Alternating direction implicit schemes for multi-dimensional PDEs in finance

Karel in 't Hout

Joint work with: Tinne Haentjens

University of Antwerp
Department of Mathematics and Computer Science

Winterschool Lunteren

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Contents

Option pricing in the HHW model

European call option gives the holder the right to buy a given asset at a prescribed *maturity* date T for a prescribed *strike* price K.

Let S_τ denote the price of the asset at time $\tau \geq 0$.

The *payoff* of the call option is $\phi(S_T) = \max(0, S_T - K)$.

For the evolution of S_τ we consider the Heston–Hull–White model:

$$\begin{cases} dS_\tau &= R_\tau\, S_\tau\, d\tau + \sqrt{V_\tau}\, S_\tau\, dW_\tau^1\,, \\ dV_\tau &= \kappa(\eta - V_\tau)\, d\tau + \sigma_1 \sqrt{V_\tau}\, dW_\tau^2\,, \\ dR_\tau &= a(b(\tau) - R_\tau)\, d\tau + \sigma_2\, dW_\tau^3 \end{cases}$$

with real parameters κ, η, σ_1, a, σ_2 and deterministic function b.

W_τ^1, W_τ^2, W_τ^3 are Brownian motions having correlation factors $\rho_{12}\,,\ \rho_{13}\,,\ \rho_{23} \in [-1, 1]$.

Let $u(s, v, r, t)$ denote the fair value of the European call option if $S_\tau = s$, $V_\tau = v$, $R_\tau = r$ where $\tau = T - t$, $0 \le t \le T$.

Financial option valuation theory yields that u satisfies a parabolic three-dimensional PDE,

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \tfrac{1}{2}s^2 v\frac{\partial^2 u}{\partial s^2} + \tfrac{1}{2}\sigma_1^2 v\frac{\partial^2 u}{\partial v^2} + \tfrac{1}{2}\sigma_2^2\frac{\partial^2 u}{\partial r^2} \\
&\quad + \rho_{12}\sigma_1 sv\frac{\partial^2 u}{\partial s\partial v} + \rho_{13}\sigma_2 s\sqrt{v}\frac{\partial^2 u}{\partial s\partial r} + \rho_{23}\sigma_1\sigma_2\sqrt{v}\frac{\partial^2 u}{\partial v\partial r} \\
&\quad + rs\frac{\partial u}{\partial s} + \kappa(\eta - v)\frac{\partial u}{\partial v} + a\left(b(T-t) - r\right)\frac{\partial u}{\partial r} - ru
\end{aligned}$$

for $s > 0$, $v > 0$, $-\infty < r < \infty$, $0 < t \le T$. This is the HHW PDE.

Note: degenerate boundary $v = 0$.

For feasibility of the numerical solution, the spatial domain is restricted to bounded set $[0, S_{\max}] \times [0, V_{\max}] \times [-R_{\max}, R_{\max}]$ with $S_{\max}$, $V_{\max}$, $R_{\max}$ taken sufficiently large.

The payoff gives the initial condition

$$u(s, v, r, 0) = \phi(s).$$

Boundary conditions:

$$\begin{aligned} u(s, v, r, t) &= 0 \quad \text{whenever} \quad s = 0, \\ \frac{\partial u}{\partial s}(s, v, r, t) &= 1 \quad \text{whenever} \quad s = S_{\max}, \\ u(s, v, r, t) &= s \quad \text{whenever} \quad v = V_{\max}, \\ \frac{\partial u}{\partial r}(s, v, r, t) &= 0 \quad \text{whenever} \quad r = \pm R_{\max}. \end{aligned}$$

Note: no assumptions are made about the Feller condition.

Semi-discretization HHW problem

The HHW PDE is semi-discretized on a Cartesian grid by replacing all spatial derivatives with suitable finite differences (FD).

Let $f : \mathbb{R} \to \mathbb{R}$ be any given function and $x_i = i \cdot \Delta x$ ($i \in \mathbb{Z}$), $\Delta x > 0$.

