (46)\\
\alpha(\tau, u) &= -\frac{\kappa\theta}{\varepsilon^2}\left(\psi_+(u)\tau + 2\ln\left(\frac{\psi_-(u)+\psi_+(u)\exp(-\zeta(u)\tau)}{2\zeta(u)}\right)\right),\\
\beta(\tau, u) &= \frac{1-\exp(-\zeta(u)\tau)}{\psi_-(u)+\psi_+(u)\exp(-\zeta(u)\tau)},\\
\gamma(\tau, u) &= \alpha(\tau, u) - \beta(\tau, u)\varpi Q(u),
\end{aligned}$$

and

$$\psi_\pm(u) = \pm\left(\rho\varepsilon\left(iu+\frac{1}{2}\right)-\kappa\right)+\zeta(u), \qquad (47)$$

Lewis-Lipton formula for Heston processes, II

The martingale condition, which is easy to verify, reads

$$\alpha\left(\tau, -\frac{i}{2}\right) = 0.$$

We note in passing that $\gamma(\tau, u)$ can be represented in the form

$$\gamma(\tau, u) = \theta\delta_1(\tau, u) + \varpi\delta_2(\tau, u),$$

which emphasizes the contributions of average and instantaneous variance, respectively.
For $\tau \longrightarrow \infty$ we can represent $\gamma(\tau, u)$ as follows

$$\gamma(\tau, u) = -\frac{\kappa\theta\psi_+(u)\tau}{\varepsilon^2} - \frac{2\kappa\theta}{\varepsilon^2}\ln\left(1 - \frac{\psi_+(u)}{2\zeta(u)}\right) - \frac{\varpi}{\varepsilon^2}\psi_+(u) + O\left(\frac{1}{\tau}\right). \tag{48}$$

It is easy to see from this formula that a Heston process with zero correlation is asymptotically equivalent to a NIGP. Naturally, in the long-time limit, to the leading order γ does not depend on ϖ.

Lewis-Lipton formula for Heston processes, III

For $\tau \to 0$ we use for inspiration the well-known duality between the Brownian motions $W(\tau)$ and $\tau W(1/\tau)$. We introduce a new variable $v = \tau u$ and notice that

$$
\begin{aligned}
\gamma\left(\tau, \frac{v}{\tau}\right) = & -\frac{i\varpi v \sinh\left(\frac{\bar{\rho}\varepsilon v}{2}\right)}{\tau\varepsilon \sinh\left(\frac{\bar{\rho}\varepsilon v}{2} + i\phi\right)} - \frac{\kappa\theta}{\varepsilon^2}\Bigg(i\rho\varepsilon v + 2\ln\Bigg(-\frac{i\sinh\left(\frac{\bar{\rho}\varepsilon v}{2} + i\phi\right)}{\bar{\rho}} \\
& + \frac{i\hat{\kappa}\varpi v\left(-\frac{\rho\bar{\rho}\varepsilon v}{2} + \sinh\left(\frac{\bar{\rho}\varepsilon v}{2}\right)\cosh\left(\frac{\bar{\rho}\varepsilon v}{2} + i\phi\right)\right)}{\bar{\rho}\varepsilon^2 \sinh^2\left(\frac{\bar{\rho}\varepsilon v}{2} + i\phi\right)^2} + O(\tau),
\end{aligned}
$$

where $\bar{\rho} = \sqrt{1-\rho^2}$, and $\phi = \arctan(\bar{\rho}/\rho)$. Thus, to the leading order a Heston process can be viewed as a Levy process but with time inverted. Naturally, in the short-time limit, to the leading order γ does not depend on θ.

Lewis-Lipton formula for Levy Processes and their asymptotics

LL formula encapsulates all the known results about the asymptotic behavior of option prices and the corresponding implied volatility in various asymptotic regimes, namely: $\tau \to 0, k = 0$; $\tau \to 0, k \neq 0$ (short-time asymptotics); $\tau \to \infty, k = \tau \bar{k}, \bar{k} \sim 1$ (long-time asymptotics); $\tau \sim 1, k \to \infty$ (wing asymptotics).

