Mathematical Behavioural Finance
A Mini Course

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Chapter 4:

Portfolio Choice under CPT

Section 1

Formulation of CPT Portfolio Choice Model

Model Primitives

- Present date $t = 0$ and a future date $t = 1$
- Randomness described by $(\Omega, \mathcal{F}, \mathrm{P})$ at $t = 1$
- An atomless *pricing kernel* $\tilde{\rho}$ so that any future payoff $\tilde{X}$ is evaluated as $E[\tilde{\rho}\tilde{X}]$ at present
- An agent with
  - initial endowment x_0 at $t = 0$
  - preference specified by CPT

... wants to choose future consumption (wealth) $\tilde{c}$

Portfolio Choice/Consumption Model under CPT

- The model

$$\begin{aligned}
\underset{\tilde{c}}{\text{Max}} \quad & V(\tilde{c}) = \int_0^\infty w_+ \left(\mathrm{P} \left(u_+ \left((\tilde{c} - \tilde{B})^+ \right) > x \right) \right) dx \\
& \quad - \int_0^\infty w_- \left(\mathrm{P} \left(u_- \left((\tilde{c} - \tilde{B})^- \right) > x \right) \right) dx \\
\text{subject to} \quad & E[\tilde{\rho}\tilde{c}] \le x_0, \ \tilde{c} \text{ is bounded below}
\end{aligned} \tag{CPT}$$

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- $u_\pm$ is assumed to be concave so overall value function $u_+(x)\mathbf{1}_{x\ge 0} - u_-(x)\mathbf{1}_{x<0}$ is S-shaped; $u_\pm(0) = 0$

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- $u_\pm$ is assumed to be concave so overall value function $u_+(x)\mathbf{1}_{x\geq 0} - u_-(x)\mathbf{1}_{x<0}$ is S-shaped; $u_\pm(0)=0$
- $w_\pm$ is in general non-convex/non-concave
- $\tilde{B}=0$ without loss of generality

CPT Preference

Write $V(\tilde{c}) = V_{+}(\tilde{c}^{+}) - V_{-}(\tilde{c}^{-})$ where

$$V_{+}(\tilde{c}) := \int_{0}^{\infty} w_{+}\left(\mathrm{P}\left(u_{+}\left(\tilde{c}\right) > x\right)\right) dx$$

$$V_{-}(\tilde{c}) := \int_{0}^{\infty} w_{-}\left(\mathrm{P}\left(u_{-}\left(\tilde{c}\right) > x\right)\right) dx$$

Mathematical Challenges

- Two difference sources

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- Probability weighting **and** S-shaped value function

Literature

- Almost none

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- Berkelaar, Kouwenberg and Post (2004): no probability weighting; two-piece power value function

Standing Assumptions

- $\tilde{\rho} > 0$ a.s., **atomless**, with $E[\tilde{\rho}] < +\infty$.
- $u_{\pm} : [0, \infty) \to \mathbb{R}$ are strictly increasing, concave, with $u_{\pm}(0) = 0$. Moreover, u_{+} is continuously differentiable on $(0, \infty)$, strictly concave, and satisfies the Inada condition: $u_{+}'(0+) = \infty$, $u_{+}'(\infty) = 0$.
- $w_{\pm} : [0, 1] \to [0, 1]$ are strictly increasing and continuously differentiable, and satisfies $w_{\pm}(0) = 0$, $w_{\pm}(1) = 1$.

Section 2

Divide and Conquer

Our Model (Again)

$$
\begin{aligned}
\underset{\tilde{c}}{\text{Max}} \quad & V(\tilde{c}) = \int_0^\infty w_+ \left(\mathrm{P}\left(u_+(\tilde{c}^+) > x\right)\right) dx \\
& \quad - \int_0^\infty w_- \left(\mathrm{P}\left(u_-(\tilde{c}^-) > x\right)\right) dx \\
\text{subject to} \quad & E[\tilde{\rho}\tilde{c}] \le x_0, \; \tilde{c} \ge 0
\end{aligned}
\tag{P}
$$

This problem admits a quantile formulation

Divide and Conquer

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- Step 1: divide into two problems: one concerns the **gain** part of $\tilde{c}$ and the other the **loss** part of $\tilde{c}$
- Step 2: combine them together via solving another problem

Step 1 – Gain Part Problem (GPP)