Three FD formulas for the first derivative:

$$
\begin{aligned}
f'(x_i) &\approx \left[\tfrac{1}{2}\, f_{i-2} - 2\, f_{i-1} + \tfrac{3}{2}\, f_i\right] / \Delta x, \\
f'(x_i) &\approx \left[-\tfrac{1}{2}\, f_{i-1} + \tfrac{1}{2}\, f_{i+1}\right] / \Delta x, \\
f'(x_i) &\approx \left[-\tfrac{3}{2}\, f_i + 2\, f_{i+1} - \tfrac{1}{2}\, f_{i+2}\right] / \Delta x.
\end{aligned}
$$

These formulas are applied for $\partial u/\partial s$, $\partial u/\partial v$, $\partial u/\partial r$.

For the second derivative:

$$
f''(x_i) \approx [f_{i-1} - 2\, f_i + f_{i+1}] / (\Delta x)^2.
$$

This FD formula is used for $\partial^2 u/\partial s^2$, $\partial^2 u/\partial v^2$, $\partial^2 u/\partial r^2$.

Next suppose $f : \mathbb{R}^2 \to \mathbb{R}$ and $y_j = j \cdot \Delta y$ ($j \in \mathbb{Z}$), $\Delta y > 0$.

For the mixed derivative:

$$\frac{\partial^2 f}{\partial x \partial y}(x_i, y_j) \approx$$

$$\left[\tfrac{1}{4}\, f_{i-1,j-1} - \tfrac{1}{4}\, f_{i-1,j+1} - \tfrac{1}{4}\, f_{i+1,j-1} + \tfrac{1}{4}\, f_{i+1,j+1}\right] / (\Delta x \Delta y).$$

This FD formula is used for $\partial^2 u/\partial s \partial v$, $\partial^2 u/\partial s \partial r$, $\partial^2 u/\partial v \partial r$.

All FD formulas above have a second-order truncation error.

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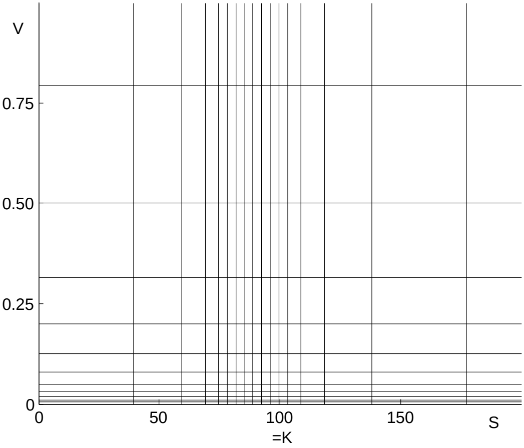
All FD formulas above have a second-order truncation error.

Actual applications: non-uniform grid in the (s, v, r)-domain.
Both for numerical and financial reasons.

Sample grid, cross-section with (s, v)-plane:



Time integration semi-discrete HHW problem

FD discretization of the HHW problem yields an initial value problem for very large system of stiff ordinary differential equations (ODEs)

$$U'(t) = A(t)\,U(t) + g(t) \quad (0 < t \leq T), \quad U(0) = U_0$$

with given matrix $A(t)$ and vectors $g(t)$, U_0.

Standard implicit numerical methods such as the trapezoidal rule (Crank–Nicolson) are often not effective.

For the numerical time integration of the ODE system we study splitting schemes of the Alternating Direction Implicit (ADI) type.

Splitting:

$$A(t) = A_0 + A_1 + A_2 + A_3(t)$$

where

- A_0 represents $\partial^2 u/\partial s \partial v$, $\partial^2 u/\partial s \partial r$, $\partial^2 u/\partial v \partial r$ terms (!)

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Assume $g(t) \equiv 0$. Let $\Delta t > 0$ and grid points $t_n = n \cdot \Delta t$.