It is shown in Andersen and Lipton (2012) that some of these results are useful but most of them are not because asymptotics works only in remote limits. For instance, wing asymptotics becomes accurate when the price of the option is 10^{-10}, etc.

Also, Andersen and Lipton contain many references to the work on asymptotics done by (many) other researchers.

Long-time Asymptotics, I

Saddle-point approximation is a method for computing integrals of the form

$$g(\tau) = \frac{1}{2\pi} \int_{\gamma} f(z)\, e^{\tau S(z)} dz \tag{49}$$

when $\tau \longrightarrow +\infty$. Here γ is a contour in the complex plane, and the amplitude and phase functions $f(z), S(z)$ are holomorphic is a domain $\mathcal{D}$ containing γ. The extremal points of S, i.e. zeroes of S' are called the saddle points of S. Under reasonable conditions, the contribution from a non-degenerate saddle point z_0 , i.e., a saddle point such that $S''(z_0) \neq 0$ is given by

$$g_{z_0}^{(0)}(\tau) = \frac{1}{\sqrt{-2\pi\tau S''(z_0)}} e^{\tau S(z_0)} f(z_0) \left(1 + O\left(\tau^{-1}\right)\right),$$

It is clear that the main contribution comes from the points where $\mathrm{Re}[S]$ attains its absolute maximum. The second-order approximation has the form

Consider $y \in (-Y_+, Y_-)$ and define $\Xi_0(y)$ and $f_0(y)$ as follows

$$\Xi_0(y) = \upsilon(iy) + \frac{1}{2}\sigma^2 R(y), \qquad f_0(y) = \frac{1}{R(y)}, \qquad R(y) = y^2 - \frac{1}{4}.$$

Also define

$$\Xi_1(y) = \Xi_0'(y), \quad \Xi_2(y) = \Xi_0''(y), \quad \Xi_3(y) = \Xi_0'''(y), \quad \Xi_4(y) = \Xi_0''''(y$$

$$\begin{aligned}
f_1(y) &= f_0'(y) = -\frac{2y}{(R(y))^2}, \quad f_2(y) = f_0''(y) = \frac{\left(6y^2 + \frac{1}{2}\right)}{(R(y))^3}, \\
c_0(y) &= \frac{f_0(y)}{\sqrt{\Xi_2(y)}} = \frac{1}{\sqrt{\Xi_2(y)}R(y)}, \\
c_1(y) &= -\frac{1}{2\Xi_2(y)}\left(\frac{\left(6y^2 + \frac{1}{2}\right)}{(R(y))^2} + \frac{2y\Xi_3(y)}{R(y)\Xi_2(y)} + \frac{5\Xi_3^2(y)}{12\Xi_2^2(y)} - \frac{\Xi_4(y)}{4\Xi_2(y)}\right)
\end{aligned}$$

Long-time Asymptotics, III

Then for $\bar{k} = -\Xi_1(y)$ the corresponding $\sigma_{imp}(\tau, \tau\bar{k})$ can be written in the form

$$\sigma_{imp}(\tau, \tau\bar{k}) = \left(a_0(y) + \frac{a_1(y)}{\tau} + \frac{a_2(y)}{\tau^2} + \ldots\right)^{1/2}, \qquad (50)$$

where

$$\begin{aligned} a_0(y) &= 2\left(s_+(y) - s_-(y)\right)^2, \qquad (5\\ a_1(y) &= \frac{2\ln\left(a_0^{1/2}(y)\, r(y)\, c_0(y)\right)}{r(y)}, \\ a_2(y) &= \frac{2c_1(y)}{r(y)} \\ &\quad + \frac{\left(r(y)\, a_1(y)\left(\left(r(y)\, a_1(y) - 3\right)\left(r(y) + \frac{1}{4}\right) - \frac{1}{4}\right) + 6r(y) + 2\right.}{r^3(y)\, a_0(y)} \end{aligned}$$

and

$r(y) \quad B\left(\Xi_1(y)\right) \quad \ldots \quad \text{sign}\left(\ldots \frac{1}{\ldots}\right) \ldots$

