Mathematical Behavioural Finance
A Mini Course

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January 2013 Winter School @ Lunteren

Chapter 2:

Portfolio Choice under RDUT - Quantile Formulation

Section 1

Formulation of RDUT Portfolio Choice Model

Model Primitives

- Present date $t = 0$ and a future date $t = 1$
- Randomness described by $(\Omega, \mathcal{F}, \mathrm{P})$ at $t = 1$
- An atomless *pricing kernel* (or *state-price density* or *stochastic discount factor*) $\tilde{\rho}$ so that any future payoff $\tilde{X}$ is evaluated as $E[\tilde{\rho}\tilde{X}]$ at present
- An agent with
  - initial endowment $x_0 > 0$ at $t = 0$
  - preference specified by RDUT pair (u, w)

... wants to choose future consumption (wealth) $\tilde{c}$

Portfolio/Consumption Choice Model under RDUT

The model

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & V(\tilde{c}) = \int_0^\infty w\left(\mathrm{P}\left(u(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & E[\tilde{\rho}\tilde{c}] \leq x_0,\ \tilde{c} \geq 0 \end{array} \qquad \text{(RDUT)}$$

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- *Feasibility*: whether there is at least one solution satisfying all the constraints

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- *Attainability*: whether a well-posed problem admits an optimal solution
- *Uniqueness*: whether an attainable problem has a unique optimal solution

EUT Model Revisited

Let $w(p) = p$

- The model

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & V(\tilde{c}) = \int_0^\infty \mathrm{P}\left(u(\tilde{c}) > x\right) dx \equiv E[u(\tilde{c})] \\ \text{subject to} & E[\tilde{\rho}\tilde{c}] \leq x_0,\ \tilde{c} \geq 0 \end{array} \qquad \text{(EUT)}$$

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- Karatzas and Shreve (1998), Jin, Xu and Zhou (2008)

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- u is assumed to be concave
- w is in general non-convex/non-concave
- **Difficulty**: due to **nonlinear** weighting function w, (RDUT) is **not** a concave maximisation problem even though u is concave!

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- Carlier and Dana (2008): necessary conditions; no explicit solution

Standing Assumptions

- $\tilde{\rho} > 0$ a.s., **atomless**, with $E[\tilde{\rho}] < +\infty$.
- $u : [0, \infty) \to \mathbb{R}$ is strictly increasing, strictly concave, continuously differentiable on $(0, \infty)$, and satisfies the **Inada** condition: $u'(0+) = \infty$, $u'(\infty) = 0$.
- $w : [0, 1] \to [0, 1]$ is strictly increasing and continuously differentiable, and satisfies $w(0) = 0$, $w(1) = 1$.

Section 2

Quantile Formulation

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- $G_{\tilde{X}}$ is non-decreasing and right-continuous

The Model Again

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- ... which $\tilde{c}'$ is the **cheapest**?
- Consider $\min_{\tilde{c}' \sim \tilde{c}} E\left[\tilde{\rho}\tilde{c}'\right]$

Hardy–Littlewood Inequality

Lemma

(Jin and Zhou 2008) *We have that* $\tilde{c}^* := G(1 - F_{\tilde{\rho}}(\tilde{\rho}))$ *solves* $\min_{\tilde{c}' \sim \tilde{c}} E\left[\tilde{\rho}\tilde{c}'\right]$, *where* G *is quantile of* $\tilde{c}$. *If in addition* $-\infty < E[\tilde{\rho}\tilde{c}^*] < +\infty$, *then* $\tilde{c}^*$ *is the unique optimal solution.*

Hardy, Littlewood and Pòlya (1952), Dybvig (1988)

Changing Decision Variable

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$$E[\tilde{\rho}\tilde{c}] \le x_0 \Leftrightarrow E\left[F_{\tilde{\rho}}^{-1}(1-\tilde{Z})G(\tilde{Z})\right] \le x_0 \Leftrightarrow \int_0^1 F_{\tilde{\rho}}^{-1}(1-z)G(z)dz \le x_0$$

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$$\int_0^{\infty} w\left(\mathrm{P}\left(u(\tilde{c}) > x\right)\right)dx = \int_0^{\infty} u(x)d\bar{w}(F_{\tilde{c}}(x))dx = \int_0^1 u(G(z))d\bar{w}(z),$$

where $\bar{w}(p) = 1 - w(1-p)$ (*dual* of w)

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- Decision variable is now changed from $\tilde{c}$ to its quantile G!