Four ADI schemes yielding $U_n \approx U(t_n)$ $(n = 1, 2, 3, \ldots)$:

Douglas (Do) scheme

$$\begin{cases} Y_0 = U_{n-1} + \Delta t \, A(t_{n-1}) U_{n-1}, \\ Y_j = Y_{j-1} + \theta \, \Delta t \, A_j \left(Y_j - U_{n-1} \right) \quad (j = 1, 2), \\ Y_3 = Y_2 + \theta \, \Delta t \left(A_3\left(t_n\right) Y_3 - A_3\left(t_{n-1}\right) U_{n-1} \right), \\ U_n = Y_3. \end{cases}$$

Classical order is 1 for all θ.

Craig–Sneyd (CS) scheme

$$
\begin{cases}
Y_0 = U_{n-1} + \Delta t\, A(t_{n-1})\, U_{n-1}, \\
Y_j = Y_{j-1} + \theta\, \Delta t\, A_j \left(Y_j - U_{n-1} \right) \quad (j = 1, 2), \\
Y_3 = Y_2 + \theta\, \Delta t \left(A_3(t_n)\, Y_3 - A_3(t_{n-1})\, U_{n-1} \right), \\
\widetilde{Y}_0 = Y_0 + \frac{1}{2}\, \Delta t\, A_0 \left(Y_3 - U_{n-1} \right), \\
\widetilde{Y}_j = \widetilde{Y}_{j-1} + \theta\, \Delta t\, A_j \left(\widetilde{Y}_j - U_{n-1} \right) \quad (j = 1, 2), \\
\widetilde{Y}_3 = \widetilde{Y}_2 + \theta\, \Delta t \left(A_3(t_n)\, \widetilde{Y}_3 - A_3(t_{n-1})\, U_{n-1} \right), \\
U_n = \widetilde{Y}_3.
\end{cases}
$$

Classical order is 2 iff $\theta = \frac{1}{2}$.

Modified Craig–Sneyd (MCS) scheme

$$
\begin{cases}
Y_0 = U_{n-1} + \Delta t\, A(t_{n-1})\, U_{n-1}, \\[2ex]
Y_j = Y_{j-1} + \theta\, \Delta t\, A_j \left(Y_j - U_{n-1} \right) \quad (j = 1, 2), \\[2ex]
Y_3 = Y_2 + \theta\, \Delta t \left(A_3(t_n)\, Y_3 - A_3(t_{n-1})\, U_{n-1} \right), \\[2ex]
\widehat{Y}_0 = Y_0 + \theta\, \Delta t\, A_0 \left(Y_3 - U_{n-1} \right), \\[2ex]
\widetilde{Y}_0 = \widehat{Y}_0 + \left(\tfrac{1}{2} - \theta \right) \Delta t \left(A(t_n)\, Y_3 - A(t_{n-1})\, U_{n-1} \right), \\[2ex]
\widetilde{Y}_j = \widetilde{Y}_{j-1} + \theta\, \Delta t\, A_j \left(\widetilde{Y}_j - U_{n-1} \right) \quad (j = 1, 2), \\[2ex]
\widetilde{Y}_3 = \widetilde{Y}_2 + \theta\, \Delta t \left(A_3(t_n)\, \widetilde{Y}_3 - A_3(t_{n-1})\, U_{n-1} \right), \\[2ex]
U_n = \widetilde{Y}_3.
\end{cases}
$$

Classical order is 2 for all θ.

Hundsdorfer–Verwer (HV) scheme

$$
\begin{cases}
Y_0 = U_{n-1} + \Delta t\, A\,(t_{n-1})\, U_{n-1}, \\
Y_j\; = Y_{j-1} + \theta\, \Delta t\, A_j\, \big(Y_j - U_{n-1}\big) \quad (j = 1, 2), \\
Y_3 = Y_2 + \theta\, \Delta t\, \big(A_3\,(t_n)\, Y_3 - A_3\,(t_{n-1})\, U_{n-1}\big), \\
\widetilde{Y}_0 = Y_0 + \frac{1}{2}\, \Delta t\, \left(A\,(t_n)\, Y_3 - A\,(t_{n-1})\, U_{n-1}\right), \\
\widetilde{Y}_j = \widetilde{Y}_{j-1} + \theta\, \Delta t\, A_j\, (\widetilde{Y}_j - Y_3) \quad (j = 1, 2), \\
\widetilde{Y}_3 = \widetilde{Y}_2 + \theta\, \Delta t\, A_3\,(t_n)\, (\widetilde{Y}_3 - Y_3), \\
U_n = \widetilde{Y}_3.
\end{cases}
$$

Classical order is 2 for all θ.