A problem with parameters (A, x_+):

$$\begin{array}{ll} \text{Max} & V_+(\tilde{c}) = \int_0^\infty w_+\left(\mathrm{P}\left(u_+(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & \begin{cases} E[\tilde{\rho}\tilde{c}] = x_+, \quad \tilde{c} \geq 0 \\ \tilde{c} = 0 \text{ on } A^C, \end{cases} \end{array} \tag{1}$$

where $x_+ \geq x_0^+ \ (\geq 0)$ and $A \in \mathcal{F}$ with $\mathrm{P}(A) \leq 1$

- Define its optimal value to be $v_+(A, x_+)$

Step 1 – Loss Part Problem (LPP)

A problem with parameters (A, x_+):

$$\begin{array}{ll} \text{Min} & V_-(\tilde{c}) = \int_0^\infty w_-\left(\mathrm{P}\left(u_-(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & \begin{cases} E[\tilde{\rho}\tilde{c}] = x_+ - x_0, \ \tilde{c} \geq 0 \\ \tilde{c} = 0 \text{ on } A, \ \tilde{c} \text{ is bounded} \end{cases} \end{array} \qquad (2)$$

where $x_+ \geq x_0^+$ and $A \in \mathcal{F}$ with $\mathrm{P}(A) \leq 1$

- Define its optimal value to be $v_-(A, x_+)$

Step 2

In Step 2 we solve

$$\begin{array}{ll} \text{Max} & v_+(A, x_+) - v_-(A, x_+) \\ \text{subject to} & \begin{cases} A \in \mathcal{F},\ x_+ \geq x_0^+, \\ x_+ = 0 \text{ when } \mathrm{P}(A) = 0, \\ x_+ = x_0 \text{ when } \mathrm{P}(A) = 1. \end{cases} \end{array} \tag{3}$$

It Works

Theorem

(Jin and Zhou 2008) *Given* $\tilde{c}^*$*, define* $A^* := \{\omega : \tilde{c}^* \geq 0\}$ *and* $x^*_+ := E[\tilde{\rho}(\tilde{c}^*)^+]$*. Then* $\tilde{c}^*$ *is optimal for the CPT portfolio choice problem (CPT) iff* (A^*, x^*_+) *are optimal for Problem (3) and* $(X^*)^+$ *and* $(X^*)^-$ *are respectively optimal for Problems (1) and (2) with parameters* (A^*, x^*_+)*.*

Proof. Direct by definitions of maximum/minimum.

Solution Flow

- Solve GPP for any parameter (A, x_+), getting optimal solution $\tilde{c}_+(A, x_+)$ and optimal value $v_+(A, x_+)$

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Solution Flow

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- Solve LPP for any parameter (A, x_+), getting optimal solution $\tilde{c}_-(A, x_+)$ and optimal value $v_-(A, x_+)$
- Solve Step 2 problem and get optimal (A^*, x_+^*)
- Then $\tilde{c}^* := \tilde{c}_+(A^*, x_+^*) - \tilde{c}_-(A^*, x_+^*)$ solves the CPT model

Simplification

Recall Step 2 problem

$$v_+(A, x_+) - v_-(A, x_+)$$

optimisation over a set of *random events* A: hard to handle

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Theorem

(Jin and Zhou 2008) *For any feasible pair* (A, x_+) *of Problem (3), there exists* $c \in [\operatorname{essinf}\ \tilde{\rho}, \operatorname{esssup}\ \tilde{\rho}]$ *such that* $\bar{A} := \{\omega : \tilde{\rho} \leq a\}$ *satisfies*

$$v_+(\bar{A}, x_+) - v_-(\bar{A}, x_+) \geq v_+(A, x_+) - v_-(A, x_+). \tag{4}$$

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optimisation over a set of *random events* A: hard to handle

Theorem

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Proof. One needs only to look for $\tilde{c} = g(\tilde{\rho})$ where g is non-increasing. Hence
$A = \{\omega : \tilde{c} \ge 0\} = \{\omega : g(\tilde{\rho}) \ge 0\} = \{\omega : \tilde{\rho} \le a\}$.