Long-time Asymptotics, IV

for the Heston model the LL exponent is not proportional to time. However, it is proportional to time to the leading order when $\tau \longrightarrow \infty$. Formal expansion in powers of $\exp\left(-\tau\zeta\left(u\right)\right)$ yields

$$\frac{E\left(\tau,u,k\right)}{Q\left(u\right)}=\frac{e^{\alpha(\tau,u)-\beta(\tau,u)\varpi Q(u)-\tau\bar{k}\left(iu-\frac{1}{2}\right)}}{Q\left(u\right)}\longrightarrow e^{\tau S(u,k)}f\left(u\right),$$

where

$$S\left(u,\bar{k}\right)=-\frac{\kappa\theta}{\varepsilon^{2}}\psi_{+}\left(u\right)-\bar{k}\left(iu-\frac{1}{2}\right),\quad f\left(u\right)=\frac{e^{-\frac{\varpi}{\varepsilon^{2}}\psi_{+}(u)}}{Q\left(u\right)\left(1-\frac{\psi_{+}(u)}{2\zeta(u)}\right)^{2\kappa\theta/\varepsilon^{2}}},$$

and $\psi_{+}\left(u\right),\zeta\left(u\right)$ are given by Eqs. (47). As before, we can use saddle-point method to obtain the asymptotic of the LL integral. It is easy to check that the corresponding saddle point has to be purely imaginary, we so that we can proceed as before.

Consider $y \in (-Y_-, Y_+)$, where

$$Y_\pm = \frac{\rho\hat{\kappa} \pm \sqrt{\hat{\kappa}^2 + \frac{1}{4}\bar{\rho}^2\varepsilon^2}}{\bar{\rho}^2\varepsilon},$$

and define $\Xi_0(y), f_0(y)$ as follows

$$\Xi_0(y) = \frac{\kappa\theta}{\varepsilon^2}(\rho\varepsilon y + \hat{\kappa} - \varsigma(y)), \qquad f_0(y) = \frac{e^{\frac{\varpi}{\varepsilon^2}(\rho\varepsilon y + \hat{\kappa} - \varsigma(y))}}{R(y)\left(\frac{1}{2} - \frac{(\rho\varepsilon y + \hat{\kappa})}{2\zeta(u)}\right)^{2\kappa\theta/\varepsilon^2}},$$

where

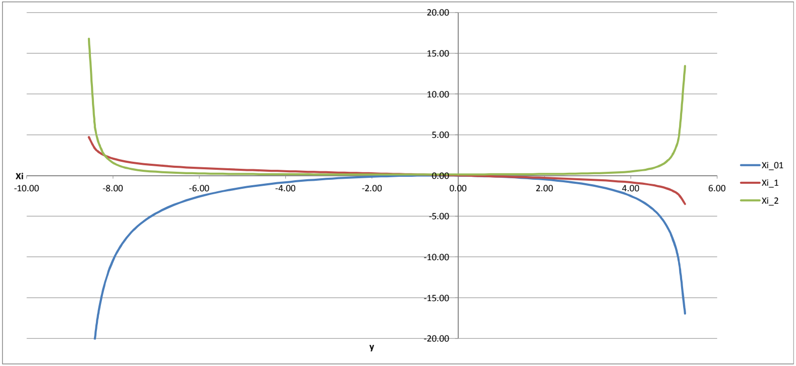
$$\varsigma(y) = \sqrt{-\bar{\rho}^2\varepsilon^2 y^2 + 2\rho\varepsilon\hat{\kappa}y + \hat{\kappa}^2 + \frac{1}{4}\varepsilon^2}.$$

