A Benchmark Approach to Investing, Pricing and Hedging

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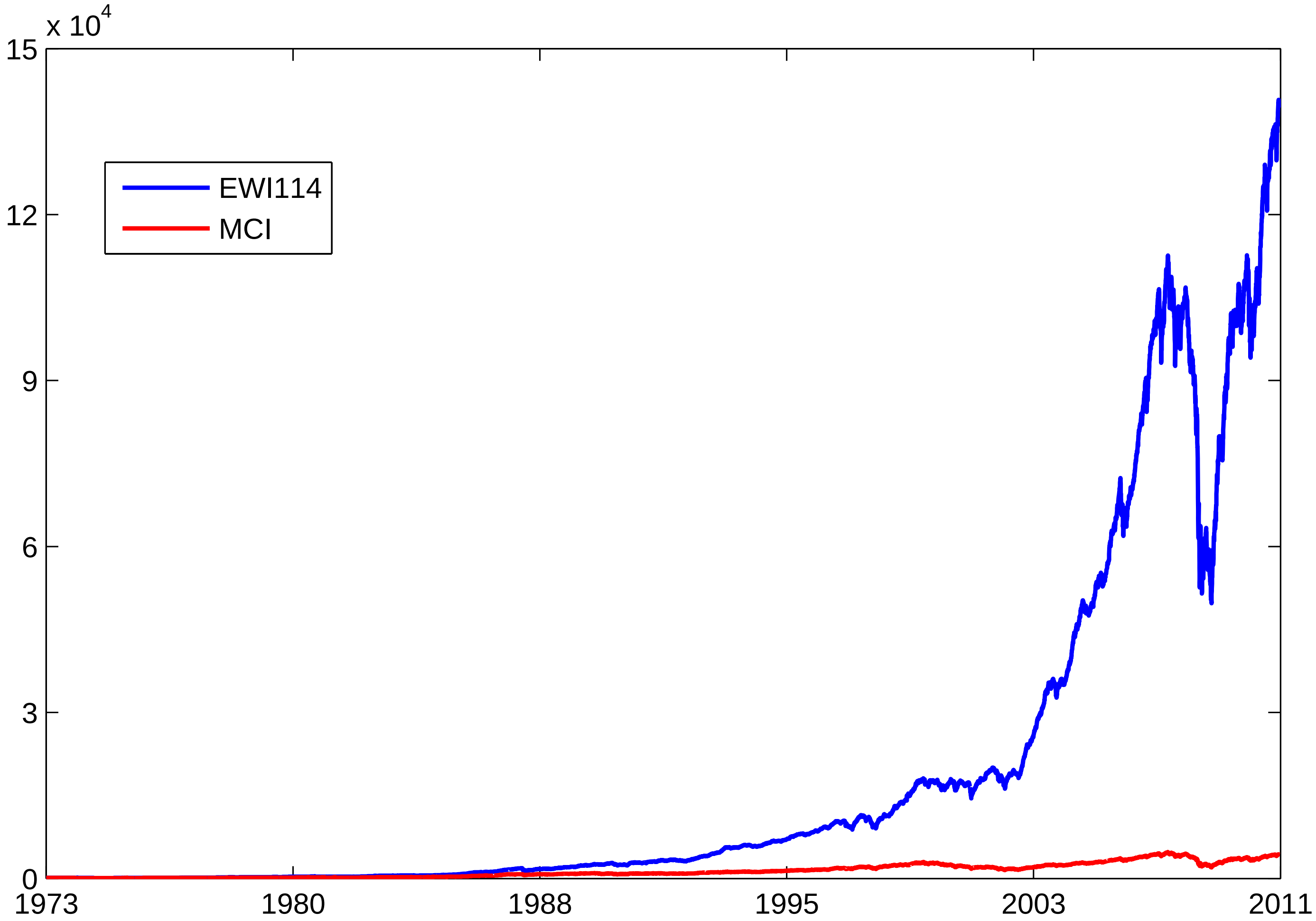
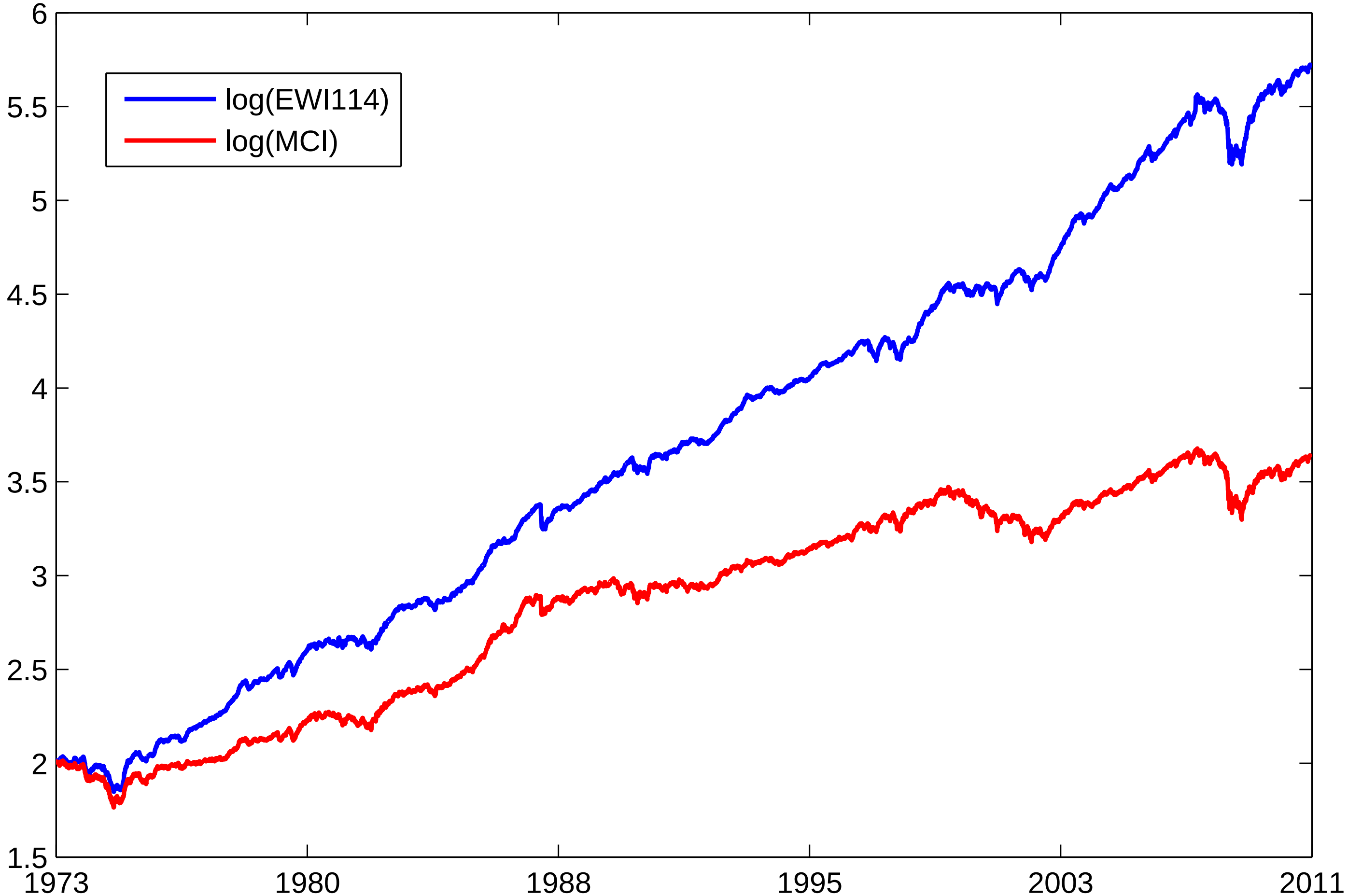
1 Best Performing Portfolio as Benchmark

- with focus on long term
- if possible pathwise
- should prefer more for less
- diversified portfolio
- robust, almost model independent construction
- sustainable (strictly positive long only)
- criterion independent from denomination
- criterion independent from sequence of drawdown events

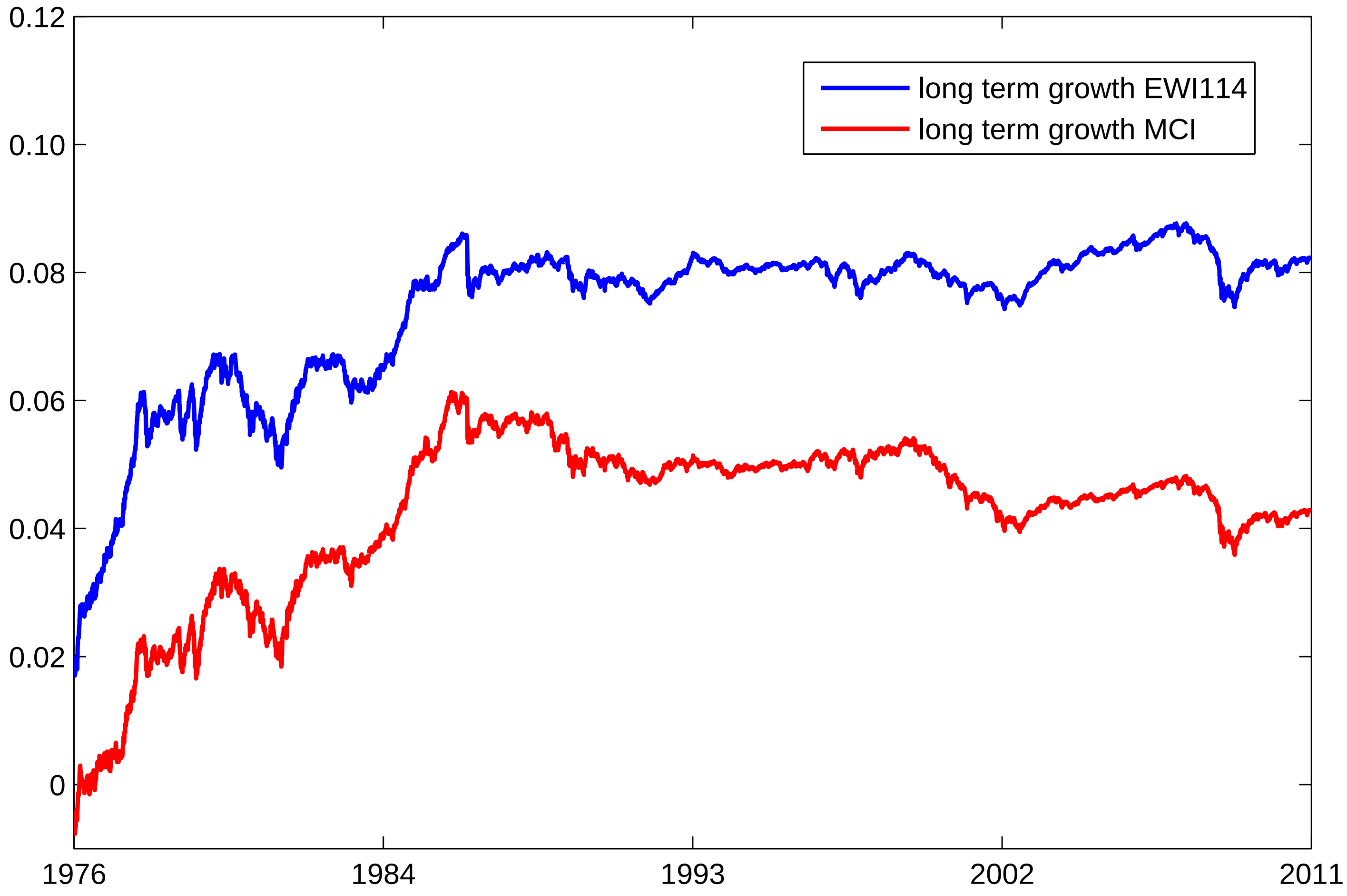
$\Rightarrow$ **maximize logarithm of portfolio**



EWI141 and MCI



logarithms of EWI114 and MCI



Long term growth of EWI114 and MCI

Key idea:

Make the **"best" performing portfolio** $S_t^{\delta_*}$ numéraire or **benchmark**

Portfolio:

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j S_t^j$$

S_t^j - jth constituent (e.g. cum dividend stocks)

Long Term Growth:

$$\lim_{t\to\infty} \frac{1}{t} \log\left(\frac{S_t^{\delta}}{S_0^{\delta}}\right) \leq \lim_{t\to\infty} \frac{1}{t} \log\left(\frac{S_t^{\delta *}}{S_0^{\delta *}}\right)$$

a.s. pathwise

$S^{\delta *}$ is the **numéraire portfolio** (NP)

Long (1990)

Main Assumption:

There exists NP $S_t^{\delta *}$,
which means

$$\frac{S_t^{\delta}}{S_t^{\delta *}} \geq E_t\left(\frac{S_s^{\delta}}{S_s^{\delta *}}\right) \qquad (*)$$

for all $0 \leq t \leq s < \infty$ and all nonnegative portfolios S_t^{δ},

$E_t(\cdot)$ - real world conditional expectation under information given at time t
$S_t^{\delta *}$ - numéraire portfolio (NP), see Long (1990)

Benchmarking:

make NP numéraire and benchmark

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta *}} \quad - \quad \text{benchmarked portfolio}$$

$$\hat{S}_t^j = \frac{S_t^j}{S_t^{\delta *}} \quad - \quad j\text{th benchmarked security}$$

Existence of NP $\Leftrightarrow$ Supermartingale Property

$$\hat{S}^{\delta}_{t} \geq E_t(\hat{S}^{\delta}_{s}) \qquad (*)$$

for all $0 \leq t \leq s < \infty$ and all nonnegative S^{δ}_{t},

- The key property of financial market!
- model independent
- makes also in the short term “best” performance of $S^{\delta *}$ precise
- existence of NP, extremely general, Karatzas and Kardaras (2007)

Outperforming Long Term Growth

- *long term growth rate*

$$g^{\delta} = \limsup_{t\to\infty} \frac{1}{t} \ln\left(\frac{S_t^{\delta}}{S_0^{\delta}}\right)$$

strictly positive portfolio S^{δ}

pathwise characteristic

Theorem ***The NP S^{δ_*} achieves the maximum long term growth rate. For any strictly positive portfolio S^{δ}***

$$g^{\delta} \leq g^{\delta_*}.$$

NP outperforms long term growth pathwise!

- Pathwise outperformance asymptotically over time!
- Ideal long term investment if no constraints!

Proof: Consider strictly positive portfolio S^δ, with $S_0^\delta = S_0^{\delta_*} = x > 0$. By supermartingale property $(*)$ of $\hat{S}^\delta$ and Doob (1953), for any $k \in \{1, 2, \ldots\}$ and $\varepsilon \in (0, 1)$ one has

$$\exp\{\varepsilon\, k\}\, P\left(\sup_{k \leq t < \infty} \hat{S}_t^\delta > \exp\{\varepsilon\, k\}\right) \leq E_0\left(\hat{S}_k^\delta\right) \leq \hat{S}_0^\delta = 1.$$

For fixed $\varepsilon \in (0,1)$

$$\sum_{k=1}^{\infty} P\left(\sup_{k\le t<\infty} \ln\left(\hat{S}_t^{\delta}\right) > \varepsilon\, k\right) \le \sum_{k=1}^{\infty} \exp\{-\varepsilon\, k\} < \infty.$$

By Lemma of Borel and Cantelli there exists k_ε such that for all $k \ge k_\varepsilon$ and $t \ge k$

$$\ln\left(\hat{S}_t^{\delta}\right) \le \varepsilon\, k \le \varepsilon\, t.$$

Therefore, for all $k > k_\varepsilon$

$$\sup_{t \geq k} \frac{1}{t} \ln\left(\hat{S}_t^\delta\right) \leq \varepsilon,$$

which implies

$$\limsup_{t\to\infty} \frac{1}{t} \ \ln\left(\frac{S_t^\delta}{S_0^\delta}\right) \leq \limsup_{t\to\infty} \frac{1}{t} \ \ln\left(\frac{S_t^{\delta_*}}{S_0^{\delta_*}}\right) + \varepsilon.$$

Since the inequality holds for all $\varepsilon \in (0, 1)$, theorem follows. □

Best Performance of NP also in the Short Term

Outperforming Expected Benchmarked Returns

$(*) \Rightarrow$ nonnegative $\hat{S}^{\delta}$ is supermartingale and $\hat{S}^{\delta *}_t = 1$

$\Longrightarrow$

- **expected benchmarked returns**

$$E_t\left(\frac{\hat{S}^{\delta}_{t+h} - \hat{S}^{\delta}_t}{\hat{S}^{\delta}_t}\right) \leq E_t\left(\frac{\hat{S}^{\delta *}_{t+h} - \hat{S}^{\delta *}_t}{\hat{S}^{\delta *}_t}\right) = 0$$

$t > 0$, $h > 0$, $\hat{S}^{\delta}$ nonnegative

$\Longrightarrow$ **No strictly positive expected benchmarked returns possible!**

Outperforming Expected Growth

- *expected growth*

$$g^{\delta}_{t,h} = E_t\left(\ln\left(\frac{S^{\delta}_{t+h}}{S^{\delta}_t}\right)\right)$$

for $t, h \geq 0$.

invest in strictly positive portfolio $S^{\underline{\delta}}$ and small fraction $\varepsilon \in (0, \frac{1}{12})$ in some nonnegative portfolio S^{δ},
which yields perturbed portfolio S^{δ_ε}

- **derivative of expected growth** in the direction of S^{δ}:

$$\frac{\partial g_{t,h}^{\delta_\varepsilon}}{\partial \varepsilon}\bigg|_{\varepsilon=0} = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \left(g_{t,h}^{\delta_\varepsilon} - g_{t,h}^{\underline{\delta}} \right)$$

Definition $S^{\underline{\delta}}$ *is called* **growth optimal** *if*

$$\left.\frac{\partial g_{t,h}^{\delta_\varepsilon}}{\partial \varepsilon}\right|_{\varepsilon=0} \leq 0$$

for all t *and* $h \geq 0$.

- classical characterization: maximization of expected logarithmic utility from terminal wealth
  Kelly (1956), Latané (1959), Breiman (1960), Hakansson (1971a), Thorp (1972), Merton (1973a), Roll (1973) and Markowitz (1976)
  equivalent

Theorem

The NP is growth optimal
if its expected growth is finite.

NP outperforms expected growth!

Proof: Pl. (2011)

For $t \geq 0$, $h > 0$, and a nonnegative portfolio S^δ we introduce the portfolio ratio $A_t^{\delta,h} = \frac{S_{t+h}^\delta}{S_t^\delta}$, which is set to 1 for $S_t^\delta = 0$. For $\varepsilon \in (0, \frac{1}{2})$; and a nonnegative portfolio S^δ, with $S_t^\delta > 0$, consider the perturbed portfolio S^{δ_ε} with $S_t^{\frac{\delta}{t}} = S_t^{\delta_*}$, yielding portfolio ratio $A_{t,h}^{\delta_\varepsilon} = \varepsilon\, A_{t,h}^\delta + (1 - \varepsilon)\, A_{t,h}^{\delta_*} > 0$. By the inequality $\ln(x) \leq x - 1$ for $x \geq 0$,

$$G_{t,h}^{\delta_\varepsilon} = \frac{1}{\varepsilon} \ln\left(\frac{A_{t,h}^{\delta_\varepsilon}}{A_{t,h}^{\delta_*}}\right) \leq \frac{1}{\varepsilon}\left(\frac{A_{t,h}^{\delta_\varepsilon}}{A_{t,h}^{\delta_*}} - 1\right) = \frac{A_{t,h}^{\delta}}{A_{t,h}^{\delta_*}} - 1$$

and

$$G_{t,h}^{\delta_\varepsilon} = -\frac{1}{\varepsilon} \ln\left(\frac{A_{t,h}^{\delta_*}}{A_{t,h}^{\delta_\varepsilon}}\right) \geq -\frac{1}{\varepsilon}\left(\frac{A_{t,h}^{\delta_*}}{A_{t,h}^{\delta_\varepsilon}} - 1\right) = \frac{A_{t,h}^{\delta} - A_{t,h}^{\delta_*}}{A_{t,h}^{\delta_\varepsilon}}.$$

Because of $A_{t,h}^{\delta_\varepsilon} > 0$ one obtains for $A_{t,h}^{\delta} - A_{t,h}^{\delta_*} \geq 0$

$$G_{t,h}^{\delta_\varepsilon} \geq 0,$$

and for $A_{t,h}^{\delta} - A_{t,h}^{\delta_*} < 0$ because of $\varepsilon \in (0, \frac{1}{2})$ and $A_{t,h}^{\delta} \geq 0$

$$G_{t,h}^{\delta_\varepsilon} \geq -\frac{A_{t,h}^{\delta_*}}{A_{t,h}^{\delta_\varepsilon}} = -\frac{1}{1-\varepsilon+\varepsilon\,\frac{A_{t,h}^{\delta}}{A_{t,h}^{\delta_*}}} \geq -\frac{1}{1-\varepsilon} \geq -2.$$

Summarizing yields

$$-2 \leq G_{t,h}^{\delta_\varepsilon} \leq \frac{A_{t,h}^{\delta}}{A_{t,h}^{\delta_*}} - 1,$$

where by $(*)$

$$E_t \left(\frac{A_{t,h}^{\delta}}{A_{t,h}^{\delta_*}} \right) \leq 1.$$

By the Dominated Convergence Theorem

$$\begin{aligned}
\left.\frac{\partial g_{t,h}^{\delta_\varepsilon}}{\partial \varepsilon}\right|_{\varepsilon=0} &= \lim_{\varepsilon \to 0+} E_t\left(G_{t,h}^{\delta_\varepsilon}\right) = E_t\left(\lim_{\varepsilon \to 0+} G_{t,h}^{\delta_\varepsilon}\right) \\
&= E_t\left(\frac{\partial}{\partial \varepsilon} \ln\left(\frac{A_{t,h}^{\delta_\varepsilon}}{A_{t,h}^{\delta_*}}\right)\Big|_{\varepsilon=0}\right) = E_t\left(\frac{A_{t,h}^{\delta}}{A_{t,h}^{\delta_*}}\right) - 1.
\end{aligned}$$

This proves that NP S^{δ_*} is growth optimal. □

Systematic Outperformance

Definition *A nonnegative portfolio* S^{δ} systematically outperforms *a strictly positive portfolio* $S^{\tilde{\delta}}$ *if*

(i) $S^{\delta}_{t_0} = S^{\tilde{\delta}}_{t_0}$;

(ii) $P(S^{\delta}_t \geq S^{\tilde{\delta}}_t) = 1$ *for some* $t > 0$ *and*

(iii) $P\left(S^{\delta}_t > S^{\tilde{\delta}}_t\right) > 0$.

- also called relative arbitrage in Fernholz & Karatzas (2005)

Theorem ***The NP cannot be systematically outperformed by any nonnegative portfolio.***

- NP can systematically outperform other portfolios.

Proof: Consider nonnegative S^{δ} with $\hat{S}_t^{\delta} = 1$ where $\hat{S}_s^{\delta} \geq 1$ almost surely at $s \in [t, \infty)$. By the supermartingale property $(*)$

$$0 \geq E_t\left(\hat{S}_s^{\delta} - \hat{S}_t^{\delta}\right) = E_t\left(\hat{S}_s^{\delta} - 1\right).$$

Since $\hat{S}_s^{\delta} \geq 1$ and $E_t(\hat{S}_s^{\delta}) \leq 1$, it can only follow $\hat{S}_s^{\delta} = 1$.
$\Longrightarrow \quad S_s^{\delta} = S_s^{\delta_*}$. □

- **NP needs shortest expected time to reach a given wealth level**
  Kardaras & Platen (2010)

Strong Arbitrage

- only market participants can exploit arbitrage with their total wealth
- **limited liability**

$\Longrightarrow$ nonnegative total wealth of each market participant

Definition *A nonnegative portfolio* S^δ *is a* strong arbitrage *if* $S_0^\delta = 0$ *and for some* $T > 0$

$$P\left(S_T^\delta > 0\right) > 0.$$

- Loewenstein & Willard (2000) - same notion of arbitrage through economic arguments
- Pl. (2002)-mathematical motivation through supermartingale property $(*)$

Theorem ***There is no strong arbitrage.***

Proof:

$(*) \Longrightarrow$

$\hat{S}^\delta$ nonnegative supermartingale

$\Longrightarrow$

$$0 = \hat{S}_0^\delta \geq E\left(\hat{S}_T^\delta \,\middle|\, \mathcal{A}_0\right) = E(\hat{S}_T^\delta)$$

$\Longrightarrow$

$$P(S_T^\delta > 0) = P(\hat{S}_T^\delta > 0) = 0. \qquad \square$$

Classical notion of arbitrage is too restrictive

- Delbaen & Schachermayer (1998)

  no free lunches with vanishing risk $\triangleq$ existence of risk neutral measure $\triangleq$ APT

- Loewenstein & Willard (2000)

  free snacks & *cheap thrills* $\triangleq$ some free lunch with vanishing risk

- exploiting weak forms of classical arbitrage requires to allow negative total wealth

- pricing via hedging by avoiding "strong arbitrage" makes no sense
  since there is no "strong arbitrage" under $(*)$.

How to approximate the NP in the real market?

- diversification leads asymptotically to the NP (Pl. 2005)
- confirm empirically by visualizing diversification effect
- simplest diversification:

  equal value weighting = naive diversification

Pl. & Rendek (2012)

- **investment universe**:

  market capitalization weighted indices generating the MSCI

- **Industry Classification Benchmark** (Datastream, Reuters ...):

  54 countries with country index

  10 industry indices

  19 supersector indices

  41 sector indices

  114 country subsector indices

  all market capitalization weighted

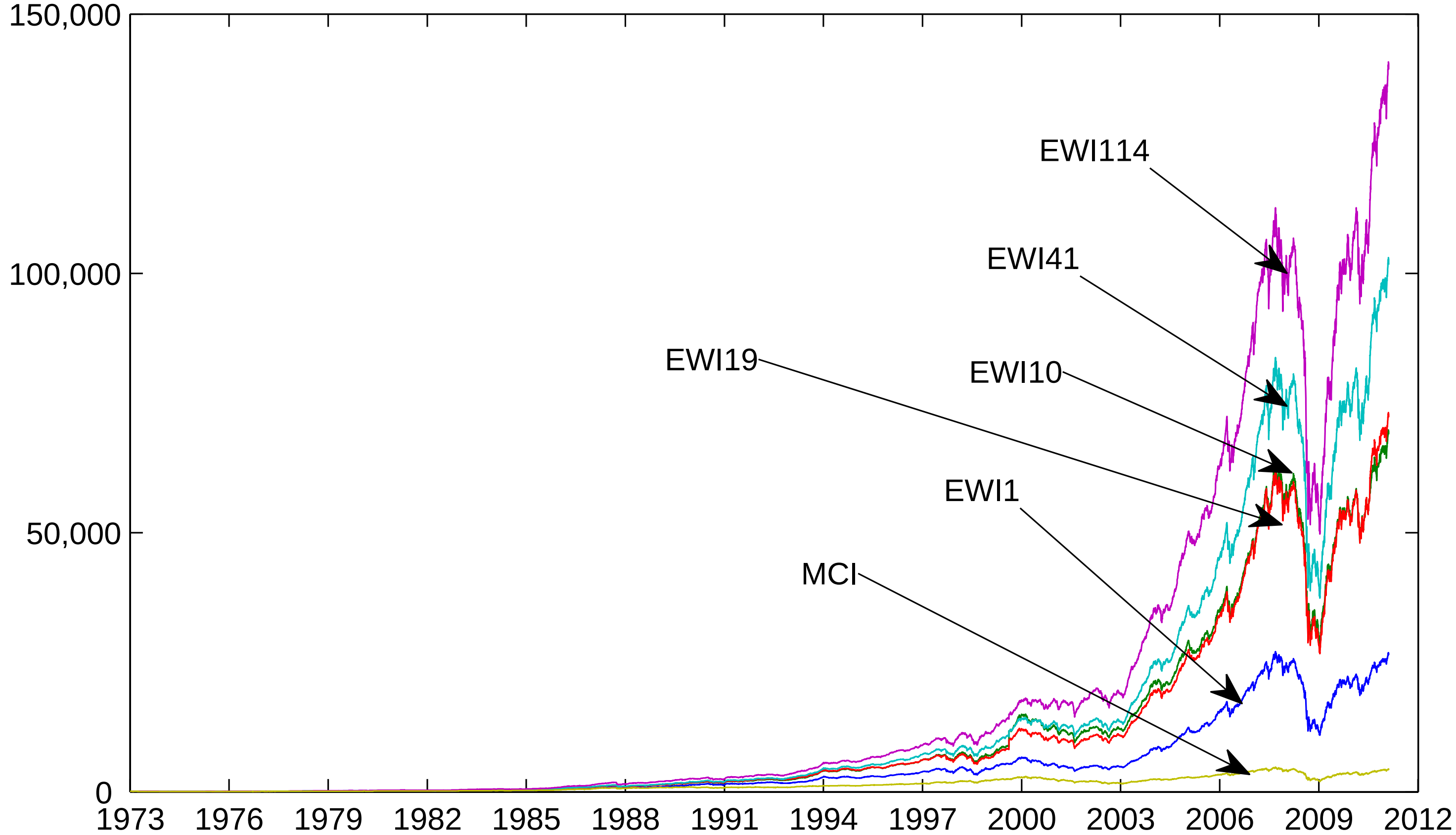
Equi-Weighted Index (EWId)

d constituents

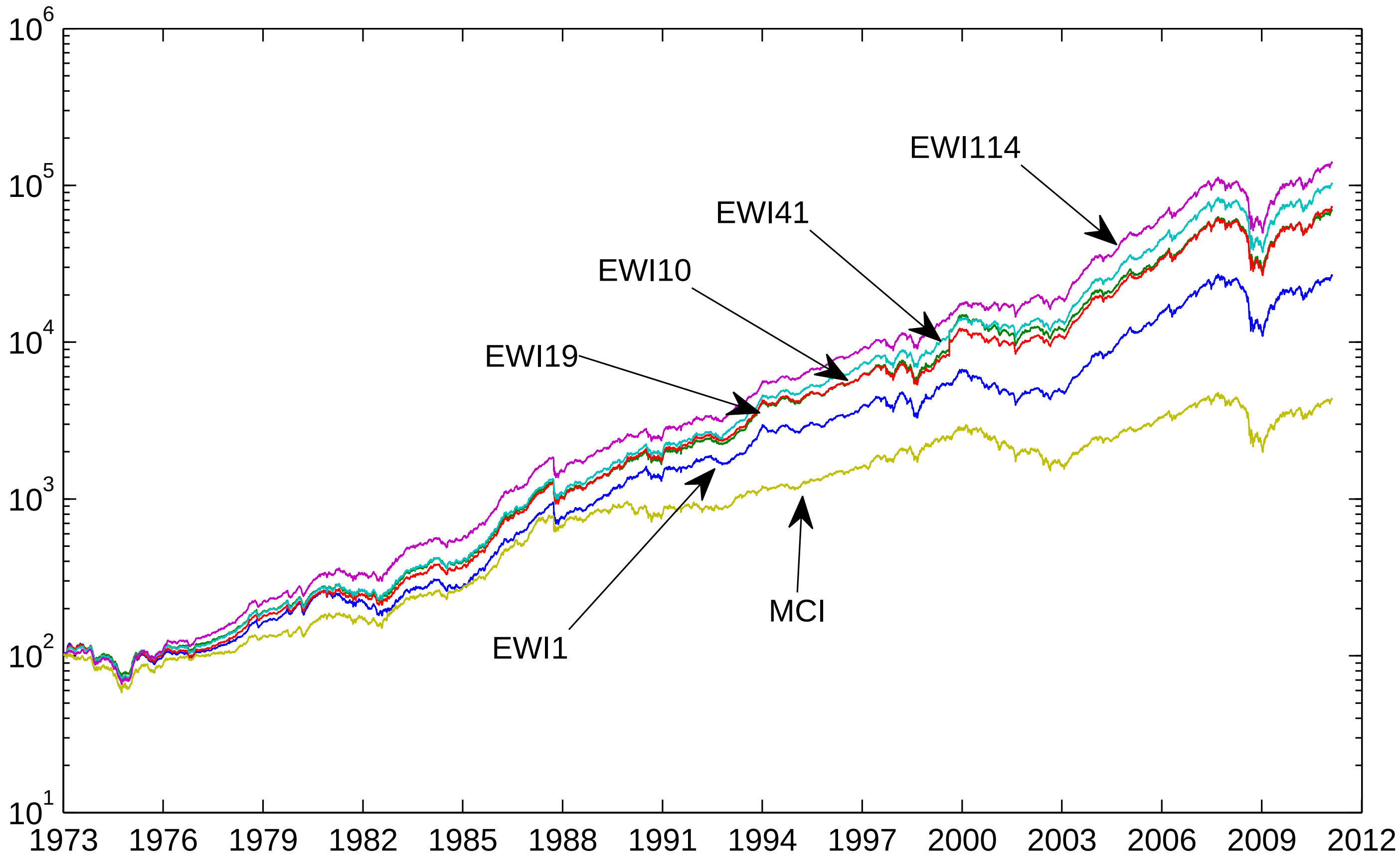
$$S_{t_n}^{\delta_{EWId}} = S_{t_{n-1}}^{\delta_{EWId}} \left(1 + \sum_{j=1}^{d} \sum_{k=1}^{\ell_{d,j}} \pi_{\delta_{EWId},t_{n-1}}^{j,k} \frac{S_{t_n}^{j,k} - S_{t_{n-1}}^{j,k}}{S_{t_{n-1}}^{j,k}} \right),$$

- Naive diversification over $\ell_{d,j}$ countries

  and then over $d =$ 10, 19, 41, 114 world industries
- equal value weighting uncertainties of different economic activities



The MCI and five equi-weighted indices: EWI1 (market), EWI10 (industry), EWI19 (supersector), EWI41 (sector), EWI114 (subsector).



The MCI and five equi-weighted indices in log-scale: EWI1 (market), EWI10 (industry), EWI19 (supersector), EWI41 (sector), EWI114 (subsector).

