Two-dimensional COS method

Marjon Ruijter

Winterschool Lunteren

22 January 2013

Introduction

- PhD student since October 2010 (Prof.dr.ir. C.W. Oosterlee).
- CWI national research center for mathematics and computer science.
- CPB Netherlands Bureau for Economic Policy Analysis.
- Impact of climate change on investments and policy decisions.
- Financial mathematics.

Centraal Planbureau

Stochastic optimization - 2D

Climate-economics problem

Two stochastic processes: temperature and capital.

Goal: maximize expected utility

$$v(t, x_1, x_2) = \max_{\{a_s, C_s\}} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} U(C_s)) ds \middle| \mathcal{F}_t\right]. \tag{1}$$

Stochastic optimization - 2D

Climate-economics problem

Two stochastic processes: temperature and capital.

Goal: maximize expected utility

$$v(t, x_1, x_2) = \max_{\{a_s, C_s\}} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} U(C_s)) ds \middle| \mathcal{F}_t\right]. \qquad (1)$$

Rainbow option pricing problem

Two stochastic processes: asset price 1 and asset price 2.

Goal: maximize expected profit

$$v(t, x_1, x_2) = \max_{\tau \in [t, T]} \mathbb{E}\left[e^{-r(\tau - t)} g(X_\tau^1, X_\tau^2) \middle| \mathcal{F}_t\right]. \qquad (2)$$

Stochastic optimization - 2D

Climate-economics problem

Two stochastic processes: temperature and capital.

Goal: maximize expected utility

$$v(t, x_1, x_2) = \max_{\{a_s, C_s\}} \mathbb{E}\left[\int_t^T e^{-\rho(s-t)} U(C_s)) ds \middle| \mathcal{F}_t\right]. \qquad (1)$$

Rainbow option pricing problem

Two stochastic processes: asset price 1 and asset price 2.

Goal: maximize expected profit

$$v(t, x_1, x_2) = \max_{\tau \in [t, T]} \mathbb{E}\left[e^{-r(\tau-t)} g(X_\tau^1, X_\tau^2) \middle| \mathcal{F}_t\right]. \qquad (2)$$

The math is (almost) the same

Table of Contents

1 Rainbow options

2 2D-COS formula

3 European options

4 Bermudan options

5 Heston stochastic volatility model

6 Conclusion

Financial mathematics

In financial markets, traders deal in assets and options. The payoff of an option depends on the value of the underlying asset price(s). Asset price X_t is stochastic.

Financial mathematics

In financial markets, traders deal in assets and options. The payoff of an option depends on the value of the underlying asset price(s). Asset price X_t is stochastic.

$$\text{Payoff call: } g^{call}(x) = \max(x - K, 0) \tag{3}$$

$$\text{Payoff put: } g^{put}(x) = \max(K - x, 0) \tag{4}$$

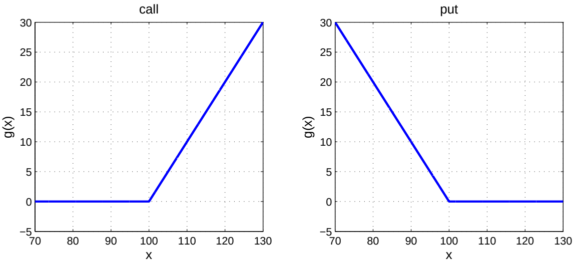


Figure 1: Payoff call and put option, strike price $K = 100$ (1D).

2 correlated asset prices

Two stochastic asset price processes, X_t^1 and X_t^2.

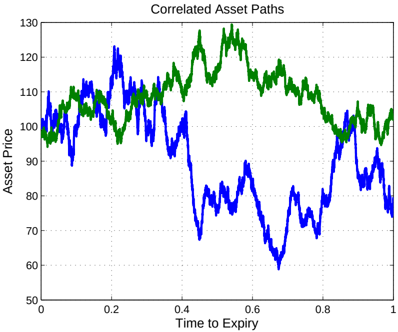
2 correlated asset prices

Two stochastic asset price processes, X_t^1 and X_t^2.
For example, correlated geometric Brownian motions:

$$dX_t^1 = \mu_1 X_t^1 dt + \sigma_1 X_t^1 dW_t^1, \quad (5)$$

$$dX_t^2 = \mu_2 X_t^2 dt + \sigma_2 X_t^2 dW_t^2, \quad (6)$$

with $dW_t^1 dW_t^2 = \rho dt$.



Payoff rainbow options

Basket option: weighted sum or average of different assets, e.g.,

$$g(x_1, x_2) = \max\left(\frac{1}{2}(x_1 + x_2) - K, 0\right). \qquad (7)$$

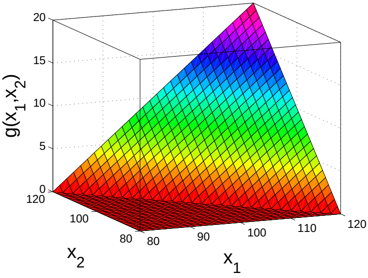


Figure 2: Basket option.

Payoff rainbow options

Basket option: weighted sum or average of different assets, e.g.,

$$g(x_1, x_2) = \max\left(\tfrac{1}{2}(x_1 + x_2) - K, 0\right). \quad (7)$$

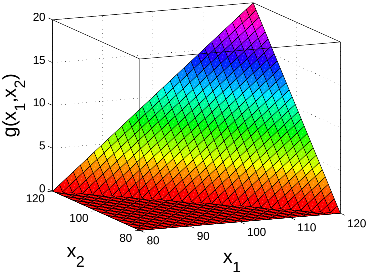


Figure 2: Basket option.