Step 2 Problem Rewritten

- Use $v_+(a, x_+)$ and $v_-(a, x_+)$ to denote $v_+(\{\omega : \tilde{\rho} \leq a\}, x_+)$ and $v_-(\{\omega : \tilde{\rho} \leq a\}, x_+)$ respectively

Step 2 Problem Rewritten

- Use $v_+(a, x_+)$ and $v_-(a, x_+)$ to denote $v_+(\{\omega : \tilde{\rho} \leq a\}, x_+)$ and $v_-(\{\omega : \tilde{\rho} \leq a\}, x_+)$ respectively
- Problem (3) is equivalent to

$$\begin{array}{ll} \text{Max} & v_+(a, x_+) - v_-(a, x_+) \\ \text{subject to} & \left\{ \begin{array}{l} \text{essinf } \tilde{\rho} \leq a \leq \text{esssup } \tilde{\rho}, \;\; x_+ \geq x_0^+, \\ x_+ = 0 \text{ when } a = \text{essinf } \tilde{\rho}, \\ x_+ = x_0 \text{ when } a = \text{esssup } \tilde{\rho} \end{array} \right. \end{array} \tag{5}$$

Section 3

Solutions to GPP and LPP

GPP

$$
\begin{array}{ll}
\text{Max} & V_+(\tilde{c}) = \int_0^\infty w_+\left(\mathrm{P}\left(u_+(\tilde{c}) > x\right)\right) dx \\
\text{subject to} & \begin{cases} E[\tilde{\rho}\tilde{c}] = x_+, \quad \tilde{c} \geq 0 \\ \tilde{c} = 0 \text{ on } A^C, \end{cases}
\end{array} \tag{6}
$$

where $x_+ \geq x_0^+$ and $A = \{\omega : \tilde{\rho} \leq a\}$ with $\text{essinf } \tilde{\rho} \leq a \leq \text{esssup } \tilde{\rho}$

We have solved this problem – RDUT portfolio choice!

Integrability Condition

- Impose the intergrability condition

$$E\left[u_+\left((u_+')^{-1}\left(\frac{\tilde{\rho}}{w_+'(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\right)w_+'(F_{\tilde{\rho}}(\tilde{\rho}))\right] < +\infty$$

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- In the following, we always assume the integrability condition holds

Solutions to GPP

Theorem

(Jin and Zhou 2008) *Assume* $M(z)=\frac{w_+'(1-z)}{F_{\tilde\rho}^{-1}(1-z)}$ *is non-decreasing on* $z\in(0,1)$.

(i) *If* $x_+=0$, *then optimal solution of (6) is* $\tilde c^*=0$ *and* $v_+(a,x_+)=0$.

(ii) *If* $x_+>0$ *and* $a=\operatorname{essinf}\tilde\rho$, *then there is no feasible solution to (6) and* $v_+(a,x_+)=-\infty$.

(iii) *If* $x_+>0$ *and* $\operatorname{essinf}\tilde\rho<a\le\operatorname{esssup}\tilde\rho$, *then optimal solution to (6) is* $\tilde c^*=(u_+')^{-1}\left(\frac{\lambda^*\tilde\rho}{w_+'(F_{\tilde\rho}(\tilde\rho))}\right)\mathbf{1}_{(\tilde\rho\le a)}$ *with the optimal value* $v_+(a,x_+)=E\left[u_+\left((u_+')^{-1}(\frac{\lambda^*\tilde\rho}{w_+'(F_{\tilde\rho}(\tilde\rho))})\right)w_+'(F_{\tilde\rho}(\tilde\rho))\mathbf{1}_{(\tilde\rho\le a)}\right]$, *where* λ^* *is determined by* $E(\tilde\rho\tilde c^*)=x_+$.

Idea of Proof

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- GPP is rewritten as

$$\begin{array}{ll} \text{Max} & V_+(\tilde{c}) = w_+(\mathrm{P}(A)) \int_0^\infty w_A\left(\mathrm{P}_A\left(u_+(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & \left\{ \; E_A[\tilde{\rho}\tilde{c}] = x_+/\mathrm{P}(A), \; \tilde{c} \geq 0 \right. \end{array}$$

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- Apply result in Chapter 2

LPP

$$\begin{array}{ll} \text{Min} & V_-(\tilde{c}) = \int_0^\infty w_-\left(\mathrm{P}\left(u_-(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & \begin{cases} E[\tilde{\rho}\tilde{c}] = x_+ - x_0, \ \tilde{c} \geq 0 \\ \tilde{c} = 0 \text{ on } A, \ \tilde{c} \text{ is bounded} \end{cases} \end{array} \tag{7}$$

where $x_+ \geq x_0^+$ and $A = \{\omega : \tilde{\rho} \leq a\}$ with $\text{essinf } \tilde{\rho} \leq a \leq \text{esssup } \tilde{\rho}$

This is a minimisation problem!

