An Equity-Interest Rate Hybrid Model with Stochastic Volatility and the Interest Rate Smile

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Empirical evidence for non-zero correlation

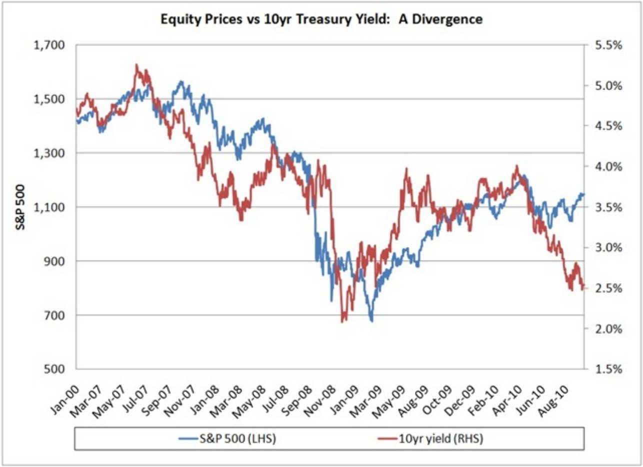

Figure: Equity prices vs 10y Treasury Yield. Source: “Shifting Correlations” in Seeking Alpha.

The Objectives of the Research

To build an Equity-Interest Rate Hybrid model which:

- $\Rightarrow$ generates a smile on the equity side;
- $\Rightarrow$ includes stochastic interest rate with interest rate smile;
- $\Rightarrow$ enables non-zero correlations between the underlying processes;
- $\Rightarrow$ allows efficient calibration;

The Heston Model and Short-Rate Interest Rate

⇒ First, the Heston-Hull-White Hybrid model:

$$
\begin{aligned}
\mathrm{d}S/S &= r\mathrm{d}t + \sqrt{\sigma}\mathrm{d}W_x^{\mathbb{Q}}, \\
\mathrm{d}\sigma &= \kappa(\bar{\sigma}-\sigma)\mathrm{d}t + \gamma\sqrt{\sigma}\mathrm{d}W_\sigma^{\mathbb{Q}}, \\
\mathrm{d}r &= \lambda(\theta - r)\mathrm{d}t + \eta\mathrm{d}W_r^{\mathbb{Q}},
\end{aligned}
$$

with correlations: $\rho_{x,\sigma} \neq 0$, $\rho_{x,r} \neq 0$ and $\rho_{\sigma,r} \neq 0$.

⇒ With the Feynman-Kac theorem, for $x = \log S$ the corresponding PDE is given by:

$$
\begin{aligned}
r\phi = \; & \phi_t + (r - 1/2\sigma)\,\phi_x + \kappa(\bar{\sigma}-\sigma)\phi_\sigma + \lambda(\theta_t - r)\phi_r \\
& + 1/2\sigma\phi_{x,x} + 1/2\gamma^2\sigma\phi_{\sigma,\sigma} + 1/2\eta^2\phi_{r,r} \\
& + \rho_{x,\sigma}\gamma\sigma\phi_{x,\sigma} + \rho_{x,r}\eta\sqrt{\sigma}\phi_{x,r} + \rho_{\sigma,r}\eta\gamma\sqrt{\sigma}\phi_{\sigma,r}.
\end{aligned}
$$

⇒ In the present form the model is not affine [Duffie *et al.* 2000].

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⇒ By linearization of the non-affine terms in the covariance matrix we find an approximation:

$$\begin{pmatrix} \sigma & \rho_{x,\sigma}\gamma\sigma & \rho_{x,r}\eta\sqrt{\sigma} \\ & \gamma^2\sigma & \rho_{\sigma,r}\eta\gamma\sqrt{\sigma} \\ & & \eta^2 \end{pmatrix} \approx \underbrace{\begin{pmatrix} \sigma & \rho_{x,\sigma}\gamma\sigma & \rho_{x,r}\eta\Psi \\ & \gamma^2\sigma & \rho_{\sigma,r}\eta\gamma\Psi \\ & & \eta^2 \end{pmatrix}}_{\mathbf{C}}.$$

⇒ We linearize the non-affine term $\sqrt{\sigma}$ by Ψ:

$$\underbrace{\Psi = \mathbb{E}(\sqrt{\sigma})}_{\text{analytic ChF}} \quad \text{or} \quad \Psi = \mathcal{N}\left(\mathbb{E}(\sqrt{\sigma}), \mathbb{V}\mathrm{ar}(\sqrt{\sigma})\right).$$

⇒ The expectation for the CIR-type process is known analytically:

⇒ Affine approximation ⇒ efficient pricing!