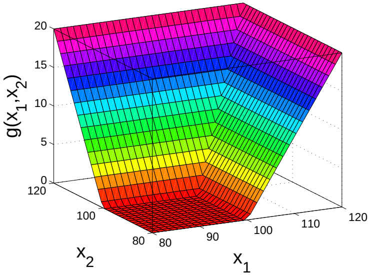


Figure 3: Call on max. option.

Call on maximum option:

$$g(x_1, x_2) = \max\left(\max(x_1, x_2) - K, 0\right). \quad (8)$$

European, American, and Bermudan-style

European-style: you buy the option now, wait until terminal time T, then the option may be exercised.

American-style: may be exercised at any time before the terminal time T.

Bermudan-style: fixed exercise dates t_m $(m = 1, \dots, M)$ at which you can exercise the option.

Financial mathematics: efficient computation of option price.

$$v(t_0, \mathbf{x_0}) = e^{-r\Delta t}\mathbb{E}\left[v(T, \mathbf{X_T})\right]. \tag{9}$$

COS method

Based on Fourier cosine series expansions.

COS method

Based on Fourier cosine series expansions.

Pricing financial and real options:

- European options (F. Fang, C.W. Oosterlee, 2008),
- Bermudan and American options (F. Fang, C.W. Oosterlee, 2009),
- Swing options, which are frequently used in energy markets (B. Zhang, C.W. Oosterlee, 2010),
- Asian-style options (B. Zhang, C.W. Oosterlee, 2011),
- Optimal dike height, (M.J. Ruijter, master thesis, 2010),
- ...

Fourier-cosine series expansion of function $h(x)$ on $[a, b]$:

$$h(x) = {\sum_{k=0}^{\infty}}' H_k \cos\left(k\pi \frac{x-a}{b-a}\right), \quad x \in [a, b], \tag{10}$$

with coefficients

$$H_k = \frac{2}{b-a} \int_a^b h(y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{11}$$

Fourier-cosine series expansion of function $h(x)$ on $[a, b]$:

$$\hat{h}(x) = {\sum_{k=0}^{N-1}}' H_k \cos\left(k\pi \frac{x-a}{b-a}\right), \quad x \in [a, b], \tag{10}$$

with coefficients

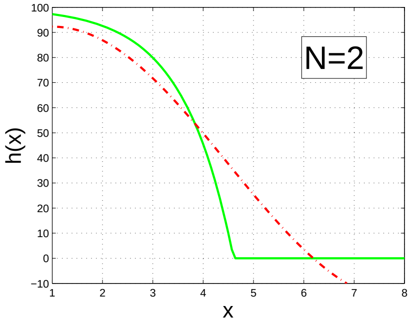
$$H_k = \frac{2}{b-a} \int_a^b h(y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{11}$$

Fourier-cosine series expansion of function $h(x)$ on $[a, b]$:

$$\hat{h}(x) = {\sum_{k=0}^{N-1}}' H_k \cos\left(k\pi\frac{x-a}{b-a}\right), \qquad x \in [a, b], \tag{10}$$

with coefficients

$$H_k = \frac{2}{b-a}\int_a^b h(y)\cos\left(k\pi\frac{y-a}{b-a}\right)dy. \tag{11}$$

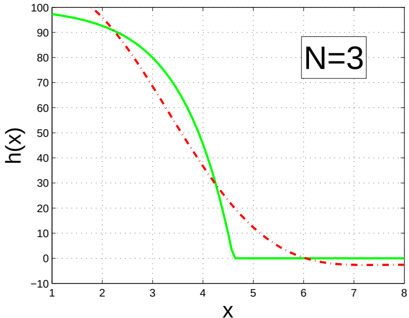


Fourier-cosine series expansion of function $h(x)$ on $[a, b]$:

$$\hat{h}(x) = {\sum_{k=0}^{N-1}}' H_k \cos\left(k\pi \frac{x-a}{b-a}\right), \qquad x \in [a, b], \tag{10}$$

with coefficients

$$H_k = \frac{2}{b-a} \int_a^b h(y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{11}$$

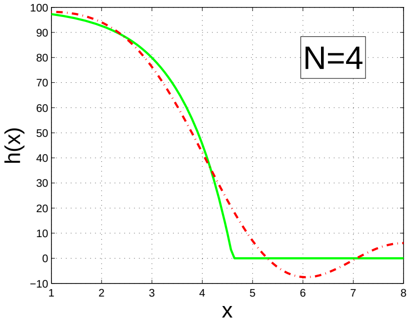


Fourier-cosine series expansion of function $h(x)$ on $[a, b]$:

$$\hat{h}(x) = {\sum_{k=0}^{N-1}}' H_k \cos\left(k\pi \frac{x-a}{b-a}\right), \qquad x \in [a, b], \tag{10}$$

with coefficients

$$H_k = \frac{2}{b-a} \int_a^b h(y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{11}$$