- all based on the same country industry subsectors of the equity market
- all resulting indices mostly driven by nondiversifiable uncertainty
- Empirically: **Better performance through better diversification!**

Illustrating Diversification Phenomenon:

- return of benchmarked constituent over small period $[t, t + h]$

$$R^j = \frac{\hat{S}^j_{t+h} - \hat{S}^j_t}{\hat{S}^j_t}$$

$j \in \{1, 2, \dots\}$

Assume for illustration purposes: $\boldsymbol{R^1}$, $\boldsymbol{R^2}$, $\cdots$ independent, and

$$\begin{aligned} \boldsymbol{E_t(R^j)} &= \boldsymbol{0} \\ \boldsymbol{E_t\left((R^j)^2\right)} &= \boldsymbol{\sigma^2 h} \end{aligned}$$

$\boldsymbol{h > 0, j \in \{1, 2, \dots\}}$

- return of portfolio equals
  sum of weighted returns of constituents
- return of benchmarked EWIℓ with ℓ constituents:

$$R^{\delta_{EWI\ell}} = \frac{\hat{S}^{\delta_{EWI\ell}}_{t+h} - \hat{S}^{\delta_{EWI\ell}}_{t}}{\hat{S}^{\delta_{EWI\ell}}_{t}} = \frac{1}{\ell} \sum_{j=1}^{\ell} R^j$$

$\Longrightarrow$

$$
\begin{aligned}
E_t(R^{\delta_{EWI\ell}}) &= 0 \\
E_t\left((R^{\delta_{EWI\ell}})^2\right) &= \frac{1}{\ell^2}\sum_{j=1}^{\ell}\sigma^2 h = \frac{1}{\ell}\sigma^2 h
\end{aligned}
$$

$\Longrightarrow$

$$\lim_{\ell\to\infty} E_t\left((R^{\delta_{EWI\ell}})^2\right) = 0$$

Strong Law of Large Numbers (Kolmogorov)

$\Rightarrow$

$$\lim_{\ell\to 0} R^{\delta_{EWI\ell}} = 0$$

almost surely

for $\ell >> 1$

$$\Longrightarrow \quad R^{\delta_{EWI\ell}} \approx 0$$

$$\Longrightarrow \quad \hat{S}_t^{\delta_{EWI\ell}} \approx 1$$

$$\Longrightarrow \quad S_t^{\delta_{EWI\ell}} = \hat{S}_t^{\delta_{EWI\ell}} S_t^{\delta *} \approx S_t^{\delta *}$$

a.s. for all t

Diversification yields approximation of the NP !

- diversification holds more generally
- model independent phenomenon
- arises naturally
- can be systematically exploited

Naive Diversification Theorem

Pl. & Rendek (2012)

- continuous market, Merton (1972)

- benchmarked constituents

$$\frac{d\hat{S}_t^j}{\hat{S}_t^j} = \sum_k \sigma_t^{j,k} dW_t^k$$

$W^1, W^2, \ldots$ - Brownian motions

- **fraction of wealth invested**

$$\pi^j_{\delta,t} = \frac{\delta^j_t \hat{S}^j_t}{\hat{S}^\delta_t} = \frac{\delta^j_t S^j_t}{S^{\delta_t}}$$

- **benchmarked portfolio**

$$\frac{d\hat{S}_t^\delta}{\hat{S}_t^\delta} = \sum_j \pi_{\delta,t}^j \frac{d\hat{S}_t^j}{\hat{S}_t^j} = \sum_j \pi_{\delta,t}^j \sum_k \sigma_t^{j,k} dW_t^k$$

- **return process of a benchmarked portfolio** $\hat{S}_t^\delta$

$$d\hat{Q}_t^\delta = \frac{1}{\hat{S}_t^\delta} d\hat{S}_t^\delta$$

$t \geq 0$ with $\hat{Q}_0^\delta = 0$

- **Quadratic Variation of Return Process**

$$\langle \hat{Q}^{\delta_\ell} \rangle_t = \int_0^t \sum_k \left(\sum_j \pi^j_{\delta_\ell, s} \sigma^{j,k}_s \right)^2 ds$$

- **Tracking rate**

$$T_{\delta_\ell}(t) = \frac{d}{dt} \langle \hat{Q}^{\delta_\ell} \rangle_t = \sum_k \left(\sum_j \pi^j_{\delta_\ell, t} \sigma^{j,k}_t \right)^2$$

- **benchmarked NP equals constant one**

  return process of benchmarked NP equals zero

$$\frac{d\hat{S}_t^{\delta_*}}{\hat{S}_t^{\delta_*}} = \sum_k \sum_j \pi_{\delta_*,t}^j \sigma_t^{j,k} dW_t^k = 0$$

$\Longrightarrow$ tracking rate

$$T_{\delta_*}(t) = \frac{d}{dt}\langle \hat{Q}^{\delta_*} \rangle_t = 0$$

- **ℓth equi-weighted index (EWIℓ)**

$$\pi^{j}_{\delta_{EWI\ell},t} = \begin{cases} \frac{1}{\ell} & \text{for} \quad j \in \{1, 2, \ldots, \ell\} \\ 0 & \text{otherwise.} \end{cases}$$

Definition:

A sequence $(\hat{S}^{\delta_\ell})_{\ell \in \{1,2,\dots\}}$ *of strictly positive benchmarked portfolios, with initial value equal to one, is called a sequence of* **benchmarked approximate NPs** *if for each* $\varepsilon > 0$ *and* $t \geq 0$ *one has*

$$\lim_{\ell \to \infty} P\left(T_{\delta_\ell}(t) = \frac{d}{dt} \left\langle \hat{Q}^{\delta_\ell} \right\rangle_t > \varepsilon \right) = 0.$$

- benchmark captures general, systematic or non-diversifiable market uncertainty
- benchmarked primary security accounts capture specific or idiosyncratic uncertainty
- different types of economic activity yield naturally different specific uncertainties
- specific uncertainty can be diversified

Particular specific uncertainty drives in reality usually only

returns of restricted number of benchmarked primary security accounts:

Definition:

A market is **well-securitized** *if there exists* $q > 0$ *and square integrable* $\underline{\sigma}^2 = \{\underline{\sigma}_t^2, t \geq 0\}$ *with finite mean for all* $\ell, k \in \{1, 2, \ldots\}$ *and* $t \geq 0$

$$\frac{1}{\ell}\left|\sum_{j=1}^{\ell} \sigma_t^{j,k}\right|^2 \leq \frac{1}{\ell^q}\underline{\sigma}_t^2$$

P-almost surely.

- **k-th uncertainty affects not too many constituents.**

Naive Diversification Theorem:

In a well-securitized market the sequence of benchmarked equi-weighted indices is a sequence of benchmarked approximate NPs.

Pl. & Rendek (2012)

- return process of ℓth benchmarked EWI

$$\hat{Q}_t^{\delta_{EWI\ell}} = \sum_{j=1}^{\ell} \frac{1}{\ell} \sum_k \int_0^t \sigma_s^{j,k} dW_s^k$$

- quadratic variation of return process

$$\left\langle \hat{Q}^{\delta_{EWI\ell}} \right\rangle_t = \frac{1}{\ell} \int_0^t \sum_k \left| \frac{1}{\sqrt{\ell}} \sum_{j=1}^{\ell} \sigma_s^{j,k} \right|^2 ds$$

- tracking rate

$$T_{\delta_{EWI\ell}}(t) = \frac{d}{dt} \left\langle \hat{Q}^{\delta_{EWI\ell}} \right\rangle_t = \frac{1}{\ell} \sum_k \left| \frac{1}{\sqrt{\ell}} \sum_{j=1}^{\ell} \sigma_t^{j,k} \right|^2$$

Lemma *Assume that for all* $\varepsilon > 0$ *and* $t \geq 0$

$$\lim_{\ell\to\infty} P\left(\frac{1}{\ell}\sum_{k}^{\ell}\left|\frac{1}{\sqrt{\ell}}\sum_{j=1}^{\ell}\sigma_t^{j,k}\right|^2 > \varepsilon\right) = 0,$$

then the sequence of equi-weighted indices is a sequence of benchmarked approximate NPs.

Proof:

For $\varepsilon > 0$ and $t > 0$

$$
\begin{aligned}
0 &= \lim_{\ell \to \infty} P\left(\frac{1}{\ell} \sum_{k} \left| \frac{1}{\sqrt{\ell}} \sum_{j=1}^{\ell} \sigma_t^{j,k} \right|^2 > \epsilon \right) \\
&= \lim_{\ell \to \infty} P\left(\frac{d}{dt} \left\langle \hat{Q}^{\delta_{EWI\ell}} \right\rangle_t > \varepsilon \right) \\
&= \lim_{\ell \to \infty} P\left(T_{\delta_{EWI\ell}}(t) > \varepsilon \right)
\end{aligned}
$$

□

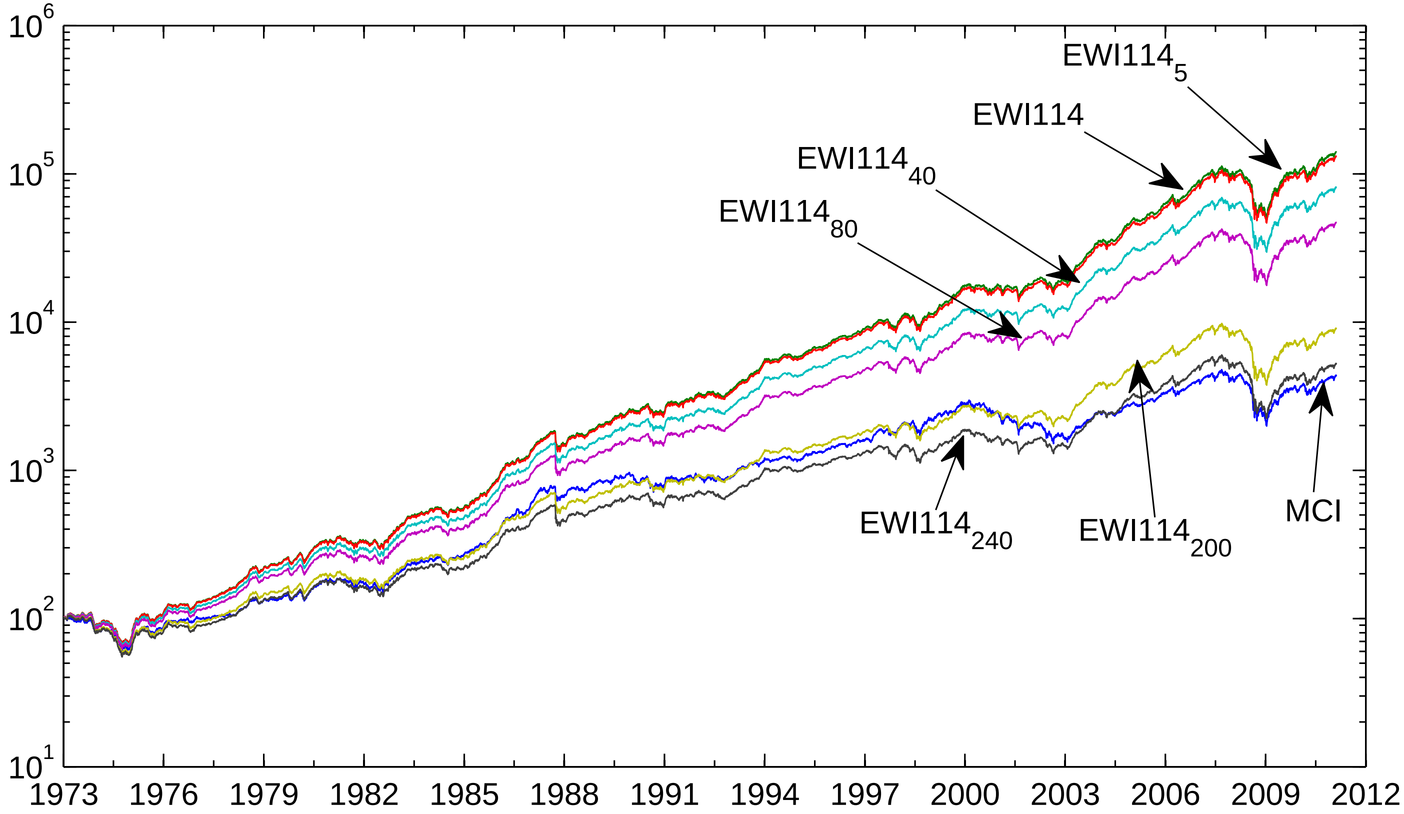
Proof of Naive Diversification Theorem:

By Markov inequality in a well-securitized market

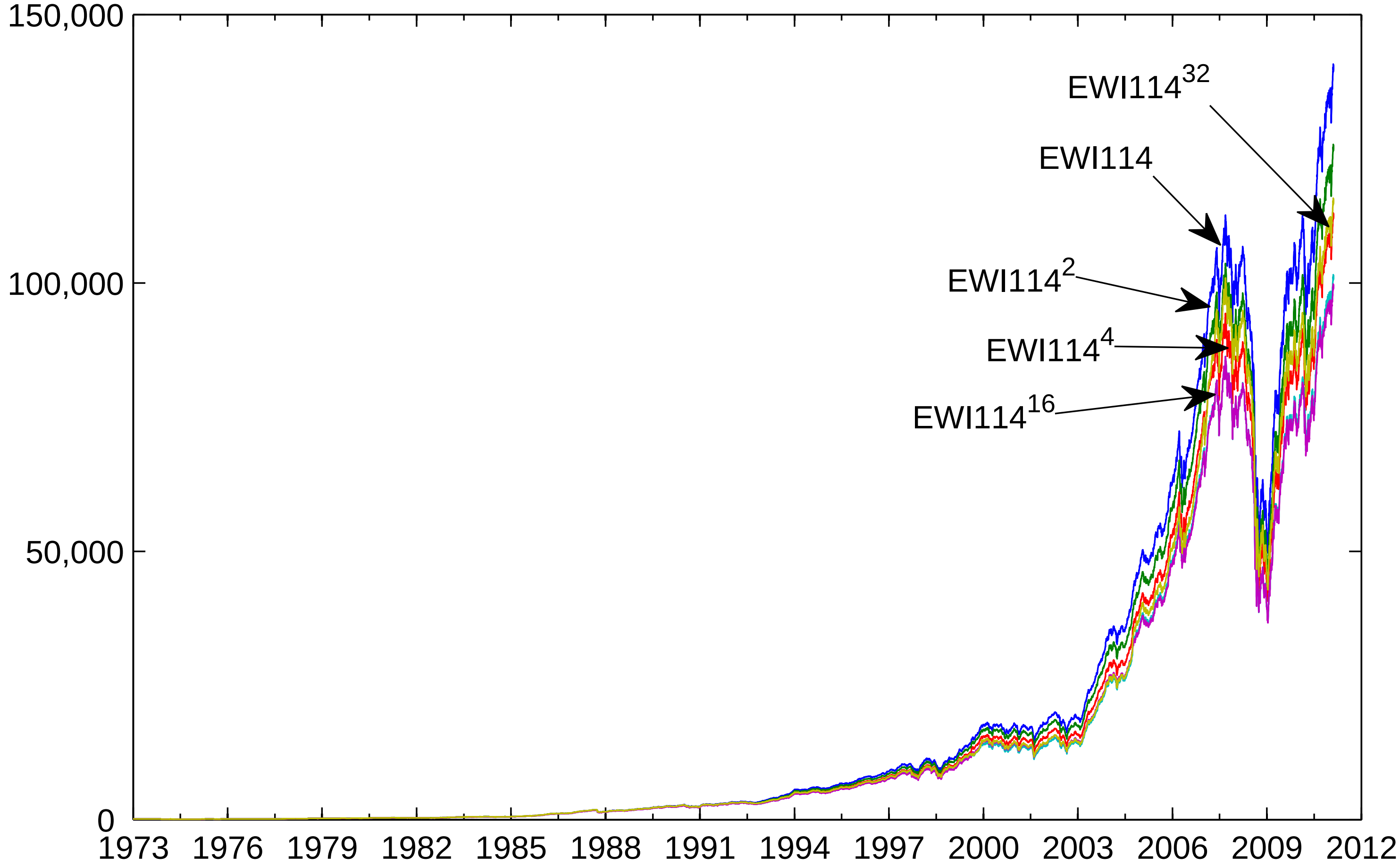
$$\lim_{\ell\to\infty} P\left(\frac{1}{\ell}\sum_{k}\left|\frac{1}{\sqrt{\ell}}\sum_{j=1}^{\ell}\sigma_t^{j,k}\right|^2 > \varepsilon\right)$$

$$= \lim_{\ell\to\infty} P\left(\frac{1}{\ell^q}\underline{\sigma}_t^2 > \varepsilon\right) \leq \lim_{\ell\to\infty}\frac{1}{\varepsilon}\frac{1}{\ell^q}E\left(\underline{\sigma}_t^2\right) = 0,$$

NDT follows from previous Lemma. □



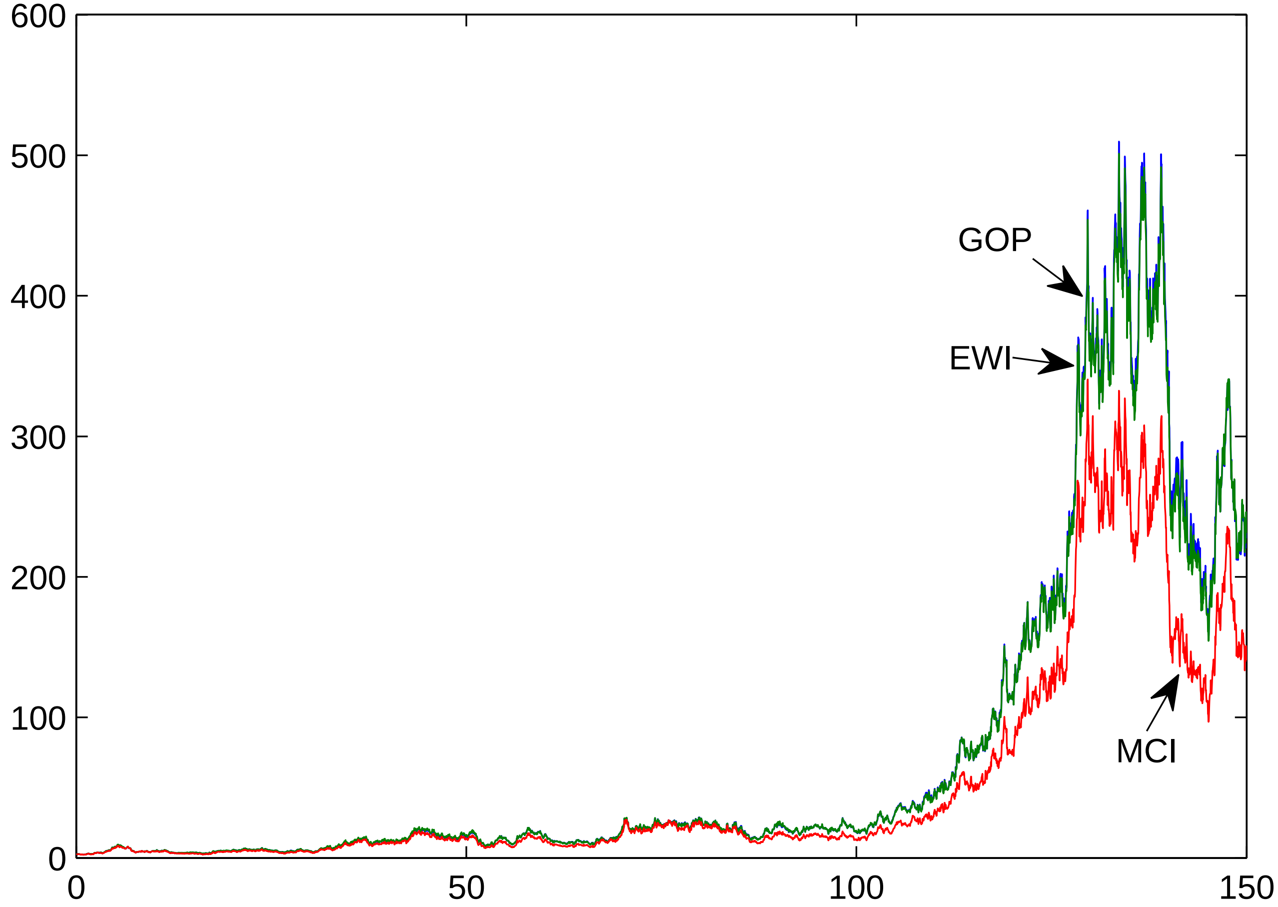
Logarithms of MCI, EWI114 without transaction cost and EWI114$_\xi$ with transaction costs of 5,40,80,200 and 240 basis points.



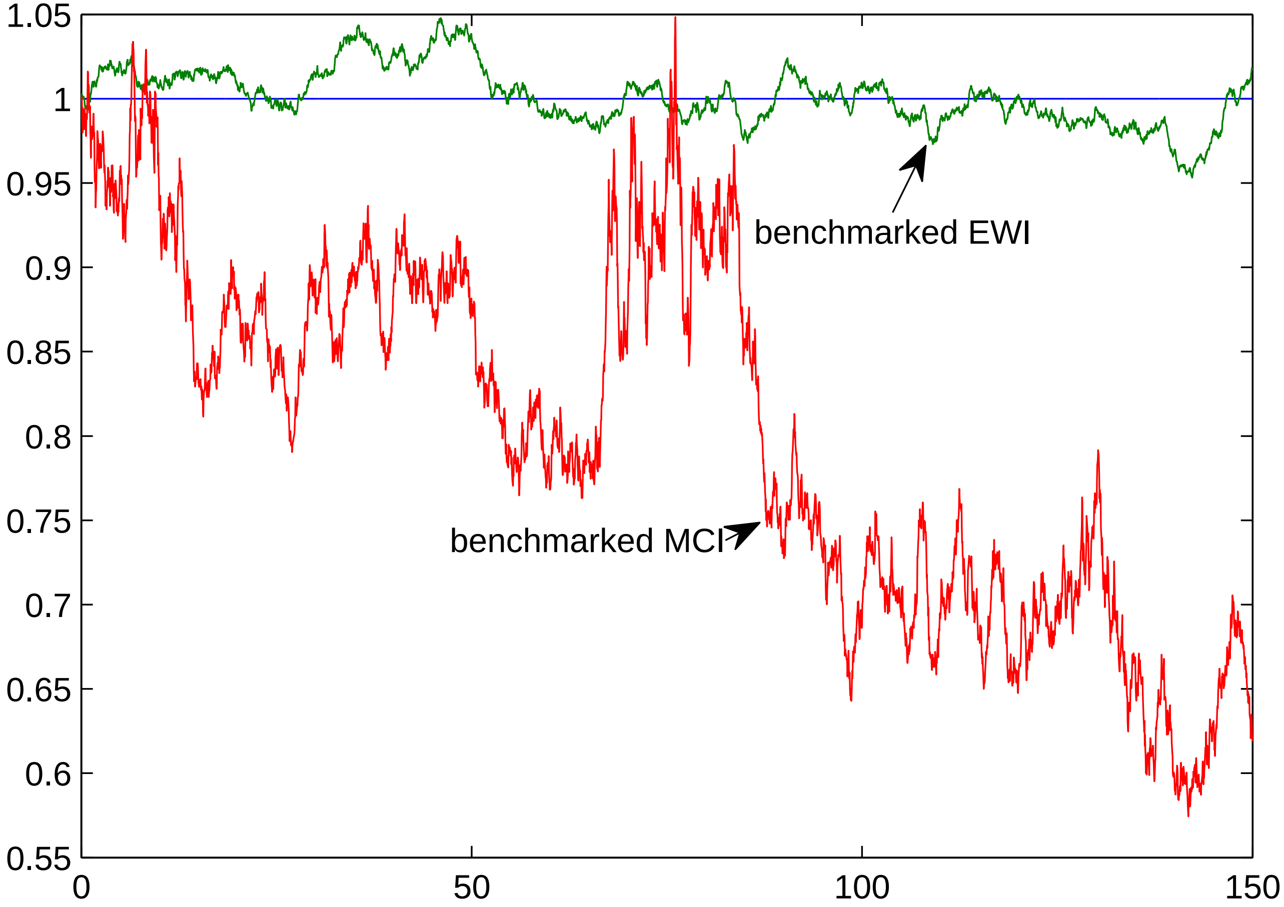
EWI114^m reallocated daily and every 2, 4, 8, 16 and 32 days.

Transaction cost	0	5	40	80	200	240
Reallocation terms	1					
Final value	139338.64	130111.93	80543.07	46555.04	8988.23	5194.46
Annualised average return	0.1979	0.1961	0.1834	0.1689	0.1254	0.1109
Annualised volatility	0.1135	0.1135	0.1135	0.1135	0.1134	0.1134
Sharpe ratio	1.4205	1.4046	1.2930	1.1654	0.7822	0.6544
Reallocation terms	2					
Final value	124542.04	119369.00	88697.63	63166.73	22808.64	16240.40
Annualised average return	0.1949	0.1938	0.1859	0.1770	0.1500	0.1411
Annualised volatility	0.1134	0.1134	0.1134	0.1134	0.1135	0.1136
Sharpe ratio	1.3955	1.3856	1.3163	1.2369	0.9987	0.9193
Reallocation terms	4					
Final value	111899.82	108230.16	85698.25	65628.82	29467.48	22562.42
Annualised average return	0.1921	0.1912	0.1850	0.1780	0.1568	0.1497
Annualised volatility	0.1135	0.1135	0.1134	0.1134	0.1134	0.1135
Sharpe ratio	1.3699	1.3622	1.3080	1.2459	1.0591	0.9967

Transaction cost	0	5	40	80	200	240
Reallocation terms	8					
Final value	100505.37	97963.66	81881.57	66705.62	36055.10	29367.02
Annualised average return	0.1892	0.1885	0.1837	0.1783	0.1621	0.1566
Annualised volatility	0.1127	0.1127	0.1127	0.1127	0.1128	0.1128
Sharpe ratio	1.3531	1.3471	1.3051	1.2569	1.1119	1.0634
Reallocation terms	16					
Final value	98775.24	96892.29	84677.43	72588.14	45711.97	39177.91
Annualised average return	0.1887	0.1882	0.1847	0.1806	0.1684	0.1643
Annualised volatility	0.1130	0.1130	0.1130	0.1130	0.1131	0.1131
Sharpe ratio	1.3463	1.3418	1.3102	1.2740	1.1647	1.1281
Reallocation terms	32					
Final value	114592.50	112929.09	101939.59	90678.85	63804.84	56744.28
Annualised average return	0.1927	0.1923	0.1896	0.1865	0.1772	0.1741
Annualised volatility	0.1131	0.1131	0.1131	0.1131	0.1133	0.1133
Sharpe ratio	1.3797	1.3763	1.3522	1.3245	1.2408	1.2127



Simulated GOP, EWI and MCI under the Black-Scholes model



Simulated benchmarked GOP, EWI and MCI under the Black-Scholes model

Example of a Multi-Asset BS Model

- jth primary security account

$$dS_t^j = S_t^j \left[\left(r + \sigma^2 \left(1 + \frac{1}{\sqrt{d}} \right) \right) dt + \frac{\sigma}{\sqrt{d}} \sum_{k=1}^{d} dW_t^k + \sigma \, dW_t^j \right]$$

$t \in [0, T], j \in \{1, 2, \ldots, d\}, \sigma > 0$

general market risk, specific market risk

- **NP fractions**

$$\pi^j_{\delta_*,t} = \left(\sqrt{d}\left(1 + \sqrt{d}\right)\right)^{-1}$$

$j \in \{1, 2, \ldots, d\}$

$$\pi^0_{\delta_*,t} = \left(1 + \sqrt{d}\right)^{-1}$$

- **NP SDE**

$$dS^{\delta_*}_t = S^{\delta_*}_t \left(\left(r + \sigma^2\right) dt + \frac{\sigma}{\sqrt{d}} \sum_{k=1}^{d} dW^k_t \right)$$

- simulate over $T = 32$ years $d = 50$ risky primary security accounts

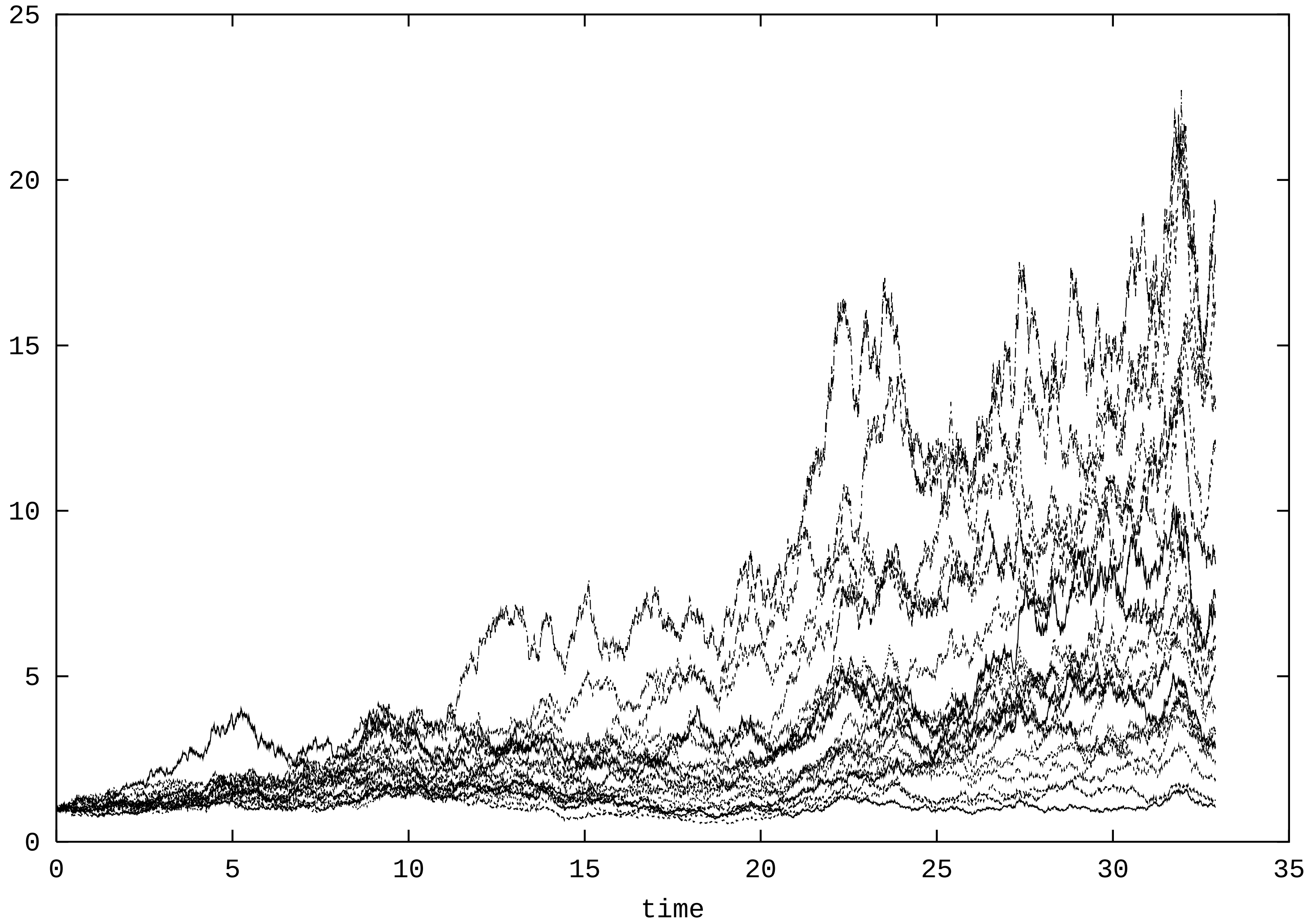
$\sigma = 0.15, r = 0.05$

- Benchmarked primary security account

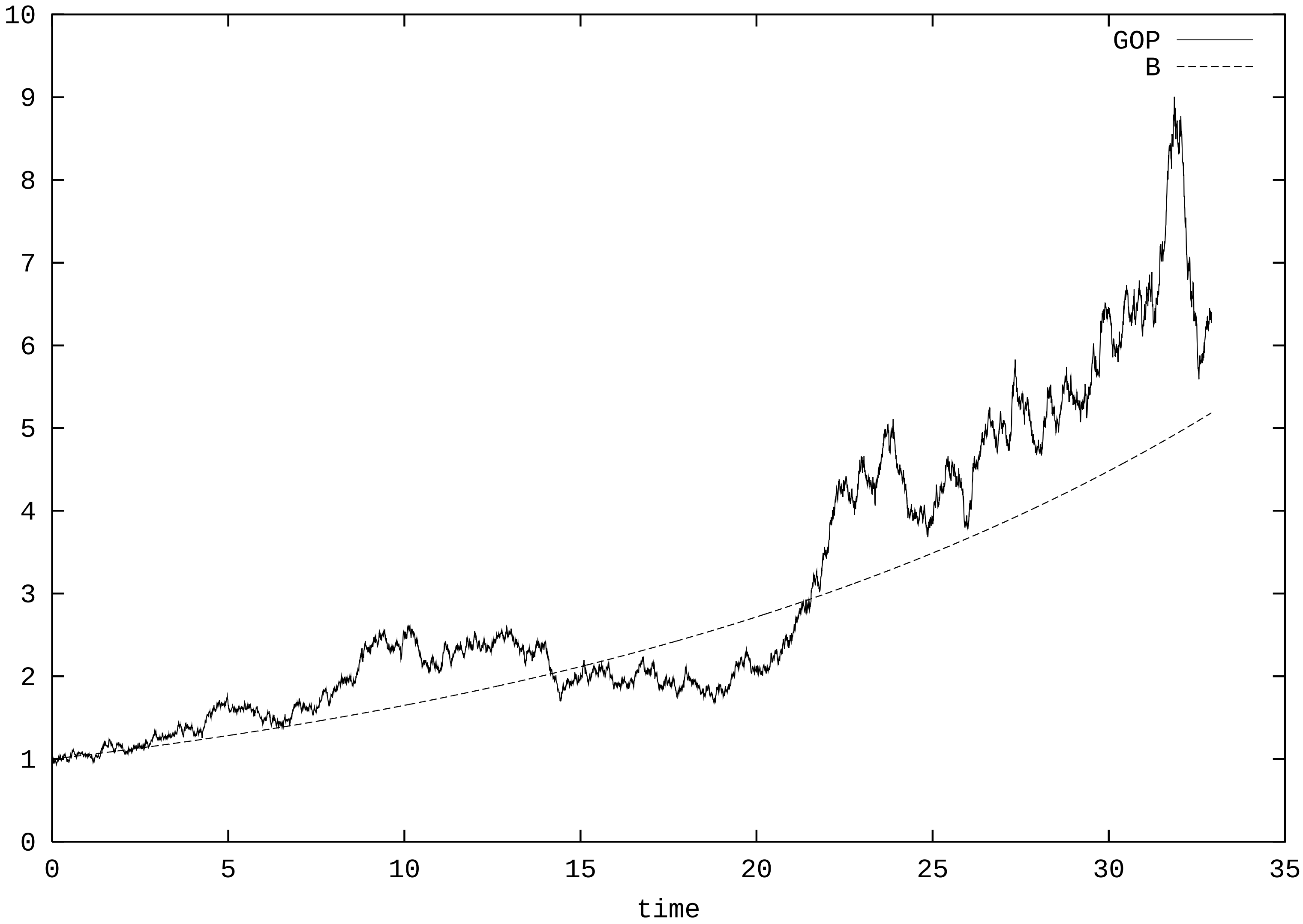
  BS - model

$$d\hat{S}_t^j = d\left(\frac{S_t^j}{S_t^{\delta *}}\right) = \hat{S}_t^j \sigma dW_t^j$$

  martingale



Simulated primary security accounts.