References

- Peaceman & Rachford (1955)
- Douglas & Rachford (1956)
- Brian (1961)
- Douglas (1962)
- McKee & Mitchell (1970)
- Van der Houwen & Verwer (1979)
- Craig & Sneyd (1988)
- McKee, Wall & Wilson (1996)
- Verwer, Spee, Blom & Hundsdorfer (1999)
- Hundsdorfer (1999, 2002)
- Lanser, Blom & Verwer (2001)
- Hundsdorfer & Verwer (2003)
- In 't H. & Welfert (2007,09)
- —— & Foulon (2010)
- —— & Mishra (2010,11)
- Haentjens & In 't H. (2012)

Stability analysis

Stability analysis based on linear scalar test equation

$$U'(t) = (\lambda_0 + \lambda_1 + \cdots + \lambda_k)U(t)$$

with complex constants λ_j $(0 \leq j \leq k)$, integer $k \geq 2$.

Application any given ADI scheme leads to iteration

$$U_n = M(z_0, z_1, \ldots, z_k)\, U_{n-1}$$

with multivariate rational function M and $z_j = \Delta t\, \lambda_j$.

Iteration stable if

$$|M(z_0, z_1, \ldots, z_k)| \leq 1.$$

Write

$$z = \sum_{j=1}^{k} z_j \quad \text{and} \quad p = \prod_{j=1}^{k} (1 - \theta z_j).$$

$M = R,\ \widetilde{S},\ S,\ T$ resp. for the Do, CS, MCS, HV schemes:

$$
\begin{aligned}
R(z_0, z_1, \ldots, z_k) &= 1 + \frac{z_0 + z}{p}\,, \\
\widetilde{S}(z_0, z_1, \ldots, z_k) &= 1 + \frac{z_0 + z}{p} + \frac{1}{2}\,\frac{z_0(z_0 + z)}{p^2}\,, \\
S(z_0, z_1, \ldots, z_k) &= 1 + \frac{z_0 + z}{p} + \theta\,\frac{z_0(z_0 + z)}{p^2} + \left(\tfrac{1}{2} - \theta\right)\frac{(z_0 + z)^2}{p^2}\,, \\
T(z_0, z_1, \ldots, z_k) &= 1 + 2\,\frac{z_0 + z}{p} - \frac{z_0 + z}{p^2} + \frac{1}{2}\,\frac{(z_0 + z)^2}{p^2}\,.
\end{aligned}
$$

Two conditions:

- $k = 2$ and all $z_j \in \mathbb{C}:\ \ \Re z_1 \le 0,\ \Re z_2 \le 0,\ |z_0| \le 2\sqrt{\Re z_1 \Re z_2}\,;$
- $k \ge 2$ and all $z_j \in \mathbb{R}:\ \ z_j \le 0\ (\forall j),\ z_0 + z \le 0,\ |z_0| \le \sum_{i \ne j} \sqrt{z_i z_j}\,.$

Stability results

Unconditional stability results - in von Neumann sense - for ADI schemes when applied to semidiscretizations of 2D convection-diffusion problems with mixed derivative term:

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- In 't H. & Welfert ('07, '09):
  - CS scheme is stable if $\theta \geq \frac{1}{2}$
  - MCS scheme is stable if $\theta \geq \frac{1}{3}$ and no convection
  - HV scheme is stable if $\theta \geq 1 - \frac{1}{2}\sqrt{2}$ and no convection
  - HV scheme is stable if $\theta \geq \frac{1}{2} + \frac{1}{6}\sqrt{3}$: *conjecture*

- In 't H. & Mishra ('11):
  - MCS scheme is stable if $\frac{1}{2} \leq \theta \leq 1$
  - MCS scheme is stable if $\theta = \frac{1}{3}$ and $|\rho_{12}| < 0.96$: *conjecture*