⇒ The model with the modified covariance structure, **C**, constitutes the affine version of non-affine model.

⇒ We set: $\kappa = 0.5$, $\gamma = 0.1$, $\lambda = 1$, $\eta = 0.01$, $\theta = 0.04$ and $\rho_{x,\sigma} = -50\%$, $\rho_{x,r} = 60\%$.

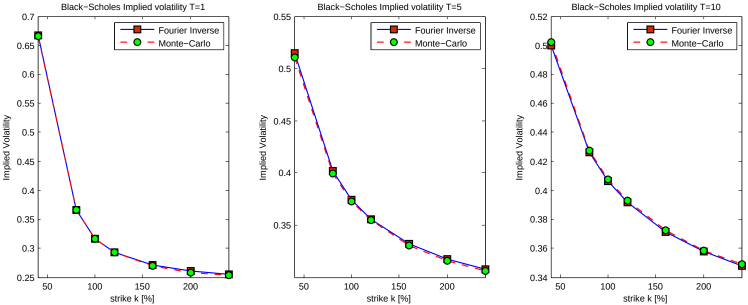

Figure: Comparison of implied Black-Scholes volatilities from Monte Carlo (40.000 paths and 500 steps) and Fourier inversion.

Intermediate Summary

⇒ The linearization method provides a high quality approximation;

⇒ The projection procedure can be simply extended to high dimensions;

⇒ The method is straightforward, and does not involve complex techniques;

⇒ Alternative methods for approximating the hybrid models are:

- Markovian projection based methods [Antonov-2008].
- Models with indirect correlation structure [Giese-2004, Andreasen-2006];

The Heston Model and the SV Libor Market Model

⇒ We now consider the Stochastic Volatility Libor Market Model [Andersen, Brotherton-Ratcliffe-2005], [Andersen, Andreasen-2000]. For $L_k := L(t, T_{k-1}, T_k)$ we define

$$L(t, T_{k-1}, T_k) \equiv \frac{1}{\tau_k}\left(\frac{P(t, T_{k-1})}{P(t, T_k)} - 1\right), \text{ for } t < T_{k-1},$$

with the dynamics under *their natural* measure given by:

$$\begin{cases} \mathrm{d}L_k = & \sigma_k\left(\beta_k L_k + (1-\beta_k)L_k(0)\right)\sqrt{V}\mathrm{d}W_k^k, \\ \mathrm{d}V = & \lambda(V(0) - V)\mathrm{d}t + \eta\sqrt{V}\mathrm{d}W_V^k, \end{cases}$$

with $\mathrm{d}W_i^k \mathrm{d}W_j^k = \rho_{i,j}\mathrm{d}t$, for $i \neq j$ and $\mathrm{d}W_V^k \mathrm{d}W_i^k = 0$.

⇒ Efficient calibration with Markovian Projection Method [Piterbarg-2005].

⇒ Fast pricing of European- style equity options:

$$\Pi(t) = B(t)\mathbb{E}^{\mathbb{Q}}\left(\frac{(S(T_N)-K)^+}{B(T_N)}\Big|\mathcal{F}(t)\right), \text{ with } t < T_N,$$

with K the strike, $S(T_N)$ the stock price at time T_N, filtration $\mathcal{F}(t)$ and a numéraire $B(T_N)$.

⇒ The money-savings account $B(T_N)$ is assumed to be correlated with stock $S(T_N)$.