Original RDUT Model

$$
\begin{array}{ll}
\underset{\tilde{c}}{\text{Max}} & \int_0^{\infty} w\left(\mathrm{P}\left(u(\tilde{c}) > x\right)\right) dx \\
\text{subject to} & E[\tilde{\rho}\tilde{c}] \leq x_0, \ \tilde{c} \geq 0
\end{array}
\qquad \text{(RDUT)}
$$

Quantile Formulation

The *quantile formulation* of (RDUT) is:

$$\begin{aligned} \max_{G\in\mathbb{G}} \quad & U(G(\cdot)) := \int_0^1 u(G(z))w'(1-z)dz \\ \text{subject to} \quad & \int_0^1 F_{\tilde{\rho}}^{-1}(1-z)G(z)dz \le x_0 \end{aligned} \qquad \text{(Q)}$$

where

$$\mathbb{G} = \{G : [0,1) \to [0,\infty] \text{ non-decreasing and right-continuous}\},$$

is the set of quantile functions of nonnegative random variables

A concave maximisation problem!

Section 3

Solutions

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  Then $G^*_{\lambda^*}$ is optimal to (Q)
- $\tilde{c}^* := G^*_{\lambda^*}(1 - F_{\tilde{\rho}}(\tilde{\rho}))$ is optimal to (RDUT)

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Unconstrained Problem

- The quantile problem is to solve

$$\begin{aligned} \operatorname*{Max}_{G\in\mathbb{G}} \quad & U(G)=\int_0^1 u(G(z))w'(1-z)dz \\ \text{subject to} \quad & \int_0^1 F_{\tilde{\rho}}^{-1}(1-z)G(z)dz \le x_0 \end{aligned} \tag{Q}$$

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- Given λ, consider

$$\underset{G\in\mathbb{G}}{\text{Max}} \quad U_\lambda(G) = \int_0^1 \left[u(G(z))w'(1-z) - \lambda F_{\tilde{\rho}}^{-1}(1-z)G(z) \right] dz \qquad (\text{Q}_\lambda)$$

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- ... provided that $\frac{F_{\tilde{\rho}}^{-1}(1-z)}{w'(1-z)}$ is non-increasing, or $M(z) := \frac{w'(1-z)}{F_{\tilde{\rho}}^{-1}(1-z)}$ is non-decreasing!

Integrability Condition

- We impose the following condition as in classical EUT model to ensure that the optimal value is finite and the optimal solution exists

$$E\left[u\left((u')^{-1}\left(\frac{\lambda\tilde{\rho}}{w'(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\right)\right] < +\infty, \quad \text{for any } \lambda > 0$$

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- In the following, we always assume the integrability condition holds

Solution under Monotonicity Condition

Theorem

(Jin and Zhou 2008) *If $M(z)$ is non-decreasing on $z \in (0,1)$, then the unique optimal solution to (RDUT) is given as*

$$\boxed{\tilde{c}^* = (u')^{-1}\left(\frac{\lambda^* \tilde{\rho}}{w'(F_{\tilde{\rho}}(\tilde{\rho}))}\right)}$$

where λ^* is determined by $E(\tilde{\rho}\tilde{c}^*) = x_0$.

Remark

When there is no probability weighting, it reduces to the classical EUT result.

The Monotonicity Condition

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$$\frac{w''(z)}{w'(z)} \leq \frac{G'_{\tilde{\rho}}(z)}{G_{\tilde{\rho}}(z)}, \quad 0 < z < 1$$

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  where $G_{\tilde{\rho}}$ is the quantile of $\tilde{\rho}$
- However: The condition is **violated** for many known weighting functions and a lognormal pricing kernel

