Efficient pricing of Asian options based on Fourier cosine expansions

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Outline

- Introduction on the COS pricing method.
- ASCOS pricing method for Asian options.
  - Recover the PDF in the risk–neutral pricing formula in terms of its characteristic function by Fourier expansions.
  - Calculation of the characteristic function recursively using Fourier expansions and Clenshaw–Curtis quadrature.
- Error convergence and computational cost.
- Extensions of our pricing method.
- Numerical results and conclusions.

COS pricing method (Fang, Oosterlee)

We start from the risk–neutral formula, where the price of an option without early–exercise features is written as an expectation of the discounted payoff at maturity time.

$$v(x, t_0) = e^{-r\Delta t} \int_{\mathbf{R}} v(y, T) f(y|x) dy.$$

Several numerical methods can be used to calculate the option price.
Our pricing method is based on the Fourier cosine expansion of the conditional density function.
First we truncate the infinite integration range of the Risk-Neutral formula

$$v(x, t_0) = e^{-r\Delta t} \int_a^b v(y, T) f(y|x) dy.$$

Here truncation range $[a, b]$ can be determined by cumulants of underlying process.

COS pricing method (Fang, Oosterlee)

Then the conditional density function of the underlying is approximated in terms of its characteristic function through Fourier cosine expansions.

$$f(y|x) \approx \frac{2}{b-a}\sum_{k=0}^{N-1}{}' Re(\varphi(\frac{k\pi}{b-a};x)\exp(-i\frac{ak\pi}{b-a}))\cos(k\pi\frac{y-a}{b-a}).$$

By replacing $f(y|x)$ by its approximation and interchanging integration and summation, we obtain the COS algorithm for option pricing

$$v(x,t_0) = e^{-r\Delta t}\sum_{k=0}^{N-1}{}' Re(\phi(\frac{k\pi}{b-a};x)e^{-ik\pi\frac{a}{b-a}})V_k,$$

where V_k is the Fourier Cosine coefficient of option value $v(y,T)$.

$$V_k = \frac{2}{b-a}\int_a^b v(y,T)\cos(k\pi\frac{y-a}{b-a}).$$

Asian options

- Asian option is written on the (arithmetic, geometric, harmonic) average of the underlying process over the time.
- For a fixed strike Asian options, the average plays the part of the underlying and for a floating strike Asian options, the average plays the part of the strike.
- The volatility inherent in the average process is less than that in the underlying process, thus an Asian option is usually cheapter than a European option. An Asian option protects us from price fluctuation in the market.
- Asian option could function as a hedging tool in place of a basket of European options with different maturities.

Pricing of Asian options

- We focus on Lévy processes, which are governed by independent and stationary increments through time.
- Our pricing method is based on the recovery of the characteristic function of the average value (or the sum) of the underlying asset through time.
- For geometric Asian options, the characteristic function of the geometric average can be calculated directly.
- For arithmetic Asian options, the characteristic function of the arithmetic average needs to be approximated recursively.
- Continuously–monitored Asian option price is derived from discretely–monitored Asian option, in combination with extrapolation.

Pricing of Arithmetic Asian Options

The payoff function of an arithmetic Asian option reads

$$v(S,T) \equiv g(S) = \max(\delta(\frac{1}{M+1}\sum_{j=0}^{M} S_j - K), 0)$$

with $\delta = 1$ for a call and $\delta = -1$ for a put.
A new stochastic process , Y_j, $j = 1, \cdots, M$, is introduced.

$$Y_1 = \log(\frac{S_M}{S_{M-1}}), Y_j = \log(\frac{S_{M+1-j}}{S_{M-j}}) + \log(1 + \exp(Y_{j-1})).$$

then

$$\frac{1}{M+1}\sum_{j=0}^{M} S_j = \frac{(1+\exp(Y_M))S_0}{M+1}.$$

Therefore, after the characteristic function of Y_M is recovered, the Asian option price is available from the COS formula.