Then

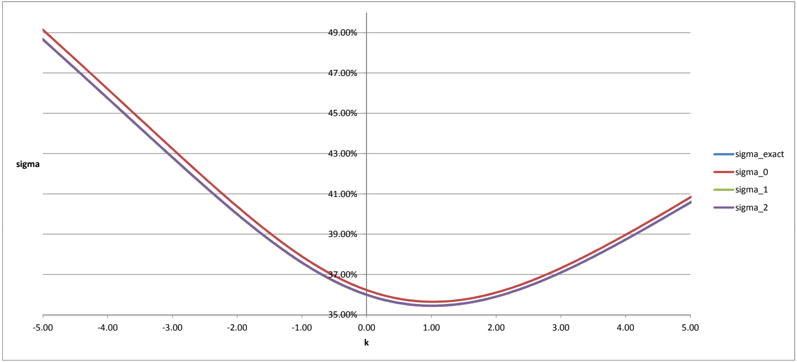
$$\sigma_{imp}(\tau, k(y)) = \left(a_0(y) + \frac{a_1(y)}{\tau} + O\left(\frac{1}{\tau^2}\right)\right)^{1/2},$$

with a_0, a_1 are given by the general Proposition. Accordingly, the zero and first order approximations have the form

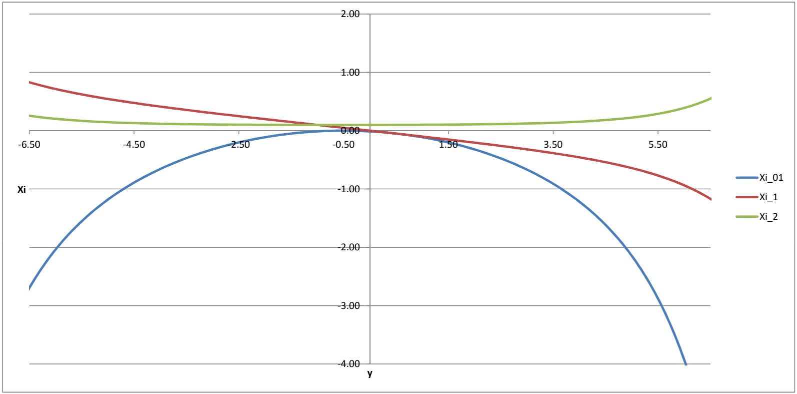
Long-time asymptotics, tempered stable processes, I



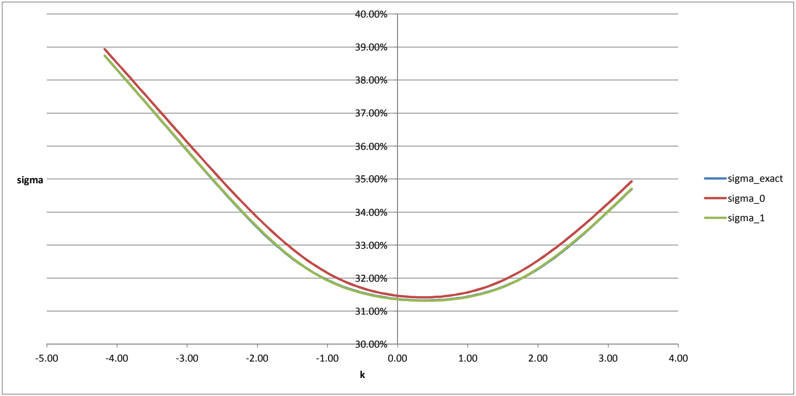
Long-time asymptotics, tempered stable processes, II



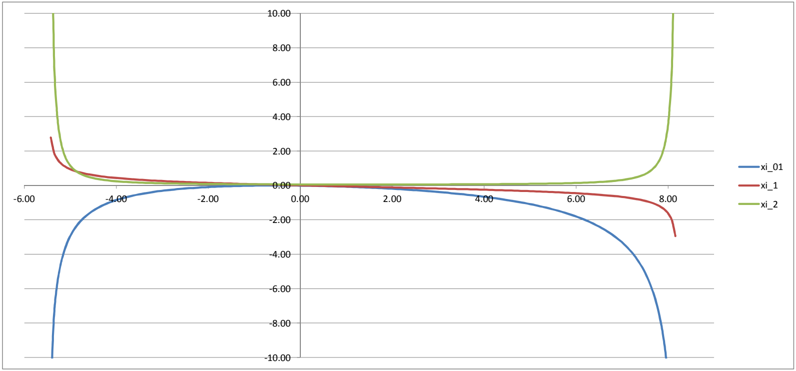
Long-time asymptotics, Merton processes, I