Simulated GOP and savings account.

Equal Weighted Index

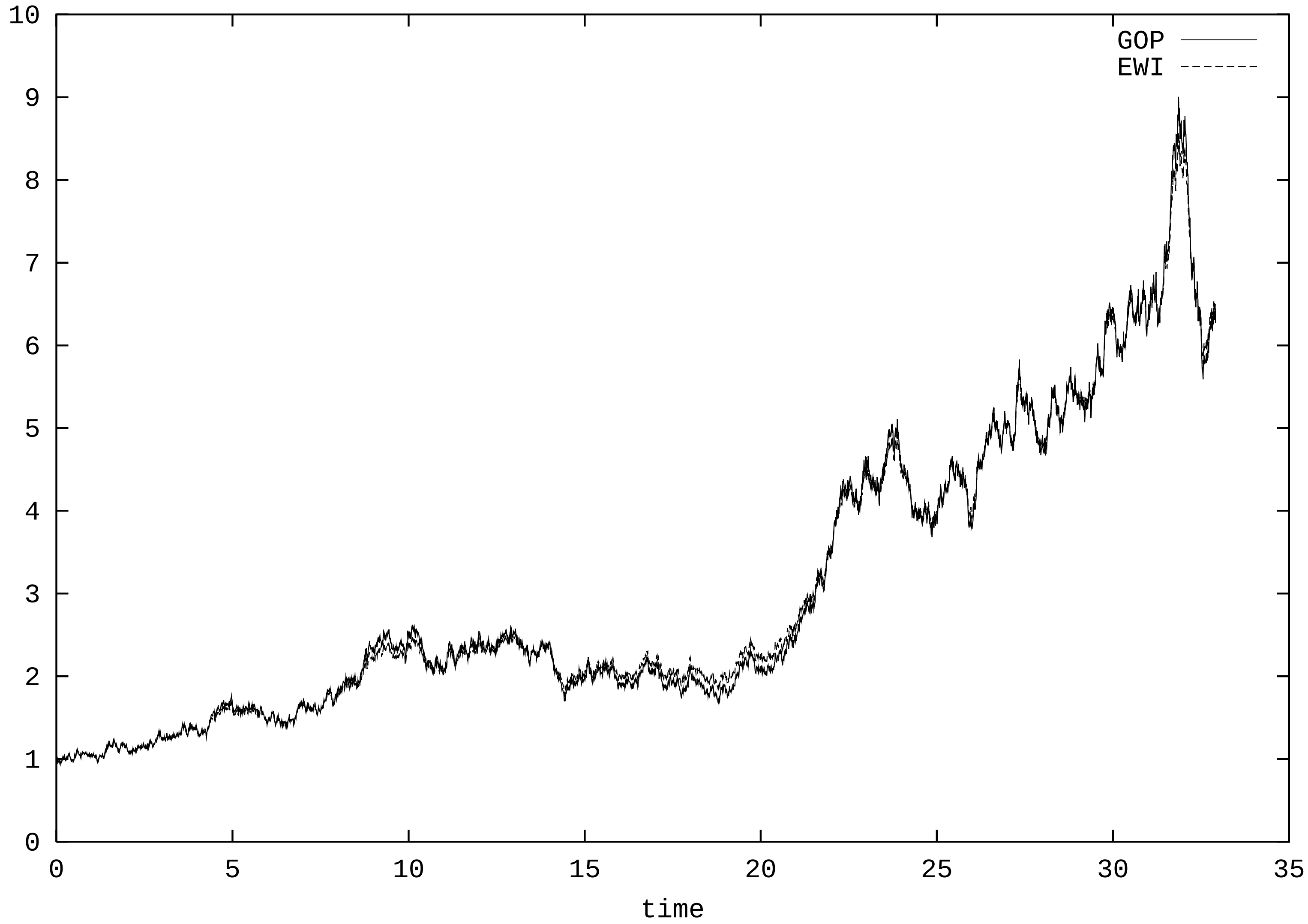
$S^{\delta_{\mathrm{EWI}d}}$ holds equal fractions

$$\pi^j_{\delta_{\mathrm{EWI}d},t} = \frac{1}{d+1}$$

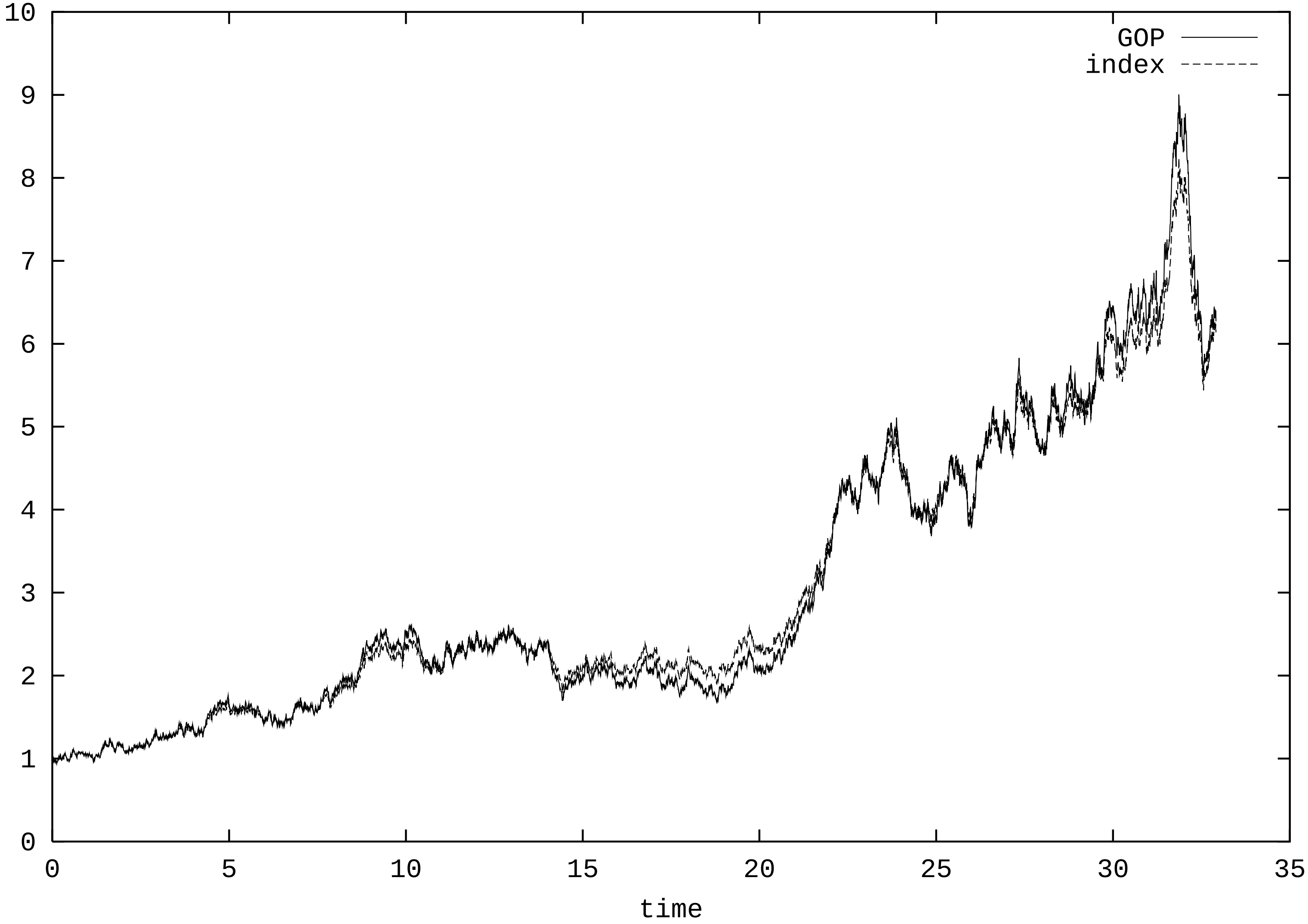
$j \in \{0, 1, \dots, d\}$

- tracking rate for BS example:

$$\begin{aligned} T^d_{\delta_{\mathrm{EWI}d}}(t) &= \sum_{k=1}^{d} \left(\frac{\sigma}{d+1} \left(\frac{1}{\sqrt{d}} - 1 \right) \right)^2 \\ &= d \left(\frac{\sigma}{d+1} \left(\frac{1}{\sqrt{d}} - 1 \right) \right)^2 \leq \frac{\sigma^2}{(d+1)} \to 0 \end{aligned}$$



Simulated NP and EWI.



Simulated diversified accumulation index and NP.

Diversification in an MMM Setting

Pl. (2001), minimal market model (MMM)

- **savings account**

$$S^0_{(d)}(t) = \exp\{r\,t\}$$

- **discounted GOP drift**

$$\alpha^{\delta_*}_t = \alpha_0 \, \exp\{\eta\,t\}$$

- j**th benchmarked primary security account**

$$\hat{S}^j_{(d)}(t) = \frac{1}{Y^j_t \, \alpha^{\delta_*}_t}$$

- **square root process**

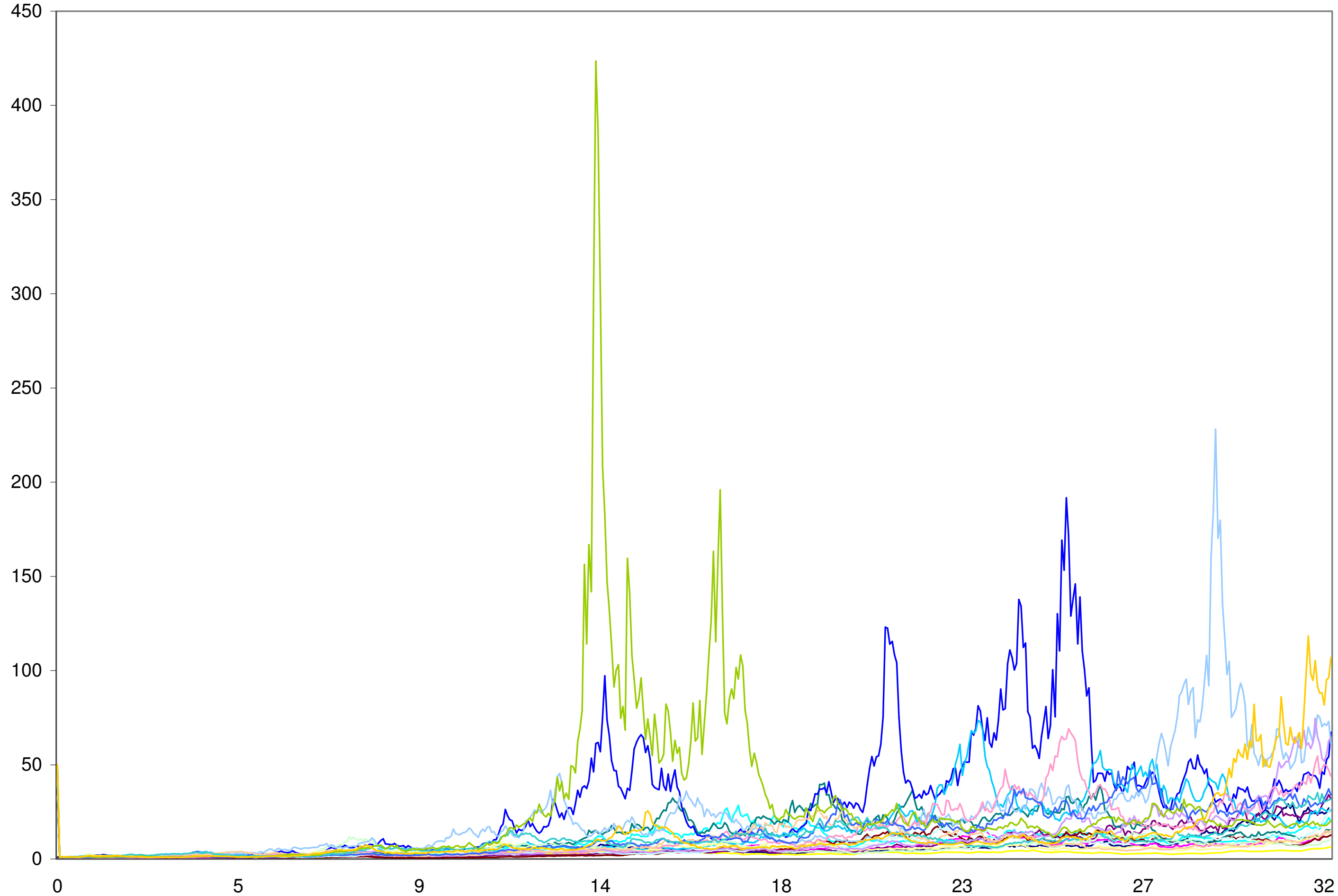
$$dY_t^j = \left(1 - \eta\, Y_t^j\right) dt + \sqrt{Y_t^j}\, dW_t^j$$

- **NP**

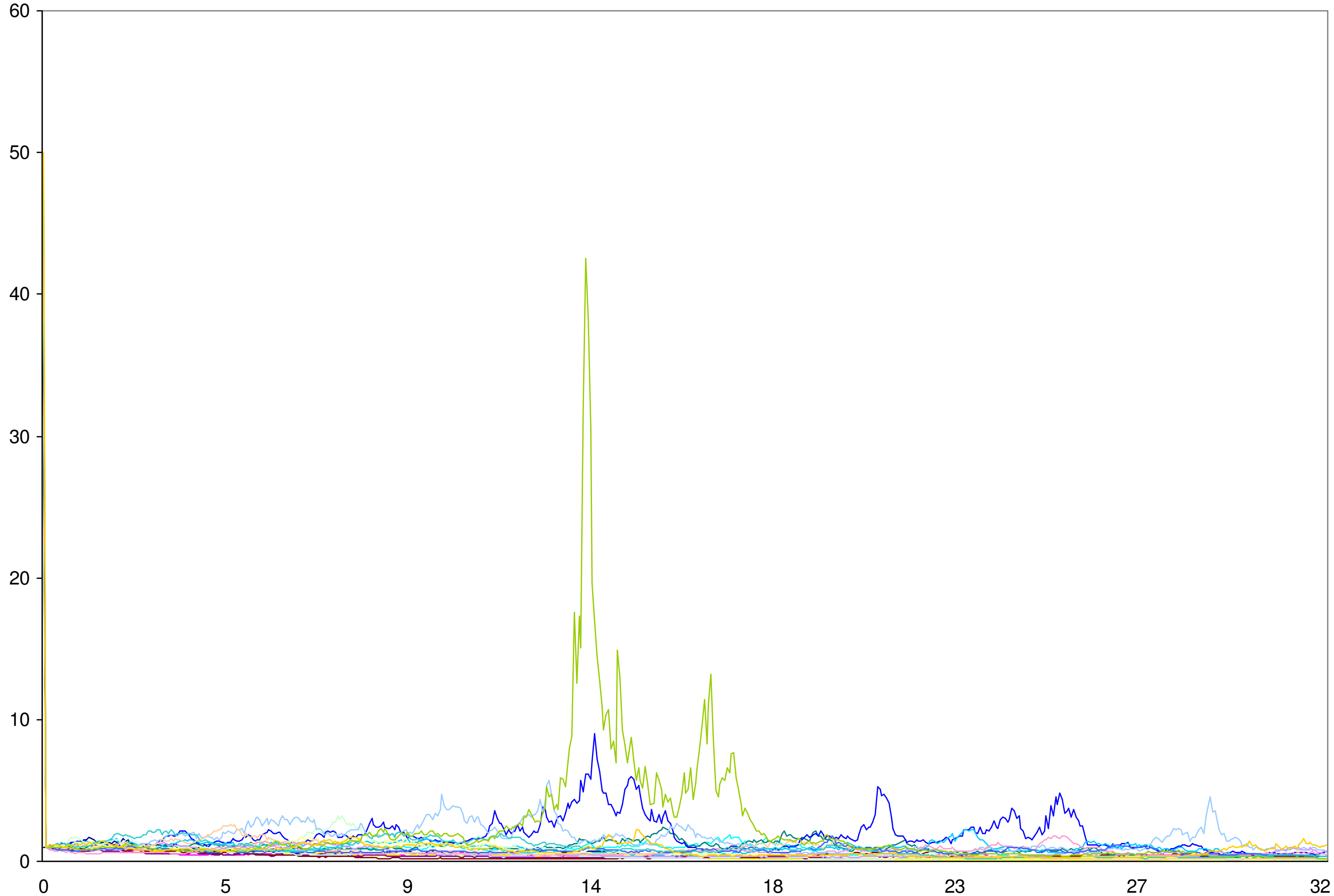
$$S_{(d)}^{\delta_*}(t) = \frac{S_{(d)}^0(t)}{\hat{S}_{(d)}^0(t)}$$

- j**th primary security account**

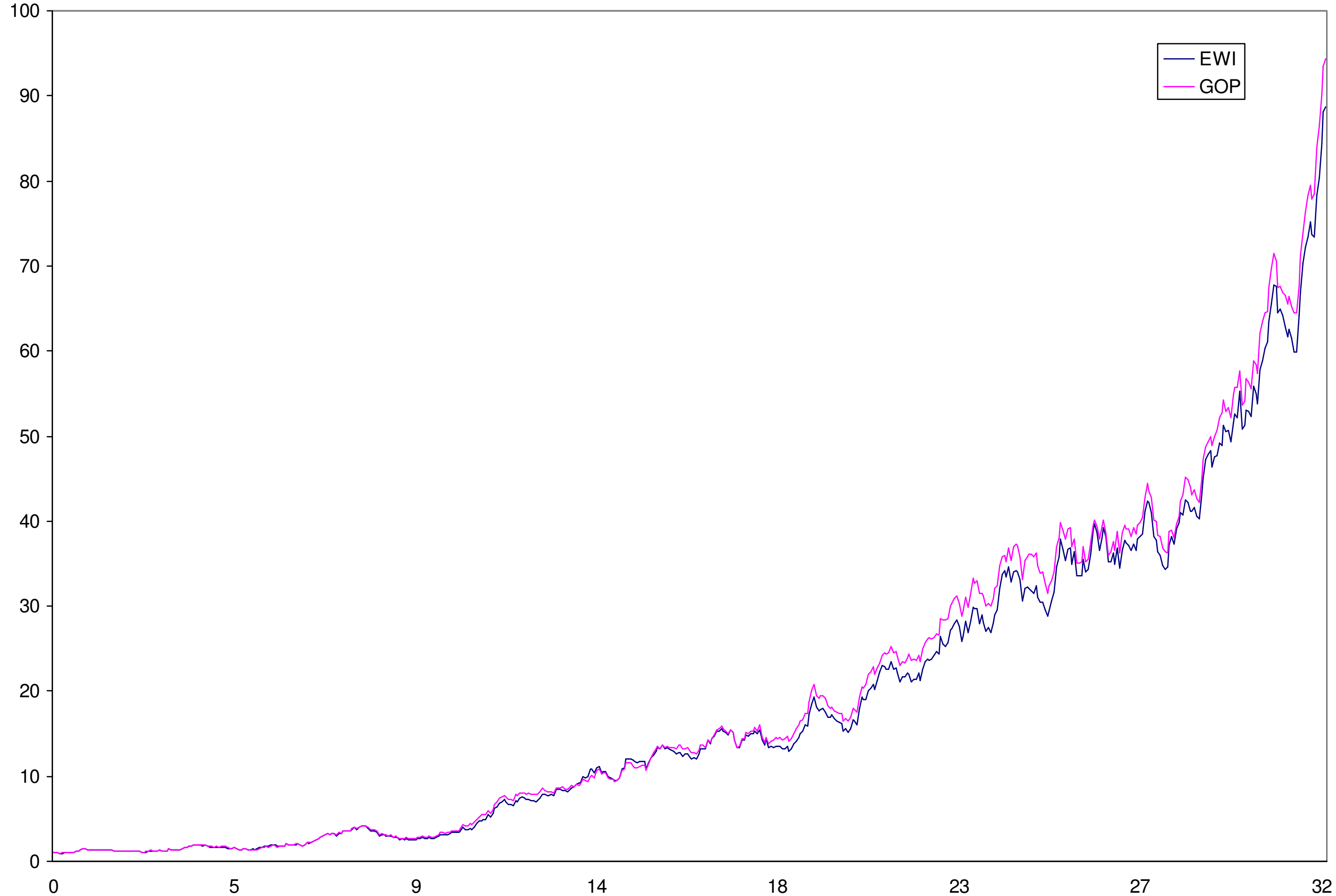
$$S_{(d)}^j(t) = \hat{S}_{(d)}^j(t)\, S_{(d)}^{\delta_*}(t)$$



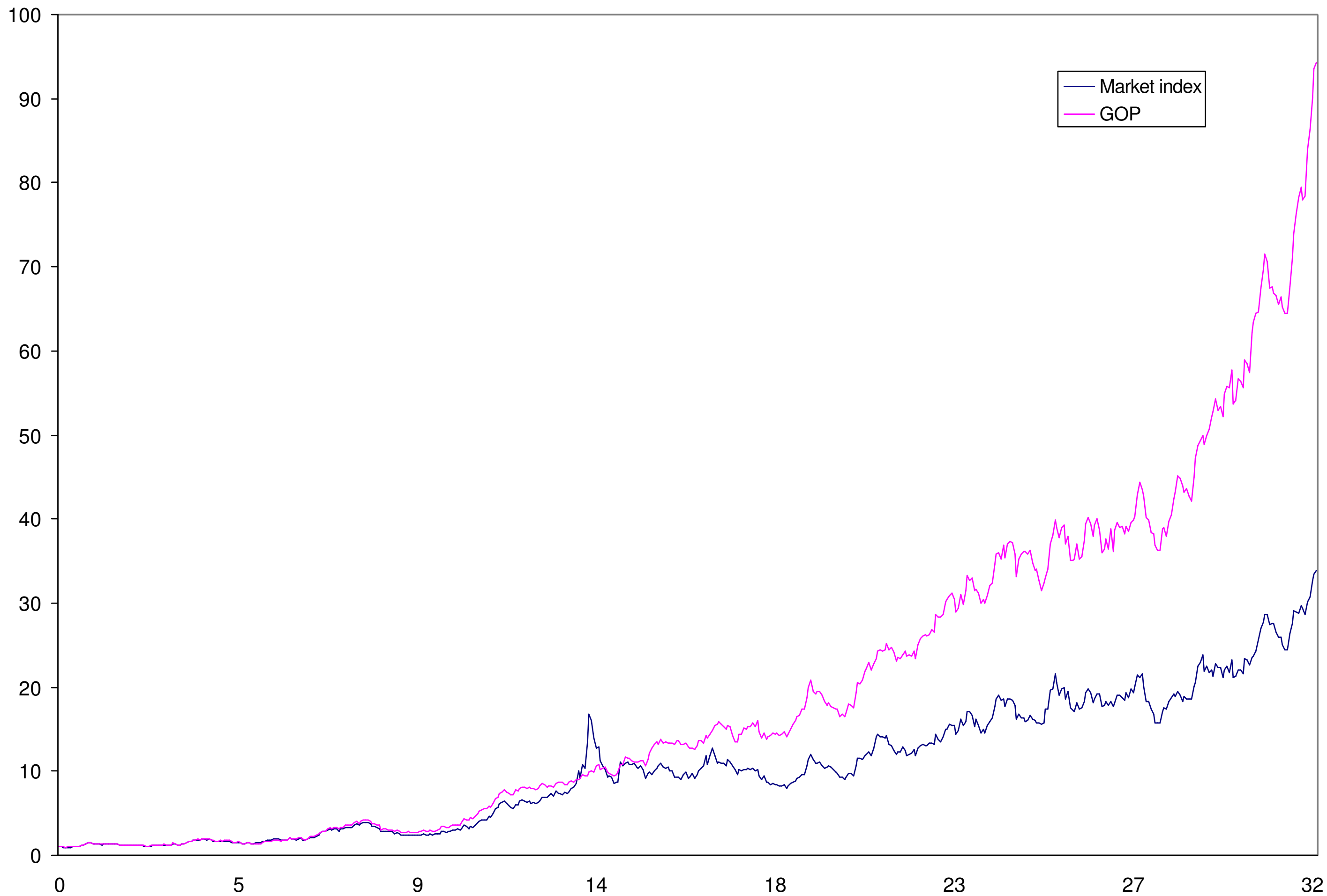
Primary security accounts under the MMM



Benchmarked primary security accounts



NP and EWI



NP and market index

Diversified Portfolios

Fundamental phenomenon of **diversification** leads naturally to **NP** also for more general diversified portfolios than the EWI:

Definition *A sequence of strictly positive portfolios* $(S^{\delta_d})_{d\in\{1,2,\dots\}}$ *is a sequence of* **diversified portfolios** *if*

$$\left|\pi^j_{\delta_d,t}\right| \le \begin{cases} \frac{K_2}{d^{\frac{1}{2}+K_1}} & \text{for } j \in \{0,1,\dots,d\} \\ 0 & \text{otherwise} \end{cases}$$

a.s. for $t \in [0,T]$.

- still small fractions but more general than EWIs

- jth benchmarked primary security account

$$\hat{S}_t^j = \frac{S_t^j}{S_t^{\delta_*}}$$

$$d\hat{S}_t^j = -\hat{S}_t^j \sum_k s_t^{j,k} \, dW_t^k$$

- $s_t^{j,k}$ $\quad (j,k)$th **specific volatility**
  measures **specific market risk** of S^j
  with respect to W^k diversifiable uncertainty

- volatility of NP measures **general market risk**
  $\triangleq$ nondiversifiable uncertainty

- ***k*th total specific volatility**

$$\hat{s}_t^k = \sum_j |s_t^{j,k}|$$

Definition *A market is called* **regular** *if*

$$E\left(\left(\hat{s}_t^k\right)^2\right) \leq K_5$$

for all $t \in [0, \infty)$, $k \in \{1, 2, \ldots\}$.

Tracking Rate

$$T_\delta(t) = \frac{d}{dt}\langle \hat{Q}^\delta \rangle_t = \sum_k \left(\sum_{j=0}^{d} \pi^j_{\delta,t}\, s_t^{j,k} \right)^2$$

$T_\delta(t)$ - quantifies "distance" between S^δ and S^{δ_*}

$$T_{\delta_*}(t) = 0$$

Approximate NP

Definition *A strictly positive portfolio* S^{δ_d} *is an* **approximate NP** *if for all* $t \in [0, T]$ *and* $\varepsilon > 0$

$$\lim_{d \to \infty} P\left(T_{\delta_d}(t) > \varepsilon\right) = 0.$$

Diversification Theorem (Pl. 2005)

In a regular market

any diversified portfolio is an approximate NP.

- **model independent**

Proof:

$$E\left(T_{\delta_d}(t)\right) \leq \sum_{k=1}^{d} E\left(\left(\sum_{j=0}^{d} |\pi_{\delta,t}^{j}|\,|s_t^{j,k}|\right)^2\right)$$

$$\leq \frac{(K_2)^2}{d^{(1+2K_1)}} \sum_{k=1}^{d} E\left(\left(\hat{s}_t^k\right)^2\right) \leq \frac{(K_2)^2 K_5}{d^{(1+2K_1)}}\, d \to 0$$

Markov inequality $\Longrightarrow$

$$\lim_{d\to\infty} P\left(T_{\delta_d}(t) > \varepsilon\right) \leq \lim_{d\to\infty} \frac{1}{\varepsilon} E\left(T_{\delta_d}(t)\right) = 0. \qquad \square$$

Summary on Diversification

- extremely robust approach to asset management
- does not rely on estimating some expected returns
- diversification can be refined
- diversified market indices are in reality similar in their dynamics
  $\Rightarrow$ proxies for the NP
  e.g. MSCI, S&P 500, EWI114, EWI142 ...

Continuous Financial Market

- forces us to be more specific

$(\Omega, \mathcal{A}, \underline{\mathcal{A}}, P)$ - filtered probability space
$\underline{\mathcal{A}} = (\mathcal{A}_t)_{t \geq 0}$ - filtration
$\mathcal{A}_t$ - information at time t
$E_t(\cdot) = E(\cdot | \mathcal{A}_t)$ - conditional expectation

- d sources of continuous **traded uncertainty**

$$W^1, W^2, \ldots, W^d, \quad d \in \{1, 2, \ldots\}$$

independent standard Brownian motions

Primary Security Accounts

S_t^j - jth primary security account at time t,

$j \in \{0, 1, \ldots, d\}$

cum-dividend share value or savings account value,

dividends or interest are reinvested

- vector process of primary security accounts

$$S = \{S_t = (S_t^0, \ldots, S_t^d)^\top, \, t \in [0, T]\}$$

- assume **unique strong solution** of SDE

$$dS_t^j = S_t^j \left(a_t^j \, dt + \sum_{k=1}^{d} b_t^{j,k} \, dW_t^k \right)$$

$t \in [0, \infty), \; S_0^j > 0, \; j \in \{1, 2, \ldots, d\}$

- predictable, integrable appreciation rate processes a^j

- predictable, square integrable volatility processes $b^{j,k}$

- **savings account**

$$S_t^0 = \exp\left\{\int_0^t r_s\, ds\right\}$$

predictable, integrable short rate process

$$r = \{r_t,\, t \in [0, \infty)\}$$

- **Market price of risk**

$$\boldsymbol{\theta}_t = (\theta_t^1, \theta_t^2, \ldots, \theta_t^d)^\top$$

unique invariant of the market

solution of relation

$$\boldsymbol{b}_t\,\boldsymbol{\theta}_t = \boldsymbol{a}_t - r_t\,\mathbf{1}$$

where

$$\boldsymbol{a}_t = (a_t^1, a_t^2, \ldots, a_t^d)^\top$$

$$\mathbf{1} = (1, 1, \ldots, 1)^\top$$

Assumption:

Volatility matrix $b_t = [b_t^{j,k}]_{j,k=1}^d$ is **invertible**

$\Longrightarrow$

- **market price of risk**

$$\theta_t = b_t^{-1}\,(a_t - r_t\,\mathbf{1})$$

- can rewrite jth primary security account SDE

$$dS_t^j = S_t^j \left\{ r_t \, dt + \sum_{k=1}^{d} b_t^{j,k} \left(\theta_t^k \, dt + dW_t^k \right) \right\}$$

$t \in [0, \infty), \; j \in \{1, 2, \ldots, d\}$

- **strategy**

$$\delta = \{\delta_t = (\delta_t^0, \ldots, \delta_t^d)^\top,\ t \in [0, T]\}$$

predictable and S-integrable,

δ_t^j number of units of jth primary security account

- **portfolio**

$$S_t^\delta = \sum_{j=0}^{d} \delta_t^j \, S_t^j$$

- S^{δ} **self-financing** $\Longleftrightarrow$

$$dS_t^{\delta} = \sum_{j=0}^{d} \delta_t^j \, dS_t^j$$

- changes in the portfolio value only due to gains from trading,

- self-financing in each denomination

- jth fraction

$$\pi^j_{\delta,t} = \delta^j_t \, \frac{S^j_t}{S^\delta_t}$$

$j \in \{0, 1, \ldots, d\}$

$$\sum_{j=0}^{d} \pi^j_{\delta,t} = 1$$

- **portfolio SDE**

$$dS_t^\delta = S_t^\delta \left(r_t \, dt + \sum_{k=1}^{d} b_{\delta,t}^k \, \left(\theta_t^k \, dt + dW_t^k \right) \right)$$

with kth portfolio volatility

$$b_{\delta,t}^k = \sum_{j=0}^{d} \pi_{\delta,t}^j \, b_t^{j,k}$$

- **logarithm** of strictly positive portfolio

$$d\ln(S_t^\delta) = g_t^\delta\, dt + \sum_{k=1}^{d} b_{\delta,t}^k\, dW_t^k$$

with **growth rate**

$$g_t^\delta = r_t + \sum_{k=1}^{d} b_{\delta,t}^k \left(\theta_t^k - \frac{1}{2}\, b_{\delta,t}^k\right)$$

$\Longrightarrow$ **maximize the growth rate**

$$\mathbf{0} = \frac{\partial g_t^\delta}{\partial b_{\delta,t}^k} = \theta_t^k - b_{\delta_*,t}^k$$

$\Longrightarrow$ for each $k \in \{1, 2, \ldots, d\}$

$$\theta_t^k = b_{\delta_*,t}^k = \sum_{\ell=1}^{d} \pi_{\delta_*,t}^\ell \, b_t^{\ell,k}$$

$\Longrightarrow$

$$\theta_t^\top = \pi_{\delta_*,t}^\top \, b_t$$

Since $\boldsymbol{b_t}$ invertible $\Longrightarrow$

$$\begin{aligned}\boldsymbol{\pi_{\delta_*,t}} &= (\boldsymbol{\pi^1_{\delta_*,t}}, \ldots, \boldsymbol{\pi^d_{\delta_*,t}})^\top \\ &= (\boldsymbol{b_t^{-1}})^\top \boldsymbol{\theta_t}\end{aligned}$$

$\Longrightarrow$ **Kelly portfolio, growth optimal portfolio, GOP, log-optimal portfolio, NP**

$$dS_t^{\delta_*} = S_t^{\delta_*} \left(\left[r_t + \sum_{k=1}^{d} (\theta_t^k)^2 \right] dt + \sum_{k=1}^{d} \theta_t^k \, dW_t^k \right)$$

finite

$\Longrightarrow$

- GOP

$$dS_t^{\delta_*} = S_t^{\delta_*} \left(r_t \, dt + \sum_{k=1}^{d} \theta_t^k \left(\theta_t^k \, dt + dW_t^k \right) \right)$$

- general portfolio

$$dS_t^{\delta} = S_t^{\delta} \left(r_t \, dt + \sum_{k=1}^{d} \sum_{j=0}^{d} \pi_{\delta,t}^j \, b_t^{j,k} \left(\theta_t^k \, dt + dW_t^k \right) \right)$$

Benchmarked Portfolio

$$\hat{S}_t^{\delta} = \frac{S_t^{\delta}}{S_t^{\delta_*}}$$

Itô formula $\Longrightarrow$

$$d\hat{S}_t^{\delta} = -\hat{S}_t^{\delta} \sum_{k=1}^{d} \sum_{j=0}^{d} \pi_{\delta,t}^{j} \left(\theta_t^k - b_t^{j,k} \right) dW_t^k$$

driftless $\Longrightarrow$ local martingale,

"zero expected return locally"

Local Efficient Market Hypothesis

similar to Fama (1965)

- nonnegative local martingale is supermartingale

$\Longrightarrow$ **Kelly portfolio (GOP) is the NP**

- NP **benchmark for investing**
- **numéraire for pricing**
- captures general market risk in **risk management**
- central in **benchmark approach**

Benchmark Approach

- provides much wider framework for modeling than classical APT
- can capture more phenomena relevant in real world
- contains as special cases classical approaches
  e.g. APT, actuarial approach, CAPM, ...