A General Problem

$$\begin{array}{ll} \textbf{Min}_{\tilde{c}} & \int_0^\infty w\left(\mathrm{P}\left(u(\tilde{c}) > x\right)\right) dx \\ \text{subject to} & E[\tilde{\rho}\tilde{c}] \geq x_0, \;\; \tilde{c} \geq 0 \end{array} \qquad \text{(G)}$$

Hardy–Littlewood Inequality (Again)

Lemma

(Jin and Zhou 2008) *We have that* $\tilde{c}^* := G(F_{\tilde{\rho}}(\tilde{\rho}))$ *solves* $\max_{\tilde{c}' \sim \tilde{c}} E\left[\tilde{\rho}\tilde{c}'\right]$, *where* G *is quantile of* $\tilde{c}$. *If in addition* $-\infty < E[\tilde{\rho}\tilde{c}^*] < +\infty$, *then* $\tilde{c}^*$ *is the unique optimal solution.*

Hardy, Littlewood and Pòlya (1952), Dybvig (1988)

Quantile Formulation

The *quantile formulation* of (G) is:

$$\begin{array}{ll} \underset{G \in \mathbb{G}}{\text{Min}} & U(G(\cdot)) := \int_0^1 u(G(z)) w'(1-z) dz \\ \text{subject to} & \int_0^1 F_{\tilde{\rho}}^{-1}(z) G(z) dz \geq x_0 \end{array} \qquad \text{(Q)}$$

Combinatorial Optimisation in Function Spaces

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- To **minimise** a **concave** functional: “wrong” direction!
- ... which originates from S-shaped value function
- Solution must have a very different structure compared with the maximisation counterpart
- Lagrange fails (positive duality gap)
- Solution should be a “corner point solution”: essentially a combinatorial optimisation in an infinite dimensional space

Characterising Corner Point Solutions

Proposition

(Jin and Zhou 2008) *Assume $u(\cdot)$ is strictly concave at 0. Then the optimal solution to (Q), if it exists, must be in the form $G^*(z) = q(b)\mathbf{1}_{(b,1)}(z)$, $z \in [0,1)$, with some $b \in [0,1)$ and $q(b) := \frac{a}{E[\tilde{\rho}\mathbf{1}_{\{F_{\tilde{\rho}}(\tilde{\rho})>b\}}]}$. Moreover, in this case, the optimal solution is $\tilde{c}^* = G^*(F_{\tilde{\rho}}(\tilde{\rho}))$.*

- One only needs to find an optimal **number** $b \in [0,1)$

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- One only needs to find an optimal **number** $b \in [0,1)$
- ... which motivates introduction of the following problem

$$\begin{array}{ll} \text{Min}_{b} & f(b) := \int_0^1 u(G(z))w'(1-z)dz \\ \text{subject to} & G(\cdot) = \frac{a}{E[\rho\mathbf{1}_{(F_{\tilde{\rho}}(\tilde{\rho})>b)}]}\mathbf{1}_{(b,1]}(\cdot), \quad 0 \le b < 1. \end{array}$$

Solving (G)

Theorem

(Jin and Zhou 2008) *Assume $u(\cdot)$ is strictly concave at 0. Then (G) admits an optimal solution if and only if the following problem*

$$\min_{0 \leq b < \text{esssup } \tilde{\rho}} u\left(\frac{x_0}{E[\tilde{\rho}\mathbf{1}_{(\tilde{\rho}>b)}]}\right) w(\mathrm{P}(\tilde{\rho} > b))$$

admits an optimal solution b^*, in which case the optimal solution to (G) is $\tilde{c}^* = \frac{x_0}{E[\tilde{\rho}\mathbf{1}_{(\tilde{\rho}>b^*)}]}\mathbf{1}_{(\tilde{\rho}>b^*)}$.