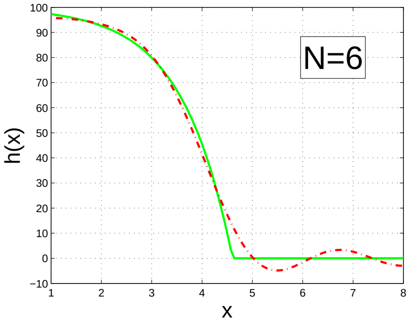


Fourier-cosine series expansion of function $h(x)$ on $[a, b]$:

$$\hat{h}(x) = {\sum_{k=0}^{N-1}}' H_k \cos\left(k\pi \frac{x-a}{b-a}\right), \qquad x \in [a, b], \tag{10}$$

with coefficients

$$H_k = \frac{2}{b-a} \int_a^b h(y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{11}$$

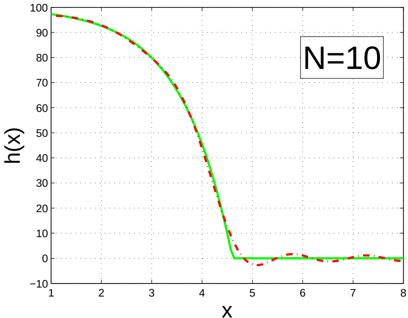


Fourier-cosine series expansion of function $h(x)$ on $[a, b]$:

$$\hat{h}(x) = {\sum_{k=0}^{N-1}}' H_k \cos\left(k\pi \frac{x-a}{b-a}\right), \quad x \in [a, b], \tag{10}$$

with coefficients

$$H_k = \frac{2}{b-a} \int_a^b h(y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{11}$$

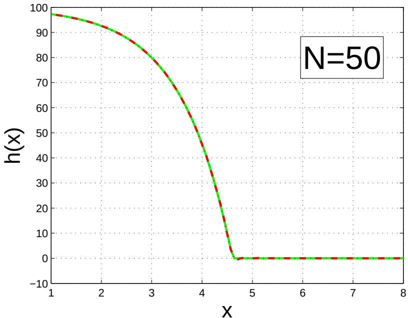


Fourier-cosine series expansion of function $h(x)$ on $[a, b]$:

$$\hat{h}(x) = {\sum_{k=0}^{N-1}}' H_k \cos\left(k\pi \frac{x-a}{b-a}\right), \quad x \in [a, b], \tag{10}$$

with coefficients

$$H_k = \frac{2}{b-a} \int_a^b h(y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{11}$$



1D-COS formula

We use the COS formula to approximate expectations.

$$v(t,x) = \mathbb{E}\left[v(T,X_T)\right] = \int_{\mathbb{R}} v(T,y) f(y|x) dy$$

1D-COS formula

We use the COS formula to approximate expectations.

$$v(t,x) = \mathbb{E}\left[v(T,X_T)\right] = \int_{\mathbb{R}} v(T,y) f(y|x) dy$$

$$\approx \int_a^b v(T,y) f(y|x) dy$$

1D-COS formula

We use the COS formula to approximate expectations.

$$
\begin{aligned}
v(t,x) = \mathbb{E}\left[v(T,X_T)\right] &= \int_{\mathbb{R}} v(T,y) f(y|x) dy \\
&\approx \int_a^b v(T,y) f(y|x) dy \\
&= \frac{b-a}{2} \sum_{k=0}^{\infty}{}' V_k(T) F_k(x),
\end{aligned}
$$

with coefficients

$$
V_k(T) := \frac{2}{b-a} \int_a^b v(T,y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy, \tag{12}
$$

$$
F_k(x) := \frac{2}{b-a} \int_a^b f(y|x) \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{13}
$$

Exponential convergence in N for smooth density $f(y|x)$.

1D-COS formula

We use the COS formula to approximate expectations.

$$v(t,x) = \mathbb{E}\left[v(T,X_T)\right] = \int_{\mathbb{R}} v(T,y)f(y|x)dy$$
$$\approx \int_a^b v(T,y)f(y|x)dy$$
$$\approx \frac{b-a}{2}\sum_{k=0}^{N-1}{}' V_k(T)F_k(x),$$

with coefficients

$$V_k(T) := \frac{2}{b-a}\int_a^b v(T,y)\cos\left(k\pi\frac{y-a}{b-a}\right)dy, \tag{12}$$
$$F_k(x) := \frac{2}{b-a}\int_a^b f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)dy. \tag{13}$$

Exponential convergence in N for smooth density $f(y|x)$.

Fourier coefficients F_k:

$$\begin{aligned} F_k(x) &= \frac{2}{b-a}\int_a^b f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &\approx \frac{2}{b-a}\int_{\mathbb{R}} f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \end{aligned} \tag{14}$$

Fourier coefficients F_k:

$$\begin{aligned} F_k(x) &= \frac{2}{b-a}\int_a^b f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &\approx \frac{2}{b-a}\int_{\mathbb{R}} f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\mathrm{Re}\left(\int_{\mathbb{R}} f(y|x)\exp\left(ik\pi\frac{y-a}{b-a}\right)dy\right) \end{aligned} \tag{14}$$

Fourier coefficients F_k:

$$\begin{aligned} F_k(x) &= \frac{2}{b-a}\int_a^b f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &\approx \frac{2}{b-a}\int_{\mathbb{R}} f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\mathrm{Re}\left(\int_{\mathbb{R}} f(y|x)\exp\left(ik\pi\frac{y-a}{b-a}\right)dy\right) \\ &= \frac{2}{b-a}\mathrm{Re}\left(\varphi\left(\frac{k\pi}{b-a}\Big|x\right)e^{-ik\pi\frac{a}{b-a}}\right) \end{aligned} \tag{14}$$

Characteristic function of random variable Y:

$$\varphi(u) = \mathbb{E}\left[\exp\left(iuY\right)\right] = \int_{\mathbb{R}} \exp(iuy)f(y)dy. \tag{15}$$

For many asset price processes the characteristic function is available.