Capital Asset Pricing Model

- **risk premium** of security $V(t)$

$$p_V(t) = \lim_{h \downarrow 0} \frac{1}{h} E_t \left(\frac{V(t+h) - V(t)}{V(t)} \right) - r_t$$

excess expected return

- **NP**

$$dS_t^{\delta_*} = S_t^{\delta_*} \left(\left(r_t + p_{S^{\delta_*}}(t) \right) dt + \sum_{k=1}^{d} \theta_t^k \, dW_t^k \right)$$

with volatility

$$|\theta_t| = \sqrt{\sum_{k=1}^{d} (\theta_t^k)^2} = \sqrt{p_{S^{\delta_*}}(t)}$$

- **underlying security**

$$dS_t = S_t \left(r_t \, dt + \sum_{k=1}^{d} \sigma_t^k \left(\theta_t^k \, dt + dW_t^k \right) \right)$$

risk premium

$$\begin{aligned} p_S(t) &= \sum_{k=1}^{d} \sigma_t^k \, \theta_t^k \\ &= \frac{d}{dt} \Big[\ln(S), \ln(S^{\delta_*}) \Big]_t \\ &= \lim_{h \downarrow 0} \frac{1}{h} \, E_t \left(\left(\frac{S_{t+h} - S_t}{S_t} \right) \left(\frac{S_{t+h}^{\delta_*} - S_t^{\delta_*}}{S_t^{\delta_*}} \right) \right) \end{aligned}$$

- **portfolio**

$$dS_t^\delta = S_t^\delta \left(r_t \, dt + \pi_{\delta,t}^1 \sum_{k=1}^{d} \sigma_t^k \left(\theta_t^k \, dt + dW_t^k \right) \right)$$

risk premium

$$\begin{aligned} p_{S^\delta}(t) &= \pi_{\delta,t}^1 \sum_{k=1}^{d} \sigma_t^k \, \theta_t^k \\ &= \lim_{h \downarrow 0} \frac{1}{h} E_t \left(\left(\frac{S_{t+h}^\delta - S_t^\delta}{S_t^\delta} \right) \left(\frac{S_{t+h}^{\delta_*} - S_t^{\delta_*}}{S_t^{\delta_*}} \right) \right) \\ &= \frac{d}{dt} \Big[\ln(S^\delta), \ln(S^{\delta_*}) \Big]_t \end{aligned}$$

Assumption: ***Market portfolio*** $\boldsymbol{S^{\delta_{MP}}}$ *is diversified and approximates NP*

$$\boldsymbol{S_t^{\delta_{MP}} \approx S_t^{\delta_*}} \quad .$$

- **systematic risk parameter beta**

$$\beta_{S^\delta}(t) = \frac{\frac{d}{dt}\left[\ln(S^\delta), \ln(S^{\delta_{\text{MP}}})\right]_t}{\frac{d}{dt}\left[\ln(S^{\delta_{\text{MP}}})\right]_t}$$

$\Longrightarrow$

$$\beta_{S^\delta}(t) \approx \frac{\frac{d}{dt}\left[\ln(S^\delta), \ln(S^{\delta_*})\right]_t}{\frac{d}{dt}\left[\ln(S^{\delta_*})\right]_t} = \frac{p_{S^\delta}(t)}{p_{S^{\delta_*}}(t)} \approx \frac{p_{S^\delta}(t)}{p_{S^{\delta_{\text{MP}}}}(t)}$$

$\Longrightarrow$

- **fundamental CAPM relation follows directly**

$$p_{S^{\delta}}(t) \approx \beta_{S^{\delta}}(t)\, p_{S^{\delta_{\text{MP}}}}(t)$$

- **CAPM relationship still holds** for $S^{\delta_{\rm MP}}$ with SDE

$$dS_t^{\delta_{\rm MP}} = S_t^{\delta_{\rm MP}} \left(r_t\, dt + \pi_t \sum_{k=1}^{d} \theta_t^k \left(\theta_t^k\, dt + dW_t^k \right) \right)$$

combination of savings account and NP, two fund separation

- $$\beta_{S^\delta}(t) = \frac{\frac{d}{dt}\left[\ln(S^\delta), \ln(S^{\delta_{\rm MP}})\right]_t}{\frac{d}{dt}\left[\ln(S^{\delta_{\rm MP}})\right]_t}$$

$$= \frac{\pi^1_{\delta,t} \sum_{k=1}^d \sigma_t^k\, \theta_t^k\, \pi_t}{(\pi_t)^2 \sum_{k=1}^d \left(\theta_t^k\right)^2}$$

$$= \frac{\pi^1_{\delta,t} \sum_{k=1}^d \sigma_t^k\, \theta_t^k}{\pi_t \sum_{k=1}^d \left(\theta_t^k\right)^2} = \frac{p_{S^\delta}(t)}{p_{S^{\delta_{\rm MP}}}(t)}$$

no utility function involved,
no principal agent,
no equilibrium assumption

follows simply by Itô calculus and BA

$\Longrightarrow$

- **capital asset pricing model (CAPM)**

  Sharpe (1964), Lintner (1965),

  Mossin (1966) and Merton (1973b)

Examples for Beta:

- savings account

$$\beta_{S^0}(t) = 0$$

- underlying security

$$\beta_S(t) = \frac{\sum_{k=1}^{d} \sigma_t^k \, \theta_t^k}{|\theta_t|^2}$$

- NP

$$\beta_{S^{\delta_*}}(t) = 1$$

- portfolio

$$\beta_{S^\delta}(t) = \frac{\pi^1_{\delta,t} \sum_{k=1}^{d} \sigma^k_t \, \theta^k_t}{|\theta_t|^2}$$

- If S^δ involves only diversifiable uncertainty, which is not in NP, then it has zero beta

$\Longrightarrow$

$$\beta_{S^\delta}(t) = 0$$

Portfolio Optimization

- **Kelly portfolio (GOP, NP) best long term investment**

  Kelly (1956), Latané (1959), Breiman (1961),
  Hakansson (1971b), Thorp (1972)

- **optimal portfolio separated into two funds**

  NP and savings account
  Tobin (1958), Sharpe (1964)

- **mean-variance efficient portfolio**

  Markowitz (1959)

- **intertemporal capital asset pricing model**

  Merton (1973a)

Discounted Portfolio

- **strictly positive portfolio** $S^{\delta} \in \mathcal{V}^{+}$

- **discounted value**

$$\bar{S}_t^{\delta} = \frac{S_t^{\delta}}{S_t^0}$$

$\Longrightarrow$

- **SDE**

$$d\bar{S}_t^{\delta} = \sum_{k=1}^{d} \psi_{\delta,t}^{k} \left(\theta_t^k \, dt + dW_t^k \right)$$

with kth diffusion coefficient

$$\psi_{\delta,t}^{k} = \sum_{j=1}^{d} \delta_t^j \, \bar{S}_t^j \, b_t^{j,k}$$

- **discounted drift**

$$\alpha_t^\delta = \sum_{k=1}^{d} \psi_{\delta,t}^k \, \theta_t^k$$

link to macro economy

- **fluctuations**

$$\bar{M}_t = \sum_{k=1}^{d} \int_0^t \psi_{\delta,s}^k \, dW_s^k$$

$\Longrightarrow$

$$\bar{S}_t^\delta = \bar{S}_0^\delta + \int_0^t \alpha_s^\delta \, ds + \bar{M}_t$$

- **aggregate diffusion coefficient** (deviation)

$$\gamma_t^\delta = \sqrt{\sum_{k=1}^{d} \left(\psi_{\delta,t}^{k} \right)^2}$$

$\Longrightarrow$ aggregate volatility

$$b_t^\delta = \frac{\gamma_t^\delta}{\bar{S}_t^\delta}$$

Locally Optimal Portfolios

Definition

$S^{\tilde{\delta}} \in \mathcal{V}^+$ **locally optimal**, *if for all* $t \in [0, \infty)$ *and* $S^\delta \in \mathcal{V}^+$ *with*

$$\gamma_t^\delta = \gamma_t^{\tilde{\delta}}$$

almost surely:

$$\alpha_t^\delta \leq \alpha_t^{\tilde{\delta}}.$$

generalization of mean-variance optimality

Markowitz (1952, 1959)

Sharpe Ratio

Sharpe (1964, 1966)

- **risk premium**

$$p_{S^\delta}(t) = \frac{\alpha_t^\delta}{\bar{S}_t^\delta}$$

- **aggregate volatility** b_t^δ

- **Sharpe ratio**

$$s_t^\delta = \frac{p_{S^\delta}(t)}{b_t^\delta} = \frac{\alpha_t^\delta}{\gamma_t^\delta} = \frac{\text{mean}}{\text{deviation}}$$

- **total market price of risk**

$$|\theta_t| = \sqrt{\sum_{k=1}^{d} \left(\theta_t^k\right)^2}$$

Assumption:

$$0 < |\theta_t| < \infty$$

and

$$\pi^0_{\delta_*,t} \neq 1$$

almost surely.

- **Portfolio Selection Theorem**

  For any $S^\delta \in \mathcal{V}^+$ it follows Sharpe ratio

$$s_t^\delta \leq |\theta_t|,$$

  where equality when S^δ **locally optimal**

$\Longrightarrow$

$$d\bar{S}_t^\delta = \bar{S}_t^\delta \frac{b_t^\delta}{|\theta_t|} \sum_{k=1}^{d} \theta_t^k \left(\theta_t^k \, dt + dW_t^k\right),$$

$\Longrightarrow$ fractions

$$\pi_{\delta,t}^j = \frac{b_t^\delta}{|\theta_t|} \, \pi_{\delta_*,t}^j$$

Markowitz (1959), Sharpe (1964), Merton (1973a)
Khanna & Kulldorff (1999), Platen (2002)
Sharpe ratio maximization

Proof of Portfolio Selection Theorem

Lagrange multiplier $\boldsymbol{\lambda}$

consider

$$\mathcal{L}(\psi_\delta^1, \ldots, \psi_\delta^d, \lambda) = \sum_{k=1}^{d} \psi_\delta^k \, \theta^k + \lambda \left(\left(\gamma^{\tilde{\delta}} \right)^2 - \sum_{k=1}^{d} \left(\psi_\delta^k \right)^2 \right)$$

suppressing time dependence

first-order conditions

$$\frac{\partial \mathcal{L}(\psi_\delta^1, \ldots, \psi_\delta^d, \lambda)}{\partial \psi_\delta^k} = \theta^k - 2\,\lambda\,\psi_\delta^k = 0$$

for all $k \in \{1, 2, \ldots, d\}$ as well as

$$\frac{\partial \mathcal{L}(\psi_{\delta}^{1}, \ldots, \psi_{\delta}^{d}, \lambda)}{\partial \lambda} = \left(\gamma^{\tilde{\delta}}\right)^2 - \sum_{k=1}^{d} \left(\psi_{\delta}^{k}\right)^2 = 0$$

$\Longrightarrow$ locally optimal portfolio $S^{(\tilde{\delta})}$ must satisfy

$$\psi_{\tilde{\delta}}^{k} = \frac{\theta^k}{2\,\lambda}$$

for all $k \in \{1, 2, \ldots, d\}$.

Then

$$\sum_{k=1}^{d} \left(\psi_{\tilde{\delta}}^{k}\right)^2 = \left(\gamma^{\tilde{\delta}}\right)^2$$

$\Longrightarrow$

$$\left(\gamma^{\tilde{\delta}}\right)^2 = \sum_{k=1}^{d} \left(\psi_{\tilde{\delta}}^{k}\right)^2 = \frac{\sum_{k=1}^{d} (\theta^k)^2}{4\,\lambda^2}$$

we have $|\theta| = \sqrt{\sum_{k=1}^{d} (\theta^k)^2} > 0$,

then

$$\psi_{\tilde{\delta}}^{k} = \frac{\gamma^{\tilde{\delta}}}{|\theta|}\,\theta^k$$

for all $k \in \{1, 2, \ldots, d\}$

$\Longrightarrow$

$$\alpha_t^{\tilde{\delta}} = \gamma_t^{\tilde{\delta}} \, \frac{|\theta_t|^2}{|\theta_t|} = \gamma_t^{\tilde{\delta}} \, |\theta_t|$$

$$d\bar{S}_t^{\tilde{\delta}} = \gamma_t^{\tilde{\delta}} \, \sum_{k=1}^{d} \frac{\theta_t^k}{|\theta_t|} \, (\theta_t^k \, dt + dW_t^k)$$

$$\psi_{\tilde{\delta},t}^k = \sum_{j=1}^{d} \tilde{\delta}_t^j \, \bar{S}_t^j \, b_t^{j,k} = \bar{S}_t^{(\tilde{\delta})} \sum_{j=1}^{d} \pi_{\tilde{\delta},t}^j \, b_t^{j,k} = \frac{\gamma_t^{\tilde{\delta}}}{|\theta_t|} \, \theta_t^k = \bar{S}_t^{(\tilde{\delta})} \, b_t^{\tilde{\delta}} \, \frac{\theta_t^k}{|\theta_t|}$$

invertibility of volatility matrix

$\Longrightarrow$

$$\pi_{\tilde{\delta},t}^j = \frac{b_t^{\tilde{\delta}}}{|\theta_t|} \sum_{k=1}^{d} \theta_t^k \, b_t^{-1\,j,k} \qquad \square$$

Two Fund Separation

- **locally optimal portfolio**

fraction of wealth in the NP

$$\frac{b_t^\delta}{|\theta_t|} = \frac{1 - \pi_{\delta,t}^0}{1 - \pi_{\delta_*,t}^0}$$

remainder in savings account

$$\pi_{\delta,t}^0 = 1 - \frac{b_t^\delta}{|\theta_t|}\left(1 - \pi_{\delta_*,t}^0\right)$$

also known as **fractional Kelly strategy:**

Kelly (1956), Latané (1959), Thorp (1972), Hakansson & Ziemba (1995)

Risk Aversion Coefficient

$$J_t^\delta = \frac{1 - \pi^0_{\delta_*,t}}{1 - \pi^0_{\delta,t}} = \frac{|\theta_t|}{b_t^\delta}$$

Pratt (1964), Arrow (1965)

- **discounted locally optimal portfolio**

$$d\bar{S}_t^\delta = \bar{S}_t^\delta \, \frac{1}{J_t^\delta} \, |\theta_t| \, (|\theta_t| \, dt + dW_t),$$

where

$$dW_t = \sum_{k=1}^{d} \frac{\theta_t^k}{|\theta_t|} \, dW_t^k$$

$\frac{1}{J_t^\delta}$ fraction in the NP

Capital Market Line

- **expected rate of return**

$$a_t^\delta = r_t + p_{S^\delta}(t)$$

- **for a locally optimal portfolio**

$$a_t^\delta = r_t + |\theta_t|\, b_t^\delta$$

- **portfolio process at the capital market line**

  has fraction $\frac{1}{J_t^\delta} = \frac{b_t^\delta}{|\theta_t|}$ in NP

  fractional Kelly strategy

Markowitz Efficient Frontier

- **aggregate volatility**

$$\boldsymbol{b_t^{\delta} = \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \, |\theta_t|}$$

Definition

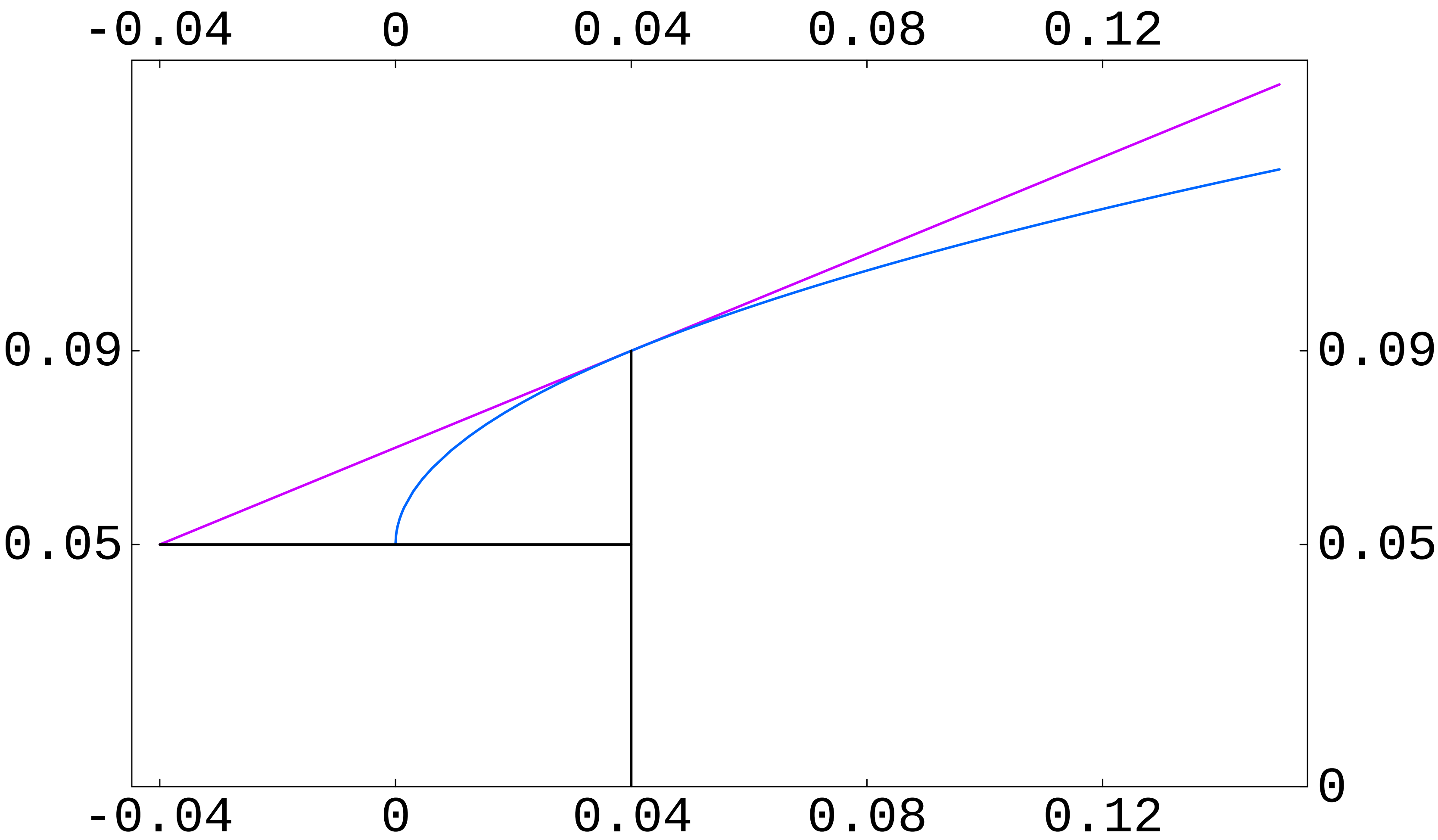
Markowitz efficient portfolio *has expected rate of return on* ***efficient frontier*** *if*

$$\boldsymbol{a_t^{\delta} = r_t + \sqrt{(b_t^{\delta})^2} \, |\theta_t|.}$$

Theorem *Any locally optimal portfolio* S^δ *is Markowitz efficient portfolio.*

$$p_{S^\delta}(t) = \frac{1 - \pi^0_{\delta,t}}{1 - \pi^0_{\delta_*,t}} \, |\theta_t|^2 = b^\delta_t \, |\theta_t|$$

Sharpe ratio maximization
risk attitude expressed via $J^\delta_t = \frac{|\theta_t|}{b^\delta_t}$
short term view



Efficient frontier

Efficient Growth Rates

- **efficient growth rate frontier**

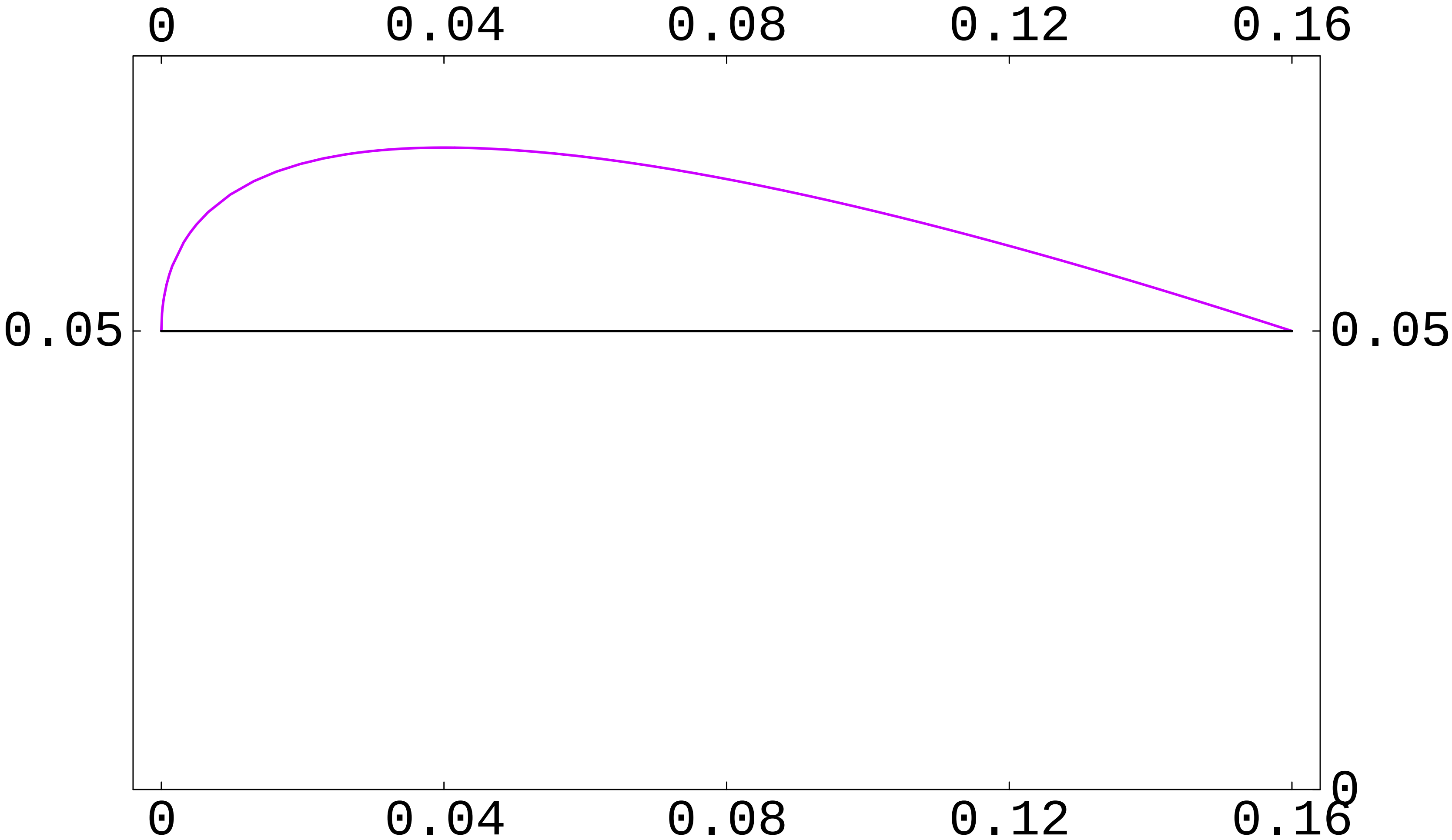
  of locally optimal portfolio

$$g_t^\delta = r_t + \sqrt{|b_t^\delta|^2}\,|\theta_t| - \frac{1}{2}\,|b_t^\delta|^2 = r_t + \frac{|\theta_t|^2}{J_t^\delta}\left(1 - \frac{1}{2\,J_t^\delta}\right)$$

- maximum

$$g_t^{\delta_*} = r_t + \frac{1}{2}\,|\theta_t|^2$$

- long term view



Efficient growth rates

Market Portfolio

$$S_t^{\delta_{\text{MP}}} = \sum_{\ell=1}^{n} S_t^{\delta_\ell}$$

portfolio of tradable wealth of ℓth investor S^{δ_ℓ}

Remark:

If each investor forms a nonnegative, locally optimal portfolio with her or his total tradable wealth, then **MP is locally optimal** .

$$\begin{aligned}
d\bar{S}_t^{\delta_{\text{MP}}} &= \sum_{\ell=1}^{n} d\bar{S}_t^{\delta_\ell} \\
&= \sum_{\ell=1}^{n} \frac{\left(\bar{S}_t^{\delta_\ell} - \delta_\ell^0\right)}{\left(1 - \pi_{\delta_*,t}^0\right)} \sum_{k=1}^{d} \theta_t^k \left(\theta_t^k \, dt + dW_t^k\right) \\
&= \bar{S}_t^{\delta_{\text{MP}}} \frac{\left(1 - \pi_{\delta_{\text{MP}},t}^0\right)}{\left(1 - \pi_{\delta_*,t}^0\right)} \sum_{k=1}^{d} \theta_t^k \left(\theta_t^k \, dt + dW_t^k\right)
\end{aligned}$$

- CAPM relationship still holds when MP locally optimal

Practical Feasibility of
Sample Based Markowitz Mean-Variance Approach:

- NP theoretically known for given model
- Markowitz (1952)
- Best & Grauer (1991)
- DeMiguel, Garlappi & Uppal (2009)

practical application
requires for 50 assets observation window of about 500 years

- not realistic to be applied
- theoretically can be reconciled under strong assumptions
  (MP locally optimal)

Maximum Drawdown Constrained Portfolios

$\mathcal{X}$ - set of nonnegative continuous discounted portfolios

- **running maximum**
  for $X = \{X_t, t \geq 0\} \in \mathcal{X}$

$$X_t^* = \sup_{u \in [0,t]} X_u$$

- **drawdown**

$$X_t^* - X_t$$

- **relative drawdown**

$$\frac{X_t}{X_t^*}$$

- **maximum relative drawdown**
- **express attitude towards risk by restricting to** $X \in {}^{\alpha}\mathcal{X}$, where

$$\frac{X_t}{X_t^*} \geq \alpha \ \ , \alpha \in [0,1)$$

pathwise criterion

- **drawdown**
  for $X \in {}^{\alpha}\mathcal{X}$
  $X_t \geq \alpha X_t^*$
  $\Longrightarrow$
$$X_t^* - X_t \leq (1-\alpha) X_t^*$$

- **maximum drawdown constrained portfolio**

  $\alpha \in [0,1)$, $X \in \mathcal{X}$

$$
\begin{aligned}
{}^{\alpha}X_t &= \alpha (X_t^*)^{1-\alpha} + (1-\alpha) X_t (X_t^*)^{-\alpha} \\
&= 1 + \int_0^t (1-\alpha)(X_s^*)^{-\alpha} dX_s \in [\alpha X_t^*, (X_t^*)^{1-\alpha}]
\end{aligned}
$$

  Grossman & Zhou (1993), Cvitanic & Karatzas (1994),
  Kardaras, Obloj & Pl. (2012)

$\Longrightarrow$ SDE

$$\frac{d^{\alpha}X_t}{{}^{\alpha}X_t} = {}^{\alpha}\pi_t \frac{dX_t}{X_t}$$

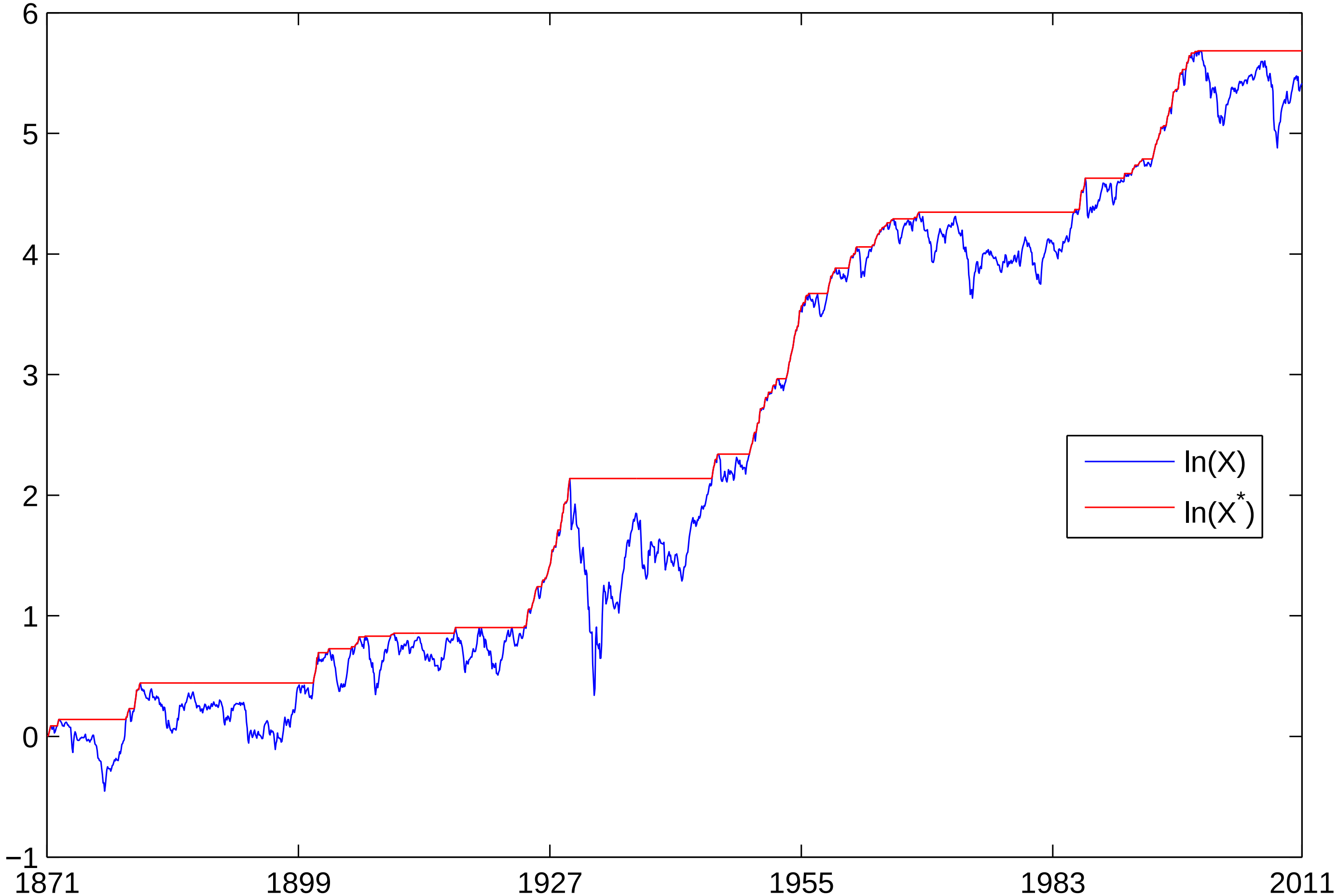
fraction

$${}^{\alpha}\pi_t = 1 - \frac{(1-\alpha)\frac{X_t}{X_t^*}}{\alpha + (1-\alpha)\frac{X_t}{X_t^*}}$$