Recursive Recovering of characteristic function $\phi_{Y_M}(u)$

The characteristic function of Y_M is recursively recovered. Y_j is rewritten as

$$Y_1 = R_M, Y_j = R_{M+1-j} + Z_{j-1}$$

where $\forall j, R_j := \log(\frac{S_j}{S_{j-1}})$ and $Z_j := \log(1+\exp(Y_j))$. For a Lévy process we have.

- $\forall u, \phi_{Y_1}(u) = \phi_R(u), j = 2, \cdots, M.$, where $R \overset{d}{=} R_j$.
- $\phi_{Y_j}(u) = \phi_{R_{M+1-j}}(u)\phi_{Z_{j-1}}(u) = \phi_R(u)\phi_{Z_{j-1}}(u), j = 2, \cdots, M.$

$\phi_R(u)$ is known analytically and

$$\phi_{Z_{j-1}}(u) = \mathbf{E}[e^{iu\log(1+\exp(Y_{j-1}))}] = \int_{-\infty}^{\infty} (e^x+1)^{iu} f_{Y_{j-1}}(x)dx,$$

is calculated from $\phi_{Y_{j-1}}$ via Fourier cosine expansions and Clenshaw–Curtis quadrature.

Fourier cosine expansion and Clenshaw–Curtis quadrature

We first truncate the integration range

$$\hat{\phi}_{Z_{j-1}}(u) = \int_a^b (e^x + 1)^{iu} f_{Y_{j-1}}(x) dx.$$

Then apply the Fourier cosine expansion on $f_{Y_{j-1}}(x)$, giving:

$$\begin{aligned} \hat{\phi}_{Z_{j-1}}(u) &= \frac{2}{b-a} {\sum_{l=0}^{N-1}}' Re\left(\hat{\phi}_{Y_{j-1}}(\frac{l\pi}{b-a}) \exp(-ia\frac{l\pi}{b-a})\right) \\ &\cdot \int_a^b (e^x + 1)^{iu} \cos\left((x-a)\frac{l\pi}{b-a}\right) dx, \end{aligned}$$

The integration can be rewritten in terms of incomplete Beta function, yet it is faster calculated with Clenshaw–Curtis quadrature.
In this setting, $\forall u, j$, $\phi_{Y_j}(u) = \phi_R(u)\phi_{Z_{j-1}}(u)$ is recovered from $\phi_{Y_{j-1}}$ and in the end we have the value of $\phi_{Y_M}(u)$.

Pricing of Arithmetic Asian Options

The value of the arithmetic Asian options reads

$$v(x_0, t_0) = e^{-r\Delta t} \int_{-\infty}^{\infty} v(y, T) f_{Y_M}(y) dy.$$

Truncate the integration range to $[a, b]$ and expand $f_{Y_M}(y)$ on the Fourier domain, we have

$$\hat{v}(x, t_0) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' Re\left(\hat{\phi}_{Y_M}\left(\frac{k\pi}{b-a}\right) e^{-ik\pi\frac{a}{b-a}}\right) V_k,$$

in which $\hat{\phi}_{Y_M}$ is the characteristic function and V_k is the Fourier Cosine coefficient of option value $v(y, T)$.

$$V_k = \frac{2}{b-a} \int_a^b v(y, T) \cos\left(k\pi \frac{y-a}{b-a}\right).$$

The value of V_k

From the the relation

$$\frac{1}{M+1}\sum_{j=0}^{M} S_j = \frac{(1+\exp(Y_M))S_0}{M+1}.$$

we have

$$V_k = \begin{cases} \frac{2}{b-a}\left(\frac{S_0}{M+1}\chi_k(x^*,b) + (\frac{S_0}{M+1} - K)\psi_k(x^*,b)\right), & \text{for a call,} \\ \frac{2}{b-a}\left((K - \frac{S_0}{M+1})\psi(a,x^*) - \frac{S_0}{M+1}\chi(a,x^*)\right), & \text{for a put.} \end{cases}$$

Here $x^* = \log(\frac{K(M+1)}{S_0} - 1)$, $\chi_k(x_1,x_2) := \int_{x_1}^{x_2} e^y \cos\left(k\pi\frac{y-a}{b-a}\right) dy$, and $\psi_k(x_1,x_2) := \int_{x_1}^{x_2} \cos\left(k\pi\frac{y-a}{b-a}\right) dy$.

