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Finite dimensional realizations for the CNKK-volatility surface model

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Outline

1 Introduction

- Implied volatility appears as a natural candidate for parametrization, since it is industry standard to quote option prices in terms of their implied volatility. However, the static and dynamic constraints on implied volatility are so awkward that it is very hard to analyse geometrically and analytically time evolutions of implied volatility surfaces, see [7, 8, 9]. Additionally it would be difficult to express stochastic interest rates or multivariate situations within this framework of the implied volatility codebook.

- Local volatility constitutes an industry standard to construct interpolations of (implied) volatility surfaces. It seems therefore natural to construct time evolutions of local volatility functions, see [1, 2]. This is even more attractive, since it is much easier to tell whether a function is a local volatility than an implied volatility. However, the description of the time evolution of local volatilities contains extremely non-linear and non-continuous operations, so that this parametrization also appears less useful. Additionally the extension towards stochastic interest rates is not well understood within the local volatility codebook.

- A last approach was independently and in parallel proposed by Carmona-Nadtochiy (see [4, 3]) and Kallsen-Krühner (see [6]), where option prices are parametrized by a time-dependent Lévy processes with characteristics absolutely continuous with respect to Lebesgue measure. From an analytic point of view it seems a bit more delicate to describe this set of parameters, however, the drift conditions are considerably less complicated in the Lévy codebook.

Following a generalization of the approaches of Carmona-Nadtochiy and Kallsen-Krühner, subsequently abbreviated by (generalized) CNKK-approach, we are equipped with tractable parameterizations. In this article we prefer the KK-approach to the CN-approach, since we see the following two advantages:

- In contrast to CN the time-inhomogenous Lévy process is encoded by its Lévy exponent, i.e. the logarithm of its Fourier-Laplace transform. CN choose the Lévy-Khintchine triplet (and assume the absense of volatility), which seems – from a purely analytic point of view – more appropriate, since the set of Lévy-Khintchine triplets is more easy to describe analytically than the set of Lévy exponents. On the other hand, and that is a main insight, the necessary martingale conditions, which express the lack of dynamic arbitrage, can be formulated again easier in the Lévy exponent parametrization.

- Dependence between increments of the underlying(s) and the increments of option prices ("leverage effect") are easily included into the KK-framework since this effect is easily expressed in the language of Lévy exponents.

Having fixed the generalized Lévy codebook the geometric and analytic approaches of [5] can be performed and due to several structural similarities the conclusions are of a very similar nature: if we assume that the term structure evolution of Lévy exponents, which describes the liquid option market prices, allows for regular finite dimensional realizations (i.e. we have a regular finite dimensional foliation on a subset of the state Hilbert space), then each leaf of this foliation is a ruled surface, i.e. an affine subspace moving transversally along a one-dimensional trajetory in Hilbert space.

2 The (generalized) CNKK-approach

Definition

The set Γ_n denotes the collection of continuous functions $\eta : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \to \mathbb{C}$ such that there exists a càdlàg process Z with finite exponential moments $\mathbb{E}(\exp((1+\varepsilon)\|Z_T\|)) < \infty$ for all maturity times $T \geq 0$, for some $\varepsilon > 0$ and

$$\mathbb{E}\left[\exp(\mathrm{i}\langle u, Z_T\rangle)\right] = \exp\left(\mathrm{i}\langle u, Z_0\rangle + \int_0^T \eta(S,u)\,dS\right) \qquad (2.1)$$

for $T \geq 0$. In particular the Fourier transform on the left hand side can be extended to an open superset of the strip $-\mathrm{i}[0,1]^n \times \mathbb{R}^n$. Any such function η will necessarily be of Lévy-Khintchine form at the short end $r = 0$.

Additionally often elements of Γ_n are subject to no-arbitrage-constraints. For instance, when the processes Z^i correspond to log-price-processes, we additionally assume that $\exp(Z^i)$ is a martingale, which translates to $\eta(r, -\mathrm{i}e_i) = 0$ with e_i being the i-th basis vector of $\mathbb{R}^n$, i.e. $\langle e_i, Z_T \rangle = Z_T^i$. We assume tacitly that such conditions are imposed if necessary. Notice in case of a components of Z corresponding to interest rates we do not need to impose such a condition.