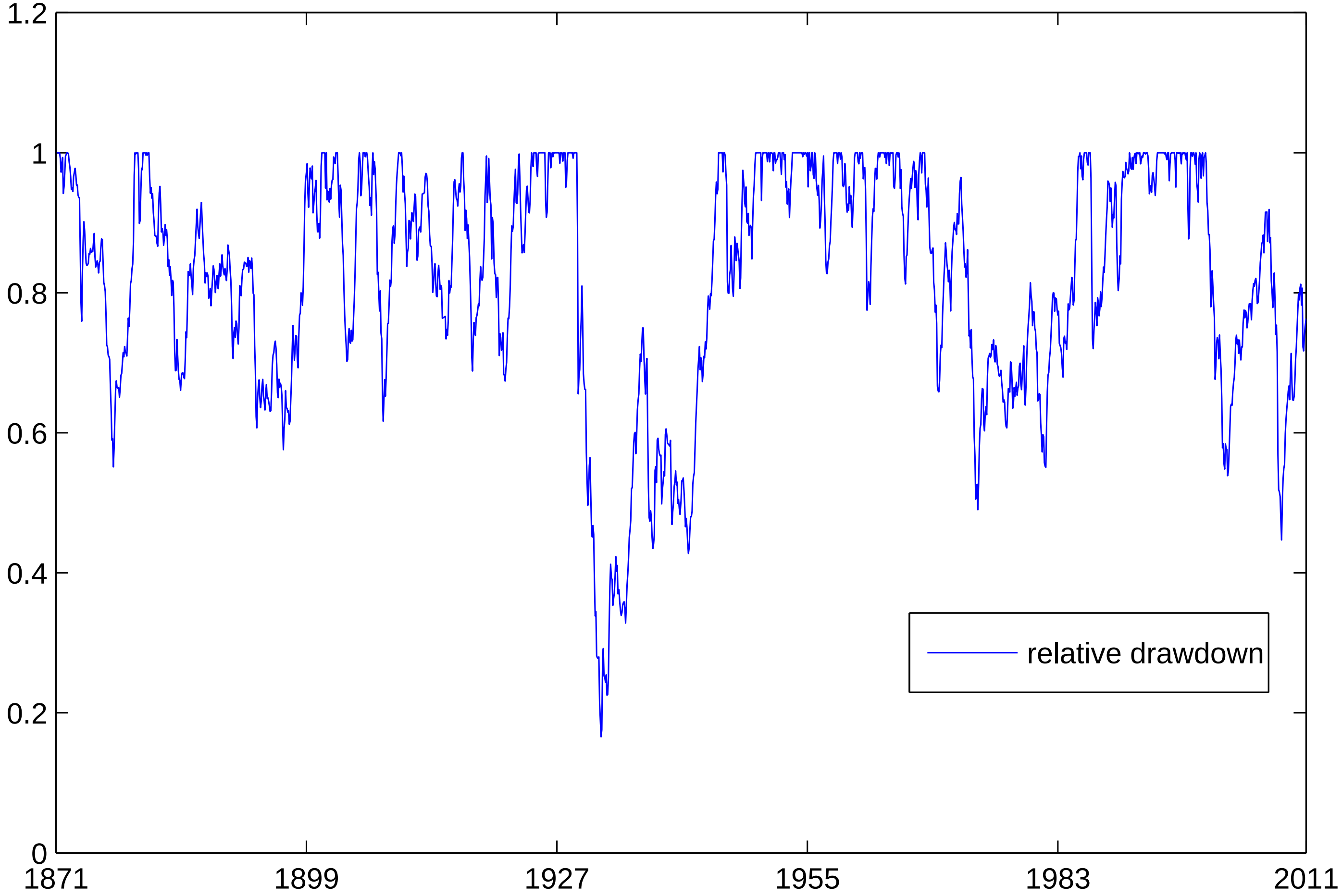
model independent

depends only on $\frac{X_t}{X_t^*}$ and α

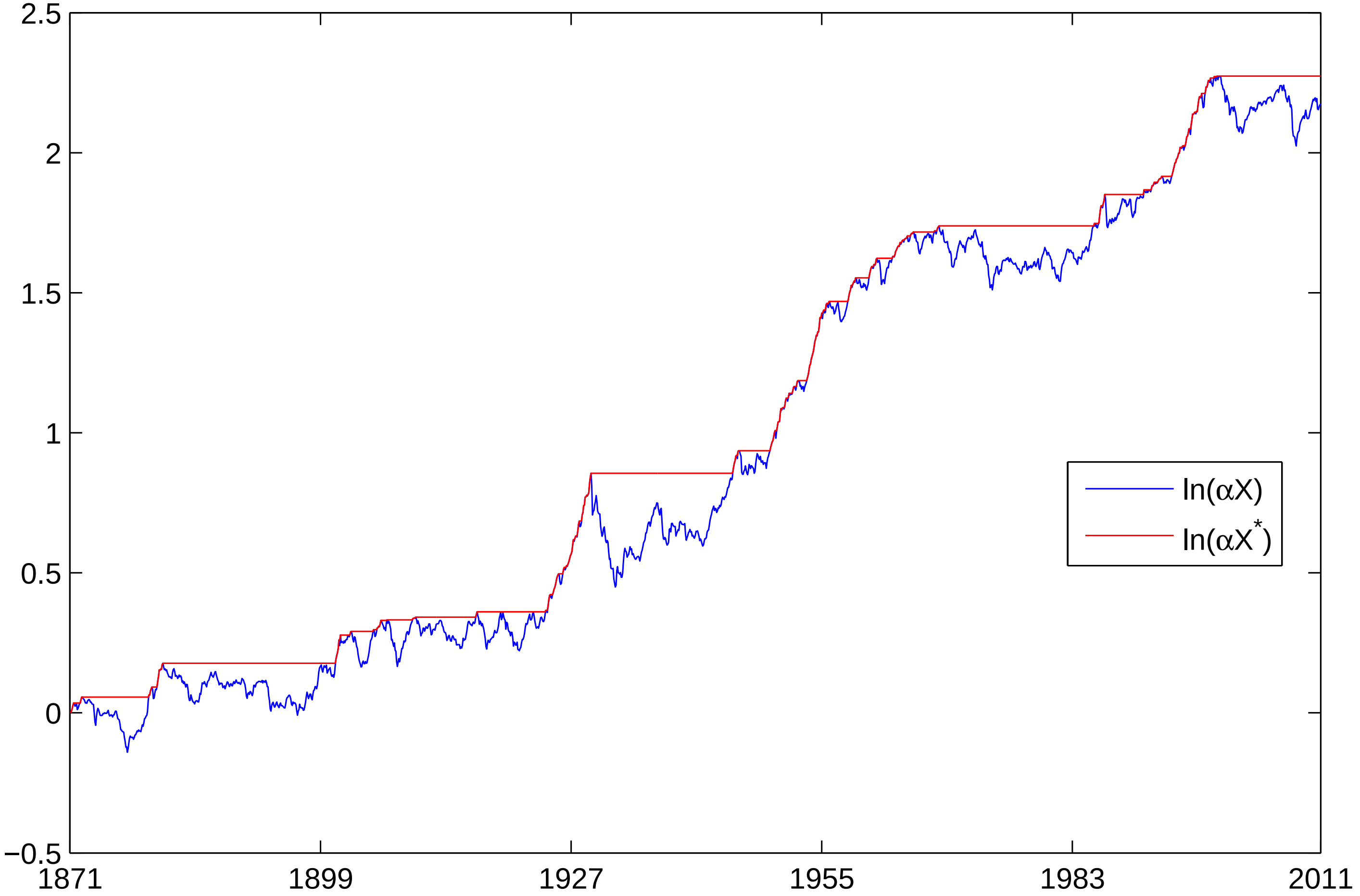
${}^{\alpha}\bar{S}_t^{\delta *}$ - locally optimal portfolio



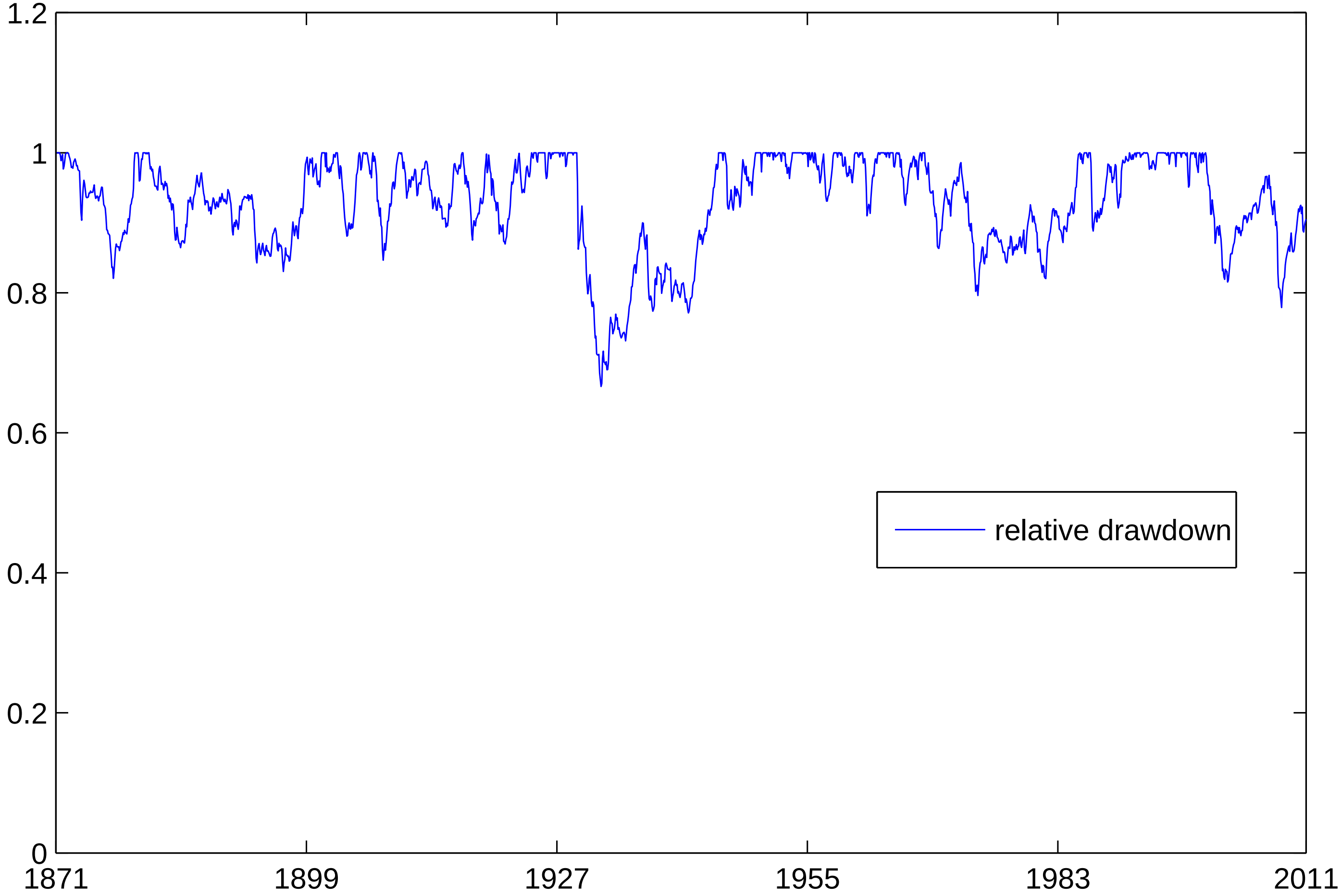
logarithm of discounted S&P500 and its running maximum



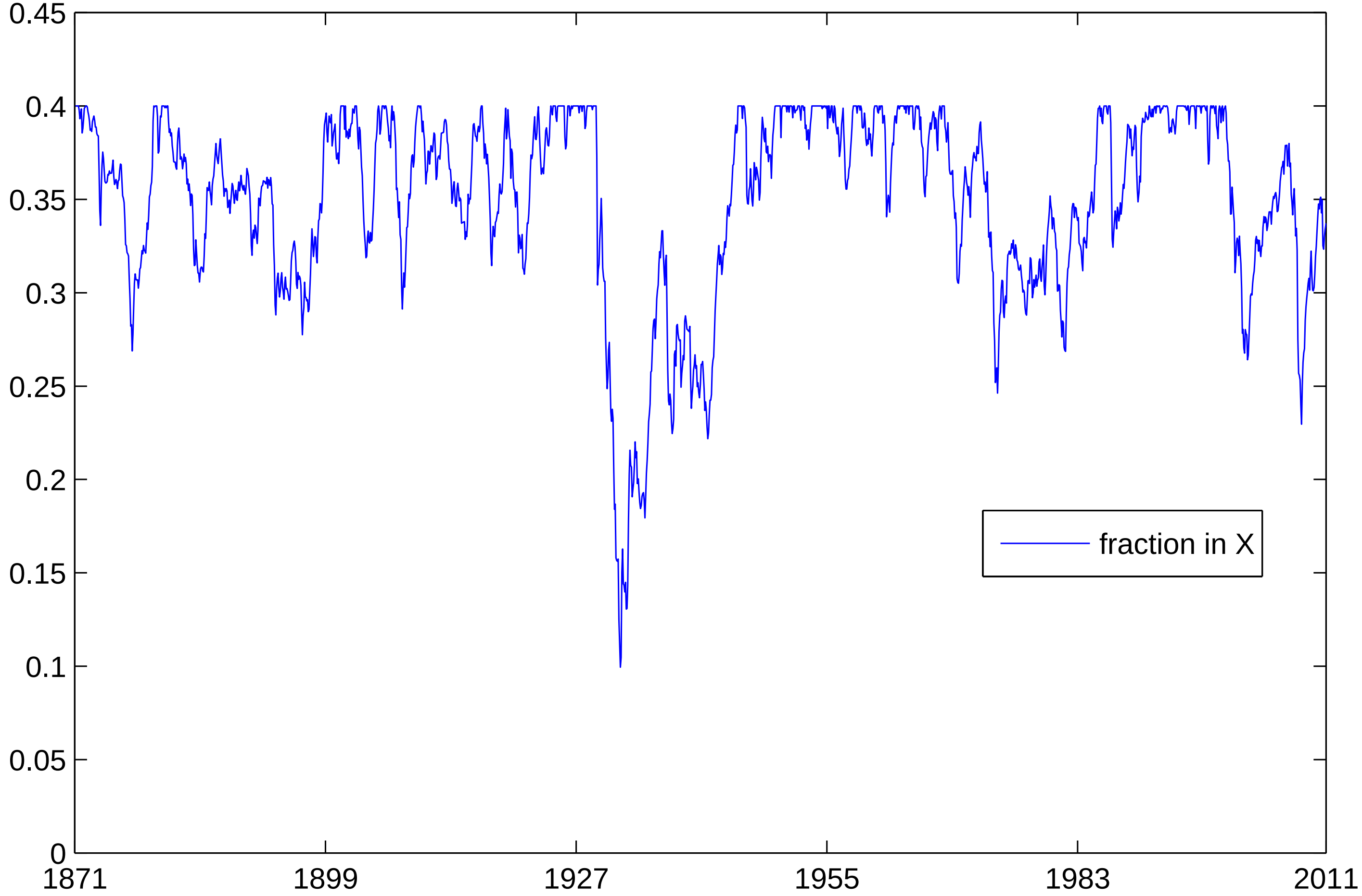
Relative drawdown of discounted S&P500



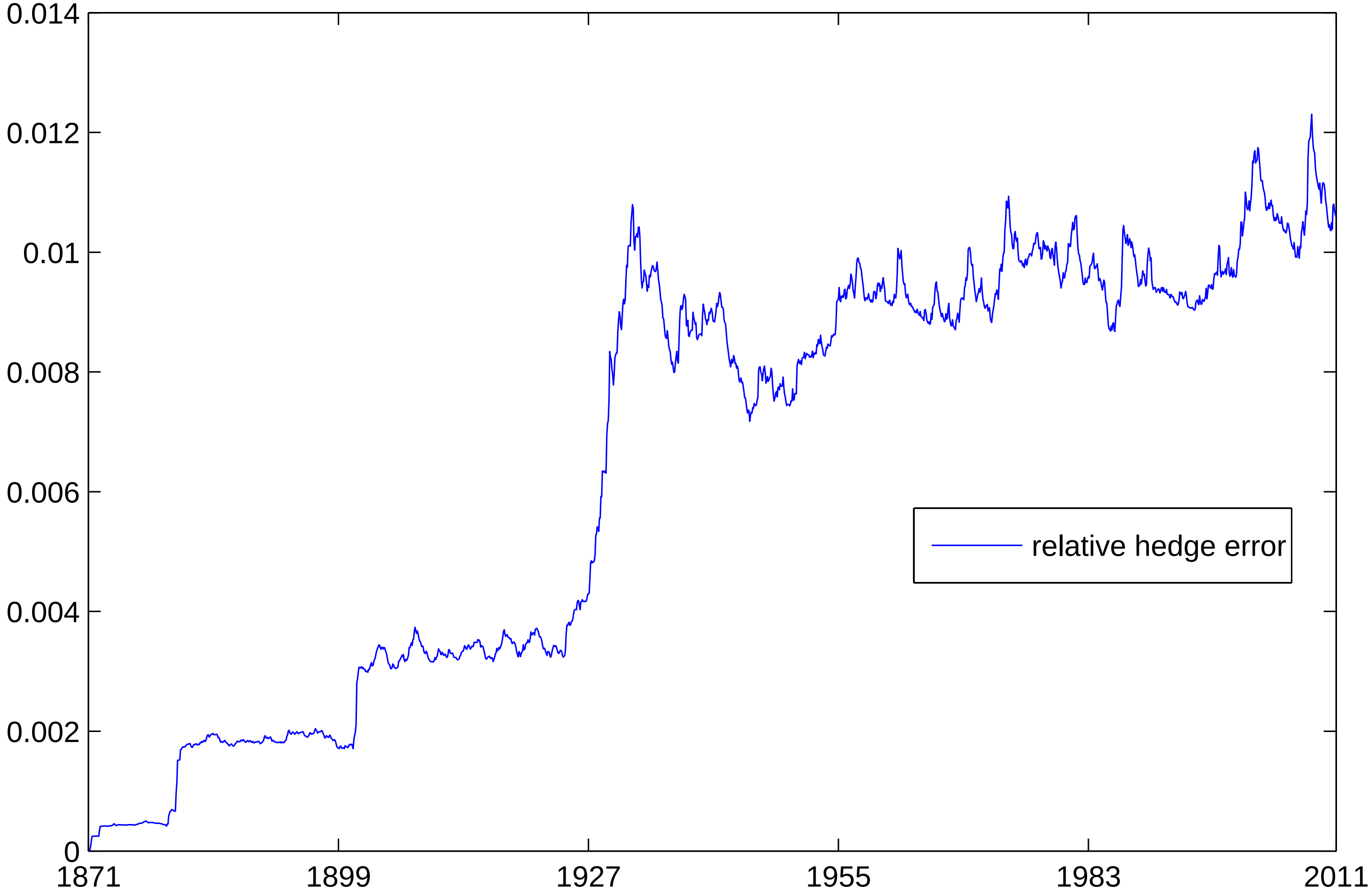
Logarithm of maximum drawdown constrained portfolio, $\boldsymbol{\alpha = 0.6}$



Relative drawdown for drawdown constrained portfolio, $\boldsymbol{\alpha = 0.6}$



Fraction of wealth in S&P500 for $\alpha = \mathbf{0.6}$



Relative hedge error, $\alpha = \mathbf{0.6}$, monthly hedging

- **Long term growth rate**

  For $\alpha \in [0, 1)$, $X \in {}^{\alpha}\mathcal{X}$

$$\lim_{t \to \infty} \left(\frac{\log(X_t)}{G_t} \right) \leq 1 - \alpha = \lim_{t \to \infty} \left(\frac{\log({}^{\alpha}\bar{S}_t^{\delta *})}{G_t} \right)$$

$$\log(\bar{S}_t^{\delta *}) = G_t + \int_0^t |\theta_s| dW_s$$

  Kardaras, Obloj & Pl. (2012)

- **asymptotic maximum long term growth rate**

$$\lim_{t\to\infty} \frac{1}{t}\log({}^{\alpha}S_t^{\delta *}) = (1-\alpha)\lim_{t\to\infty} \frac{1}{t}\log(S_t^{\delta *})$$

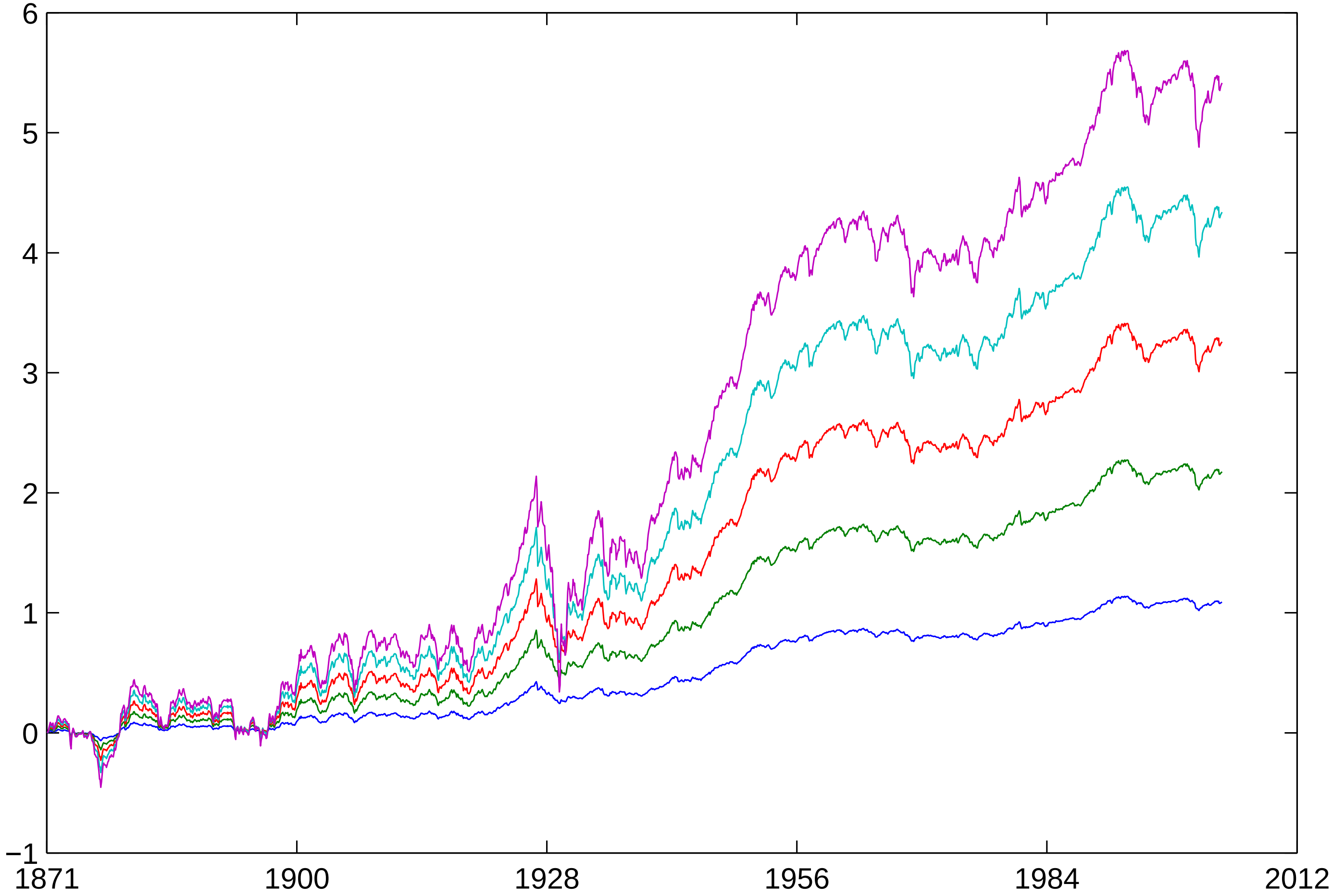
$$^{\alpha}g = (1-\alpha)g$$

restricted drawdown $\Longrightarrow$ reduced maximum growth rate

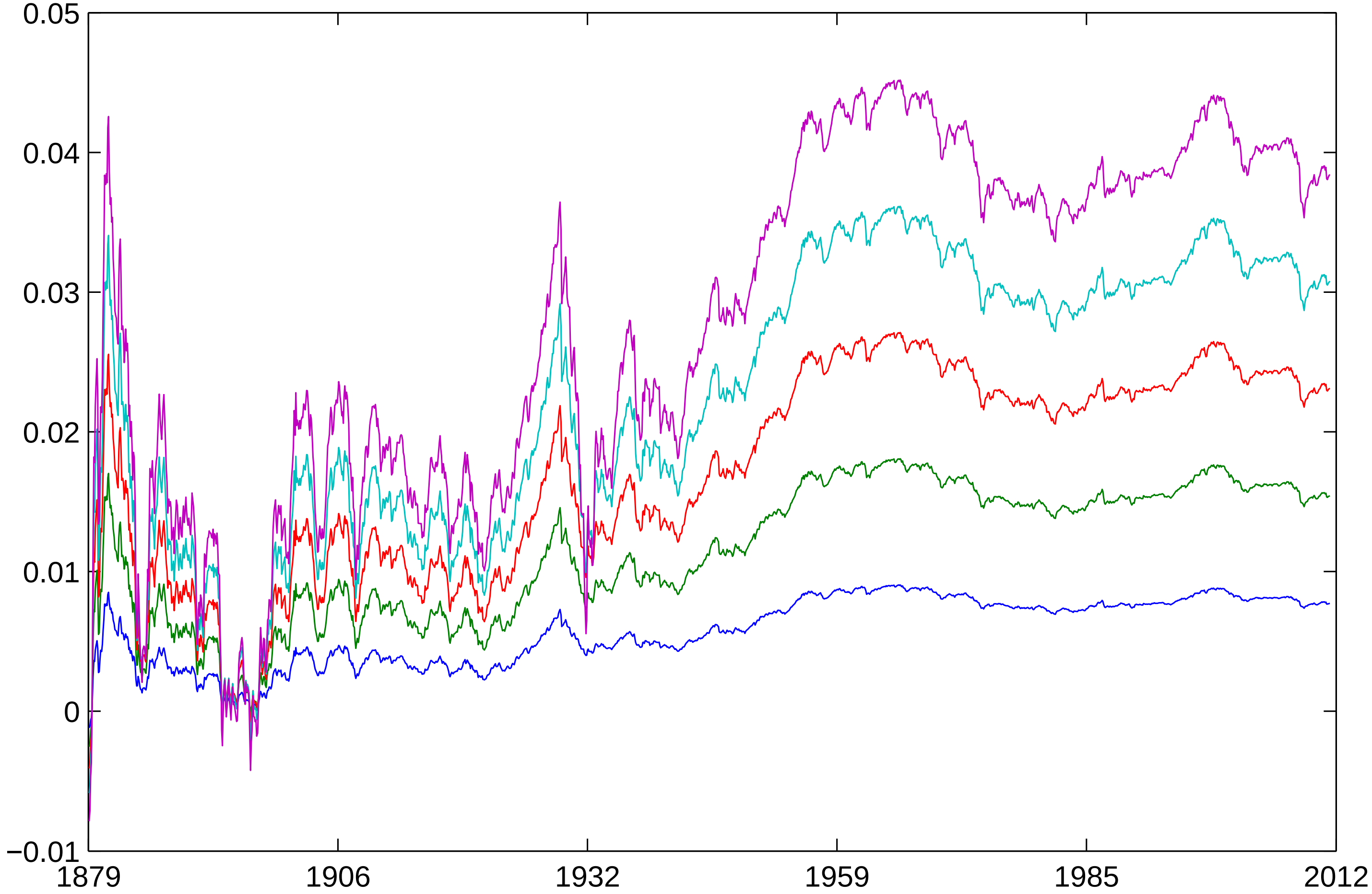
long term view with short term attitude towards risk

realistic alternative to Markowitz mean-variance approach

and utility maximization



Logarithm of drawdown constrained portfolios



Long term growth of drawdown constrained portfolios

Expected Utility Maximization

- **utility function** $U(\cdot),\ U'(\cdot) > 0 \qquad U''(\cdot) < 0$

- **examples**

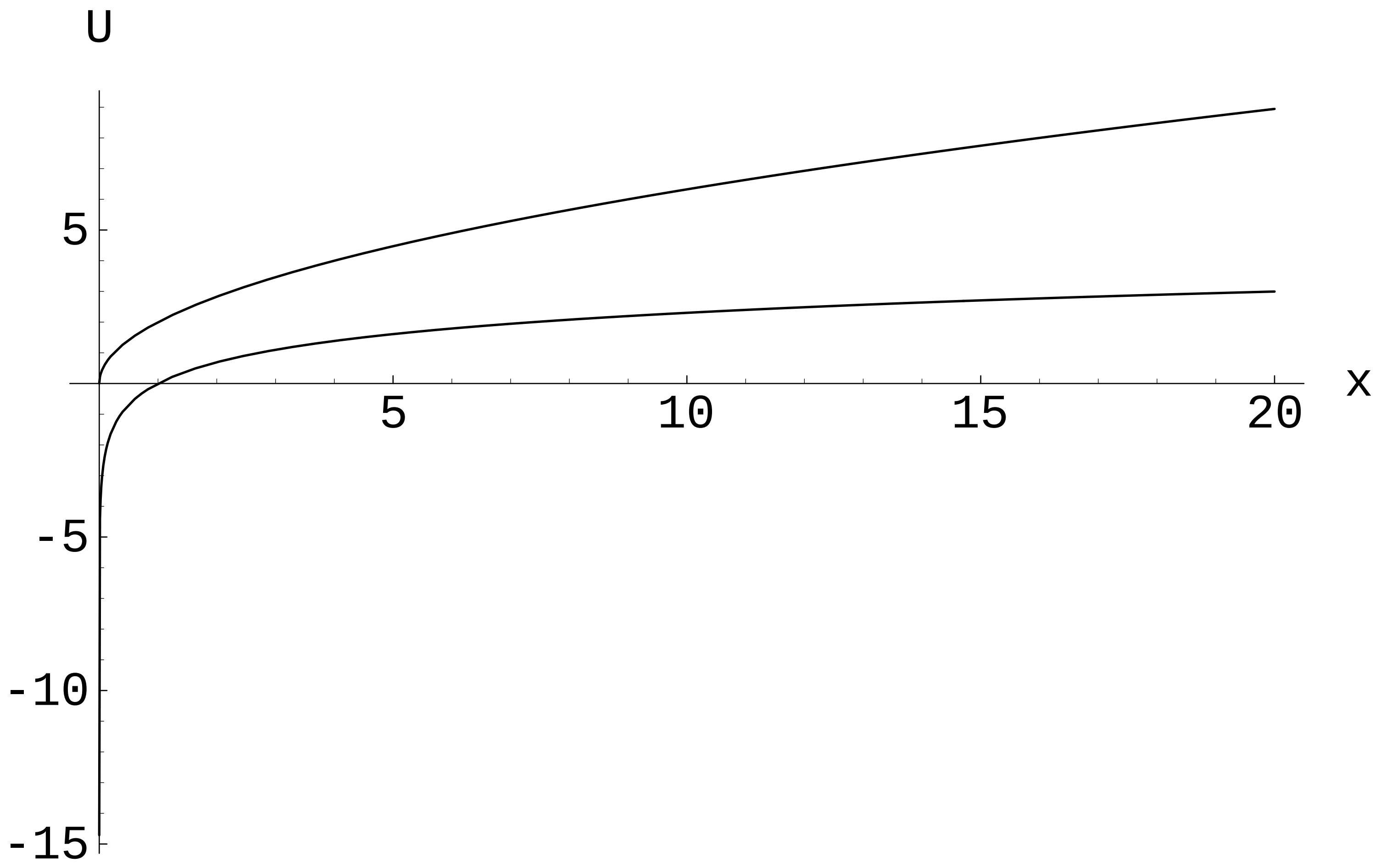
  power utility

$$U(x) = \frac{1}{\gamma}\, x^{\gamma}$$

  for $\gamma \neq 0$ and $\gamma < 1$

  log-utility

$$U(x) = \ln(x)$$



Examples for power utility (upper graph) and log-utility (lower graph)

- assume $\hat{S}^0$ is **scalar Markov process**

- **maximize expected utility**

$$v^{\tilde{\delta}} = \max_{\bar{S}^\delta \in \bar{\mathcal{V}}^+_{S_0}} E\left(U\left(\bar{S}^\delta_T\right) \,\Big|\, \mathcal{A}_0\right)$$

$\bar{\mathcal{V}}^+_{S_0}$ set of strictly positive, discounted, fair portfolios

$$\bar{S}^\delta_0 = S_0 > 0$$

Theorem

Benchmarked, expected utility maximizing portfolio:

$$\hat{S}_t^{\tilde{\delta}} = \hat{u}(t, \hat{S}_t^0) = E\left(U'^{-1}\left(\lambda\, \hat{S}_T^0\right) \hat{S}_T^0 \,\middle|\, \mathcal{A}_t\right),$$

$$\lambda \quad \text{s.t.} \qquad S_0 = \hat{S}_0^{\tilde{\delta}}\, S_0^{\delta_*} = \hat{u}(0, \hat{S}_0)\, S_0^{\delta_*}$$

two fund separation *with risk aversion coefficient*

$$J_t^{\tilde{\delta}} = \frac{1}{1 - \frac{\hat{S}_t^0}{\hat{u}(t,\hat{S}_t^0)} \frac{\partial \hat{u}(t,\hat{S}_t^0)}{\partial \hat{S}_t^0}}.$$

$\Longrightarrow$ locally optimal portfolio

Proof:

$$
\begin{aligned}
v^{\delta} &= E\left(U(\bar{S}_T^{\delta})\right) - \lambda \left(E(\hat{S}_T^{\delta}) - \hat{S}_0^{\delta}\right) \\
&= E\left(F(\bar{S}_T^{\delta})\right)
\end{aligned}
$$

$$
\begin{aligned}
F(\bar{S}_T^{\delta}) &= U(\bar{S}_T^{\delta}) - \lambda \left(\frac{\bar{S}_T^{\delta}}{\bar{S}_T^{\delta *}} - \hat{S}_0^{\delta}\right) \\
F'(\bar{S}_T^{\delta}) &= U'(\bar{S}_T^{\delta}) - \frac{\lambda}{\bar{S}_T^{\delta *}} = 0
\end{aligned}
$$

$\Longrightarrow$

$$
\begin{aligned}
U'(\bar{S}_T^{\delta}) &= \frac{\lambda}{\bar{S}_T^{\delta *}} \\
\bar{S}_T^{\delta} &= U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta *}}\right) = U'^{-1}(\lambda \hat{S}_T^{0}) \\
\hat{S}_T^{\delta} &= U'^{-1}(\lambda \hat{S}_T^{0})\hat{S}_T^{0}
\end{aligned}
$$

$\square$

- **log-utility function** $U(x) = \ln(x)$

$$U'(x) = \frac{1}{x}$$

$$U'^{-1}(y) = \frac{1}{y}$$

$$U''(x) = -\frac{1}{x^2}$$

utility concave

$$\hat{u}(t, \hat{S}^0_t) = E\left(U'^{-1}\left(\lambda\, \hat{S}^0_T\right) \hat{S}^0_T \,\middle|\, \mathcal{A}_t\right) = E\left(\frac{1}{\lambda\, \hat{S}^0_T}\, \hat{S}^0_T \,\middle|\, \mathcal{A}_t\right) = \frac{1}{\lambda}$$