Violation of Monotonicity Condition

Proposition

(He and Zhou 2012) *Suppose $\tilde{\rho}$ is lognormally distributed, i.e.,*

$$F_{\tilde{\rho}}(x) = \Phi\left(\frac{\ln x - \mu}{\sigma}\right)$$

for some μ and $\sigma > 0$, where $\Phi(\cdot)$ is the CDF of standard Normal. For any weighting function in K-T, T-F, P with $0 < \gamma < 1$, there exists $\varepsilon > 0$ such that

$$\frac{w''(z)}{w'(z)} > \frac{G'_{\tilde{\rho}}(z)}{G_{\tilde{\rho}}(z)}, \quad 1 - \varepsilon < z < 1.$$

Probability Weighting Functions

- Kahneman and Tversky (1992) weighting

$$w(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}},$$

- Tversky and Fox (1995) weighting

$$w(p) = \frac{\delta p^\gamma}{\delta p^\gamma + (1-p)^\gamma},$$

- Prelec (1998) weighting

$$w(p) = e^{-\delta(-\ln p)^\gamma}$$

- Jin and Zhou (2008) weighting

$$w(z) = \begin{cases} y_0^{b-a} k e^{a\mu + \frac{(a\sigma)^2}{2}} \Phi\left(\Phi^{-1}(z) - a\sigma\right) & z \leq 1 - z_0, \\ C + k e^{b\mu + \frac{(b\sigma)^2}{2}} \Phi\left(\Phi^{-1}(z) - b\sigma\right) & z \geq 1 - z_0 \end{cases}$$

Endogenous Portfolio Insurance

Theorem

(He and Zhou 2012) *If there exists* $\varepsilon > 0$ *such that*

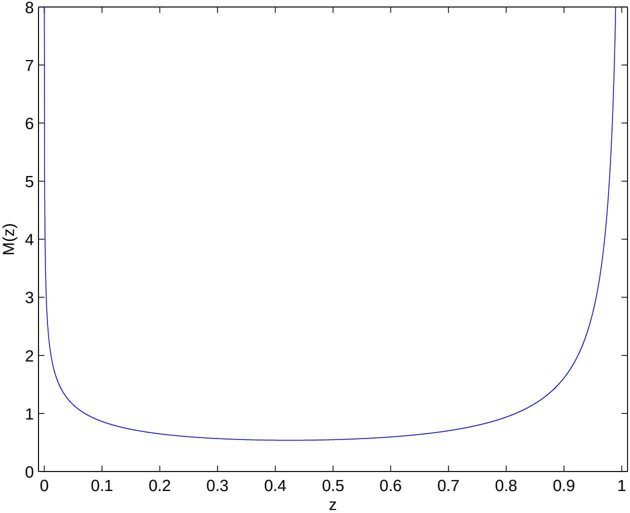
$$\frac{w''(z)}{w'(z)} > \frac{G'_{\tilde{\rho}}(z)}{G_{\tilde{\rho}}(z)}, \quad 1-\varepsilon < z < 1,$$

then for any optimal solution $\tilde{c}^*$ *to (RDUT), we have* $\operatorname{essinf}\ \tilde{c}^* > 0$.

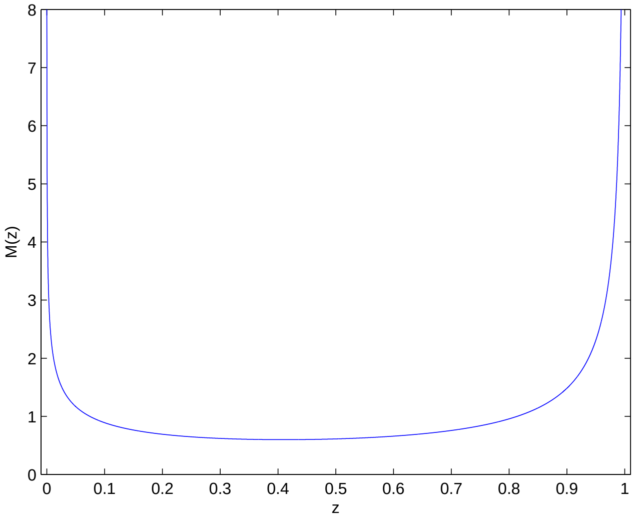
Remark

- Agent will set a positive floor (portfolio/consumption insurance) **endogenously** if $\frac{w''(z)}{w'(z)}$ is sufficiently large when z is near 1
- *Fear index*: $\frac{w''(z)}{w'(z)}$ when z is near 1