Remark

The set $\mathbb{R}^n \times \Gamma_n$ is the "chart" or "codebook" for all liquid market prices at one moment in time, since the knowledge of a tuple (Z_0, η) allows to construct all marginal distributions of a process Z_T. There are two assumptions implicitly involved: first it is assumed that having the liquid market prices is equivalent to having the marginal distributions of several underlying processes. Second it is assumed that the so given marginal distributions have differentiable (forward) characteristics absolutely continuous with respect to Lebesgue measure (with continuous derivative).

Remark

Notice that we extend the definitions of CNKK since we only assume the Lévy-Khintchine form of η at the short end.

Definition

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space. A stochastic process η is called a Γ_n-valued semimartingale if $(\eta_t(T, u))_{0 \leq t \leq T}$ is a complex-valued semimartingale for $T \geq 0$ and $u \in \mathbb{R}^n$ and if

$$\big((r, u) \mapsto \eta_t(r + t, u)\big) \in \Gamma_n \, .$$

In particular all trajectories are assumed to be càdlàg.

We say that η allows for a *regular decomposition* with respect to a d-dimensional semimartingale M if there exist predictable processes $(\alpha_t(T,u))_{0\leq t\leq T}$ and $(\beta_t(T,u))_{0\leq t\leq T}$ taking values in $\mathbb{C}$ and $\mathbb{C}^d$ for $T\geq 0$ and $u\in\mathbb{R}^n$ such that

$$\eta_t(T,u)=\eta_0(T,u)+\int_0^t \alpha_s(T,u)\,ds+\sum_{i=1}^d\int_0^t \beta_s^i(T,u)\,dM_s^i \tag{2.2}$$

for $0\leq t\leq T$, and $\left(\sqrt{\int_t^T \|\beta_t(S,u)\|^2\,dS}\right)_{t\geq 0}\in L(M)$.

Definition (Conditional expectation condition)
Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space, then we say that a tuple (X, η) of an n-dimensional semimartingale X and of a Γ_n-valued semimartingale η satisfies the *conditional expectation condition* if

$$\mathbb{E}\left[\exp(\mathrm{i}\langle u, X_t\rangle) \middle| \mathcal{F}_s\right] = \exp\left(\mathrm{i}\langle u, X_s\rangle + \int_s^t \eta_s(r, u)\, dr\right) \tag{2.3}$$

for $t \geq s \geq 0$.

Remark

Shortly we shall call a tuple (X, η) of an n-dimensional semimartingale X and of a Γ_n-valued semimartingale η satisfying the conditional expectation condition a *term structure model for derivatives' prices*.

Remark

We call the stochastic process η the process of *forward characteristics of the process X*.

Remark

Let (X, η) satisfy the conditional expectation condition: if $n = 1$ and $\eta(., -\mathrm{i}) = 0$, then $\exp(X)$ describes a peacock, i.e. a process with marginals increasing in convex order, since it is a martingale. To consider peacocks as chart for option prices is the most general point of view.

Not every process η qualifies as a forward characteristics process, but it can be easily characterized whether it is the case.

Theorem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space together with a tuple (X, η) of an n-dimensional semimartingale X and of a Γ_n-valued semimartingale η satisfying the *conditional expectation condition*, then

- the differentiable, predictable characteristic κ^X of the n-dimensional semimartingale X exists and is given by $\kappa_t^X(u) = \eta_{t-}(t,u)$ for $t \geq 0$ and $u \in \mathbb{R}^n$, i.e. the process

$$\exp\left(i\langle u, X_t\rangle - \int_0^t \eta_{s-}(s,u)ds \right) \tag{2.4}$$

is a local martingale.

- If η allows for a regular decomposition (2.2) with respect to a d-dimensional semimartingale M, then the drift condition

$$\int_t^T \alpha_t(r,u)\,dr = \eta_{t-}(t,u) - \kappa_t^{(X,M)}(u, -i\int_t^T \beta_s(r,u)\,dr) \tag{2.5}$$

holds for $T \geq t \geq 0$ and $u \in -i[0,1]^n \times \mathbb{R}^n$.