Fourier coefficients F_k:

$$\begin{aligned} F_k(x) &= \frac{2}{b-a}\int_a^b f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &\approx \frac{2}{b-a}\int_{\mathbb{R}} f(y|x)\cos\left(k\pi\frac{y-a}{b-a}\right)dy \\ &= \frac{2}{b-a}\mathrm{Re}\left(\int_{\mathbb{R}} f(y|x)\exp\left(ik\pi\frac{y-a}{b-a}\right)dy\right) \\ &= \frac{2}{b-a}\mathrm{Re}\left(\varphi\left(\frac{k\pi}{b-a}\middle|x\right)e^{-ik\pi\frac{a}{b-a}}\right) \\ &= \frac{2}{b-a}\mathrm{Re}\left(\varphi_{levy}\left(\frac{k\pi}{b-a}\right)e^{ik\pi\frac{x-a}{b-a}}\right). \end{aligned} \tag{14}$$

Characteristic function of random variable Y:

$$\varphi(u) = \mathbb{E}\left[\exp\left(iuY\right)\right] = \int_{\mathbb{R}} \exp(iuy)f(y)dy. \tag{15}$$

For many asset price processes the characteristic function is available.

2D-COS formula

In 1D:

$$v(t,x) = e^{-r\Delta t}\mathbb{E}\left[v(T,X_T)\right] \approx \frac{b-a}{2}e^{-r\Delta t}{\sum_{k=0}^{N-1}}' V_k(T)F_k(x). \qquad (16)$$

In 2D:

$$v(t,\mathbf{x}) = e^{-r\Delta t}\mathbb{E}\left[v(T,\mathbf{X}_T)\right]$$

2D-COS formula

In 1D:

$$v(t,x) = e^{-r\Delta t}\mathbb{E}\left[v(T,X_T)\right] \approx \frac{b-a}{2}e^{-r\Delta t}\sum_{k=0}^{N-1}{}' V_k(T)F_k(x). \qquad (16)$$

In 2D:

$$v(t,\mathbf{x}) = e^{-r\Delta t}\mathbb{E}\left[v(T,\mathbf{X}_T)\right]$$

2D-COS formula

In 1D:

$$\begin{aligned} v(t,x) &= e^{-r\Delta t}\mathbb{E}\left[v(T,X_T)\right] \\ &\approx \frac{b-a}{2}e^{-r\Delta t}\sum_{k=0}^{N-1}{}' V_k(T)F_k(x). \end{aligned} \tag{16}$$

In 2D:

$$\begin{aligned} v(t,\mathbf{x}) &= e^{-r\Delta t}\mathbb{E}\left[v(T,\mathbf{X}_T)\right] \\ &= e^{-r\Delta t}\int_{\mathbb{R}} v(T,\mathbf{y})f(\mathbf{y}|\mathbf{x})d\mathbf{y} \\ &\approx \frac{b_1-a_1}{2}\frac{b_2-a_2}{2}e^{-r\Delta t}\sum_{k_1=0}^{N_1-1}{}'\sum_{k_2=0}^{N_2-1}{}' V_{k_1,k_2}(T)F_{k_1,k_2}(\mathbf{x}). \end{aligned} \tag{17}$$

This can be extended to higher dimensions.

Coefficients - most difficult part.

$$F_{k_1,k_2}(\mathbf{x}) \approx \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) dy_1 dy_2. \tag{18}$$

Coefficients - most difficult part.

$$F_{k_1,k_2}(\mathbf{x}) \approx \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) dy_1 dy_2. \tag{18}$$

We use the following goniometric relation:

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta) + \cos(\alpha-\beta). \tag{19}$$

Coefficients - most difficult part.

$$F_{k_1,k_2}(\mathbf{x}) \approx \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)dy_1dy_2. \tag{18}$$

We use the following goniometric relation:

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta)+\cos(\alpha-\beta). \tag{19}$$

Then

$$2F_{k_1,k_2}(\mathbf{x}) = F^{+}_{k_1,k_2}(\mathbf{x}) + F^{-}_{k_1,k_2}(\mathbf{x}), \tag{20}$$

where

$$F^{\pm}_{k_1,k_2}(\mathbf{x}) := \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\pm k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)dy_1dy_2 \tag{21}$$

Coefficients - most difficult part.

$$F_{k_1,k_2}(\mathbf{x}) \approx \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)dy_1dy_2. \tag{18}$$

We use the following goniometric relation:

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta)+\cos(\alpha-\beta). \tag{19}$$

Then

$$2F_{k_1,k_2}(\mathbf{x}) = F^{+}_{k_1,k_2}(\mathbf{x}) + F^{-}_{k_1,k_2}(\mathbf{x}), \tag{20}$$

where

$$\begin{aligned} F^{\pm}_{k_1,k_2}(\mathbf{x}) &:= \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1} \pm k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)dy_1dy_2 \\ &= \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\mathrm{Re}\left(\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x})\exp\left(ik_1\pi\frac{y_1-a_1}{b_1-a_1} \pm ik_2\pi\frac{y_2-\pm a_2}{b_2-a_2}\right)dy_1dy_2\right) \end{aligned} \tag{21}$$