⇒ We switch between the measures: From risk neutral $\mathbb{Q}$ to the T_N-forward $\mathbb{Q}^{T_N}$:

$$\Pi(t) \quad = \quad P(t,T_N)\mathbb{E}^{T_N}\left(\left(F^{T_N}(T_N)-K\right)^+\Big|\mathcal{F}(t)\right), \text{with } t < T_N,$$

with $F^{T_N}(t)$ the forward of the stock $S(t)$, defined as:

$$F^{T_N}(t) = \frac{S(t)}{P(t,T_N)}.$$

⇒ The ZCB $P(t,T_N)$ is not well-defined for all t!

⇒ Since $P(T_{k-1}, T_{k-1}) = 1$ we find for the ZCB $P(t, T_k)$:

$$P(t, T_k) = (1 + \tau_k L(t, T_{k-1}, T_k))^{-1}.$$

⇒ For $t \neq T_{k-1}$ we use the interpolation from [Schlögl-2002]:

$$P(t, T_k) \approx (1 + (T_k - t)L(t, T_{k-1}, T_k))^{-1}, \text{ for } T_{k-1} \leq t \leq T_k.$$

⇒ This ZCB interpolation is sufficient for calibration purposes but for pricing callable exotics more attention is needed [Piterbarg-2004, Davis *et al.*-2009, Beveridge & Joshi-2009].

Derivation of the Hybrid Model

Under the T_N-forward measure we have:

⇒ An equity part is driven by the Heston model:

$$\begin{aligned} \mathrm{d}S/S &= (\dots)\mathrm{d}t + \sqrt{\xi}\mathrm{d}W_x^N, \\ \mathrm{d}\xi &= \kappa(\bar{\xi} - \xi)\mathrm{d}t + \gamma\sqrt{\xi}\mathrm{d}W_\xi^N. \end{aligned}$$

⇒ The SV Libor Market Model under the T_N-measure is given by:

$$\begin{aligned} \mathrm{d}L_k &= -\phi_k\sigma_k V \sum_{j=k+1}^{N} \frac{\tau_j\phi_j\sigma_j}{1+\tau_j L_j}\rho_{k,j}\mathrm{d}t + \sigma_k\phi_k\sqrt{V}\mathrm{d}W_k^N, \\ \mathrm{d}V &= \lambda(V(0) - V)\mathrm{d}t + \eta\sqrt{V}\mathrm{d}W_V^N, \end{aligned}$$

with $\phi_k = \beta_k L_k + (1-\beta_k)L_j(0)$.

⇒ We assume non-zero correlation between asset $S(t)$ and Libor rates $L_j(t)$.

Deriving the Forward Dynamics

⇒ The forward F^{T_N} is a martingale under the T_N-forward measure:

$$\mathrm{d}F^{T_N}(t) = \frac{1}{P(t,T_N)}\mathrm{d}S(t) - \frac{S(t)}{P^2(t,T_N)}\mathrm{d}P(t,T_N).$$

⇒ Dynamics for $S(t)$ are known (the Heston model), for ZCB $P(t,T_N)$ we find:

$$\frac{1}{P(t,T_N)} = \underbrace{\left(1+\left(T_{m(t)}-t\right)L_{m(t)}\left(T_{m(t)-1}\right)\right)}_{\text{interpolation}}\underbrace{\prod_{j=m(t)+1}^{N}\left(1+\tau_j L(t,T_{j-1},T_j)\right)}_{\text{rolling}}.$$

with $m(t) = \min\{k : t \le T_k\}$.

⇒ For the ZCB $P(t, T_N)$ we are only interested in diffusion coefficients:

$$\frac{\mathrm{d}P(t,T_N)}{P(t,T_N)} = (\dots)\mathrm{d}t - \sqrt{V}\sum_{j=m(t)+1}^{N}\frac{\tau_j\sigma_j\phi_j}{1+\tau_j L_j}\mathrm{d}W_j^N.$$

⇒ The forward $F^{T_N}(t)$ dynamics are now given by:

$$\frac{\mathrm{d}F^{T_N}}{F^{T_N}} = \underbrace{\sqrt{\xi}\mathrm{d}W_x^N}_{\text{asset}} + \underbrace{\sqrt{V}\sum_{j=m(t)+1}^{N}\frac{\tau_j\sigma_j\phi_j}{1+\tau_j L_j}\mathrm{d}W_j^N}_{\text{interest rate}}.$$

⇒ The model is not affine !