Computing cost

Assuming that

- M is the number of monitoring dates.
- N is the number of terms in the Fourier cosine expansions.
- n_q is the number of terms in the Clenshaw–Curtis quadrature.

Then the total computational complexity for arithmetic Asian option is $O((n_q + M)N^2)$.

Error Convergence

Assuming that

- N is the number of terms in the Fourier cosine expansions.
- n_q is the number of terms in the Clenshaw–Curtis quadrature.

Then if the underlying process is governed by a smooth probability density function, the error in the Asian option price is bounded by

$$|\epsilon| \leq \bar{P}(N, n_q)(\exp(-(N-1)\nu_F) + \exp(-(n_q-1)\nu_q)),$$

where $\bar{P}(N, n_q)$ is a term which varies less than exponentially with respect to N and n_q, and $\nu_F > 0, \nu_q > 0$. That is, the error decays exponentially with respect to N and n_q.

If the density function of the underlying process or it derivative is not continuous then the error converges exponentially with respect to n_q and algebraically with respect to N.

Asian options on the harmonic average

Asian options with a payoff based on the harmonic average $M/(\sum_{j=1}^{m} 1/S_j)$ can be priced in a similar fashion by our method. Denote $\bar{R}_j = \log(S_{j-1}/S_j), j = 1, \cdots, M$ and let

$$Y_1 := \bar{R}_M, \text{and for } j = 2, \cdots, M, Y_j := \bar{R}_{M+1-j} + Z_{j-1},$$

where $\forall j, Z_j := \log(1 + \exp(Y_j))$, then $Y_M = \log(\sum_{j=1}^{m} S_0/S_j)$. Apply the COS method to the risk–neutral pricing formula and the harmonic Asian option value is then given by

$$\hat{v}(x, t_0) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' Re\left(\hat{\phi}_{Y_M}(\frac{k\pi}{b-a}) e^{-ik\pi\frac{a}{b-a}}\right) V_k.$$

The Fourier coefficient V_k is known analytically and the characteristic function of Y_M is recursively computed.

Asian options on the harmonic average

- For Lévy processes, we have that $\forall u \in \mathbf{R}$,

  $\phi_{Y_1}(u) = \phi_{\bar{R}_M}(u)$and for $j = 2, \cdots, M, \phi_{Y_j}(u) = \phi_{\bar{R}_{M+1-j}}(u)\phi_{Z_{j-1}}(u)$.

  Here $\forall j, u$, $\phi_{\bar{R}_j}(u) = \phi_{R_j}(-u)$, where $R_j = \log(S_j/S_{j-1})$, and $\phi_{Z_{j-1}}(u)$ is calculated via Clenshaw–Curtis quadrature.
- For harmonic Asian options

$$V_k = \begin{cases} \frac{2}{b-a}(MS_0\bar{\chi}_k(x^*, b) - K\psi_k(x^*, b)), & \text{for a call}, \\ \frac{2}{b-a}(K\psi(a, x^*) - MS_0\bar{\chi}(a, x^*)), & \text{for a put}, \end{cases}$$

with $x^* = \log(MS_0/K)$, $\quad \bar{\chi}(x_1, x_2) := \int_{x_1}^{x_2} e^{-y}\cos(k\pi\frac{y-a}{b-a})dy$, and $\psi_k(x_1, x_2) := \int_{x_1}^{x_2} \cos\left(k\pi\frac{y-a}{b-a}\right)dy$.