3D pure diffusion problems with mixed derivative terms:

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- In 't H. & Mishra ('10):
  - MCS scheme is stable if $\theta \geq \max\{\frac{1}{4}\,,\,\frac{2}{13}(2\gamma+1)\}$ with $\gamma = \max_{ij} |\rho_{ij}|$

Numerical experiments

ADI schemes:

- Do with $\theta = \frac{2}{3}$
- CS with $\theta = \frac{1}{2}$
- MCS with $\theta = \max\{\frac{1}{3}\,,\ \frac{2}{13}(2\gamma + 1)\}$
- HV with $\theta = \frac{1}{2} + \frac{1}{6}\sqrt{3}$

HHW data (Bloomberg):

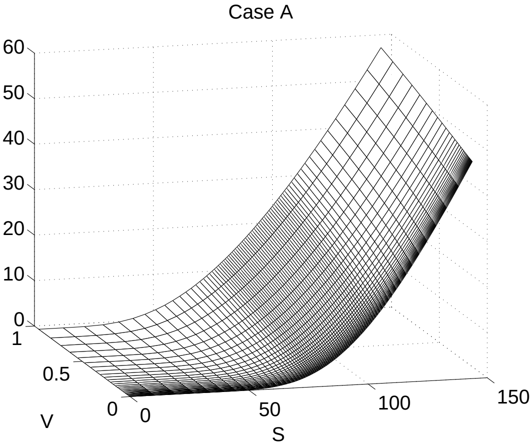
$$\kappa = 3,\ \eta = 0.12,\ \sigma_1 = 0.04,\ a = 0.2,\ \sigma_2 = 0.03$$

$$b(\tau) = c_1 - c_2 e^{-c_3\tau},\ c_1 = 0.05, c_2 = 0.01, c_3 = 1$$

$$\rho_{12} = 0.6,\ \rho_{13} = 0.2,\ \rho_{23} = 0.4$$

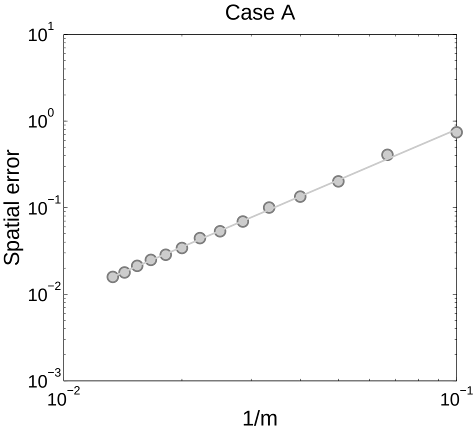
European call option, $T = 1$, $K = 100$

$r \approx 0.025$



Number of grid points (s, v, r) : $2m \times m \times m$. Nonuniform grid.

FD discretization errors if $\rho_{13} = \rho_{23} = 0$:



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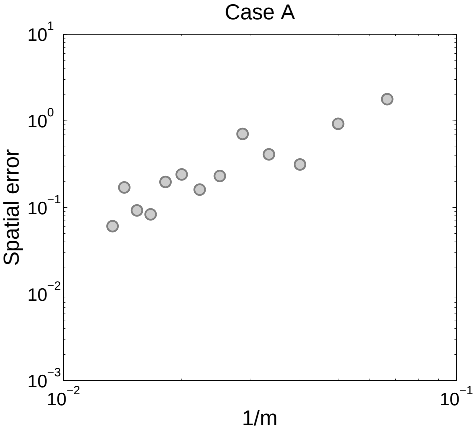
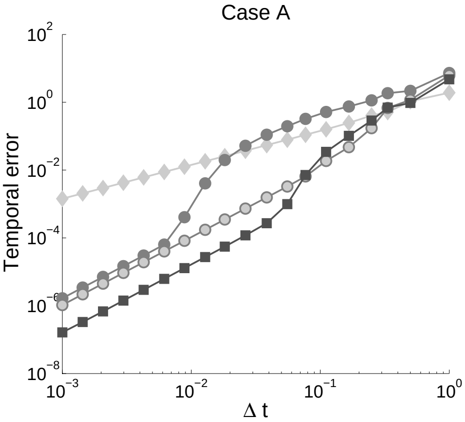
FD discretization errors if $\rho_{13} = \rho_{23} = 0$:



Number of grid points (s, v, r): $100 \times 50 \times 50$.