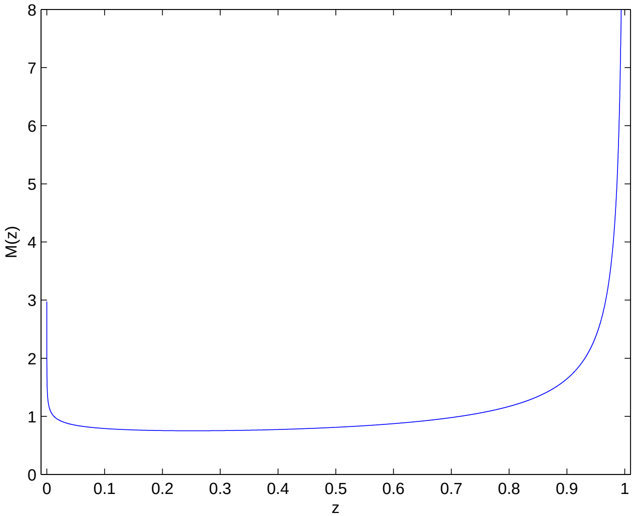
Tversky and Kahneman 1992



Tversky and Fox 1995



Prelec 1998



Monotonicity Condition

Assumption

$M(\cdot)$ is continuously differentiable on $(0,1)$ and there exists $0 < z_0 < 1$ such that $M(\cdot)$ is strictly decreasing on $(0, z_0)$ and strictly increasing on $(z_0, 1)$. Furthermore, $\lim_{z\uparrow 1} M(z) = +\infty$.

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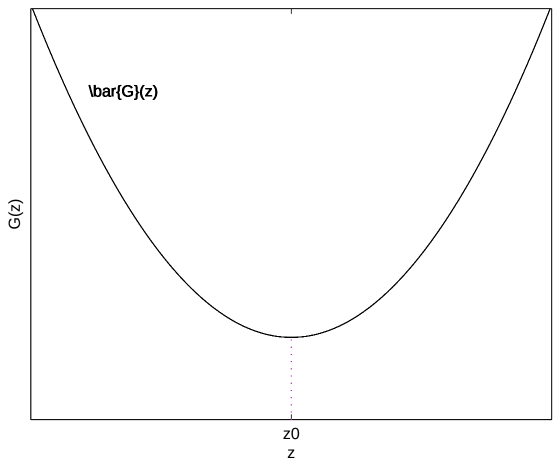
- Under this assumption, $\bar{G}(z) = (u')^{-1}\left(\frac{\lambda F_{\tilde{\rho}}^{-1}(1-z)}{w'(1-z)}\right) \equiv (u')^{-1}(\lambda/M(z))$ is no longer non-decreasing, so the brutal force (point-wise maximization) fails

Way Out: An Illustration



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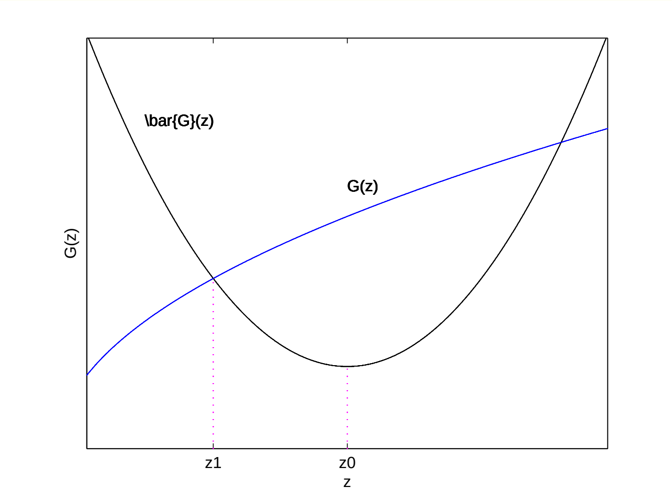