Theorem

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space together with a tuple (X, η) of an n-dimensional semimartingale X and of a Γ_n-valued semimartingale η. Assume furthermore that η allows for a regular decomposition (2.2) with respect to a d-dimensional semimartingale M such that the predictable characteristics of X satisfy (2.4) and such that the drift condition (2.5) hold, then the *conditional expectation condition* holds true.

Corollary

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$ be a filtered probability space together with a tuple (X, η) of an n-dimensional semimartingale X and of a Γ_n-valued semimartingale η satisfying the *conditional expectation condition*. Assume furthermore that η allows for a regular decomposition (2.2) with respect to a d-dimensional semimartingale M and that the processes X and M are locally independent, i.e.

$$\kappa_t^{(X,M)}(u_1, u_2) = \kappa_t^X(u_1) + \kappa_t^M(u_2)$$

for $u_1 \in \mathbb{R}^n$ and $u_1 \in \mathbb{R}^d$. Then

$$\int_t^T \alpha_t(r,u)\,dr = -\kappa_t^M(-\mathrm{i}\int_t^T \beta_s(r,u)\,dr)$$

for $T \geq t \geq 0$ and $u \in -\mathrm{i}[0,1]^n \times \mathbb{R}^n$, and furthermore the conditional expectation condition (2.3) reads in this case

$$\mathbb{E}\left[\exp(\int_s^t \eta_{r-}(r,u)dr)\middle|\, \mathcal{F}_s\right] = \exp\left(\int_s^t \eta_s(r,u)\,dr\right)$$

for $t \geq s \geq 0$.

3 Affine processes as generic example for the CNNK-approach

In this section we build a generic example for term structure models for derivatives' prices: consider a proper convex cone $C \subset \mathbb{R}^m$ (the stochastic covariance structures) and a homogenous affine process (X, Y) taking values in $\mathbb{R}^n \times C$, i.e. (X, Y) is a time-homogeneous Markov process relative to some filtration $(\mathcal{F}_t)$ and with state space $D = \mathbb{R}^n \times C$ such that

- it is stochastically continuous, that is, $\lim_{s \to t} p_s(x, y, \cdot) = p_t(x, y, \cdot)$ weakly on D for every $t \geq 0$ and $(x, y) \in D$, and

- its Fourier-Laplace transform has exponential affine dependence on the initial state. This means that there exist functions $\Phi : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}$ and $\psi_C : \mathbb{R}_{\geq 0} \times \mathcal{U} \to \mathbb{C}^m$ with

$$\mathbb{E}_{x,y}\left[e^{\langle u, X_t \rangle + \langle v, Y_t \rangle}\right] = \Phi(t, u, v) e^{\langle u, x \rangle + \langle \psi_C(t,u,v), y \rangle},$$

for all $x \in D$ and $(t, u, v) \in \mathbb{R}_{\geq 0} \times \mathcal{U}$, where

$$\mathcal{U} := \{(u, v) \in \mathbb{C}^{m+n} \mid e^{\langle u, . \rangle + \langle v, . \rangle} \in L^\infty(D)\}.$$

Remark

In line with the standard literature on affine processes there is a $\mathbb{C}^{m+n}$-valued function ψ, whose projection onto the X-directions is u. Whence we only need the projection in the C-directions, which we denote by ψ_C.

We shall need the following results on affine processes on general state spaces:

Proposition

Let (X, Y) be a homogenous affine process taking values in $\mathbb{R}^n \times C$, then we have that $\Phi(t, u, 0) = \exp(\phi(t, u, 0))$,

$$\phi(t, u, 0) = \int_0^t F(u, \psi(s, u, 0))\, ds$$

and

$$\psi_C(t, u, 0) = \int_0^t R_C(u, \psi(s, u, 0))\, ds,$$

where $(u, v) \mapsto F(u, v)$ and $(u, v) \mapsto \langle R_C(u, v), y \rangle$ are of Lévy-Khintchine form.