Coefficients - most difficult part.

$$F_{k_1,k_2}(\mathbf{x}) \approx \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)dy_1dy_2. \tag{18}$$

We use the following goniometric relation:

$$2\cos(\alpha)\cos(\beta) = \cos(\alpha+\beta)+\cos(\alpha-\beta). \tag{19}$$

Then

$$2F_{k_1,k_2}(\mathbf{x}) = F^{+}_{k_1,k_2}(\mathbf{x}) + F^{-}_{k_1,k_2}(\mathbf{x}), \tag{20}$$

where

$$\begin{aligned} F^{\pm}_{k_1,k_2}(\mathbf{x}) &:= \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1} \pm k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)dy_1dy_2 \\ &= \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\mathrm{Re}\left(\iint_{\mathbb{R}^2} f(\mathbf{y}|\mathbf{x})\exp\left(ik_1\pi\frac{y_1-a_1}{b_1-a_1} \pm ik_2\pi\frac{y_2-\pm a_2}{b_2-a_2}\right)dy_1dy_2\right) \\ &= \frac{2}{b_1-a_1}\frac{2}{b_2-a_2}\mathrm{Re}\left(\varphi\left(\frac{k_1\pi}{b_1-a_1}, \pm\frac{k_2\pi}{b_2-a_2}\middle|\mathbf{x}\right)\exp\left(-ik_1\pi\frac{a_1}{b_1-a_1} \mp ik_2\pi\frac{a_2}{b_2-a_2}\right)\right) \end{aligned} \tag{21}$$

Approximate the terminal coefficients $V_{k_1,k_2}(T)$ with DCTs. Take $Q \geq \max[N_1, N_2]$ grid-points and

$$y_i^{n_i} := a_i + (n_i + \tfrac{1}{2})\frac{b_i - a_i}{Q} \quad \text{and} \quad \Delta y_i := \frac{b_i - a_i}{Q}, \quad i = 1, 2. \quad (22)$$

Approximate the terminal coefficients $V_{k_1,k_2}(T)$ with DCTs. Take $Q \geq \max[N_1, N_2]$ grid-points and

$$y_i^{n_i} := a_i + (n_i + \tfrac{1}{2})\frac{b_i - a_i}{Q} \quad \text{and} \quad \Delta y_i := \frac{b_i - a_i}{Q}, \quad i = 1, 2. \quad (22)$$

The midpoint-rule integration gives us

$$\begin{aligned} &V_{k_1,k_2}(T) \\ &= \tfrac{2}{b_1-a_1}\tfrac{2}{b_2-a_2}\int_{a_2}^{b_2}\int_{a_1}^{b_1} g(\mathbf{y})\cos\left(k_1\pi\tfrac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\tfrac{y_2-a_2}{b_2-a_2}\right)dy_1 dy_2 \\ &\approx \sum_{n_1=0}^{Q-1}\sum_{n_2=0}^{Q-1} g(y_1^{n_1}, y_2^{n_2})\cos\left(k_1\pi\tfrac{2n_1+1}{2Q}\right)\cos\left(k_2\pi\tfrac{2n_2+1}{2Q}\right)\tfrac{b_1-a_1}{Q}\tfrac{b_2-a_2}{Q}. \end{aligned}$$

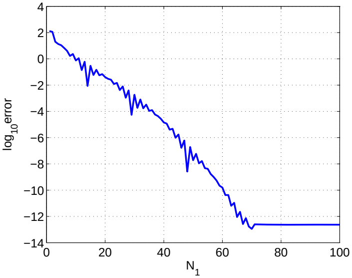
The above 2D-DCT can be calculated efficiently by, for example, MATLAB's function `dct2`.

Results - European options

Results geometric basket call under correlated geometric Brownian motion, $v(t_0, \mathbf{x}_0) = 8.8808$.

Table 1: $(N_2 = N_1)$.

N_1	10	20	40	60	80
Error	7.60e-1	4.07e-2	1.42e-5	1.59e-10	2.34e-13
CPU (ms)	1.65	1.99	3.15	5.52	7.46



Jump-diffusion process

The log-jump-diffusion process

$$dS_t^i = (r - \lambda \mathbb{E}[e^{J_i} - 1])S_t^i dt + \sigma_i S_t^i dW_t^i + S_t^i (e^{J_i} - 1) dq_t, \quad (23)$$

with q_t a Poisson process with intenstity λ, and $\mathbf{J} = (J_1, J_2)$ bivariate normally distributed jumps.

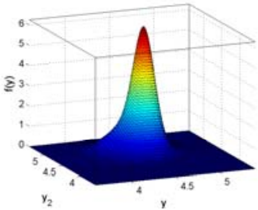


Figure 4: Density recovery $\hat{f}(\mathbf{X}_T | \mathbf{x}_0)$.

Jump-diffusion process

The log-jump-diffusion process

$$dS_t^i = (r - \lambda \mathbb{E}[e^{J_i} - 1])S_t^i dt + \sigma_i S_t^i dW_t^i + S_t^i(e^{J_i} - 1)dq_t, \quad (23)$$

with q_t a Poisson process with intenstity λ, and $\mathbf{J} = (J_1, J_2)$ bivariate normally distributed jumps.