Solutions to LPP

Theorem

(Jin and Zhou 2008) *Assume $u(\cdot)$ is strictly concave at 0.*

(i) *If $a = \text{esssup}\ \tilde{\rho}$ and $x_+ = x_0$, then optimal solution of (7) is $\tilde{c}^* = 0$ and $v_-(a, x_+) = 0$.*

(ii) *If $a = \text{esssup}\ \tilde{\rho}$ and $x_+ \neq x_0$, then there is no feasible solution to (7) and $v_-(a, x_+) = +\infty$.*

(iii) *If $\text{essinf}\ \tilde{\rho} \leq a < \text{esssup}\ \tilde{\rho}$, then*
$v_-(a, x_+) = \inf_{b \in [a, \text{esssup}\ \tilde{\rho})} u_-\left(\frac{x_+ - x_0}{E[\tilde{\rho}\mathbf{1}_{(\tilde{\rho}>b)}]}\right) w_-\left(1 - F_{\tilde{\rho}}(b)\right)$.
Moreover, Problem (7) admits an optimal solution $\tilde{c}^*$ iff the following problem

$$\min_{b \in [a, \text{esssup}\ \tilde{\rho})} u_-\left(\frac{x_+ - x_0}{E[\tilde{\rho}\mathbf{1}_{(\tilde{\rho}>b)}]}\right) w_-\left(1 - F_{\tilde{\rho}}(b)\right) \tag{8}$$

admits an optimal solution b^*, in which case $\tilde{c}^* = \frac{x_+ - x_0}{E[\tilde{\rho}\mathbf{1}_{(\tilde{\rho}>b^*)}]}\mathbf{1}_{\tilde{\rho}>b^*}$.

Section 4

Grand Solution

A Mathematical Programme

Consider a mathematical programme in (a, x_+):

$$\mathrm{Max}_{(a,x_+)} \quad E\left[u_+\left((u'_+)^{-1}\left(\frac{\lambda(a,x_+)\tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\right)w'_+(F_{\tilde{\rho}}(\tilde{(}\rho))\mathbf{1}_{(\tilde{\rho}\le a)}\right]$$

$$-u_-\left(\frac{x_+-x_0}{E[\tilde{\rho}\mathbf{1}_{\tilde{\rho}>a}]}\right)w_-(1-F(a))$$

$$\text{subject to} \quad \begin{cases} \text{essinf } \tilde{\rho} \le a \le \text{esssup } \tilde{\rho}, \quad x_+ \ge x_0^+, \\ x_+ = 0 \text{ when } a = \text{essinf } \tilde{\rho}, \quad x_+ = x_0 \text{ when } a = \text{esssup } \tilde{\rho}), \end{cases} \tag{MP}$$

where $\lambda(a, x_+)$ satisfies $E\left[(u'_+)^{-1}\left(\frac{\lambda(a,x_+)\tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\tilde{\rho}\mathbf{1}_{(\tilde{\rho}\le a)}\right] = x_+$

Grand Solution

Theorem

(Jin and Zhou 2008) *Assume $u_-(\cdot)$ is strictly concave at 0 and M is non-decreasing. Let (a^*, x_+^*) solves (MP). Then the optimal solution to (CPT) is*

$$\tilde{c}^* = \left[(u_+')^{-1}\left(\frac{\lambda\tilde{\rho}}{w_+'(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\right]\mathbf{1}_{(\tilde{\rho}\le a^*)} - \left[\frac{x_+^* - x_0}{E[\tilde{\rho}\mathbf{1}_{(\tilde{\rho}>a^*)}]}\right]\mathbf{1}_{(\tilde{\rho}>a^*)}.$$

Interpretations and Implications

$$\tilde{c}^* = \left[(u'_+)^{-1}\left(\frac{\lambda\tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\right]\mathbf{1}_{(\tilde{\rho}\le a^*)} - \left[\frac{x_+^* - x_0}{E[\tilde{\rho}\mathbf{1}_{(\tilde{\rho}>a^*)}]}\right]\mathbf{1}_{(\tilde{\rho}>a^*)}$$

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- Future world divided by "good" states (where you have gains) and "bad" ones (losses), *completely* determined by whether $\tilde{\rho} \le a^*$ or $\tilde{\rho} > a^*$

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- Future world divided by “good” states (where you have gains) and “bad” ones (losses), *completely* determined by whether $\tilde{\rho} \le a^*$ or $\tilde{\rho} > a^*$
- Agent buy claim $\left[(u'_+)^{-1}\left(\frac{\lambda\tilde{\rho}}{w'_+(F_{\tilde{\rho}}(\tilde{\rho}))}\right)\right]\mathbf{1}_{(\tilde{\rho}\le a^*)}$ at cost $x_+^* \ge x_0$ and sell $\left[\frac{x_+^*-x_0}{E[\tilde{\rho}\mathbf{1}_{(\tilde{\rho}>a^*)}]}\right]\mathbf{1}_{(\tilde{\rho}>a^*)}$ to finance shortfall $x_+^* - x_0$