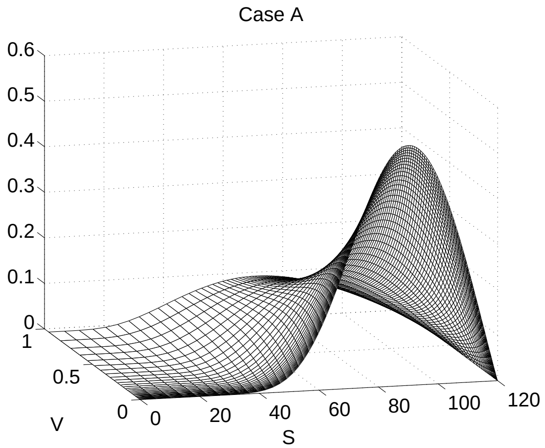
ADI discretization errors:



Do: ◇
CS: ●
MCS: ○
HV: ■

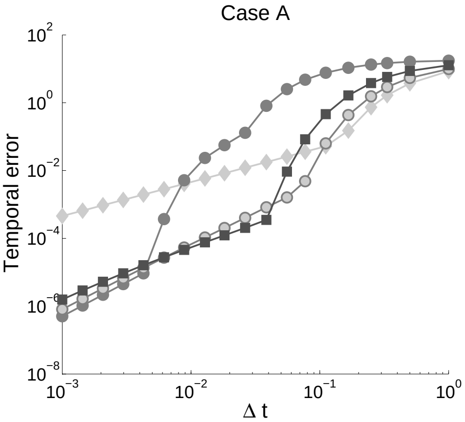
European up-and-out call option, $T = 1$, $K = 100$, $B = 120$

$r \approx 0.025$



Number of grid points (s, v, r): $100 \times 50 \times 50$.

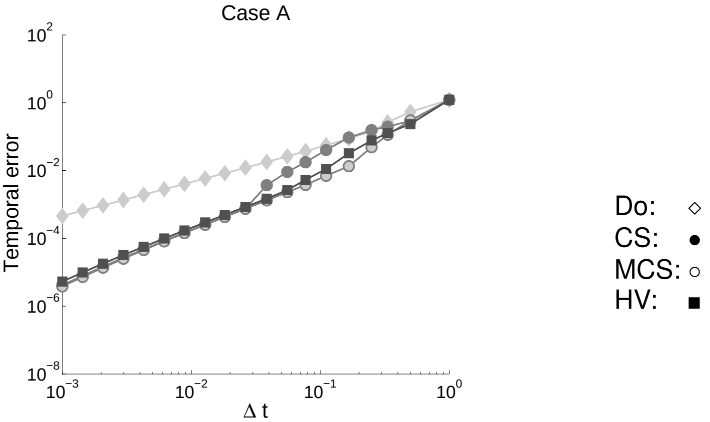
ADI discretization errors without damping:



Do: ◇
CS: ●
MCS: ○
HV: ■

Number of grid points (s, v, r): $100 \times 50 \times 50$.

ADI discretization errors with Do ($\theta = 1$) damping:



Conclusions and future research

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- MCS and HV schemes, with proper θ, preferable.
- CS with $\theta = \frac{1}{2}$ and damping good second choice.

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Current and future research:

- Application of ADI FD approach to exotic options and more advanced asset price models.
- Approximation of hedging parameters (“the Greeks”).
- Theoretical stability analysis of FD and ADI schemes.

Reference

T. Haentjens & K. J. in 't Hout, *Alternating direction implicit finite difference schemes for the Heston–Hull–White partial differential equation*, Journal of Computational Finance **16**, 83–110 (2012).