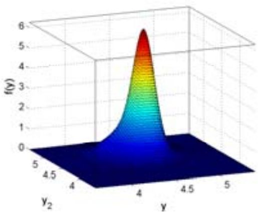


Figure 4: Density recovery $\hat{f}(\mathbf{X}_T|\mathbf{x}_0)$.

Table 2: Put-on-min option values $\hat{v}(t_0, \mathbf{x}_0)$ ($N_1 = N_2 = 125$), they correspond to value in (Clift, 2008).

S_0^2 \ S_0^1	90	100	110
90	15.6916	13.4073	12.1305
100	12.1918	9.1360	7.5175
110	10.3853	6.7274	4.8337

Spark spread option - 3D

3-dimensional GBM $\mathbf{S}_t = [S_t^{power}, S_t^{gas}, S_t^{CO_2}]$.
Spark spread: net revenue from selling power.
Payoff :

$$g(\mathbf{S}_T) = \Omega \max\left(S_T^{power} - \alpha^g S_T^{gas} - \alpha^{CO_2} S_T^{CO_2} - K, 0\right).$$

Spark spread option - 3D

3-dimensional GBM $\mathbf{S}_t = [S_t^{power}, S_t^{gas}, S_t^{CO_2}]$.
Spark spread: net revenue from selling power.
Payoff :

$$g(\mathbf{S}_T) = \Omega \max \left(S_T^{power} - \alpha^g S_T^{gas} - \alpha^{CO_2} S_T^{CO_2} - K, 0 \right).$$

Spark spread option - 3D

3-dimensional GBM $\mathbf{S}_t = [S_t^{power}, S_t^{gas}, S_t^{CO_2}]$.
Spark spread: net revenue from selling power.
Payoff :

$$g(\mathbf{S}_T) = \Omega \max\left(S_T^{power} - \alpha^g S_T^{gas} - \alpha^{CO_2} S_T^{CO_2} - K, 0\right).$$

	$v(t_0, \mathbf{S}_0)$			CPU time (s)		
	N			N		
Q	20	40	60	20	40	60
50	294830.89	294893.01	n/a	0.02	0.07	n/a
100	294818.38	294883.93	294883.93	0.16	0.20	0.32
150	294816.68	294882.82	294882.82	0.58	0.60	0.78
200	294816.30	294882.65	294882.65	1.72	1.74	1.87
250	294816.45	294882.88	294882.88	3.88	3.87	3.98

Calculation of the option's Greeks is straightforward.

Bermudan options

Bermudan option: fixed exercise dates t_m $(m = 1, \ldots, M)$ at which you can either exercise the option or continue. Option value in 1D:

$$v(t_m, x) = \max[g(x), c(t_m, x)]. \tag{24}$$

Bermudan options

Bermudan option: fixed exercise dates t_m $(m = 1, \ldots, M)$ at which you can either exercise the option or continue. Option value in 1D:

$$v(t_m, x) = \max[g(x), c(t_m, x)]. \tag{24}$$

Coefficients V_k at time t_m:

$$V_k(t_m) := \frac{2}{b-a} \int_a^b v(t_m, y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{25}$$

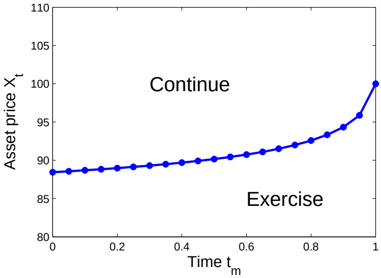
Bermudan options

Bermudan option: fixed exercise dates t_m $(m = 1, \ldots, M)$ at which you can either exercise the option or continue. Option value in 1D:

$$v(t_m, x) = \max[g(x), c(t_m, x)]. \tag{24}$$

Coefficients V_k at time t_m:

$$V_k(t_m) := \frac{2}{b-a} \int_a^b v(t_m, y) \cos\left(k\pi \frac{y-a}{b-a}\right) dy. \tag{25}$$



Bermudan option - 2D

Coefficients V_{k_1,k_2} at time t_m:

$$V_{k_1,k_2}(t_m) := \int_{a_2}^{b_2}\int_{a_1}^{b_1} v(t_m,\mathbf{y})\cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right)dy_1\,dy_2$$

with

$$v(t_m,\mathbf{x}) = \max[g(\mathbf{x}),\ c(t_m,\mathbf{x})]. \tag{26}$$

Bermudan option - 2D

Coefficients V_{k_1,k_2} at time t_m:

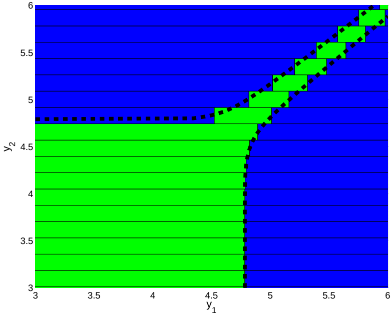
$$V_{k_1,k_2}(t_m) := \int_{a_2}^{b_2}\int_{a_1}^{b_1} v(t_m,\mathbf{y}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) dy_1 dy_2$$

with

$$v(t_m,\mathbf{x}) = \max[g(\mathbf{x}),\ c(t_m,\mathbf{x})]. \quad (26)$$

Left: Optimal exercise domains (blue) and continuation domains (green) at initial time t_0.

J rectangular sub-domains.