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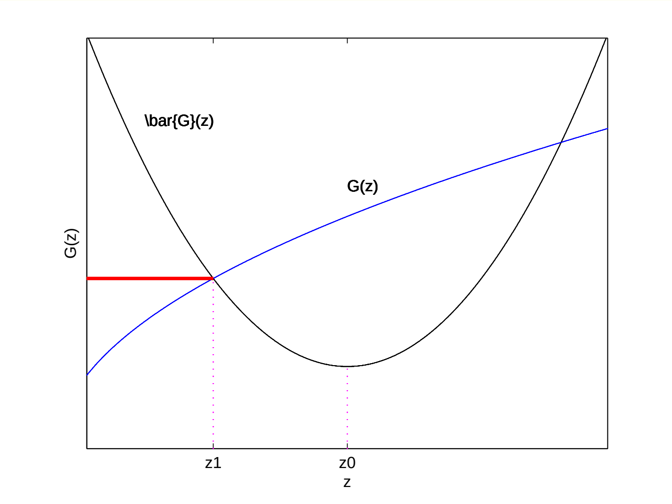

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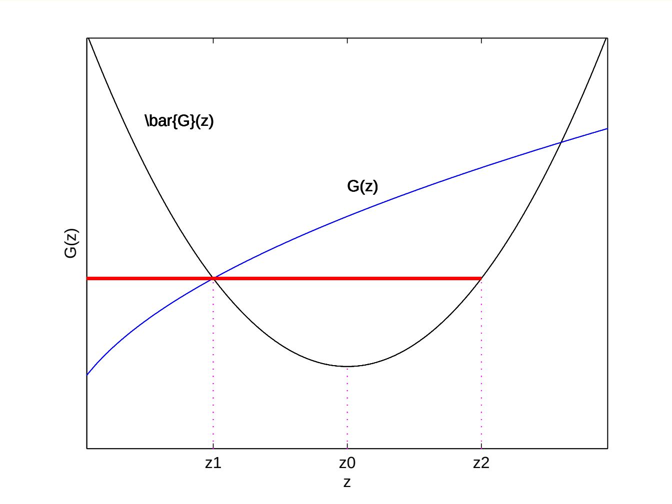

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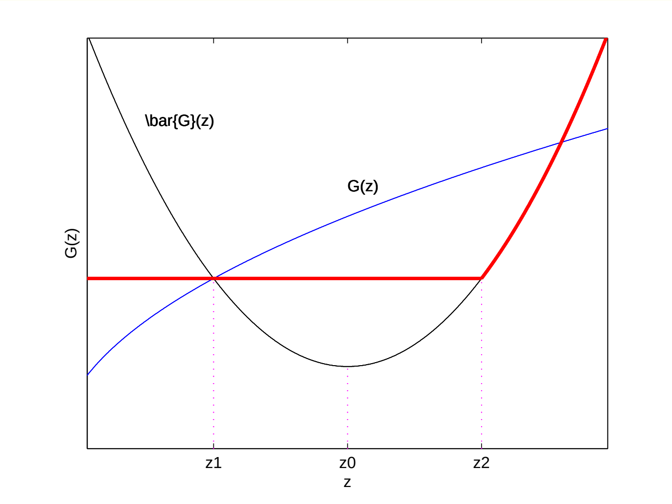

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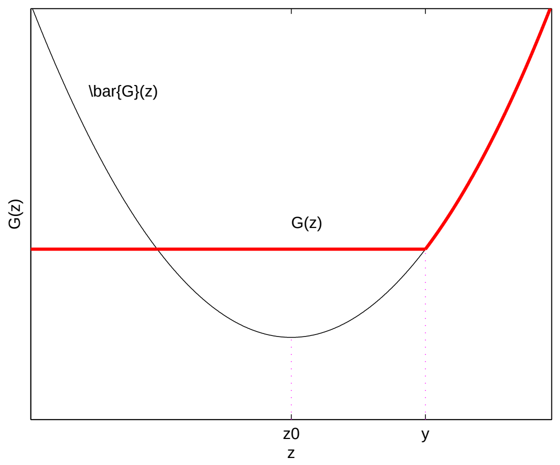

One Dimensional Optimisation

- We only need to consider quantiles in the form of

$$G(z) := \bar{G}(y)\mathbf{1}_{0<z\leq y} + \bar{G}(z)\mathbf{1}_{y<z<1}$$

for $z_0 \leq y < 1$

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- Substitute above G into

$$U_\lambda(G) = \int_0^1 \left[u(G(z))w'(1-z) - \lambda F_{\tilde{\rho}}^{-1}(1-z)G(z) \right] dz$$

and find optimal y!