Lagrange multiplier

$$\lambda = \frac{S_0^{\delta_*}}{S_0}$$

risk aversion coefficient

$$J_t^{\tilde{\delta}} = 1$$

- expected log-utility

$$\begin{aligned} v^{\tilde{\delta}} &= E\left(\ln\left(\bar{S}_T^{\delta_*}\right) \,\Big|\, \mathcal{A}_0\right) \\ &= \ln(\lambda) + \ln(S_0) + \frac{1}{2}\int_0^T E\left(\left(\theta(s, \bar{S}_s^{\delta_*})\right)^2 \,\Big|\, \mathcal{A}_0\right) ds \end{aligned}$$

if local martingale part forms a martingale

- **power utility**

$$U(x) = \frac{1}{\gamma}\, x^{\gamma}$$

for $\gamma < 1, \gamma \neq 0$

$$\begin{aligned} U'(x) &= x^{\gamma-1} \\ U'^{-1}(y) &= y^{\frac{1}{\gamma-1}} \\ U''(x) &= (\gamma-1)\, x^{\gamma-2} \end{aligned}$$

concave

$$\hat{u}(t, \hat{S}_t^0) = E\left(\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)^{\frac{1}{\gamma-1}} \frac{1}{\bar{S}_T^{\delta_*}} \,\middle|\, \mathcal{A}_t\right) = \lambda^{\frac{1}{\gamma-1}}\, E\left(\left(\bar{S}_T^{\delta_*}\right)^{\frac{\gamma}{1-\gamma}} \,\middle|\, \mathcal{A}_t\right)$$

$\bar{S}^{\delta_*}$ geometric Brownian motion

$$\hat{u}(t, \hat{S}_t^0) = \lambda^{\frac{1}{\gamma-1}} \left(\bar{S}_t^{\delta_*}\right)^{\frac{\gamma}{1-\gamma}}$$

$$\times E\left(\exp\left\{\frac{\gamma}{1-\gamma}\left(\frac{\theta^2}{2}(T-t) + \theta\,(W_T - W_t)\right)\right\} \Big| \mathcal{A}_t\right)$$

$$= \lambda^{\frac{1}{\gamma-1}} \left(\bar{S}_t^{\delta_*}\right)^{\frac{\gamma}{1-\gamma}} \exp\left\{\frac{\theta^2}{2}\,\frac{\gamma}{(1-\gamma)^2}\,(T-t)\right\}$$

Lagrange multiplier

$$\lambda = S_0^{\gamma - 1}\, \bar{S}_0^{\delta_*}\, \exp\left\{ \frac{\theta^2}{2} \frac{\gamma}{1-\gamma} T \right\}$$

$$\hat{S}_t^0 = (\bar{S}_t^{\delta_*})^{-1}$$

$$\frac{\partial \hat{u}(t, \hat{S}_t^0)}{\partial \hat{S}^0} = \frac{\hat{u}(t, \hat{S}_t^0)}{\hat{S}_t^0} \frac{\gamma}{\gamma - 1}$$

risk aversion coefficient

$$J_t^{\tilde{\delta}} = 1 - \gamma$$

expected utility

$$v^{\tilde{\delta}} = E\left(\frac{1}{\gamma} \left(\bar{S}_T^{\tilde{\delta}} \right)^{\gamma} \,\middle|\, \mathcal{A}_0 \right) = \exp\left\{ -\frac{\theta^2}{2} \frac{\gamma}{1-\gamma} T \right\} (S_0)^{\gamma}$$

Various Approaches to Asset Pricing

- Actuarial Pricing Approach (APA)
- Capital Asset Pricing Model (CAPM)
- Arbitrage Pricing Theory (APT)
- Utility Indifference Pricing
- Benchmark Approach (BA)

Classical

Law of One Price

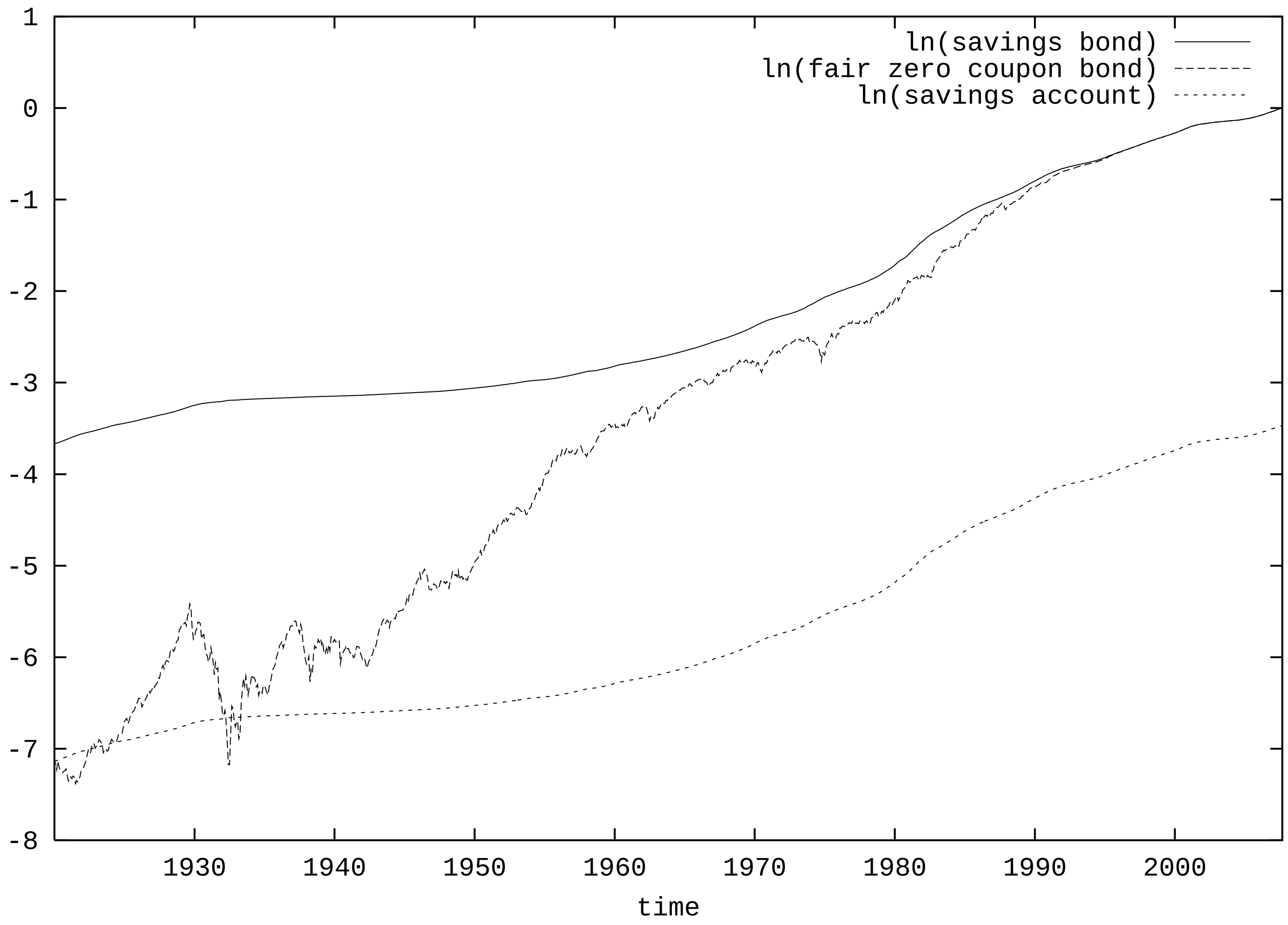
"All replicating portfolios of a payoff have the same price!"

Debreu (1959), Sharpe (1964), Lintner (1965),

Merton (1973a, 1973b), Ross (1976), Harrison & Kreps (1979),

Cochrane (2001), ...

will be, in general, violated under the benchmark approach.



Logarithms of savings bond, fair zero coupon bond and savings account

- **benchmarked value**

$$\hat{S}_t^\delta = \frac{S_t^\delta}{S_t^{\delta_*}}$$

Supermartingale Property

For nonnegative S^δ

$$\hat{S}_t^\delta \geq E_t\left(\hat{S}_s^\delta\right)$$

$0 \leq t \leq s < \infty$

$\hat{S}^\delta$ supermartingale

- One needs consistent pricing concept

- All benchmarked nonnegative securities are supermartingales (no upward trend)

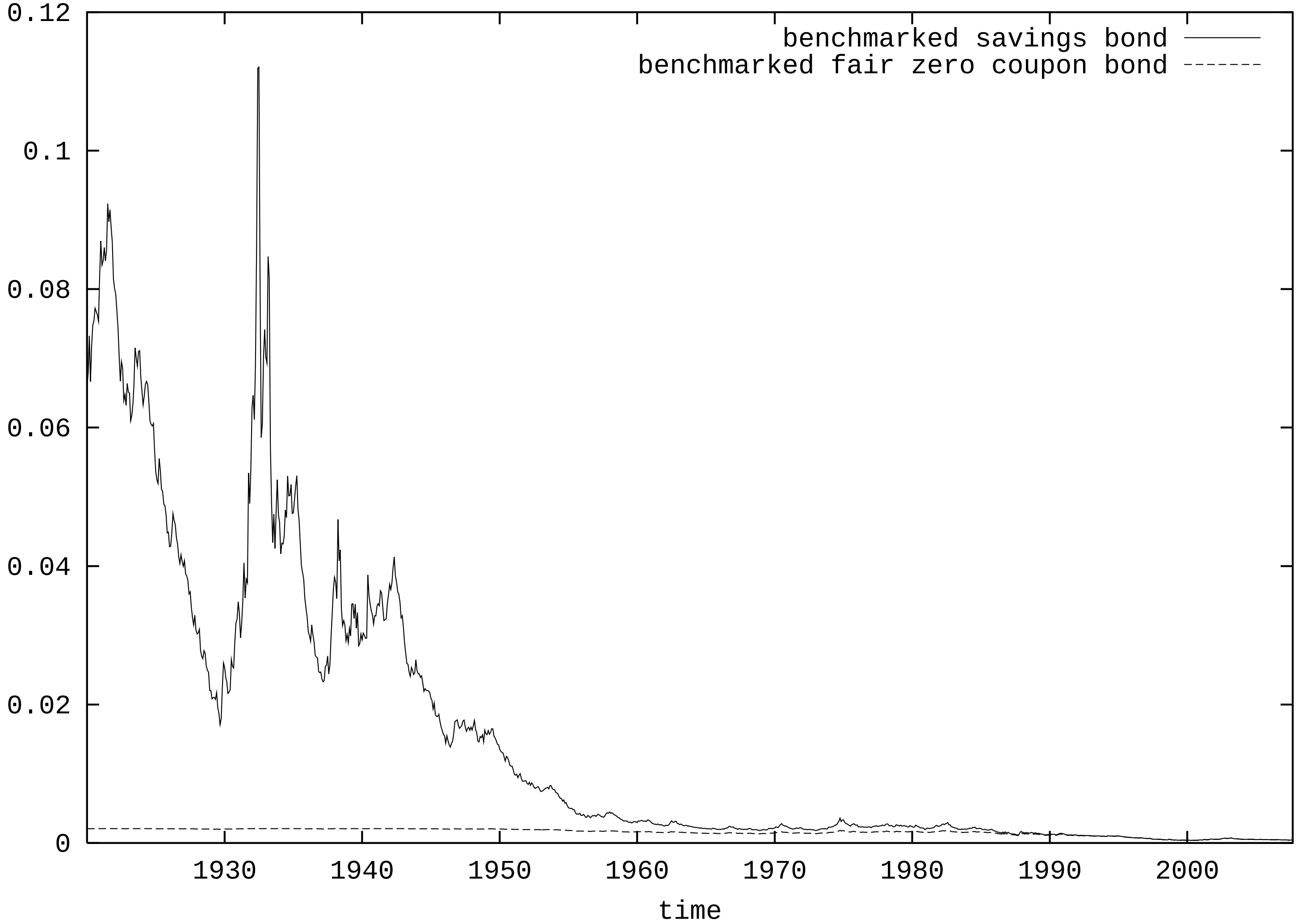
Definition *Price is* fair *if, when benchmarked, forms martingale (no trend)*

$$\hat{S}_t^{\delta} = E_t\left(\hat{S}_s^{\delta}\right)$$

$0 \leq t \leq s < \infty$.

Lemma *The **minimal nonnegative supermartingale** that reaches a given benchmarked contingent claim **is a martingale**.*

see Revuz & Yor (1999)



Benchmarked savings bond and benchmarked fair zero coupon bond

Benchmark Approach:

Law of the Minimal Price

Pl. (2008)

Theorem *If a fair portfolio replicates a nonnegative payoff, then this represents the minimal replicating portfolio.*

- least expensive
- minimal possible hedge
- economically correct price in a competitive market

- **contingent claim**

  H_T

$$E_0\left(\frac{H_T}{S_T^{\delta_*}}\right) < \infty$$

- $S_t^{\delta_H}$ **fair** if

$$\hat{S}_t^{\delta_H} = \frac{S_t^{\delta_H}}{S_t^{\delta_*}} = E_t\left(\frac{H_T}{S_T^{\delta_*}}\right)$$

- real world expectation

  best performing portfolio as numéraire

  $\Longrightarrow$ **Direct link with real world and economy !**

Law of the Minimal Price $\Longrightarrow$

Corollary

Minimal price for replicable $\boldsymbol{H_T}$ is fair and given by

real world pricing formula

$$S_t^{\delta_H} = S_t^{\delta_*} \, E_t\left(\frac{H_T}{S_T^{\delta_*}}\right).$$

- Du & Pl. (2013) benchmarked risk minimization for nonreplicable claims

- most pricing concepts become unified and generalized by real world pricing

$\Longrightarrow$

actuarial pricing
risk neutral pricing
pricing with stochastic discount factor
pricing with numeraire change
pricing with deflator
pricing with state pricing density
pricing with pricing kernel
utility indifference pricing
pricing with numeraire portfolio
classical risk minimization
benchmarked risk minimization

- **Equivalent to risk neutral pricing are pricing via:**

  stochastic discount factor, Cochrane (2001);

  deflator, Duffie (2001);

  pricing kernel, Constantinides (1992);

  state price density, Ingersoll (1987);

  numeraire portfolio as in Long (1990)

Actuarial Pricing

When H_T independent of $S^{\delta_*}_T$

$\Longrightarrow$ rigorous derivation of

- **actuarial pricing formula**

$$S^{\delta_H}_t = P(t,T)\, E_t(H_T)$$

with **zero coupon bond** as discount factor

$$P(t,T) = S^{\delta_*}_t\, E_t\left(\frac{1}{S^{\delta_*}_T}\right)$$

Risk Neutral Pricing

- **real world pricing formula** $\Longrightarrow$

$$S_0^{\delta_H} = E_0\left(\Lambda_T \frac{S_0^0}{S_T^0} H_T\right)$$

Radon-Nikodym derivative (density)

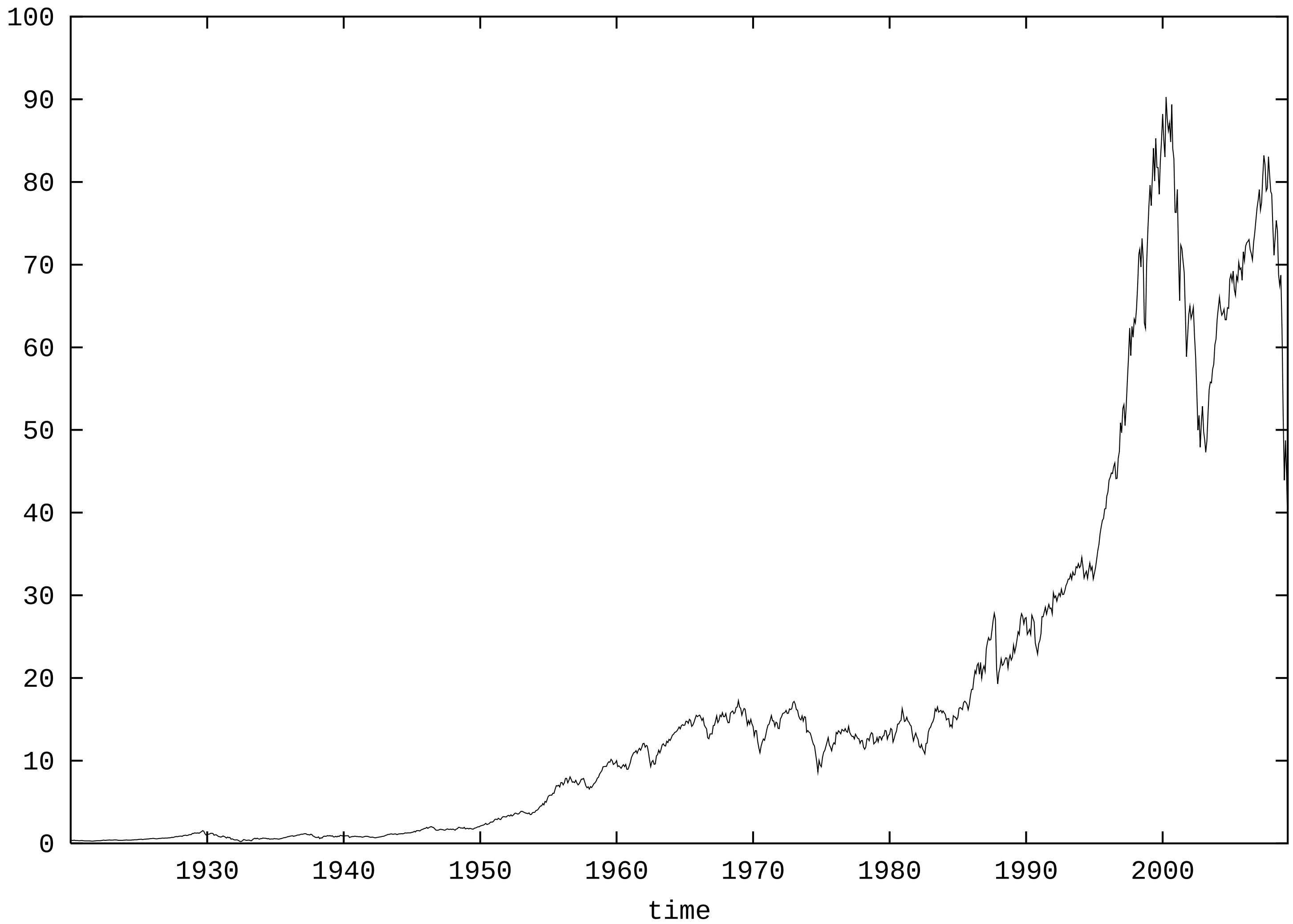
$$\Lambda_t = \frac{\hat{S}_t^0}{\hat{S}_0^0} \quad \text{supermartingale}$$

$$1 = \Lambda_0 \geq E_0(\Lambda_T)$$

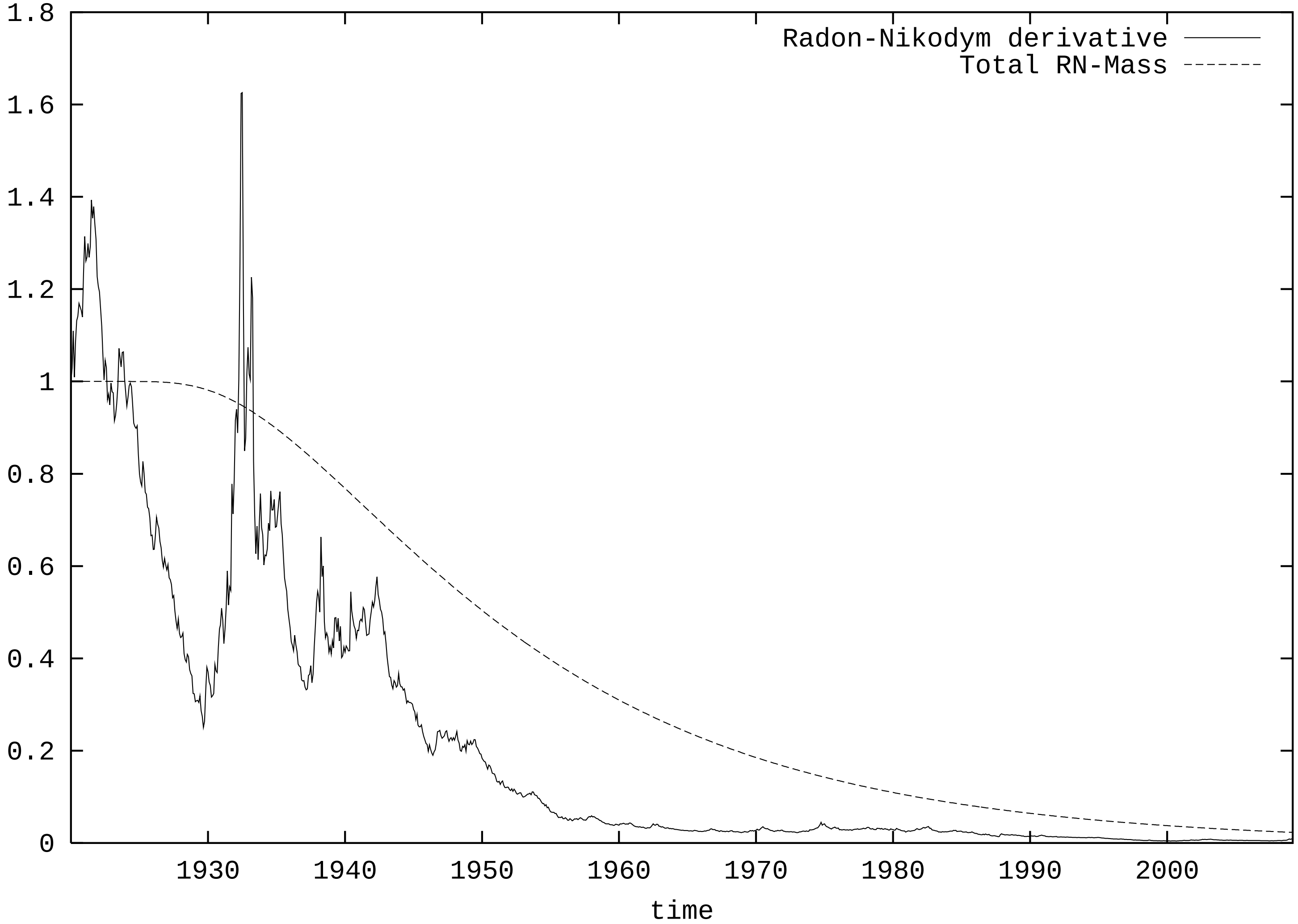
$\Longrightarrow$

$$S_0^{\delta_H} \leq \frac{E_0\left(\Lambda_T \frac{S_0^0}{S_T^0} H_T\right)}{E_0(\Lambda_T)}$$

similar for any numeraire (here S^0 savings account)



Discounted S&P500 total return index



Radon-Nikodym derivative and total mass of putative risk neutral measure

- **special case when savings account is fair:**

$\Longrightarrow \qquad \Lambda_T = \frac{dQ}{dP}$ forms martingale; $\quad E_0(\Lambda_T) = 1;$

equivalent risk neutral probability measure Q exists;

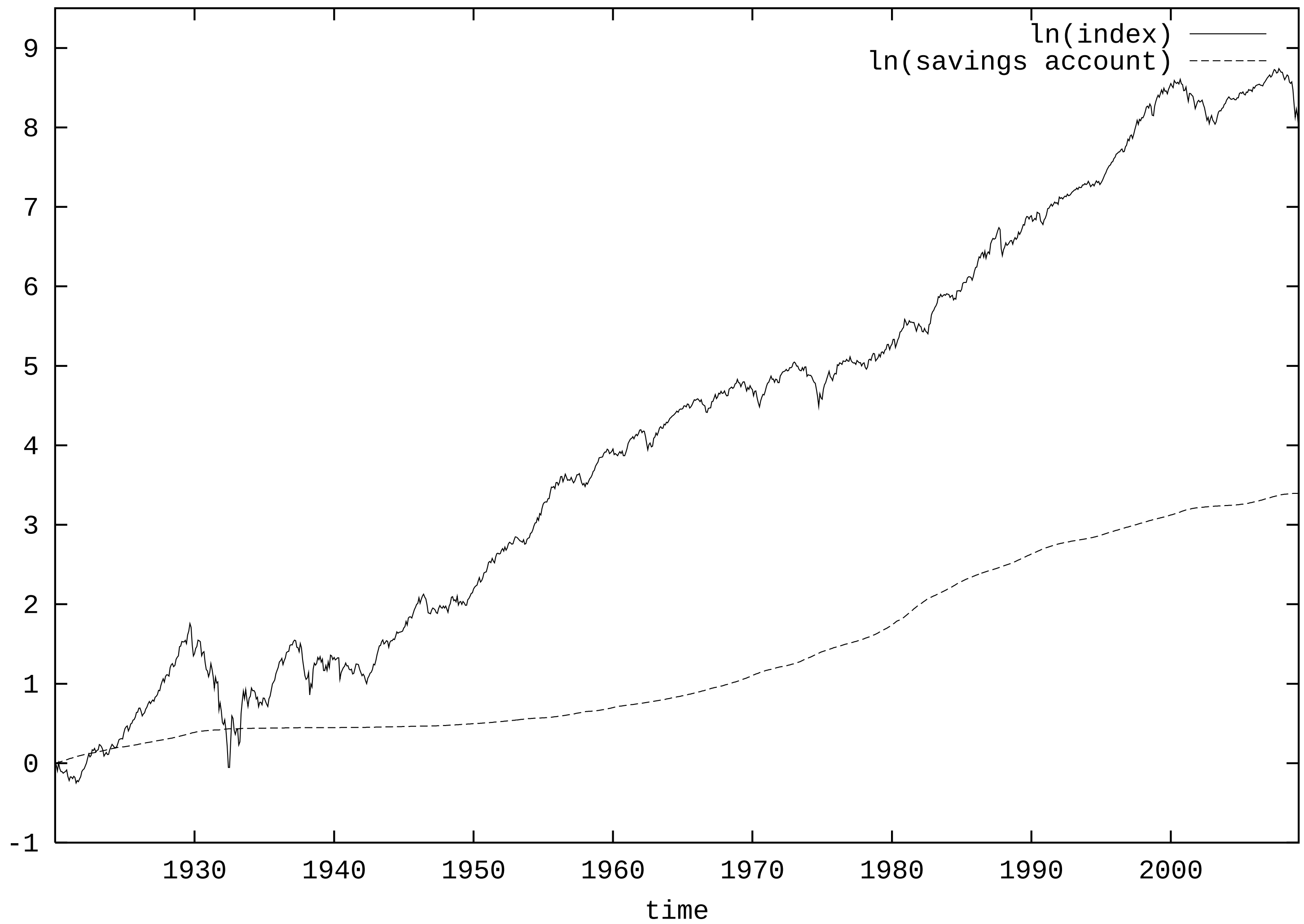
Bayes' formula $\quad \Longrightarrow$
risk neutral pricing formula

$$S_0^{\delta_H} = E_0^Q \left(\frac{S_0^0}{S_T^0} H_T \right)$$

Harrison & Kreps (1979), Ingersoll (1987),

Constantinides (1992), Duffie (2001), Cochrane (2001), $\ldots$

- otherwise "risk neutral price" $\geq$ real world price



ln(S&P500 accumulation index) and ln(savings account)

Risk Neutral Pricing under BS Model

$\theta_t = \theta,\ a_t = a,\ r_t = r,\ \sigma_t = \sigma$

Drifted Wiener Process

- **drifted Wiener process** W_θ

$$W_\theta(t) = W_t + \theta\, t$$

- **market price of risk**

$$\theta = \frac{a - r}{\sigma}$$

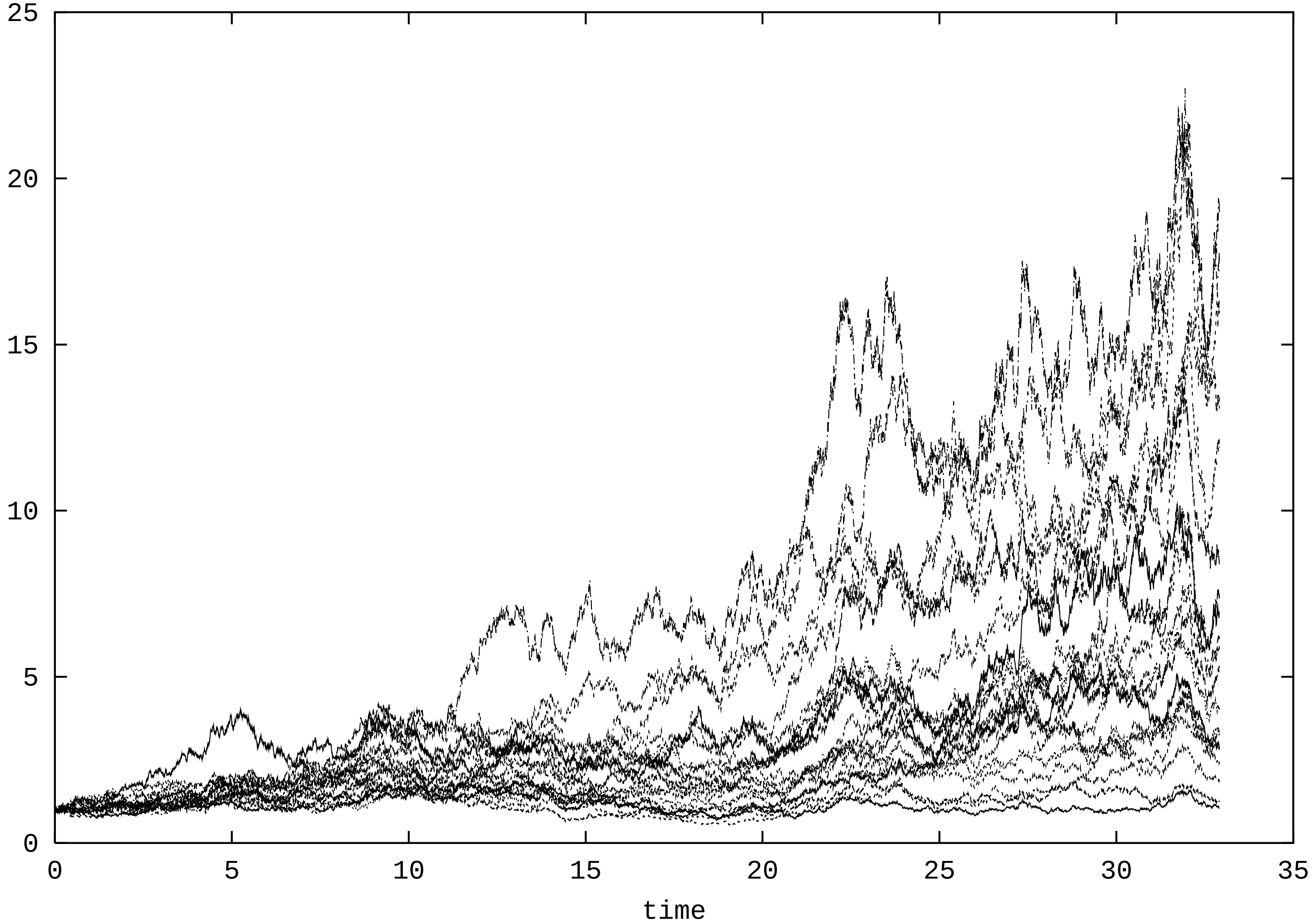
- **underlying security**

$$\begin{aligned} dS_t &= (a - \sigma\,\theta)\, S_t\, dt + \sigma\, S_t\, (\theta\, dt + dW_t) \\ &= (a - \sigma\,\theta)\, S_t\, dt + \sigma\, S_t\, dW_\theta(t) \\ &= r\, S_t\, dt + \sigma\, S_t\, dW_\theta(t) \end{aligned}$$

$$S_t = S_0\, \exp\left\{ \left(r - \frac{1}{2}\,\sigma^2 \right) t + \sigma\, W_\theta(t) \right\}$$

W_θ is **not** a Wiener process under the real world probability P,

however, will be under some P_θ



Simulated primary security accounts.

- **Radon-Nikodym derivative**

$$\Lambda_\theta(T) = \frac{dP_\theta}{dP} = \frac{\hat{S}_T^0}{\hat{S}_0^0}$$

- **risk neutral measure** P_θ

$$\begin{aligned} P_\theta(A) &= \int_\Omega 1_{\{A\}}\, dP_\theta(\omega) = \int_A \frac{dP_\theta(\omega)}{dP(\omega)}\, dP(\omega) \\ &= \int_A \Lambda_\theta(T)\, dP(\omega) \end{aligned}$$

for all subsets $A \in \Omega$

Definition $\tilde{P}$ *is* **equivalent** *to* $\hat{P}$ *if both have the same sets of events that have probability zero.*

- **equivalence of P and P_θ**

  fundamental for risk neutral pricing

Key Risk Neutral assumption:

Λ_θ is $(\underline{\mathcal{A}}, P)$ martingale

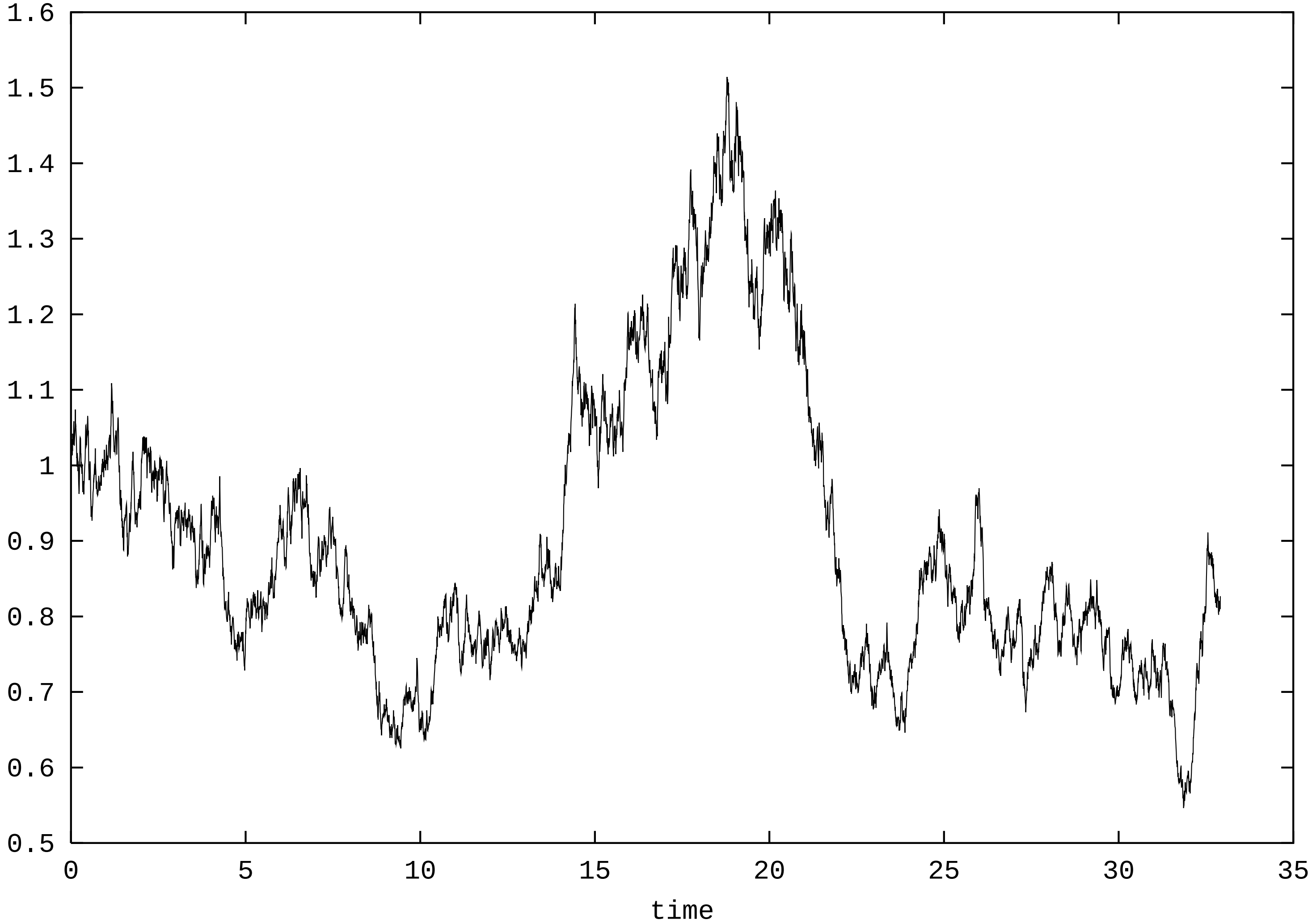
- **under BS model**

  Radon-Nikodym derivative

$$\Lambda_{\theta}(t) = \frac{\hat{S}_t^0}{\hat{S}_0^0}$$

  is $(\underline{\mathcal{A}}, P)$-martingale,

  $\Longrightarrow$ there exists equivalent risk neutral probability measure P_{θ} for BS model



Benchmarked savings account under BS model, martingale.