The Hybrid Model Approximation

⇒ We *freeze* the Libor rates [Glasserman,Zhao-1999], [Hull,White-1996], [Jäckel,Rebonato-2000], i.e.:

$$L_j(t) \approx L_j(0) \quad \Rightarrow \quad \phi_j(t) \approx L_j(0).$$

⇒ Now, the linearized dynamics are given by:

$$\frac{\mathrm{d}F^{T_N}}{F^{T_N}} \approx \sqrt{\xi}\mathrm{d}W_x^N + \sqrt{V}\sum_{j=m(t)+1}^{N}\frac{\tau_j\sigma_j L_j(0)}{1+\tau_j L_j(0)}\mathrm{d}W_j^N.$$

⇒ The model does not depend on the Libor processes ! It is fully described by the volatility structure.

⇒ The model is now given by:

$$\begin{aligned}\mathrm{d}F^{T_N}/F^{T_N} &\approx \sqrt{\xi}\mathrm{d}W_x^N + \sqrt{V}\mathbf{\Sigma}^\mathsf{T}\mathrm{d}\mathbf{W}^N,\\ \mathrm{d}\xi &= \kappa(\bar{\xi}-\xi)\mathrm{d}t + \gamma\sqrt{\xi}\mathrm{d}W_\xi^N,\\ \mathrm{d}V &= \lambda(V(0)-V)\mathrm{d}t + \eta\sqrt{V}\mathrm{d}W_V^N,\end{aligned}$$

with appropriate column vectors $\mathbf{\Sigma}$ and $\mathrm{d}\mathbf{W}^N$.

⇒ Under the log-transform, $x = \log F^{T_N}$, we find:

$$\mathrm{d}x \approx -\frac{1}{2}\left(\sqrt{\xi}\mathrm{d}W_x^N + \sqrt{V}\mathbf{\Sigma}^\mathsf{T}\mathrm{d}\mathbf{W}^N\right)^2 + \sqrt{\xi}\mathrm{d}W_x^N + \sqrt{V}\mathbf{\Sigma}^\mathsf{T}\mathrm{d}\mathbf{W}^N.$$

⇒ Since $\mathrm{d}W_x^N$ is correlated with $\mathrm{d}\mathbf{W}^N$ cross terms are still not affine!

⇒ We set: $\mathcal{A} = m(t)+1,\ldots,N$ and $\psi_j = \frac{\tau_j\sigma_j L_j(0)}{1+\tau_j L_j(0)}$.

⇒ The dynamics for $x = \log F^{T_N}$ are given by:

$$\mathrm{d}x \approx -\frac{1}{2}\left(\xi + A_1(t)V + 2\sqrt{V}\sqrt{\xi}A_2(t)\right)\mathrm{d}t + \sqrt{\xi}\mathrm{d}W_x^N + \sqrt{V}\mathbf{\Sigma}^{\mathrm{T}}\mathrm{d}\mathbf{W}^N.$$

⇒ $A_1(t)$ and $A_2(t)$ are deterministic piecewise constant functions!

⇒ The drift and covariance matrix include the non-affine term $\sqrt{V}\sqrt{\xi}$, we linearize it by:

$$\begin{aligned}\sqrt{\xi}\sqrt{V} &\approx \mathbb{E}(\sqrt{\xi}\sqrt{V}) \\ &\stackrel{\perp\!\!\!\perp}{=} \mathbb{E}(\sqrt{\xi})\mathbb{E}(\sqrt{V}) =: \vartheta(t).\end{aligned}$$