Corollary

Let (X, Y) be a homogeneous affine process taking values in $\mathbb{R}^n \times C$ and assume that $\mathbb{E}\left[\exp((1+\varepsilon)\|X_t\|)\right] < \infty$, for some $\varepsilon > 0$, then for $T \geq t \geq 0$

$$\eta_t(T, u) = F(\mathrm{i}u, \psi(T-t, \mathrm{i}u, 0)) + \langle R_C(\mathrm{i}u, \psi(T-t, \mathrm{i}u, 0)), Y_t \rangle$$

defines a Γ_n-valued semimartingale and the tuple (X, η) satisfies the conditional expectation condition.

The analogue of Hull-White extensions from interest rate theory is described in the following theorem. Instead of making a drift time-dependent we make the whole constant part of the affine process, which is encoded in F time-dependent:

Corollary

Let (X, Y) be a time-inhomogenous, homogeneous affine process taking values in $\mathbb{R}^n \times C$ with time-dependent $T \mapsto F_T$, and assume that the finite moment condition $\mathbb{E}\left[\exp((1+\varepsilon)\|X_t\|)\right] < \infty$ holds true for some $\varepsilon > 0$, then $T \geq t \geq 0$

$$\eta_t(T,u) = F_T(\mathrm{i}u, \psi(T-t,u,0)) + \langle R_C(\mathrm{i}u, \psi(T-t,u,0)), Y_t \rangle$$

defines a Γ_n-valued semimartingale and the tuple (X, η) satisfies the conditional expectation condition.

Remark

Here time-inhomogenous, homogenous affine processes appear as generic realization of the CNKK-approach, since we can calibrate any initial term structure into $T \mapsto F_T$. In the next section we shall argue that there is a a hard mathematical reason for this generic property.

Let us describe several more concrete examples:

3.1 Deterministic term structure of forward characteristics

Deterministic forward term structure models correspond to time-dependent Lévy processes. More precisely let (X, η) be a tuple satisfying the conditional expectation condition and assume that η is a deterministic, then X is an additive process and $\eta_t(T, u) = \eta_0(T, u)$ is of Lévy-Khintchine form for every $T \geq t \geq 0$. The processes X are then time-dependent Lévy models.

3.2 Interest rate models

If the process X is one-dimensional, pure-drift and absolutely continuous with respect to Lebesgue measure, then

$$\int_t^T \alpha_t(r,u)\,dr = -\kappa_t^M(-\mathrm{i}\int_t^T \beta_s(r,u)\,dr)$$

and

$$uX_t = uX_0 - \int_0^t \eta_{s-}(s,u)ds,$$

which then yields the well-known formula of interest rate theory

$$\mathbb{E}\left[\exp(-\int_t^T \eta_{s-}(s,u)ds)\middle|\mathcal{F}_t\right] = \exp(-\int_t^T \eta_t(S,u)\,dS)$$

for $T \geq t \geq 0$ and $u \in \mathbb{R}$. Notice that $(\eta_{s-}(s,u))_{s\geq 0}$ is linear in u, since X is pure drift.

3.3 Stochastic volatility models with jumps in log-prices

Consider a jump-extended Heston stochastic volatility model

$$dX_t = -\frac{Y_t}{2}\,dt + \sqrt{Y_t}\,dW_t^1 + dL_t \tag{3.1}$$

$$dY_t = \lambda(b - Y_t)\,dt + \sigma\sqrt{Y_t}\,dW_t^2, \tag{3.2}$$

where the two Brownian motions W^1 and W^2 are correlated, $d\langle W^1, W^2\rangle_t = \rho dt$ and where L is an additive process with Lévy exponent F^L. Then for $T \geq t \geq 0$

$$\eta_t(T,u) = F_T(\mathrm{i}u, \psi(T-t, \mathrm{i}u, 0)) + \langle R_C(\mathrm{i}u, \psi(T-t, \mathrm{i}u, 0)), Y_t\rangle$$

defines a Γ_1-valued semimartingale and the tuple (X, η) satisfies the conditional expectation condition. C denotes here the non-negative real numbers. We have

$$F_T(u, v) = F_T^L(u) + \lambda b v, \tag{3.3}$$

$$R_C(u, v) = \frac{u^2}{2} + \rho\sigma uv + \frac{\sigma^2 v^2}{2} + \frac{u}{2} + \lambda v \tag{3.4}$$

for $u, v \in \mathbb{R}$.