The forward contract

A forward contract, as often encountered in the commodity market, may be defined by the payoff:

$$g(S) = \frac{1}{M+1}\sum_{j=0}^{M} S_j - K.$$

Let $Y_1 := \log(\frac{S_M}{S_{M-1}})$, $Y_j := \log(\frac{S_{M+1-j}}{S_{M-j}}) + \log(1+\exp(Y_{j-1}))$, then $\frac{1}{M+1}\sum_{j=0}^{M} S_j = \frac{(1+\exp(Y_M))S_0}{M+1}$ and the value of the forward contract, obtained from risk–neutral formula, is given by

$$\begin{aligned} v(x_0, t_0) &= e^{-r\Delta t}\mathbf{E}(\frac{1}{M+1}\sum_{j=0}^{M} S_j - K) \\ &= e^{-r\Delta t}\left(\frac{S_0}{M+1}\mathbf{E}[e^{Y_M}] + (\frac{S_0}{M+1} - K)\right). \end{aligned}$$

The forward contract

The expected value of Y_M is derived recursively via

$$\mathbf{E}[e^{Y_1}] \equiv \mathbf{E}[e^R], \text{and } \mathbf{E}[e^{Y_j}] = \mathbf{E}[e^R](1 + \mathbf{E}[e^{Y_{j-1}})], \ \forall j$$

where R is the increments between (any) two consecutive monitoring dates of the Lévy process. We have

$$\mathbf{E}[e^R] = \int_{-\infty}^{\infty} e^y f_R(y) dy = {\sum_{k=0}^{N-1}}' Re \left(\phi_R(\frac{k\pi}{b-a}) e^{-ik\pi \frac{a}{b-a}} \right) \chi_k(a, b),$$

where $\bar{\chi}(x_1, x_2) := \int_{x_1}^{x_2} e^y \cos(k\pi \frac{y-a}{b-a}) dy$.

Exponential Convergence

- x–axis is index d with $N = 64d$, $n_q = 100d$
- y–axis is the logarithm with base 10 of the error, i.e. $\log_{10}(\epsilon)$.

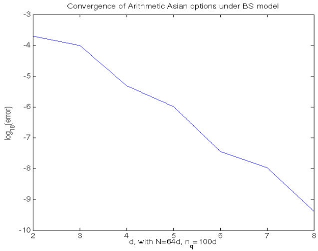

Figure: Error convergence of arithmetic Asian option, BS model, $M = 50$.

Time and error

NIG model M	time and error	$N = 128$ $n_q = 200$	$N = 256$ $n_q = 400$	$N = 384$ $n_q = 600$
12	abs.error	2.0e-3	1.71e-4	5.16e-6
	CPU time	2.41	15.13	46.09
50	abs.error	2.26e-4	6.94e-5	2.17e-6
	CPU time	2.43	15.16	46.22
250	time and error	$N = 128$ $n_q = 200$	$N = 256$ $n_q = 400$	$N = 512$ $n_q = 800$
	abs.error	7.8e-3	9.33e-5	6.94e-7
	CPU time	2.42	15.23	104.28

- Exponential convergence and robustness for all M.
- M has no significant influence on convergence behaviour, nor on the CPU time.
- Works well for other Lévy processes (e.g. CGMY) and in particular advantageous for frequently–monitored Asian options.

Continuously–monitored Asian options

Let $\hat{v}(M)$ denote the computed value of a discretely–monitored Asian option with M monitoring dates. The continuously–monitored Asian option value, denoted by $\hat{v}_\infty$, is approximated by a four-point Richardson extrapolation scheme, as follows:

$$\hat{v}_\infty(d) = \frac{1}{21}(64\hat{v}(2^{d+3}) - 56\hat{v}(2^{d+2}) + 14\hat{v}(2^{d+1}) - \hat{v}(2^d)).$$

Thanks to the non–increased computing time w.r.t M, accurate continuously–monitored arithmetic Asian option can be obtained with a large value of d without the lose of computing time.

d	$K = 90$		$K = 100$	
	Option value	CPU time	Option value	CPU time
4	12.6748	60.05	5.1191	60.01
5	12.6744	60.13	5.1186	59.94
6	12.6743	60.35	5.1185	60.17

Conclusions

- Our pricing method for Asian options works well for different underlying processes and different types of averages.
- For all underlying processes with smooth density functions, the option price converges exponentially with respect to the number of terms used in the Fourier cosine expansions and that used in the Clenshaw–Curtis quadrature.
- The computing time does not change significantly as the number of monotoring dates goes up, which makes our pricing method advantageous for frequently monotored Asian options.
- Our pricing method is extended to harmonic Asian options, forward contract with Asian payoff, and Asian options with early–exercise feature.