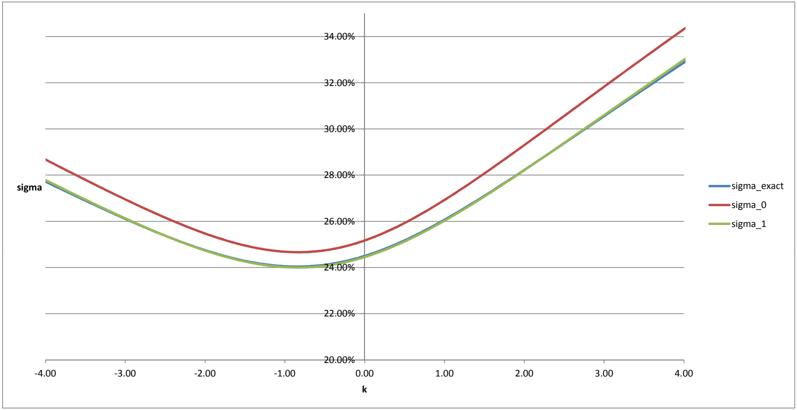
Long-time asymptotics, Merton processes, II



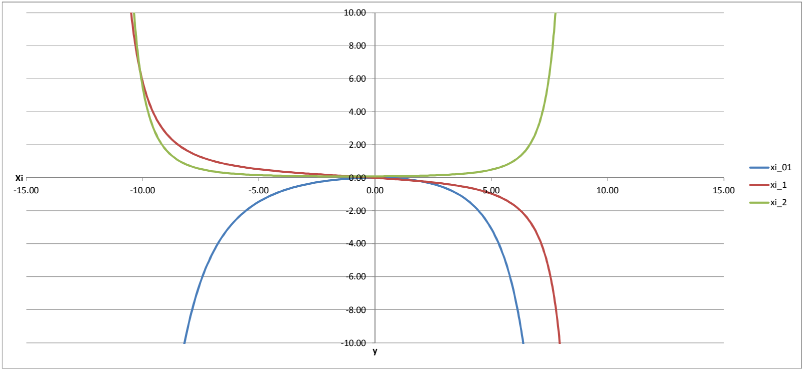
Long-time asymptotics, Heston processes, I



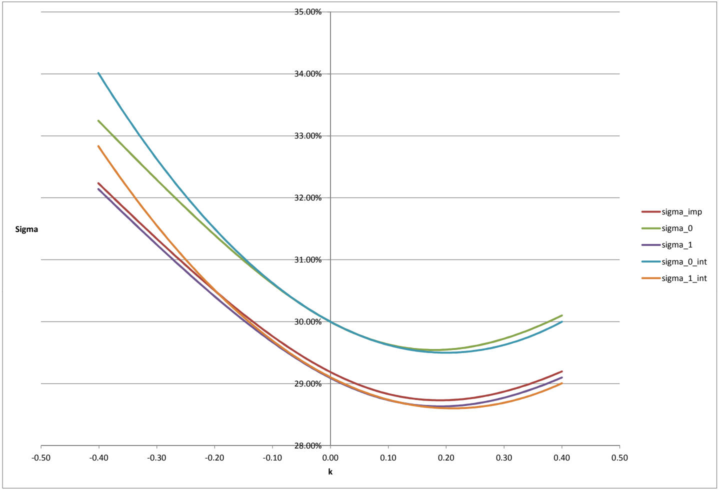
Long-time asymptotics, Heston processes, II



Short-time asymptotics, Heston processes, I



Short-time asymptotics, Heston processes, II



Wing asymptotics, tempered-stable processes