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- Agent not only invests in stocks, but also generally takes a **leverage** to do so
- Optimal strategy is a *gambling* policy, betting on the good states while accepting a **known** loss on the bad

Section 5

Continuous Time and Time Inconsistency

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- m stocks whose price processes $S_1(t), \cdots S_m(t)$ satisfy stochastic differential equation (SDE)

$$dS_i(t) = S_i(t)\left(\mu_i(t)dt + \sum_{j=1}^{m}\sigma_{ij}(t)dW^j(t)\right);\ S_i(0) = s_i$$

Tame Portfolios

- Let

$$\sigma(t) := (\sigma_{ij}(t))_{m\times m}$$
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- A portfolio $\pi(\cdot)$ is *admissible* if
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Wealth Equation

- Wealth process $x(\cdot)$ follows the *wealth equation*

$$\begin{cases} dx(t) &= [r(t)x(t) + B(t)'\pi(t)]dt + \pi(t)'\sigma(t)dW(t) \\ x(0) &= x_0 \end{cases}$$

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- An admissible portfolio $\pi(\cdot)$ is called *tame* if the corresponding wealth process $x(\cdot)$ is uniformly lower bounded

Market Assumptions

Market assumptions:

(i) There exists $k \in \mathbb{R}$ such that $\int_0^T r(t)dt \geq k$,

(ii) $\int_0^T [\sum_{i=1}^m |b_i(t)| + \sum_{i,j=1}^m |\sigma_{ij}(t)|^2] dt < +\infty$,

(iii) $\mathrm{Rank}\ (\sigma(t)) = m$, $t \in [0, T]$,

(iv) There exists an $\mathbb{R}^m$-valued, uniformly bounded, $\mathcal{F}_t$-progressively measurable process $\theta(\cdot)$ such that $\sigma(t)\theta(t) = B(t)$

Pricing Kernel

- Define

$$\rho(t) := \exp\left\{-\int_0^t \left[r(s) + \frac{1}{2}|\theta(s)|^2\right] ds - \int_0^t \theta(s)' dW(s)\right\}$$

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- Assume that $\tilde{\rho}$ is atomless

Continuous-Time Portfolio Choice under EUT

$$\begin{array}{ll} \text{Max} & E[u(x(T))] \\ \text{subject to} & (x(\cdot), \pi(\cdot)): \ \text{tame and admissible pair} \end{array} \tag{9}$$

where u is a concave utility function satisfying the usual assumptions

Forward Approach: Dynamic Programming

- Let v be the *value function* corresponding to (9): $v(t,x)$ is the optimal value of (9) if the initial time is t (instead of 0) and the initial budget is x (instead of x_0)

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- *Verification theorem*: optimal portfolio

$$\pi^*(t,x) = -(\sigma(t)')^{-1}\theta(t)\frac{v_x(t,x)}{v_{xx}(t,x)} \tag{11}$$

Backward Approach: Replication

- One solves first a static optimization problem in terms of terminal wealth, $\tilde{c}$:

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- Setting $\pi^*(t) = (\sigma(t)')^{-1}z^*(t)$ and $(x^*(\cdot), \pi^*(\cdot))$ is optimal pair

Time Inconsistency under Probability Weighting

- Choquet expectation
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- Dynamic programming falls apart
- Consider a weak notion of "optimality" - equilibrium portfolio in other settings (Ekeland and Pirvu 2008, Hu, Jin and Zhou 2012, Bjork, Murgoci and Zhou 2012)

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- Such a portfolio is an optimal *pre-committed* strategy (Jin and Zhou 2008, He and Zhou 2011)

Section 6

Summary and Further Readings

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- Inherent time inconsistency for continuous-time behavioural problems

Essential Readings

- A. Berkelaar, R. Kouwenberg and T. Post. Optimal portfolio choice under loss aversion, Review of Economics and Statistics, 86:973–987, 2004.
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Other Readings

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Section 7

Final Words

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  - Neoclassical (maximising) finance starting 1960s: *Expected utility maximisation, CAPM, efficient market theory, option pricing*
  - Behavioural finance starting 1980s: *Cumulative prospect theory, SP/A theory, regret and self-control, heuristics and biases*

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  - It seeks to explore the consistency and predictability in human flaws so that such flaws can be avoided or even exploited for profit

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  - Robert Shiller (2006), "the two ... have always been interwined, and some of the most important applications of their insights will require the use of both approaches"

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- *Mathematical behavioural finance*: research is in its infancy, yet potential is unlimited – or so we believe