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- Optimal y exists and is unique, and independent of λ
- Denote optimal y by z^*, which is shown to be the unique root of

$$\varphi(y) = \int_0^y w'(1-z)dz - M(y)\int_0^y F_{\tilde{\rho}}^{-1}(1-z)dz, \quad z_0 \leq y < 1$$

Solution under Two-Piece Monotonicity Condition

Theorem

(He and Zhou 2012) *Under the specified condition on M, (RDUT) has a unique optimal solution*

$$\tilde{c}^* = (u')^{-1}\left(\frac{\lambda^*\tilde{\rho}}{w'(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\mathbf{1}_{(\tilde{\rho}\le a^*)} + (u')^{-1}\left(\frac{\lambda^* a^*}{w'(F_{\tilde{\rho}}(a^*))}\right)\mathbf{1}_{(\tilde{\rho}>a^*)}$$

where $a^* > 0$ is the root of

$$\varphi(x) := x(1 - w(F_{\tilde{\rho}}(x))) - w'(F_{\tilde{\rho}}(x))\int_x^\infty s dF_{\tilde{\rho}}(x)$$

on $(F_{\tilde{\rho}}^{-1}(z_0), +\infty)$, and $\lambda^* > 0$ is such that $E(\tilde{\rho}\tilde{c}^*) = x_0$.

Section 4

Quantile Formulation as a General Approach

A Generic Model

$$
\begin{array}{ll}
\underset{\tilde{c}}{\text{Max}} & V(\tilde{c}) \\
\text{subject to} & E[\tilde{\rho}\tilde{c}] \le x_0,\ \tilde{c} \ge 0
\end{array}
\qquad \text{(P)}
$$

Basic Assumptions

- V is law invariant

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- "The more money the better": $v(x_0) > v(x_0')$ whenever $x_0 > x_0'$, where $v(x_0)$ is the supremum of (P)

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- "The more money the better": $v(x_0) > v(x_0')$ whenever $x_0 > x_0'$, where $v(x_0)$ is the supremum of (P)
- $\tilde{\rho}$ is atomless

Quantile Formulation

- Quantile formulation

$$\begin{array}{ll} \underset{G \in \mathbb{G}}{\text{Max}} & V(G(\tilde{Z})) \\ \text{subject to} & E[F_{\tilde{\rho}}^{-1}(1-\tilde{Z})G(\tilde{Z})] \leq x_0 \end{array} \qquad \text{(Q)}$$

where $\tilde{Z} \sim U(0,1)$

Quantile Formulation

- Quantile formulation

$$\begin{array}{ll} \underset{G\in\mathbb{G}}{\text{Max}} & V(G(\tilde{Z})) \\ \text{subject to} & E[F_{\tilde{\rho}}^{-1}(1-\tilde{Z})G(\tilde{Z})] \le x_0 \end{array} \qquad \text{(Q)}$$

  where $\tilde{Z} \sim U(0,1)$
- If G^* is optimal to (Q) then $\tilde{c}^* := G^*(1 - F_{\tilde{\rho}}(\tilde{\rho}))$ is optimal to (P)

Quantile Formulation

- Quantile formulation

$$\begin{array}{ll} \text{Max}_{G \in \mathbb{G}} & V(G(\tilde{Z})) \\ \text{subject to} & E[F_{\tilde{\rho}}^{-1}(1-\tilde{Z})G(\tilde{Z})] \leq x_0 \end{array} \qquad \text{(Q)}$$

  where $\tilde{Z} \sim U(0,1)$
- If G^* is optimal to (Q) then $\tilde{c}^* := G^*(1 - F_{\tilde{\rho}}(\tilde{\rho}))$ is optimal to (P)
- So $\tilde{c}^*$ is **always** anti-comonotonic with $\tilde{\rho}$

Goal Achieving

$$\begin{array}{ll} \underset{\tilde{c}}{\text{Max}} & \mathrm{P}(\tilde{c} \geq b) \\ \text{subject to} & E[\tilde{\rho}\tilde{c}] \leq x_0, \quad \tilde{c} \geq 0 \end{array}$$

where b: the *goal*

Kulldorff (1993), Heath (1993), Browne (1999), Föllmer and Leukert (1999), Spivak and Cvitanić (1999), etc.