4 The CNKK-equation as SPDE

In order to analyse the structure of finite factor models in the CNKK-approach we have to set up a framework, where the CNKK-equation appears as SPDE, in particular as a Markov process taking values in a Hilbert space of system states.

Definition

Let G be a Hilbert space of continuous complex-valued functions defined on the strip $-\mathrm{i}[0,1]^n \times \mathbb{R}^n$, i.e.

$$G \subset C((-\mathrm{i}[0,1]^n) \times \mathbb{R}^n; \mathbb{C}).$$

H is called a *Lévy codebook Hilbert space* if H is a Hilbert space of continuous functions $\eta : \mathbb{R}_{\geq 0} \to G$, i.e. $H \subset C(\mathbb{R}_{\geq 0}; G)$ such that

- we have a continuous embedding

$$H \subset C(\mathbb{R}_{\geq 0} \times (-\mathrm{i}[0,1]^n) \times \mathbb{R}^n; \mathbb{C}).$$

- The shift semigroup $(S_t\eta)(x,u) := \eta(t+x,u)$ acts as strongly continuous semigroup of linear operators on H.
- Continuous functions of finite activity Lévy-Khintchine type

$$(t,u) \mapsto \mathrm{i}a(t)u - \frac{u^T b(t) u}{2} + \int_{\mathbb{R}^n} (\exp(\mathrm{i}u\xi) - 1)\nu_t(d\xi)$$

lie in H, where a, b, ν are continuous functions defined on $\mathbb{R}_{\geq 0}$ taking values in $\mathbb{R}^n$, the positive-semidefinite matrices on $\mathbb{R}^n$ and the finite positive measures on $\mathbb{R}^n$ (this corresponds to processes with independent increments and finite activity).

Remark
Notice that we do not assume that there are additional stochastic factors outside the considered parametrization of liquid market prices.

Remark
Notice that elements of the Hilbert space H are understood in Musiela parametrization and therefore denoted by a different letter in the sequel. We have the relationship $\eta_t(t + x, u) = \theta_t(x, u)$, with $T - t =: x$.

In the sequel we are defining stochastic partial differential equations, which express the conditions of the CNKK-approach in the corresponding Musiela parametrization.

Definition
Let H be a Lévy codebook Hilbert space. We call the following stochastic partial differential equation

$$d\theta_t = (A\theta_t + \mu_{CNKK}(\theta_t))dt + \sum_{i=1}^{d} \sigma_i(\theta_t)\, dB_t^i \tag{4.1}$$

a *CNKK-equation* $(\theta_0, \kappa, \sigma)$ with initial term structure θ_0 and characteristics σ and κ,

- if $A = \frac{d}{dx}$ is the generator of the shift semigroup on H,
- if $\sigma_i : U \subset H \to H$, U an open subset of H, are locally Lipschitz vector fields, and
- if $\mu_{CNKK} : U \to H$ is locally Lipschitz and satisfies that for all $\eta \in \Gamma_n$ we have

$$\int_0^{T-t} \mu_{CNKK}(\theta)(r,u)\,dr = \theta(0,u) - \kappa_\theta(0,u,-\mathrm{i}\int_0^{T-t} \sigma(\theta)(r,u)\,dr), \tag{4.2}$$

where $(\kappa_\theta)_{\theta \in U}$ is Γ_{n+d}-valued for each $\theta \in \Gamma_n$, such that $\kappa_\theta(0,u,0) = \theta(0,u)$ and $\kappa_\theta(0,0,v) = -\frac{\|v\|}{2}$, for $u \in \mathbb{R}^n$, $v \in \mathbb{R}^d$.

Remark

We do not require that all solutions of equation (4.2) are Γ_n-valued, which would be too strong a condition and difficult to characterize.

Proposition

Let θ be a Γ_n-valued solution of a CNKK-equation and let X be a semimartingale such that the predictable characteristics satisfy

$$\kappa_t^{(X,W)}(u,v) = \kappa_{\theta_t}(t,u,v)$$

for $u \in \mathbb{R}^n$, $v \in \mathbb{R}^d$ and $t \geq 0$, then the tuple (X,θ) satisfies the conditional expectation condition.