Iterative Characteristic Function

⇒ With Feynman-Kac theorem we find the corresponding PDE:

$$
\begin{aligned}
0 &= \phi_t + 1/2\left(\xi + A_1 V + 2A_2\vartheta(t)\right)\left(\phi_{x,x} - \phi_x\right) \\
&+ \kappa(\bar{\xi} - \xi)\phi_\xi + \lambda(V(0) - V)\phi_V + 1/2\eta^2 V\phi_{V,V} \\
&+ 1/2\gamma^2\xi\phi_{\xi,\xi} + \rho_{x,\xi}\gamma\xi\phi_{x,\xi},
\end{aligned}
$$

subject to $\phi(u, \mathbf{X}(T), 0) = \exp(iux(T_N))$.

⇒ The corresponding characteristic function is given by:

$$\phi(u, \mathbf{X}(t), \tau) = \exp(A(u,\tau) + iux(t) + B(u,\tau)\xi(t) + C(u,\tau)V(t)),$$

with $\tau = T_N - t$.

⇒ The ODEs for $A(u,\tau)$, $B(u,\tau)$, $C(u,\tau)$ are of Heston-type and can be solved recursively [Andersen,Andreasen-2000].

Quality of the Approximations

⇒ We price an equity call option and investigate the accuracy of the approximation.

⇒ For equity we take:
$\kappa = 1.2, \quad \bar{\xi} = 0.1, \quad \gamma = 0.5, \quad S(0) = 1, \quad \xi(0) = 0.1.$

⇒ For the interest rate model we take term structure:
$P(0,T) = \exp\left(-0.05T\right)$, with
$\beta_k = 0.5, \quad \sigma_k = 0.25, \quad \lambda = 1, \quad V(0) = 1, \quad \eta = 0.1.$

⇒ The correlation structure is given by:

$$\left(\begin{array}{c|cccc} 1 & \rho_{x,\xi} & \rho_{x,1} & \dots & \rho_{x,N} \\ \hline \rho_{\xi,x} & 1 & \rho_{\xi,1} & \dots & \rho_{\xi,N} \\ \rho_{1,x} & \rho_{1,\xi} & 1 & \dots & \rho_{1,N} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{N,x} & \rho_{N,\xi} & \rho_{N,1} & \dots & 1 \end{array}\right) = \left(\begin{array}{c|cccc} 1 & -30\% & 50\% & \dots & 50\% \\ \hline -30\% & 1 & 0 & \dots & 0 \\ 50\% & 0 & 1 & \dots & 98\% \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 50\% & 0 & 98\% & \dots & 1 \end{array}\right).$$

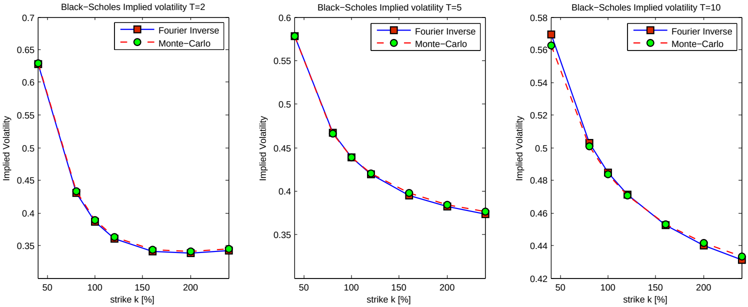

Figure: Comparison of implied Black-Scholes volatilities for the European equity option, obtained by Fourier inversion of approximation and by Monte Carlo simulation.

Equity Options and IR skew

⇒ We investigate the effect of β on equity implied vol. with Monte Carlo simulation of the full-scale model:

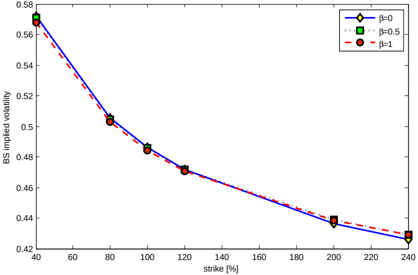

Figure: The effect of the interest rate skew, controlled by β_k, on the equity implied volatilities. The Monte Carlo simulation was performed with for maturity $T = 10$.