Quantile Formulation

- $\mathrm{P}(\tilde{c} \geq b) = \int_0^\infty \mathbf{1}_{(x \geq b)} dF_{\tilde{c}}(x) = \int_0^1 \mathbf{1}_{(F_{\tilde{c}}^{-1}(z) \geq b)} dz$

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- $\mathrm{P}(\tilde{c} \geq b) = \int_0^\infty \mathbf{1}_{(x \geq b)} dF_{\tilde{c}}(x) = \int_0^1 \mathbf{1}_{(F_{\tilde{c}}^{-1}(z) \geq b)} dz$
- Quantile formulation

$$\begin{array}{ll} \underset{G \in \mathbb{G}}{\text{Max}} & U(G) = \int_0^1 \mathbf{1}_{(G(z) \geq b)} dz \\ \text{Subject to} & \int_0^1 F_{\tilde{\rho}}^{-1}(1-z) G(z) dz \leq x_0 \end{array}$$

Solution

Theorem

(He and Zhou 2009) *The unique optimal solution to goal-achieving problem is* $\tilde{c}^* = b\mathbf{1}_{(\tilde{\rho} \le a)}$ *where* $a > 0$ *is such that* $E[\mathbf{1}_{(\tilde{\rho} \le a)}\tilde{\rho}] = x_0/b$. *The optimal value is* $F_{\tilde{\rho}}(a)$.

Proof.

Lagrange – pointwise maximisation – binding budget constraint ☐

SP/A Portfolio Choice Model

$$\begin{array}{ll} \text{Max}_{\tilde{c}} & V(\tilde{c}) = \int_0^\infty w\left(\mathrm{P}\left(u(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & E[\tilde{\rho}\tilde{c}] \leq x_0,\ \tilde{c} \geq 0, \\ & \mathrm{P}(\tilde{c} \geq A) \geq \alpha \end{array} \qquad \text{(SPA)}$$

where

- $A \geq 0$: aspiration level
- α: confidence level

Lopes and Oden (1999)

Quantile Formulation

$$
\begin{array}{ll}
\underset{G\in\mathbb{G}}{\text{Max}} & U(G) := \int_0^1 u(G(z))w'(1-z)dz \\
\text{Subject to} & \int_0^1 F_{\tilde{\rho}}^{-1}(1-z)G(z)dz \le x_0, \;\; G(1-\alpha) \ge A
\end{array}
\qquad \text{(Q)}
$$

Solution

Theorem

(He and Zhou 2012) *Assume that* $x_0 \geq AE\left[\tilde{\rho}\mathbf{1}_{(\tilde{\rho}\leq F_{\tilde{\rho}}^{-1}(\alpha))}\right]$, *and* M *is non-decreasing on* $(0,1)$. *Then the unique optimal solution to* (SPA) *is given as*

$$\begin{aligned}\tilde{c}^* = \quad & (u')^{-1}\left(\frac{\lambda^*\tilde{\rho}}{w'(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\mathbf{1}_{(\tilde{\rho}\geq F_{\tilde{\rho}}^{-1}(\alpha))} \\ & + \left[(u')^{-1}\left(\frac{\lambda^*\tilde{\rho}}{w'(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\vee A\right]\mathbf{1}_{(\tilde{\rho}< F_{\tilde{\rho}}^{-1}(\alpha))}\end{aligned}$$

where λ^* *is the one binding the initial budget constraint, i.e.,* $E(\tilde{\rho}\tilde{c}^*) = x_0$.

Section 5

Summary and Further Readings

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- Portfolio choice under RDUT - probability weighting

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Summary

- Portfolio choice under RDUT - probability weighting
- Technical challenge arising from probability weighting: non-convex optimisation in infinite dimension
- Approach – quantile formulation
- Think of **distribution/quantile** of future consumption!
- A monotonicity condition - its economic interpretation
- Quantile formulation can treat a much broader class of problems

Essential Readings

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