Equidistant vs. non-equidistant grid

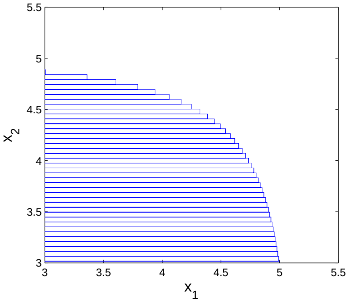


Figure 5: Equidistant grid.

Equidistant vs. non-equidistant grid

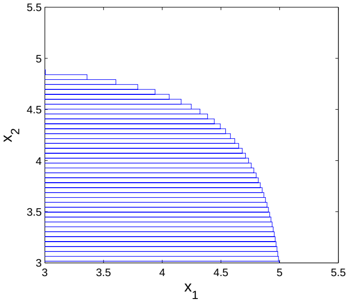


Figure 5: Equidistant grid.

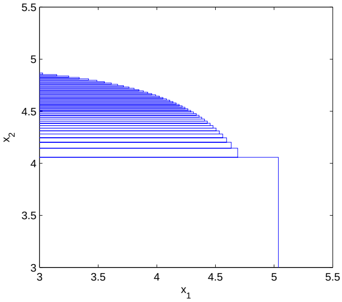


Figure 6: Non-equidistant grid.

Recursive recovery

$$
\begin{aligned}
V_{k_1,k_2}(t_m) &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} v(t_m,\mathbf{y}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) dy_1\, dy_2 \\
&= \sum_p \iint_{\mathcal{G}^p} g(\mathbf{y}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) d\mathbf{y} \\
&+ \sum_q \iint_{\mathcal{C}^q} c(t_m,\mathbf{y}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) d\mathbf{y}
\end{aligned}
$$

Recursive recovery

$$
\begin{aligned}
V_{k_1,k_2}(t_m) &= \int_{a_2}^{b_2}\int_{a_1}^{b_1} v(t_m,\mathbf{y}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) dy_1\, dy_2 \\
&= \sum_p \iint_{\mathcal{G}^p} g(\mathbf{y}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) d\mathbf{y} \\
&+ \sum_q \iint_{\mathcal{C}^q} \hat{c}(t_m,\mathbf{y}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) d\mathbf{y}
\end{aligned}
$$

Recursive recovery

$$V_{k_1,k_2}(t_m) = \int_{a_2}^{b_2}\int_{a_1}^{b_1} v(t_m,\mathbf{y}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) dy_1\, dy_2$$

$$= \sum_p \iint_{\mathcal{G}^p} g(\mathbf{y}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) d\mathbf{y}$$

$$+ \sum_q \iint_{\mathcal{C}^q} \hat{c}(t_m,\mathbf{y}) \cos\left(k_1\pi\frac{y_1-a_1}{b_1-a_1}\right)\cos\left(k_2\pi\frac{y_2-a_2}{b_2-a_2}\right) d\mathbf{y}$$

The resulting matrix-vector products $M\mathbf{u}$ can be computed efficiently by a Fourier-based algorithm. The computation time achieved is $O(N\log_2 N)$.

Algorithm

2D-COS method for pricing Bermudan rainbow options

Initialisation: Calculate coefficients $V_{k_1,k_2}(t_\mathcal{M})$.
Main loop to recover $\hat{V}(t_m)$:
For $m = \mathcal{M} - 1$ to 1:

- Determine the optimal continuation regions $\mathcal{C}^q$ and early-exercise regions $\mathcal{G}^p$.
- Compute $\hat{V}(t_m)$ with the help of the FFT algorithm.

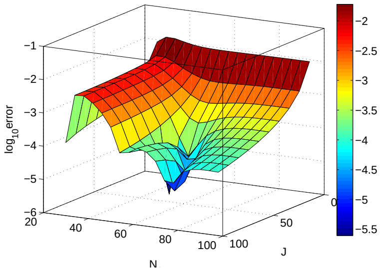
Final step: Compute $\hat{v}(t_0, \mathbf{x}_0)$ by inserting $\hat{V}_{k_1,k_2}(t_1)$ into COS formula.

Results Bermudan option

Error geometric basket option under GBM.
N_1 is number of terms in series expansion ($N_2 = N_1$).
J is number of sub-domains.

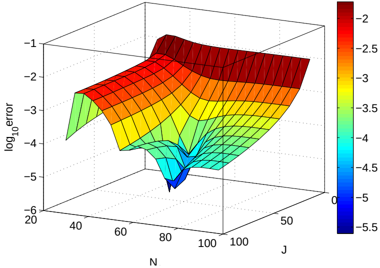
Results Bermudan option

Error geometric basket option under GBM.
N_1 is number of terms in series expansion ($N_2 = N_1$).
J is number of sub-domains.



Results Bermudan option

Error geometric basket option under GBM.
N_1 is number of terms in series expansion ($N_2 = N_1$).
J is number of sub-domains.



Call on maximum option

Table 3: Results call-on-max (GBM).