We can construct one particular example, which corresponds literally to the HJM-equation: consider a situation without leverage, i.e. we assume that

$$\kappa_\theta(0, u, v) = \theta(0, u) - \frac{\|v\|}{2}$$

for $u \in \mathbb{R}^n$, $v \in \mathbb{R}^d$ and $t \geq 0$.

This means that the CNKK-equation is a parameter-dependent HJM-equation, i.e.

$$d\theta_t = (A\theta_t + \mu_{CNKK}(\theta_t))dt + \sum_{i=1}^{d} \sigma_i(\theta_t)\, dB_t^i, \tag{4.3}$$

where

$$\mu_{CNKK}(\theta)(x,u) = -\sum_{i=1}^{d} \sigma^i(\theta)(x,u) \int_0^x \sigma^i(\theta)(y,u)dy \tag{4.4}$$

for $x \geq 0$ and $u \in \mathbb{R}^n$.

5 Finite dimensional realizations for CNKK-equations

In the sequel we shall consider particular vector fields σ, which only depend on the state θ via a tenor $0 \leq x_1, \ldots, x_n$ of times-to-maturity. As in interest rate theory such vector fields allow for a geometric analysis of solutions of CNKK-equations.

This idea is (generalized and) expressed in terms of the following regularity and non-degeneracy assumptions. Recall that G is a Hilbert space of continuous complex-valued functions defined on the strip $-\mathrm{i}[0,1]^n \times \mathbb{R}^n$, i.e. $G \subset C((-\mathrm{i}[0,1]^n) \times \mathbb{R}^n; \mathbb{C})$.

Definition

We call the volatility vector fields $\sigma_1, \dots, \sigma_d$ of a CNKK-equation *tenor-dependent* if

- we have that

$$\sigma_i(\theta) = \phi_i(\ell(\theta)), \quad 1 \leq i \leq d,$$

  where $\ell \in L(H, G^p)$, for some $p \in \mathbb{N}$, and $\phi_1, \dots, \phi_d : G^p \to D(A^\infty)$ are smooth and pointwise linearly independent maps. Moreover

$$\mu_{CNKK}(\theta) = \phi_0(\ell(\eta)),$$

  where $\phi_0 : G^p \to D(A^\infty)$ is smooth. We usually have to assume $\ell_1(\eta) = \eta(0, .))$;

- for every $q \geq 0$, the map

$$\left(\ell, \ell \circ (d/dx), \dots, \ell \circ (d/dx)^q\right) : D\left((d/dx)^\infty\right) \to G^{p(q+1)}$$

  is open; and

- A is an unbounded linear operator; that is, $D(A)$ is a strict subset of H. Equivalently, $A : D(A^\infty) \to D(A^\infty)$ is not a Banach map.

Theorem

Let $\sigma_1, \ldots, \sigma_d$ be a tenor-dependent volatility structure of a CNKK-equation. Assume furthermore that for initial values in a large enough subset of Γ_n the local mild solutions θ of the CNKK-equation leave leaves of a given foliation with constant dimension $N \geq 2$ locally invariant (regular finite dimensional realization).

Then there exist $\lambda_1, \ldots, \lambda_{N-1}$ such that $\sigma_i(\theta) \in \mathrm{span}(\lambda_1, \ldots, \lambda_{N-1})$. This means in particular that

$$\theta_t(x, u) = A_t(x, u) + \sum_{i=1}^{N-1} \lambda_i(x, u) Y_t^i$$

up to some stopping time τ, for $x \geq 0$ and $u \in \mathbb{R}^n$.

Remark

The affine character of the representation of the solution process θ is apparent. In particular this representation leads via the conditional expectation formula (in case of global solution of the CNKK-equation) to affine factor processes Y and a homogenous, time-inhomogenous affine process (X, Y).

- formulate the CNKK-equation as SPDE.
- show the literatal equivalence to the HJM-equation in case of "no leverage".
- apply the geometric reasonings from interest rate theory to this general case and conclude the central importance of affine processes.

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