Risk Neutral Measure Transformation

$$\left.\frac{dP_\theta}{dP}\right|_{\mathcal{A}_t} = \Lambda_\theta(t) = \exp\left\{-\frac{1}{2}\theta^2\, t - \theta\, W_t\right\}$$

- **total risk neutral measure** at $t = 0$:

$$\left.\frac{dP_\theta}{dP}\right|_{\mathcal{A}_0} = E(\Lambda_\theta(T) \,|\, \mathcal{A}_0) = \Lambda_\theta(0) = 1$$

$\Longrightarrow \quad P_\theta$ is a probability measure

Let us show that W_θ is Wiener process under P_θ:

For fixed $\tilde{y} \in \Re$,

$t \in [0, T]$ and $s \in [0, t]$

let A be the event

$$A = \{\omega \in \Omega : \; W_\theta(t, \omega) - W_\theta(s, \omega) < \tilde{y}\}$$

$\Longrightarrow$

$$A = \{\omega \in \Omega : \; W_t(\omega) - W_s(\omega) < \tilde{y} - \theta\,(t - s)\}$$

E_θ denoting expectation under P_θ $\Longrightarrow$

$$\begin{aligned}
P_\theta(A) &= E_\theta(1_A|\mathcal{A}_0) = E\left(\left.\frac{dP_\theta}{dP}\right|_{\mathcal{A}_T} 1_A \,\middle|\, \mathcal{A}_0\right) \\
&= E(\Lambda_\theta(T)\,1_A|\mathcal{A}_0) = E\left(\Lambda_\theta(t)\,1_A\,\frac{\Lambda_\theta(T)}{\Lambda_\theta(t)} \,\middle|\, \mathcal{A}_0\right) \\
&= E\left(\Lambda_\theta(t)\,1_A\,E\left(\frac{\Lambda_\theta(T)}{\Lambda_\theta(t)} \,\middle|\, \mathcal{A}_t\right) \,\middle|\, \mathcal{A}_0\right) \\
&= E\left(\Lambda_\theta(t)\,1_A|\mathcal{A}_0\right) = E\left(\Lambda_\theta(s)\,\frac{\Lambda_\theta(t)}{\Lambda_\theta(s)}\,1_A \,\middle|\, \mathcal{A}_0\right) \\
&= E\left(\Lambda_\theta(s)|\mathcal{A}_0\right)\;E\left(\frac{\Lambda_\theta(t)}{\Lambda_\theta(s)}\,1_A \,\middle|\, \mathcal{A}_0\right) \\
&= E\left(\frac{\Lambda_\theta(t)}{\Lambda_\theta(s)}\,1_A \,\middle|\, \mathcal{A}_0\right)
\end{aligned}$$

since $y = W_t - W_s$ is Gaussian $\sim \mathcal{N}(0, t-s)$ under P $\Longrightarrow$

$$P_\theta(A) \;=\; E\left(\frac{\Lambda_\theta(t)}{\Lambda_\theta(s)}\, 1_A\right)$$

$$= \int_{-\infty}^{\tilde{y}-\theta(t-s)} \exp\left\{-\frac{\theta^2}{2}\,(t-s) - \theta\, y\right\} \frac{1}{\sqrt{2\,\pi\,(t-s)}}$$

$$\exp\left\{-\frac{y^2}{2\,(t-s)}\right\} dy$$

$$= \int_{-\infty}^{\tilde{y}-\theta(t-s)} \frac{1}{\sqrt{2\,\pi\,(t-s)}}\; \exp\left\{-\frac{(y+\theta\,(t-s))^2}{2\,(t-s)}\right\} dy$$

$$= \int_{-\infty}^{\tilde{y}} \frac{1}{\sqrt{2\,\pi\,(t-s)}}\; \exp\left\{-\frac{z^2}{2\,(t-s)}\right\} dz$$

for $z = y + \theta(t-s)$

$\Longrightarrow$

Theorem (Girsanov) *Under the above BS model the process*

$$W_\theta = \{W_\theta(t) = W_t + \theta\, t,\ t \geq 0\}$$

is a standard Wiener process in the filtered probability space $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P_\theta)$.

- W_θ is $(\underline{\mathcal{A}}, P_\theta)$-Wiener process

- simply a transformation of variable

  (not always possible, exploits here BS model)

Real World Pricing $\Longrightarrow$ Risk Neutral Pricing

$$\begin{aligned}
V(0,S_0) &= S_0^{\delta_*}\, E\left(\frac{H(S_T)}{S_T^{\delta_*}} \,\Big|\, \mathcal{A}_0\right) \\
&= E\left(\frac{\hat{S}_T^0}{\hat{S}_0^0}\, \frac{H(S_T)}{S_T^0} \,\Big|\, \mathcal{A}_0\right) \\
&= E\left(\Lambda_\theta(T) \left(\frac{H(S_T)}{S_T^0}\right) \,\Big|\, \mathcal{A}_0\right) \\
&\leq \frac{E\left(\Lambda_\theta(T)\, \frac{H(S_T)}{S_T^0} \,\big|\, \mathcal{A}_0\right)}{E\left(\Lambda_\theta(T) \,\big|\, \mathcal{A}_0\right)}
\end{aligned}$$

$\Longrightarrow$ **Risk Neutral Pricing** if Λ_θ is $(\underline{\mathcal{A}}, P)$-martingale

$$
\begin{aligned}
V(0, S_0) &= E\left(\exp\left\{-\frac{\theta^2}{2}\, T - \theta\, W_T\right\} \exp\{-r\, T\}\, H(S_T) \,\middle|\, \mathcal{A}_0\right) \\
&= \exp\{-r\, T\} \int_{-\infty}^{\infty} H\left(S_0 \, \exp\left\{\left(a - \frac{1}{2}\, \sigma^2\right) T + \sigma\, y\right\}\right) \\
&\qquad \times \exp\left\{-\frac{\theta^2}{2}\, T - \theta\, y\right\} \frac{1}{\sqrt{T}}\, N'\left(\frac{y}{\sqrt{T}}\right) dy \\
&= \exp\{-r\, T\} \int_{-\infty}^{\infty} \left(H\left(S_0 \, \exp\left\{\left(r - \frac{1}{2}\, \sigma^2\right) T + \sigma\, z\right\}\right)\right) \\
&\qquad \times \frac{1}{\sqrt{T}}\, N'\left(\frac{z}{\sqrt{T}}\right) dz \\
&= \exp\{-r\, T\}\, E_\theta\left(H(S_T) \,\middle|\, \mathcal{A}_0\right) = E_\theta\left(\frac{H(S_T)}{S_T^0} \,\middle|\, \mathcal{A}_0\right)
\end{aligned}
$$

$z = y + \theta\, T \quad \Longrightarrow \quad E_\theta$ - expectation with respect to P_θ

Risk Neutral SDEs under BS model

- **discounted underlying security price**

$$d\bar{S}_t = \sigma\, \bar{S}_t\, dW_\theta(t)$$

driftless under P_θ, $\sigma \bar{S} \in \mathcal{L}_T^2 \quad \Longrightarrow$

$(\underline{\mathcal{A}}, P_\theta)$**-martingale**

- **discounted option price**

$$d\bar{V}(t,\bar{S}_t) = \frac{\partial \bar{V}(t,\bar{S}_t)}{\partial \bar{S}} \, \sigma \, \bar{S}_t \, dW_\theta(t)$$

driftless under P_θ, $\frac{\partial \bar{V}}{\partial \bar{S}} \, \sigma \bar{S} \in \mathcal{L}_T^2 \quad \Longrightarrow$

($\underline{\mathcal{A}}$, P_θ)-martingale

- **discounted savings account** $\bar{S}_t^0 = 1 \quad \Longrightarrow \quad$ **($\underline{\mathcal{A}}$, P_θ)-martingale**

- **discounted portfolio**

$$\begin{aligned} d\bar{S}_t^\delta &= d(\bar{S}_t^{\delta_*}\,\hat{S}_t^\delta) \\ &= \bar{S}_t^\delta\,\pi_\delta^1(t)\,\sigma\,(\theta\,dt + dW_t) \\ &= \bar{S}_t^\delta\,\pi_\delta^1(t)\,\sigma\,dW_\theta(t) \end{aligned}$$

$(\underline{\mathcal{A}}, P_\theta)$**-local martingale**

- **if** $\bar{S}^\delta\,\pi_\delta^1\,\delta = \delta^1\,\bar{S}\,\sigma \in \mathcal{L}_T^2 \quad \Longrightarrow \quad \bar{S}^\delta$ is an $(\underline{\mathcal{A}}, P_\theta)$**-martingale**,

$\Longrightarrow$ **risk neutral pricing formula** holds

$\Longrightarrow$

$$\bar{S}_t^\delta = E_\theta\left(\bar{S}_T^\delta \,\middle|\, \mathcal{A}_t\right) = E_\theta\left(\frac{S_T^\delta}{S_T^0} \,\middle|\, \mathcal{A}_t\right)$$

- $\bar{S}^{\delta_*}$ is $(\underline{\mathcal{A}}, P_\theta)$-martingale under BS model

- **strict supermartingale portfolio** under BS model

  if Z_t **strict $(\underline{\mathcal{A}}, P_\theta)$-supermartingale**, independent of $\bar{S}^{\delta_*}_t$ and

$$\bar{S}^{\delta}_t = \bar{S}^{\delta_*}_t \, Z_t$$

  $\Longrightarrow \quad \bar{S}^{\delta}_t$ strict $(\underline{\mathcal{A}}, P_\theta)$-supermartingale

- risk neutral pricing formula does **not** hold for such S^{δ}_t

$$\bar{S}^{\delta}_t > E_\theta\left(\bar{S}^{\delta}_T \,\middle|\, \mathcal{A}_t\right) \text{ for } T > t$$

  $\Longrightarrow \quad$ **not** all discounted portfolios are $(\underline{\mathcal{A}}, P_\theta)$-martingales

  even when Radon-Nikodym derivative Λ_θ is $(\underline{\mathcal{A}}, P)$-martingale

Change of Variables

- we performed a change of variables from W_t to $W_\theta(t)$

  W **Wiener processes** under P

  W_θ **Wiener processes** under P_θ

- relies on existence of an

  equivalent risk neutral probability measure P_θ

General Girsanov Transformation

- m**-dimensional standard Wiener process**

$$W = \{W_t = (W_t^1, \ldots, W_t^m)^\top,\ t \in [0, T]\}$$

$(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P)$

- $\underline{\mathcal{A}}$-adapted, predictable m-dimensional stochastic process

$$\theta = \{\theta_t = (\theta_t^1, \ldots,\ \theta_t^m)^\top,\ t \in [0, T]\}$$

$$\int_0^T \sum_{i=1}^m (\theta_t^i)^2 \, dt < \infty$$

- assume **Radon-Nikodym derivative**

$$\Lambda_\theta(t) = 1 - \sum_{i=1}^{m} \int_0^t \Lambda_\theta(s)\, \theta_s^i \, dW_s^i$$

$$= \exp\left\{ -\int_0^t \theta_s^\top \, dW_s - \frac{1}{2} \int_0^t \theta_s^\top \, \theta_s \, ds \right\} < \infty$$

forms $(\underline{\mathcal{A}}, P)$**-martingale**

- $\boldsymbol{\Lambda_\theta}$ $(\underline{\boldsymbol{\mathcal{A}}}, \boldsymbol{P})$-martingale

$\Longrightarrow$

$$\boldsymbol{E(\Lambda_\theta(t) \,|\, \mathcal{A}_s) = \Lambda_\theta(s)}$$

for $\boldsymbol{t \in [0, T]}$ and $\boldsymbol{s \in [0, t]}$

$\Longrightarrow$

$$\boldsymbol{E(\Lambda_\theta(t) \,|\, \mathcal{A}_0) = \Lambda_\theta(0) = 1}$$

- define a candidate risk neutral measure P_θ with

$$\left.\frac{dP_\theta}{dP}\right|_{\mathcal{A}_T} = \Lambda_\theta(T)$$

by setting

$$\begin{aligned} P_\theta(A) &= E(\Lambda_\theta(T)\,1_A) \\ &= E_\theta(1_A) \end{aligned}$$

- since Λ_θ is an $(\underline{\mathcal{A}}, P)$-martingale

$$P_\theta(\Omega) = E(\Lambda_\theta(T)) = E\left(\Lambda_\theta(T) \,\middle|\, \mathcal{A}_0\right) = \Lambda_\theta(0) = 1$$

$\Longrightarrow$ P_θ is a **probability measure**

Girsanov Theorem

Theorem (Girsanov) *If a given strictly positive Radon-Nikodym derivative process* Λ_θ *is an* $(\underline{\mathcal{A}}, P)$*-martingale, then the* m*-dimensional process* $W_\theta = \{W_\theta(t),\ t \in [0,T]\}$*, given by*

$$W_\theta(t) = W_t + \int_0^t \theta_s \, ds$$

for all $t \in [0,T]$*, is an* m*-dimensional standard Wiener process on the filtered probability space* $(\Omega, \mathcal{A}_T, \underline{\mathcal{A}}, P_\theta)$.

- W is $(\underline{\mathcal{A}}, P)$-Wiener process

  W_θ is $(\underline{\mathcal{A}}, P_\theta)$-Wiener process

Novikov Condition

When is the strictly positive $(\underline{\mathcal{A}}, P)$-local martingale Λ_θ an $(\underline{\mathcal{A}}, P)$-martingale?

- a **sufficient** condition is the **Novikov condition**

  Novikov (1972)

$$E\left(\exp\left\{\frac{1}{2}\int_0^T |\theta_s^\top \theta_s|\, ds\right\}\right) < \infty$$

  (is satisfied for the BS model)

- for other sufficient conditions see Revuz & Yor (1999)

Bayes's Theorem

Theorem (Bayes) *Assume that* Λ_θ *is an* $(\underline{\mathcal{A}}, P)$*-martingale, then for any given stopping time* $\tau \in [0, T]$ *and any* $\mathcal{A}_\tau$*-measurable random variable* Y*, satisfying the integrability condition*

$$E_\theta(|Y|) < \infty,$$

one has the Bayes rule

$$E_\theta(Y \mid \mathcal{A}_s) = \frac{E(\Lambda_\theta(\tau)\, Y \mid \mathcal{A}_s)}{E(\Lambda_\theta(\tau) \mid \mathcal{A}_s)}$$

for $s \in [0, \tau]$.

Proof of Bayes's Theorem (*)

For each $\mathcal{A}_\tau$-measurable random variable Y and a set $A \in \mathcal{A}_s$ with $s \in [0, \tau]$, $\tau \leq T$

$$
\begin{aligned}
\mathbf{1}_A\, E_\theta\,(Y|\mathcal{A}_s) &= E_\theta\left(\mathbf{1}_A\, Y \,\middle|\, \mathcal{A}_s\right) = E\left(\mathbf{1}_A\, Y\, \Lambda_\theta(T) \,\middle|\, \mathcal{A}_s\right) \\
&= E\,(\mathbf{1}_A\, Y\, \Lambda_\theta(\tau)|\mathcal{A}_s) = E\,(\mathbf{1}_A\, E\,(Y\, \Lambda_\theta(\tau)|\mathcal{A}_s)\,|\mathcal{A}_s) \\
&= E\left(\Lambda_\theta(s)\left\{\frac{\mathbf{1}_A}{\Lambda_\theta(s)}\, E\,(Y\, \Lambda_\theta(\tau)\,|\,\mathcal{A}_s)\right\} \,\middle|\, \mathcal{A}_s\right) \\
&= E_\theta\left(\frac{\mathbf{1}_A}{\Lambda_\theta(s)}\, E\,(Y\, \Lambda_\theta(\tau)\,|\,\mathcal{A}_s) \,\middle|\, \mathcal{A}_s\right) \\
&= E_\theta\left(\mathbf{1}_A\, \frac{E\,(\Lambda_\theta(\tau)\, Y\,|\,\mathcal{A}_s)}{E\,(\Lambda_\theta(\tau)\,|\,\mathcal{A}_s)} \,\middle|\, \mathcal{A}_s\right) \\
&= \mathbf{1}_A\, \frac{E\,(\Lambda_\theta(\tau)\, Y\,|\,\mathcal{A}_s)}{E\,(\Lambda_\theta(\tau)\,|\,\mathcal{A}_s)}. \qquad \square
\end{aligned}
$$

Change of Numeraire Technique

Geman, El Karoui & Rochet (1995)

Jamshidian (1997)

- **numeraire**

$$S^{\bar{\delta}} = \{S^{\bar{\delta}}_t,\, t \in [0, T]\}$$

  normalizes all other portfolios

- may lead to simplifications in calculations

- relative price of a portfolio

$$\frac{S^{\delta}_t}{S^{\bar{\delta}}_t}$$

Self-financing under Numeraire Change

$$S_t^{\delta} = \delta_t^0 \, S_t^0 + \delta_t^1 \, S_t$$

$$dS_t^{\delta} = \delta_t^0 \, dS_t^0 + \delta_t^1 \, dS_t$$

$$dS_t^{\bar{\delta}} = \bar{\delta}_t^0 \, dS_t^0 + \bar{\delta}_t^1 \, dS_t$$

Itô formula $\Longrightarrow$

$$
\begin{aligned}
d\left(\frac{S_t^{\delta}}{S_t^{\bar{\delta}}}\right) &= \frac{1}{S_t^{\bar{\delta}}}\, dS_t^{\delta} + S_t^{\delta}\, d\left(\frac{1}{S_t^{\bar{\delta}}}\right) + d\left[\frac{1}{S^{\bar{\delta}}}, S^{\delta}\right]_t \\
&= \delta_t^0 \left(\frac{1}{S_t^{\bar{\delta}}}\, dS_t^0 + S_t^0\, d\left(\frac{1}{S_t^{\bar{\delta}}}\right)\right) \\
&\quad + \delta_t^1 \left(\frac{1}{S_t^{\bar{\delta}}}\, dS_t + S_t\, d\left(\frac{1}{S_t^{\bar{\delta}}}\right) + d\left[\frac{1}{S^{\bar{\delta}}}, S\right]_t\right)
\end{aligned}
$$

$$
[X, Y]_t \approx \sum_{i=0}^{i_t} \left(X_{t_{i+1}} - X_{t_i}\right)\left(Y_{t_{i+1}} - Y_{t_i}\right) \quad \text{covariation}
$$

Itô formula for

$$\frac{S_t^0}{S_t^{\bar{\delta}}} \quad \text{and} \quad \frac{S_t}{S_t^{\bar{\delta}}}$$

$\Longrightarrow$

$$d\left(\frac{S_t^{\delta}}{S_t^{\bar{\delta}}}\right) = \delta_t^0 \, d\left(\frac{S_t^0}{S_t^{\bar{\delta}}}\right) + \delta_t^1 \, d\left(\frac{S_t}{S_t^{\bar{\delta}}}\right)$$

$\Longrightarrow$

- in denomination of another numeraire **always self-financing**

Numeraire Pairs

- numeraire pair $(S^{\bar{\delta}}, P_{\theta_{\bar{\delta}}})$

  real world pricing formula:

  $$S_t = E\left(\frac{S_t^{\delta_*}}{S_T^{\delta_*}} \, S_T \,\middle|\, \mathcal{A}_t \right)$$

  existence of GOP

  $\Longrightarrow$ numeraire pair (S^{δ_*}, P)

if $\boldsymbol{\Lambda_\theta}$ is $(\underline{\boldsymbol{\mathcal{A}}}, \boldsymbol{P})$-martingale $\quad \Longrightarrow$

- **risk neutral** pricing formula:

$$\boldsymbol{S_t = E_\theta \left(\frac{S_t^0}{S_T^0} \, S_T \,\middle|\, \mathcal{A}_t \right)}$$

$\Longrightarrow \quad$ numeraire pair $(\boldsymbol{S^0}, \boldsymbol{P_\theta})$

- for fair price process S

  from real world pricing formula

$$\frac{S_t}{S_t^{\delta_*}} = E\left(\frac{S_T}{S_T^{\delta_*}} \,\middle|\, \mathcal{A}_t \right)$$

$\Longrightarrow$ for any strictly positive numeraire $S^{\bar{\delta}}$

$$\begin{aligned} \frac{S_t}{S_t^{\bar{\delta}}} &= E\left(\frac{S_t^{\delta_*}}{S_t^{\bar{\delta}}} \, \frac{S_T^{\bar{\delta}}}{S_T^{\delta_*}} \, \frac{S_T}{S_T^{\bar{\delta}}} \,\middle|\, \mathcal{A}_t \right) \\ &= E\left(\frac{\Lambda_{\theta_{\bar{\delta}}}(T)}{\Lambda_{\theta_{\bar{\delta}}}(t)} \, \frac{S_T}{S_T^{\bar{\delta}}} \,\middle|\, \mathcal{A}_t \right) \end{aligned}$$

with **benchmarked normalized numeraire**

$\Longrightarrow$

$$\Lambda_{\theta_{\bar{\delta}}}(t) = \frac{\hat{S}_t^{\bar{\delta}}}{\hat{S}_0^{\bar{\delta}}}$$

$$\begin{aligned} d\Lambda_{\theta_{\bar{\delta}}}(t) &= d\left(\frac{\hat{S}_t^{\bar{\delta}}}{\hat{S}_0^{\bar{\delta}}}\right) \\ &= \Lambda_{\theta_{\bar{\delta}}}(t)\left(\pi^1_{\bar{\delta}}(t)\,\sigma_t - \theta_t\right) dW_t \end{aligned}$$

$(\underline{\mathcal{A}}, P)$-local martingale

$\Longrightarrow$ **Radon-Nikodym derivative** for candidate measure $P_{\theta_{\bar{\delta}}}$

$$\frac{dP_{\theta_{\bar{\delta}}}}{dP} = \Lambda_{\theta_{\bar{\delta}}}(T)$$

- **drifted Wiener process**

$$dW_{\theta_{\bar{\delta}}}(t) = dW_t + \theta_{\bar{\delta}}(t)\, dt,$$

where

$$\theta_{\bar{\delta}}(t) = \theta_t - \pi^{1}_{\bar{\delta}}(t)\, \sigma_t$$

- **If** $\Lambda_{\theta_{\bar{\delta}}}$ is true **$(\underline{\mathcal{A}}, P)$-martingale**

$\Longrightarrow$ $P_{\theta_{\bar{\delta}}}$ **equivalent to** P and probability measure

allows to apply **Girsanov transformation**

$\Longrightarrow$ $W_{\theta_{\bar{\delta}}}$ **standard Wiener process under** $P_{\theta_{\bar{\delta}}}$

$\Longrightarrow$ **numeraire pair** $(S^{\bar{\delta}}, P_{\theta_{\bar{\delta}}})$

Some Examples

- **GOP** S^{δ_*} as numeraire $\quad\Longrightarrow\quad \theta_{\delta_*}(t) = 0$

- **savings account** S^0 as numeraire

  $$\pi^1_{\bar{\delta}}(t) = 0 \quad\Longrightarrow\quad \theta_{\bar{\delta}}(t) = \theta_t$$

- **underlying security** S as numeraire

  $$\pi^1_{\bar{\delta}}(t) = 1 \quad\Longrightarrow\quad \theta_{\bar{\delta}}(t) = \theta_t - \sigma_t$$

- **unfair portfolio** $S_t^{\bar{\delta}} = S_t^{\delta_*} Z_t$ as numeraire

  with Z an $(\underline{\mathcal{A}}, P)$-strict supermartingale, independent of S^{δ_*}

$\Longrightarrow$ $\hat{S}^{\bar{\delta}}$ is **not** an $(\underline{\mathcal{A}}, P)$-martingale

Girsanov Theorem **can not** be applied !

$\Longrightarrow$ **If savings account is not fair,**

then the change of numeraire approach is not applicable !

Pricing Formula for General Numeraire $S^{\underline{\delta}}$

$$V(0, S_0) = E\left(\frac{S_0^{\delta_*}}{S_T^{\delta_*}} H(S_T) \,\middle|\, \mathcal{A}_0 \right)$$

$$= E\left(\Lambda_{\theta_{\bar{\delta}}}(T) \frac{H(S_T)}{S_T^{\bar{\delta}}} \,\middle|\, \mathcal{A}_0 \right)$$

the quantity

$$\frac{\Lambda_{\theta_{\bar{\delta}}}(T)}{S_T^{\bar{\delta}}} = \frac{S_0^{\delta_*}}{S_T^{\delta_*}}$$

remains **numeraire invariant**

- if $\Lambda_{\theta_{\bar{\delta}}}$ forms an $(\underline{\mathcal{A}}, P)$-martingale

  Bayes's Theorem $\Longrightarrow$

$$E\left(\Lambda_{\theta_{\bar{\delta}}}(T)\,\frac{H(S_T)}{S_T^{\bar{\delta}}}\,\bigg|\,\mathcal{A}_0\right) = E_{\theta_{\bar{\delta}}}\left(\frac{H(S_T)}{S_T^{\bar{\delta}}}\,\bigg|\,\mathcal{A}_0\right)$$

$\Longrightarrow$

pricing formula with **general numeraire**

$$V(0, S_0) = E_{\theta_{\bar{\delta}}}\left(\frac{H(S_T)}{S_T^{\bar{\delta}}}\,\bigg|\,\mathcal{A}_0\right)$$

- represents a **transformation of variables** in integration
- may provide **computational advantages**

Utility Indifference Pricing

discounted payoff $\bar{H} = \frac{H}{S_T^0}$ (not perfectly hedgable)

$$v^{\tilde{\delta}}_{\varepsilon, V} = E\left(U \left((S_0 - \varepsilon\, V)\, \frac{\bar{S}^{\tilde{\delta}}_T}{S_0} + \varepsilon\, \bar{H} \right) \,\Big|\, \mathcal{A}_0 \right)$$

- assume $\hat{S}^0$ is **scalar diffusion**

Definition *Utility indifference price V s.t.*

$$\lim_{\varepsilon \to 0} \frac{v^{\tilde{\delta}}_{\varepsilon, V} - v^{\tilde{\delta}}_{0, V}}{\varepsilon} = 0$$

e.g. Davis (1997)

- Taylor expansion

$$v^{\tilde{\delta}}_{\varepsilon,V} \approx E\left(U\left(\bar{S}^{\tilde{\delta}}_T\right) + \varepsilon\, U'\left(\bar{S}^{\tilde{\delta}}_T\right)\left(\bar{H} - \frac{V\,\bar{S}^{\tilde{\delta}}_T}{S_0}\right) \,\middle|\, \mathcal{A}_0\right)$$

$$= v^{\tilde{\delta}}_{0,V} + \varepsilon\, E\left(U'\left(\bar{S}^{\tilde{\delta}}_T\right)\bar{H} \,\middle|\, \mathcal{A}_0\right) - V\,\varepsilon\, E\left(U'\left(\bar{S}^{\tilde{\delta}}_T\right)\frac{\bar{S}^{\tilde{\delta}}_T}{S_0} \,\middle|\, \mathcal{A}_0\right)$$

$\Longrightarrow$

- **utility indifference price**

$$V = \frac{E\left(U'\left(\bar{S}_T^{\tilde{\delta}}\right)\bar{H}\,\middle|\,\mathcal{A}_0\right)}{E\left(U'\left(\bar{S}_T^{\tilde{\delta}}\right)\frac{\bar{S}_T^{\tilde{\delta}}}{S_0}\,\middle|\,\mathcal{A}_0\right)} = \frac{E\left(U'\left(U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)\right)\bar{H}\,\middle|\,\mathcal{A}_0\right)}{E\left(U'\left(U'^{-1}\left(\frac{\lambda}{\bar{S}_T^{\delta_*}}\right)\right)\frac{\bar{S}_T^{\tilde{\delta}}}{S_0}\,\middle|\,\mathcal{A}_0\right)}$$

$$= \frac{E\left(\frac{H}{S_T^{\delta_*}}\,\middle|\,\mathcal{A}_0\right)}{\frac{1}{S_0}\,\frac{S_0}{S_0^{\delta_*}}}$$

independent of U and independent of particular model $\Longrightarrow$

- **real world pricing formula**

$$V = S_0^{\delta_*}\,E\left(\frac{H}{S_T^{\delta_*}}\,\middle|\,\mathcal{A}_0\right)$$

Pricing via Hedging

- **underlying security**

$$S = \{S_t,\, t \in [0,T]\}$$

$$dS_t = a_t\, S_t\, dt + \sigma_t\, S_t\, dW_t$$

$t \in [0,T]$, $S_0 > 0$

appropriate stochastic appreciation rate a_t

and volatility σ_t

Wiener process $W = \{W_t,\, t \in [0,T]\}$

- **savings account**

$$dS_t^0 = r_t\, S_t^0\, dt$$

$t \in [0,T]$, $S_0^0 = 1$

appropriate stochastic short rate

- **replicate payoff**

$$f(S_T) \geq 0$$

assuming

$$E(f(S_T)) < \infty$$

- **hedge portfolio**

$$V(t, S_t) = \delta_t^0 \, S_t^0 + \delta_t^1 \, S_t$$

δ_t^0 units of savings account

δ_t^1 units of underlying security

- **hedging strategy**

$$\delta = \{\delta_t = (\delta_t^0, \delta_t^1)^\top, \; t \in [0, T]\}$$

$\delta^0 = \{\delta_t^0, \; t \in [0, T]\}$

$\delta^1 = \{\delta_t^1, \; t \in [0, T]\}$

predictable processes

- **replicating portfolio**

$$V(T, S_T) = f(S_T)$$

- **SDE for hedge portfolio**

$$\begin{aligned} dV(t, S_t) &= \delta_t^0 \, dS_t^0 + \delta_t^1 \, dS_t \\ &\quad + S_t^0 \, d\delta_t^0 + S_t \, d\delta_t^1 + d\left[\delta^1, S\right]_t \end{aligned}$$

- **self-financing**

  portfolio changes are caused by gains from trade

$$dV(t, S_t) = \delta_t^0 \, dS_t^0 + \delta_t^1 \, dS_t$$

$\Longrightarrow$

self-financing condition:

$$S_t^0 \, d\delta_t^0 + S_t \, d\delta_t^1 + d\left[\delta^1, S\right]_t = 0$$

- **discounted underlying security**

$$\bar{S}_t = \frac{S_t}{S_t^0}$$

$t \in [0, T]$

Itô formula $\Longrightarrow$

$$d\bar{S}_t = (a - r)\,\bar{S}_t\,dt + \sigma_t\,\bar{S}_t\,dW_t$$

$t \in [0, T]$, $\bar{S}_0 = S_0$

- **discounted value function**

$\bar{V} : [0, T] \times [0, \infty) \to [0, \infty)$

$$\bar{V}(t, \bar{S}_t) = \frac{V(t, S_t)}{S_t^0}$$

- **profit and loss** (P&L)

  $C_t =$ price – gains from trade – initial price

- **discounted P&L**

$$\bar{C}_t = \frac{C_t}{S_t^0}$$
$$= \bar{V}(t, \bar{S}_t) - I_{\delta^1, \bar{S}}(t) - \bar{V}(0, \bar{S}_0)$$

- **discounted gains from trade**

$$
\begin{aligned}
I_{\delta^1,\bar{S}}(t) &= \int_0^t \delta_u^1 \, d\bar{S}_u \\
&= \int_0^t \delta_u^1 \, (a - r) \, \bar{S}_u \, du \\
&\quad + \int_0^t \delta_u^1 \, \sigma \, \bar{S}_u \, dW_u
\end{aligned}
$$

$$
I_{\delta^0,\bar{S}^0}(t) = \int_0^t \delta_u^0 \, d\bar{S}_u^0 = 0
$$

- **initial payment**

$$V(0, S_0) = \bar{V}(0, \bar{S}_0),$$

$\Longrightarrow$

$$\bar{C}_0 = 0$$

- **identify hedging strategy δ**

  for which discounted P&L zero

$$\bar{C}_t = 0$$

  for all $t \in [0, T]$

$\Longrightarrow$ perfect hedge

$V(t, S_t)$ - **replicating portfolio**

- **increments of discounted P&L**

$$0 = \bar{C}_t - \bar{C}_s \;=\; \bar{V}(t, \bar{S}_t) - \bar{V}(s, \bar{S}_s) - \int_s^t \delta_u^1 \, d\bar{S}_u$$

$$= \int_s^t \left[\frac{\partial \bar{V}(u, \bar{S}_u)}{\partial t} + \frac{1}{2} \sigma_u^2 \, \bar{S}_u^2 \, \frac{\partial^2 \bar{V}(u, \bar{S}_u)}{\partial \bar{S}^2} \right.$$

$$\left. + (a_u - r_u) \, \bar{S}_u \left(\frac{\partial \bar{V}(u, \bar{S}_u)}{\partial \bar{S}} - \delta_u^1 \right) \right] du$$

$$+ \int_s^t \sigma_u \, \bar{S}_u \left(\frac{\partial \bar{V}(u, \bar{S}_u)}{\partial \bar{S}} - \delta_u^1 \right) dW_u$$

$\Longrightarrow$

- **hedge ratio**

$$\delta_u^1 = \frac{\partial \bar{V}(u, \bar{S}_u)}{\partial \bar{S}}$$

- **discounted PDE**

$$\frac{\partial \bar{V}(u, \bar{S})}{\partial t} + \frac{1}{2}\,\sigma_u^2\,\bar{S}^2\,\frac{\partial^2 \bar{V}(u, \bar{S})}{\partial \bar{S}^2} = 0$$

$u \in [0, T)$, $\bar{S} \in (0, \infty)$

with **terminal condition**

$$\bar{V}(T, \bar{S}) = \frac{f(\bar{S}\, S_T^0)}{S_T^0} = \frac{f(S)}{S_T^0}$$

$\Longrightarrow$

- **PDE**

$$\frac{\partial V(u,S)}{\partial t} + r_u\, S\, \frac{\partial V(u,S)}{\partial S}$$

$$+ \frac{1}{2}\,\sigma_u^2\, S^2\, \frac{\partial^2 V(u,S)}{\partial S^2} - r_u\, V(u,S) \quad = \quad 0$$

$u \in [0,T)$, $S \in (0,\infty)$

with **terminal condition**

$$V(T,S) = f(S)$$

- special boundary condition not specified
  PDE may **not** provide minimal possible price

$$\Longrightarrow$$

- **units in savings account**

$$\delta_t^0 = \frac{V(t, S_t) - \delta_t^1 \, S_t}{S_t^0} = \bar{V}(t, \bar{S}_t) - \delta_t^1 \, \bar{S}_t$$

- **option price**

$$\begin{aligned} V(t, S_t) &= \bar{V}(t, \bar{S}_t) \, S_t^0 \\ &= \delta_t^0 \, S_t^0 + \delta_t^1 \, S_t \end{aligned}$$

$t \in [0, T]$

$\Longrightarrow$ **strategy**

$$\delta_t = (\delta_t^0, \delta_t^1)^\top$$

such that

$$0 = d\bar{C}_t = d\bar{V}(t, \bar{S}_t) - \delta_t^1 \, d\bar{S}_t$$

$\Longrightarrow$

$$d\bar{V}(t, \bar{S}_t) = \delta_t^0 \, d\bar{S}_t^0 + \delta_t^1 \, d\bar{S}_t$$

$\Longrightarrow$ **self-financing**

P&L vanishes

$$C_t = \bar{C}_t \, S_t^0 = 0$$

for all $t \in [0, T]$

If $V(t, S_t)$ is not fair, then there may exist less expensive hedge portfolio!