$\mathbf{S}_0$	2D-COS	Andersen (2004) Binomial	Shashi SGM Direct
90	8.073	8.075	8.079 (0.005)
100	13.902	13.902	13.907(0.007)
110	21.344	21.345	21.356(0.011)

Heston model

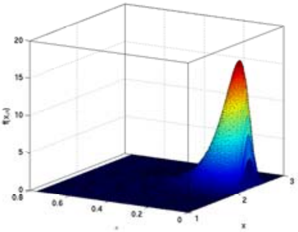
X_t represents the asset price process and ν_t is the variance process, with $(dW_t^1 dW_t^2 = \rho dt)$

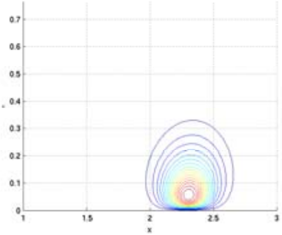
$$dX_t = (r - \tfrac{1}{2}\nu_t)X_t dt + \sqrt{\nu_t} X_t dW_t^1,$$
$$d\nu_t = \kappa(\bar{\nu} - \nu_t)dt + \eta\sqrt{\nu_t} dW_t^2. \tag{27}$$

Heston model

X_t represents the asset price process and ν_t is the variance process, with $(dW_t^1 dW_t^2 = \rho dt)$

$$dX_t = (r - \tfrac{1}{2}\nu_t)X_t dt + \sqrt{\nu_t}X_t dW_t^1,$$
$$d\nu_t = \kappa(\bar{\nu} - \nu_t)dt + \eta\sqrt{\nu_t}dW_t^2. \tag{27}$$

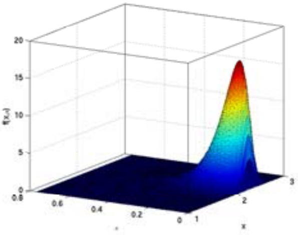


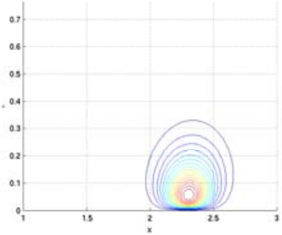


Heston model

X_t represents the asset price process and ν_t is the variance process, with $(dW_t^1 dW_t^2 = \rho dt)$

$$dX_t = (r - \tfrac{1}{2}\nu_t)X_t dt + \sqrt{\nu_t} X_t dW_t^1,$$
$$d\nu_t = \kappa(\bar{\nu} - \nu_t)dt + \eta\sqrt{\nu_t} dW_t^2. \qquad (27)$$





The variance process remains strictly positive if the Feller condition is satisfied, $2\kappa\bar{u}/\eta^2 - 1 := q_{Feller} \geq 0$, otherwise it may reach zero.

European option, with Bermudan framework

$\mathcal{M} = 12$ time steps.

European option, with Bermudan framework

$\mathcal{M} = 12$ time steps.

Feller satisfied ($q_{Feller} = 0.98$), $v = 0.50147$.

		N_2			
		50	100	150	200
N_1	50	-1.01e-4	-1.03e-5	-1.20e-6	1.02e-6
	100	-1.07e-4	-1.63e-5	-7.13e-6	-4.90e-6
	150	-1.07e-4	-1.62e-5	-7.09e-6	-4.86e-6

European option, with Bermudan framework

$\mathcal{M} = 12$ time steps.

Feller satisfied ($q_{Feller} = 0.98$), $v = 0.50147$.

		N_2			
		50	100	150	200
N_1	50	-1.01e-4	-1.03e-5	-1.20e-6	1.02e-6
	100	-1.07e-4	-1.63e-5	-7.13e-6	-4.90e-6
	150	-1.07e-4	-1.62e-5	-7.09e-6	-4.86e-6

Feller not satisfied ($q_{Feller} = -0.47$), $v = 3.1325$.

		N_2			
		50	100	150	200
N_1	50	3.83e-4	2.05e-4	1.27e-4	8.94e-5
	100	3.75e-4	1.95e-4	1.17e-4	7.90e-5
	150	3.75e-4	1.95e-4	1.17e-4	7.90e-5

Feller not satisfied ($q_{Feller} = -0.84$), $v = 6.271$.

		N_2			
		400	600	800	1000
N_1	50	5.53e-2	3.35e-2	2.29e-2	1.67e-2
	100	5.56e-2	3.45e-2	2.44e-2	1.84e-2
	150	5.61e-2	3.51e-2	2.49e-2	1.90e-2

Bermudan put option

Feller satisfied ($q_{Feller} = 0.98$)

		N_2			
		40	60	80	100
N_1	40	0.517765	0.517868	0.517893	0.517902
	60	0.517175	0.517284	0.517311	0.517320
	80	0.517020	0.517129	0.517155	0.517165
	100	0.517007	0.517116	0.517142	0.517152

Bermudan put option

Feller satisfied ($q_{Feller} = 0.98$)

		N_2 = 40	60	80	100
N_1	40	0.517765	0.517868	0.517893	0.517902
	60	0.517175	0.517284	0.517311	0.517320
	80	0.517020	0.517129	0.517155	0.517165
	100	0.517007	0.517116	0.517142	0.517152

Feller not satisfied ($q_{Feller} = -0.47$)

		N_2 = 40	60	80	100
N_1	40	3.200829	3.200768	3.200705	3.200660
	60	3.199089	3.199032	3.198971	3.198929
	80	3.199124	3.199068	3.199008	3.198966
	100	3.199101	3.199046	3.198986	3.198944

Summary and conclusion

- COS method is based on Fourier-cosine series expansion.
- Exponential convergence for $f \in C^\infty$.
- Can be extended to higher dimensions for pricing rainbow options.
- Experiments with financial and spark options.
- Heston stochastic volatility model.

Future research

- Higher dimensions,
- Gibbs phenomenon and filters,
- Asian options.

Contact: m.j.ruijter@cwi.nl