⇒ The prices of the European style options are rather insensitive to skew parameter β!

Example: Pricing a Hybrid Product

- ⇒ We consider an investor who is willing to take some risk in one asset class in order to obtain a participation in a different asset class.
- ⇒ An example of such hybrid product is *minimum of several assets* [Hunter-2005] with payoff defined as:

$$\text{Payoff} = \max\left(0, \min\left(C_n(T), k\% \times \frac{S(T)}{S(t)}\right)\right),$$

where $C_n(T)$ is an n-years CMS, and $S(T)$ is a stock.

- ⇒ By taking $\mathcal{T} = \{1, 2, ..., 10\}$ and the payment date $T_N = 5$ we get:

$$\frac{\Pi_{\mathrm{H}}(t)}{P(t, T_5)} = \mathbb{E}^{T_5}\left[\max\left(0, \min\left(\frac{1 - P(T_5, T_{10})}{\sum_{k=6}^{10} P(T_5, T_k)}, k\% \times \frac{S(T_5)}{S(t)}\right)\right) \Big| \mathcal{F}(t)\right].$$

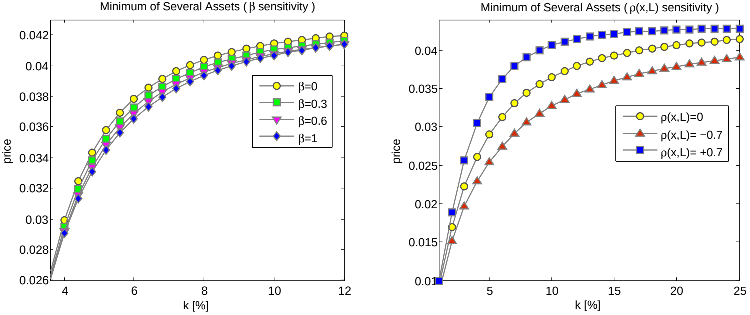

Figure: The value for a *minimum of several assets* hybrid product. The prices are obtained by Monte Carlo simulation with 20.000 paths and 20 intermediate points. Left: Influence of β; Right: Influence of $\rho_{x,L}$.

Now, we compare the results with Heston-Hull-White model

- ⇒ From calibration routine we have: $\lambda = 0.0614$, $\eta = 0.0133$, $r_0 = 0.05$ and $\kappa = 0.65$, $\gamma = 0.469$, $\bar{\xi} = 0.090$, $\rho_{x,\xi} = -0.222$ and $\xi_0 = 0.114$.
- ⇒ Calibration ensures that prices on the equities are the same, so the hybrid price differences can only result from the interest rate component!

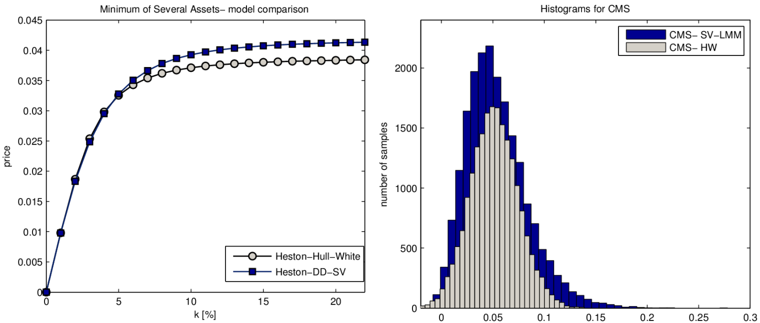

Figure: LEFT: Hybrid prices obtained by two different hybrid models, H-LMM and HHW. The models were calibrated to the same data set., RIGHT: CMS rate for the SV LMM and the Hull-White models.

Conclusion

⇒ We have developed an efficient approximation method projecting non-affine models on affine versions;

⇒ The models with modified covariance structure are affine *by construction*;

⇒ We have presented an extension of the Heston model with stochastic interest rates:

- Short-rate processes;
- SV LMM;

⇒ The model can be easily generalized to FX and Inflation;

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