Numeraire Invariance

$\bar{V}$ self-financing

$$\begin{aligned} dV(t, S_t) &= d(\bar{V}(t, \bar{S}_t)\, S_t^0) \\ &= S_t^0\, d\bar{V}(t, \bar{S}_t) + \bar{V}(t, \bar{S}_t)\, dS_t^0 + d[S^0, \bar{V}(\cdot, \bar{S}.)]_t \\ &= S_t^0\, \delta_t^1\, d\bar{S}_t + (\delta_t^0 + \delta_t^1\, \bar{S}_t)\, dS_t^0 + \delta_t^1\, d[S^0, \bar{S}]_t \\ &= \delta_t^0\, dS_t^0 + \delta_t^1\, \left(S_t^0\, d\bar{S}_t + \bar{S}_t\, dS_t^0 + d[S^0, \bar{S}]_t\right) \\ &= \delta_t^0\, dS_t^0 + \delta_t^1\, d(S_t^0\, \bar{S}_t) \\ &= \delta_t^0\, dS_t^0 + \delta_t^1\, dS_t \end{aligned}$$

$\Longrightarrow$ V also **self-financing**

The Black-Scholes Formula

- solution $c_{T,K}(t, S)$ of PDE

  with call payoff function

$$f(S) = (S - K)^+$$

- $a_t = a,\ \sigma_t = \sigma,\ r_t = r$

  Black & Scholes (1973)
  Merton (1973a)

Black-Scholes Formula

$$c_{T,K}(t, S_t) = S_t\, N(d_1(t)) - K\, \frac{S_t^0}{S_T^0}\, N(d_2(t))$$

with

$$d_1(t) = \frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\,\sigma^2\right)\,(T - t)}{\sigma\,\sqrt{T - t}}$$

$$d_2(t) = d_1(t) - \sigma\,\sqrt{T - t}$$

$t \in [0, T)$

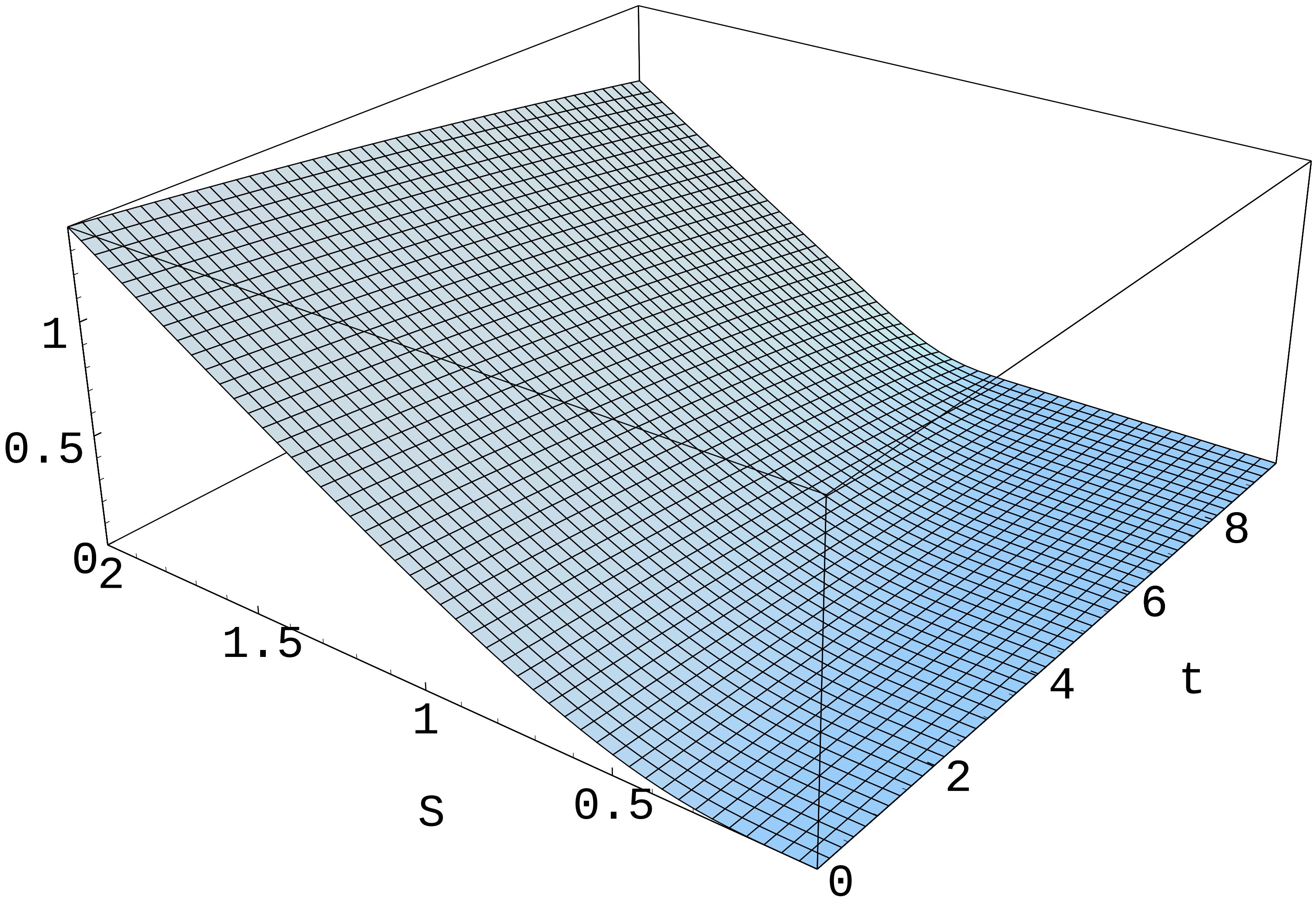
$$\hat{c}_{T,K}(t, S_t) = \frac{c_{T,K}(t, S_t)}{S_t^{\delta *}} \qquad (P, \mathcal{A}) - \text{martingale}$$

minimal possible price

- $N(\cdot)$ standard **Gaussian distribution function**

  with density

$$N'(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$$



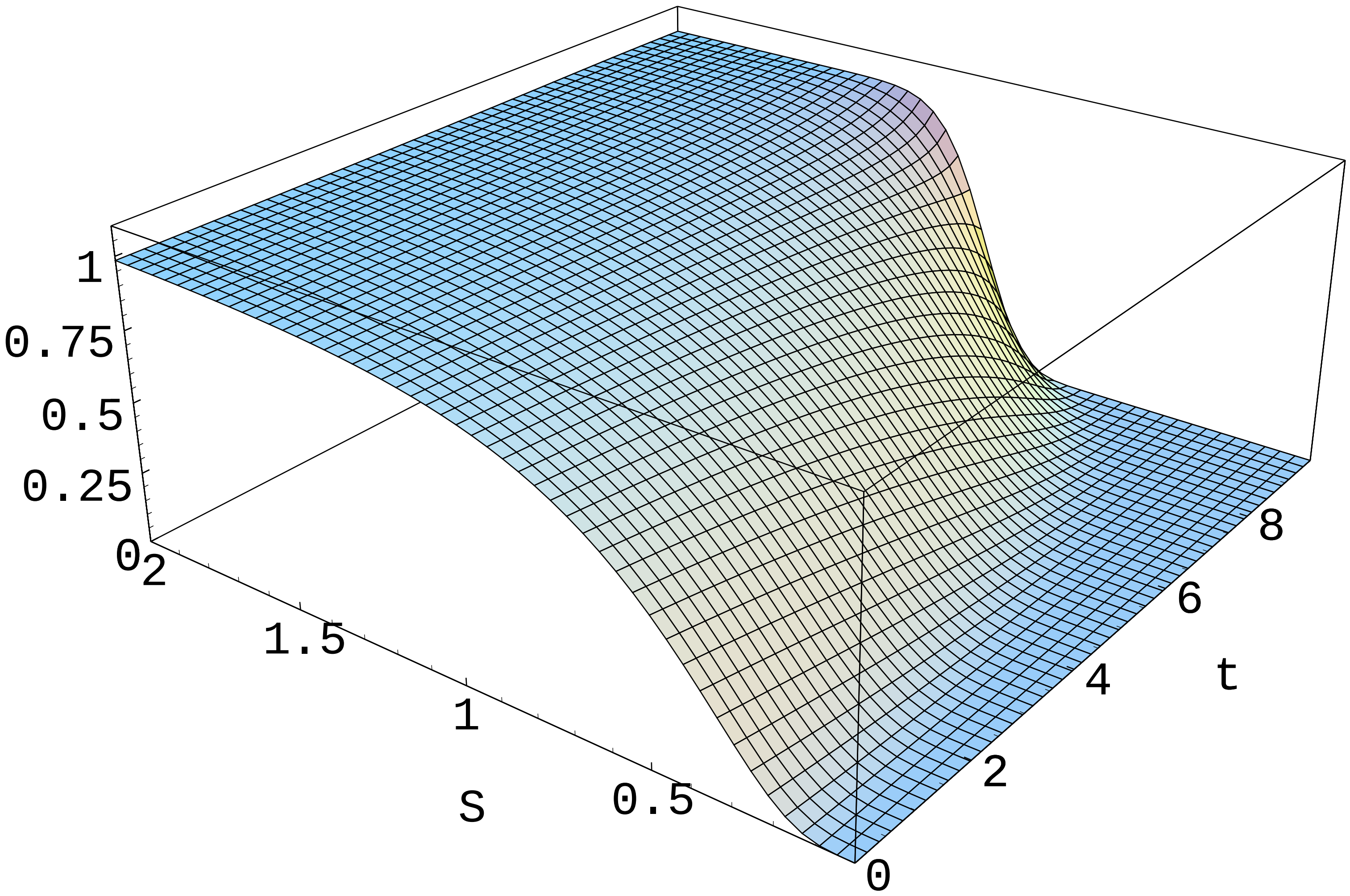
Black-Scholes European call option price.

Delta

- sensitivity with respect to the underlying S_t

$$\Delta = \frac{\partial V(t, S_t)}{\partial S} = \delta_t^1$$

$$\Delta = N(d_1(t))$$



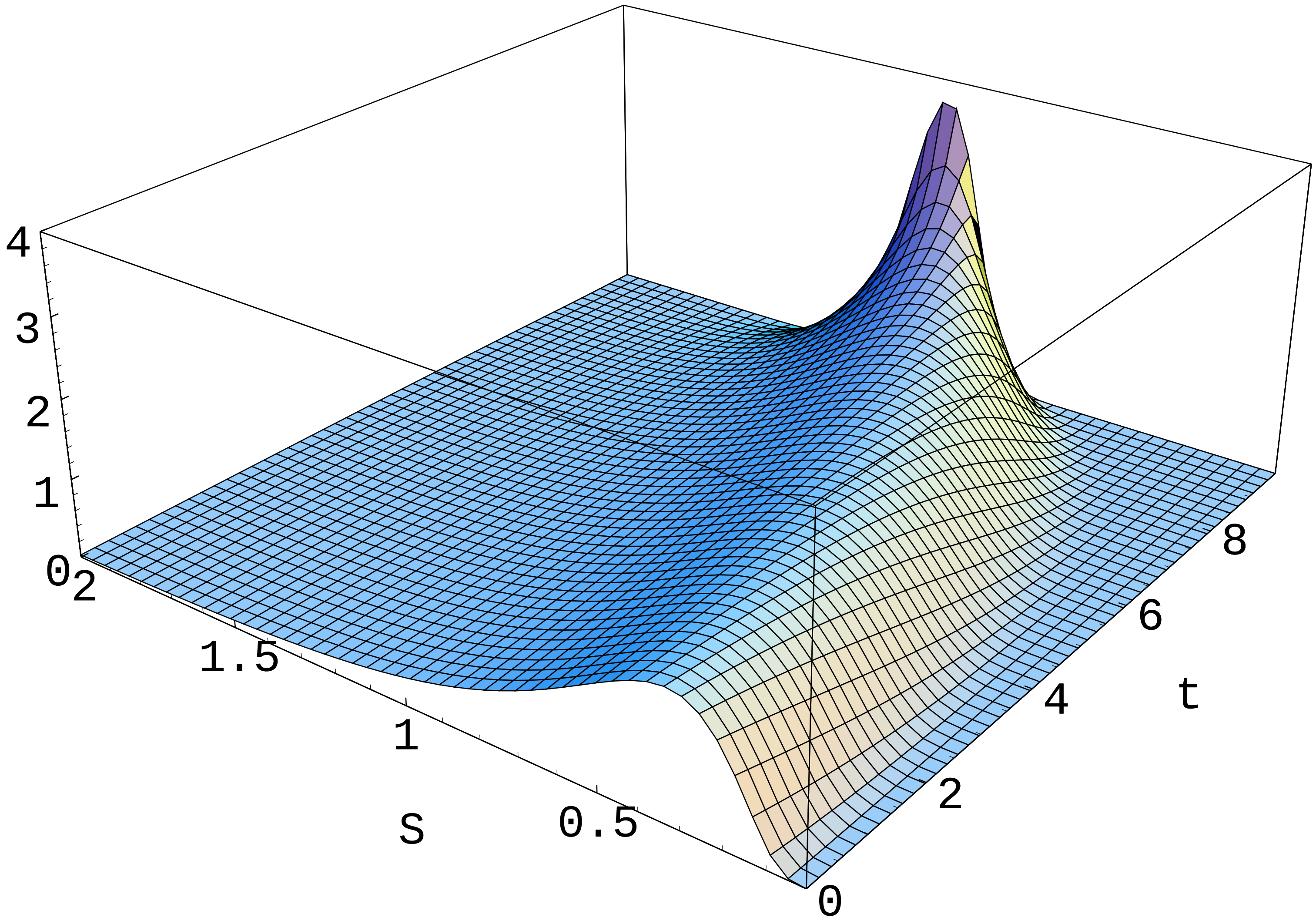
Delta as a function of t and S_t.

Gamma

- sensitivity of the hedge ratio with respect to underlying

$$\Gamma = \frac{\partial \Delta}{\partial S} = \frac{\partial^2 V(t, S_t)}{\partial S^2}$$

$$\Gamma = N'(d_1(t)) \frac{1}{S_t \, \sigma \, \sqrt{T - t}}$$



Gamma as a function of t and S_t.

Simulation of Delta Hedging

- simulate scenario along an equidistant time discretization

- **price of call option**

$$c_{T,K}(t, S_t) = \delta_t^1\, S_t + \delta_t^0\, S_t^0$$

- **hedging strategy**

$$\delta = \left\{\delta_t = (\delta_t^0, \delta_t^1)^\top,\ t \in [0, T]\right\}$$

- **hedge ratio, units in the underlying**

$$\delta_t^1 = N(d_1(t))$$

$$= N\left(\frac{\ln\left(\frac{S_t}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma\sqrt{T - t}}\right)$$

- **units in domestic savings account**

$$
\begin{aligned}
\delta_t^0 &= -\frac{K}{S_T^0}\, N(d_2(t)) \\
&= -\frac{K}{S_T^0}\, N\left(\frac{\ln\left(\frac{S_t}{K}\right) + \left(r - \frac{1}{2}\,\sigma^2\right)(T-t)}{\sigma\,\sqrt{T-t}}\right)
\end{aligned}
$$

- **discounted P&L**

$$\bar{C}_t = \bar{V}(t, \bar{S}_t) - \int_0^t \delta_s^1 \, d\bar{S}_s - \bar{V}(0, \bar{S}_0)$$

$$\bar{V}(t, \bar{S}_t) = \bar{V}(0, \bar{S}_0) + \int_0^t \delta_s^1 \, d\bar{S}_s$$

$\Longrightarrow$ theoretically under continuous hedging

$$\bar{C}_t = 0$$

for all $t \in [0, T]$

In-the-Money Scenario

- security price ends up in-the-money $S_T > K$

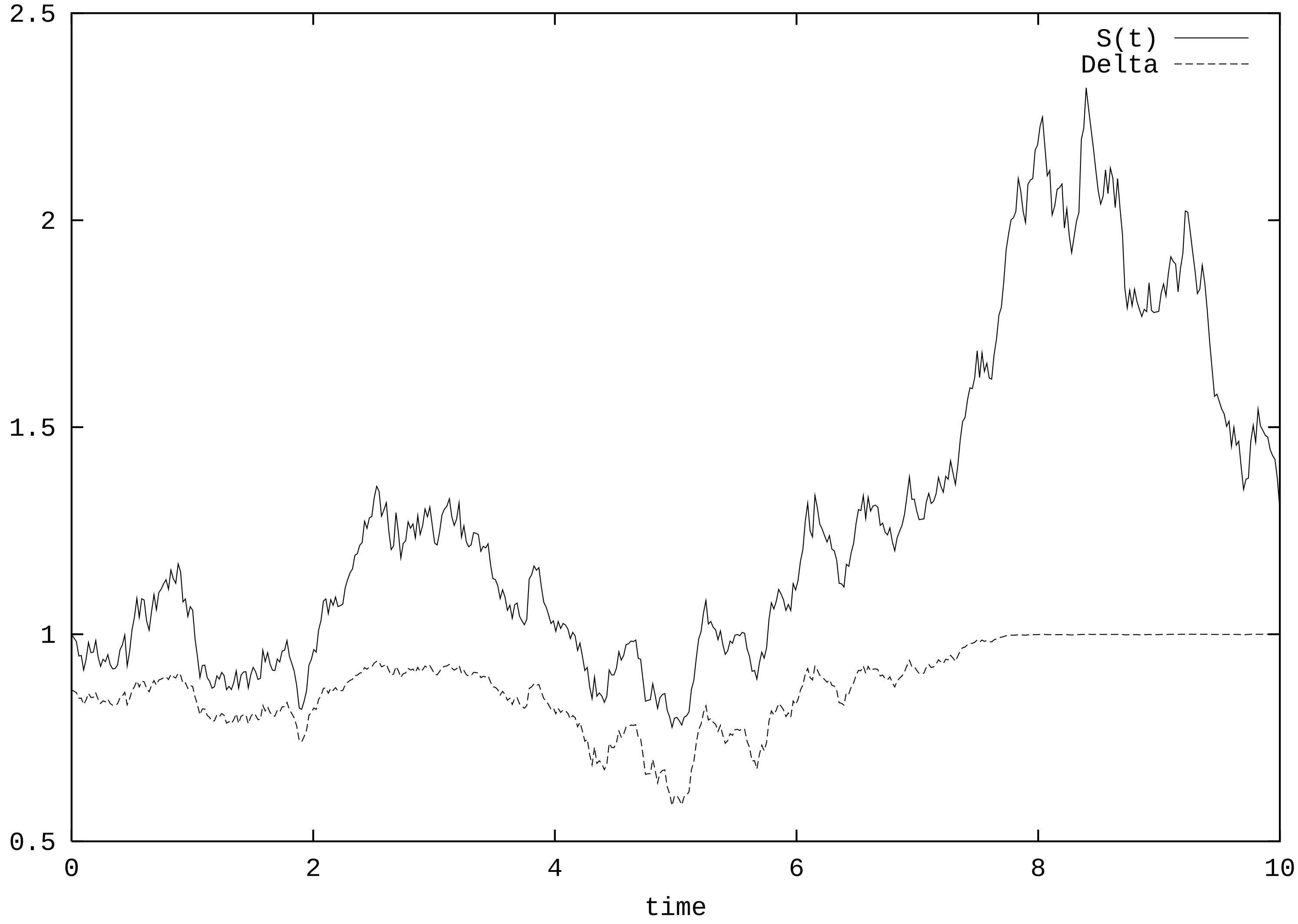
  $\Longrightarrow$ hedge ratio converges to $\delta_T^1 = 1$

- self-financing strategy $\delta_t = (\delta_t^0, \delta_t^1)^\top$

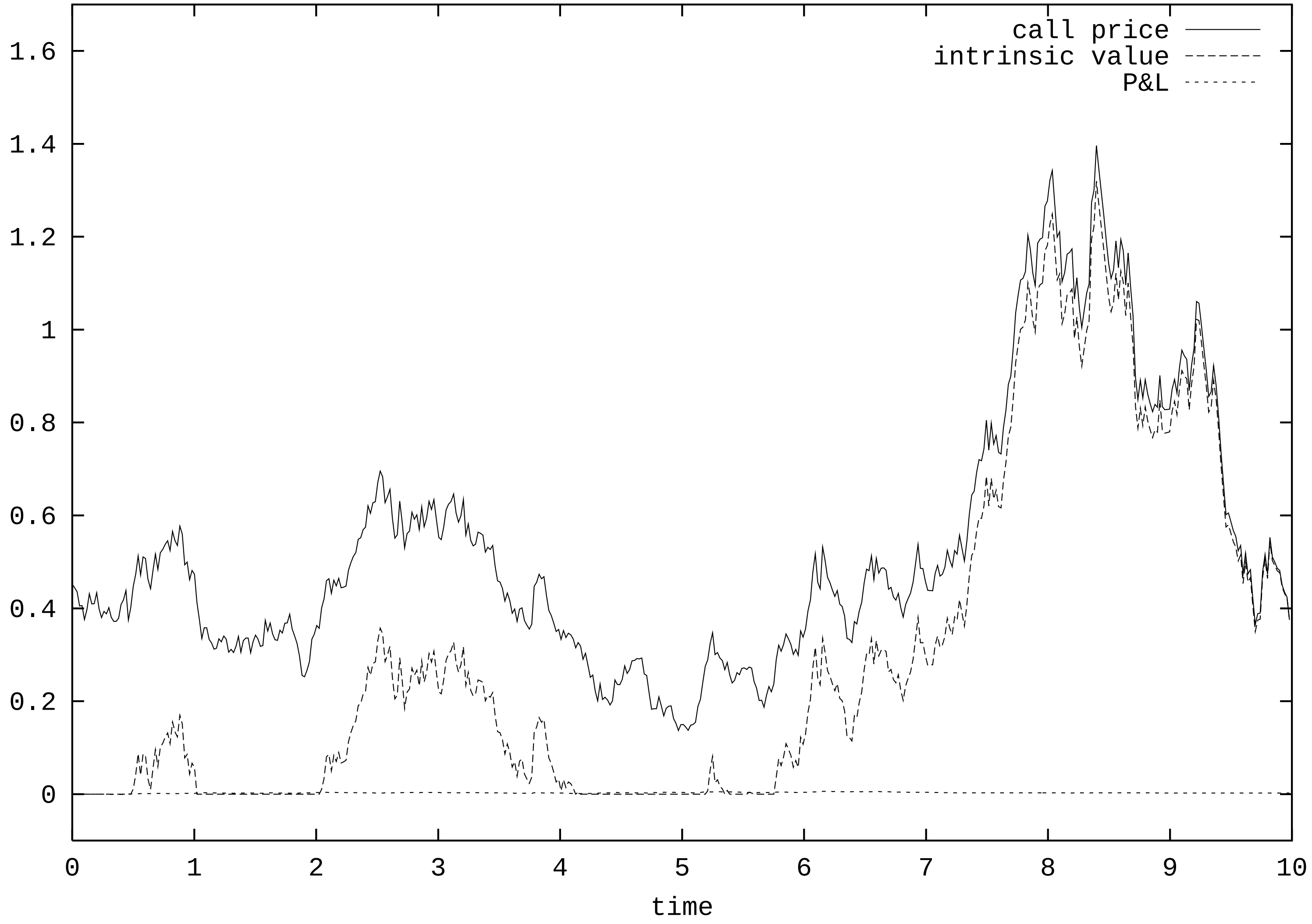
- **intrinsic value**

$$f(S_t) = (S_t - K)^+$$

- discounted P&L $\bar{C}_t$ remains almost perfectly zero

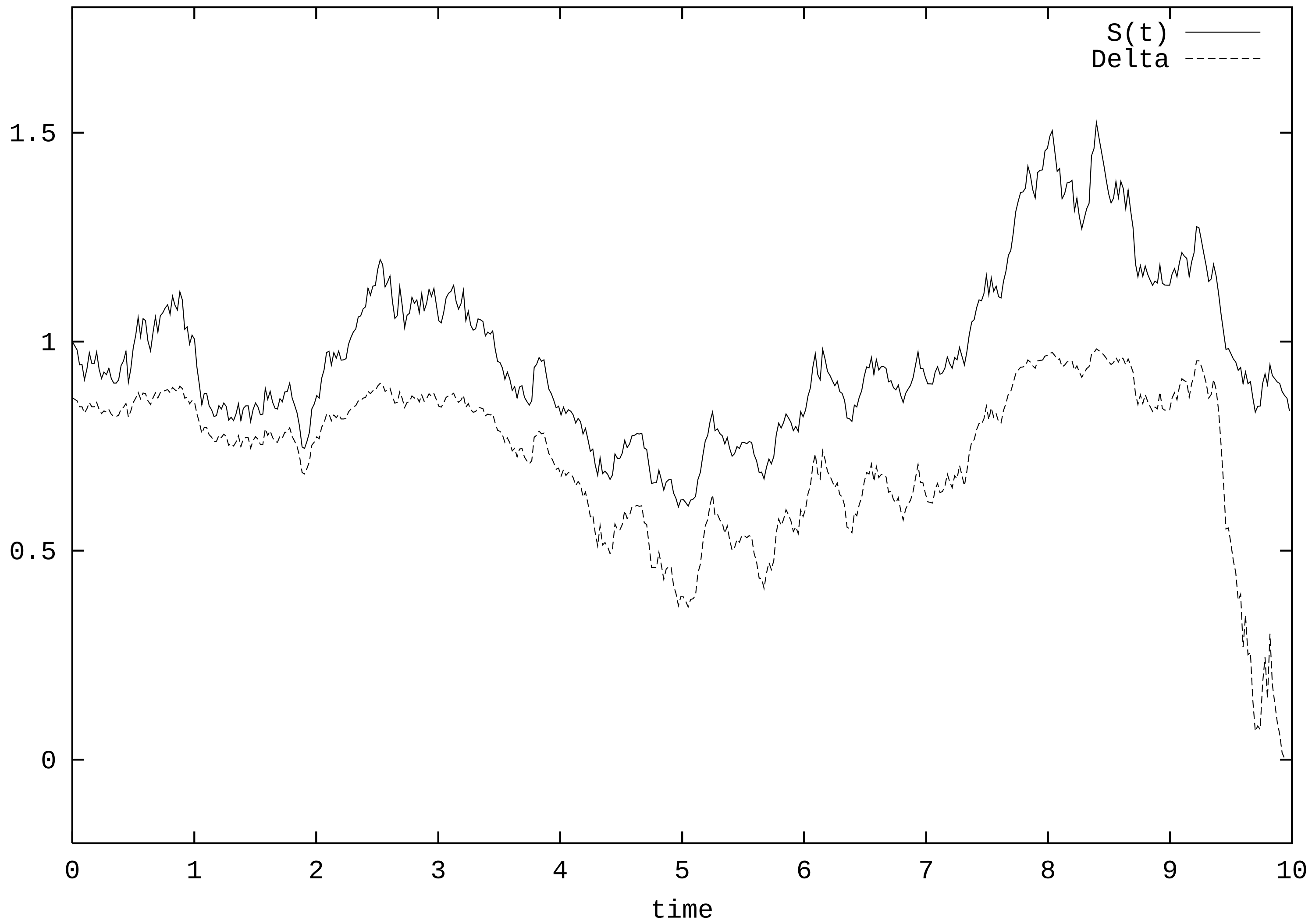


Underlying security and hedge ratio for in-the-money call.

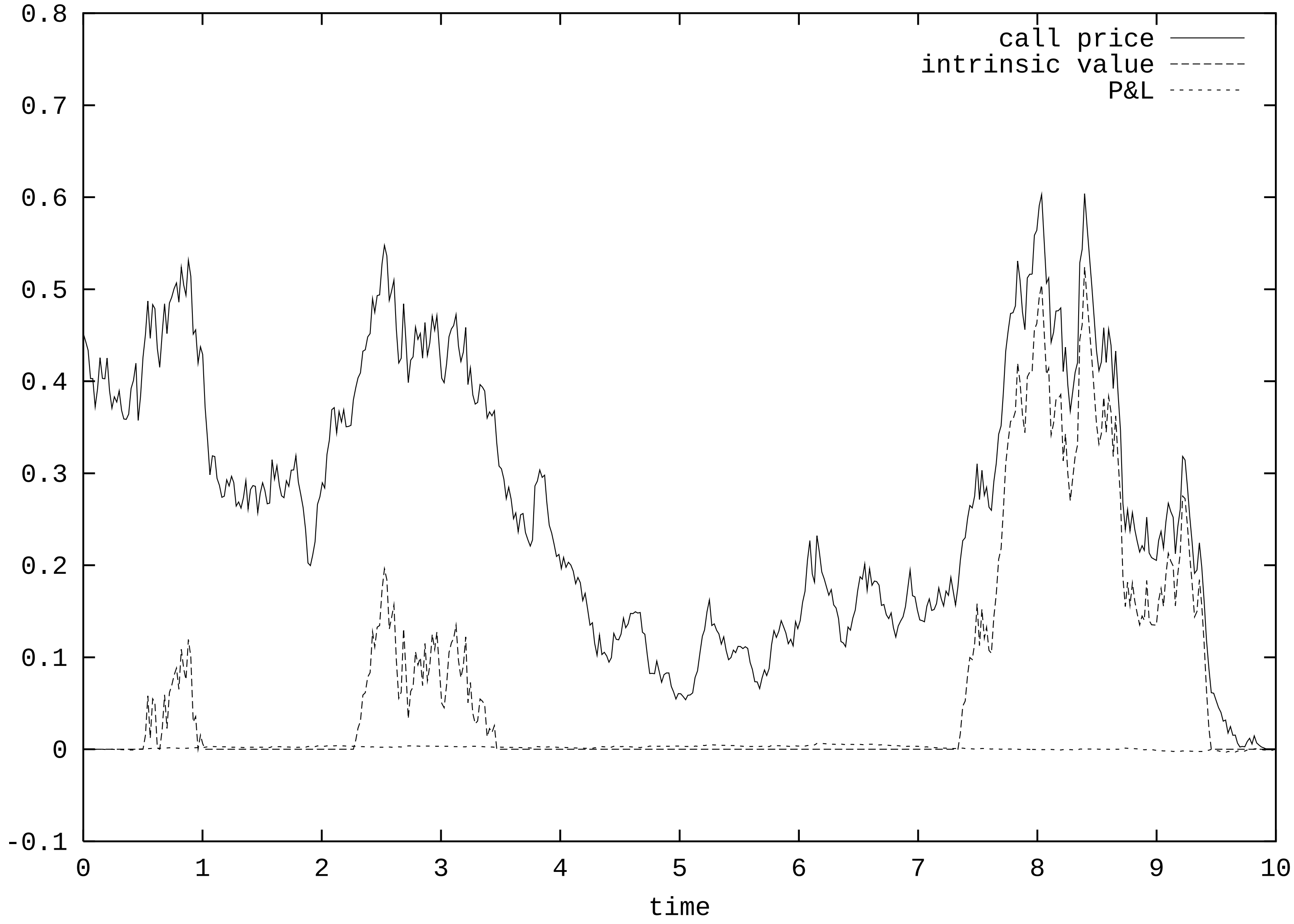


Price, intrinsic value and P&L for in-the-money call.

Out-of-the Money Scenario

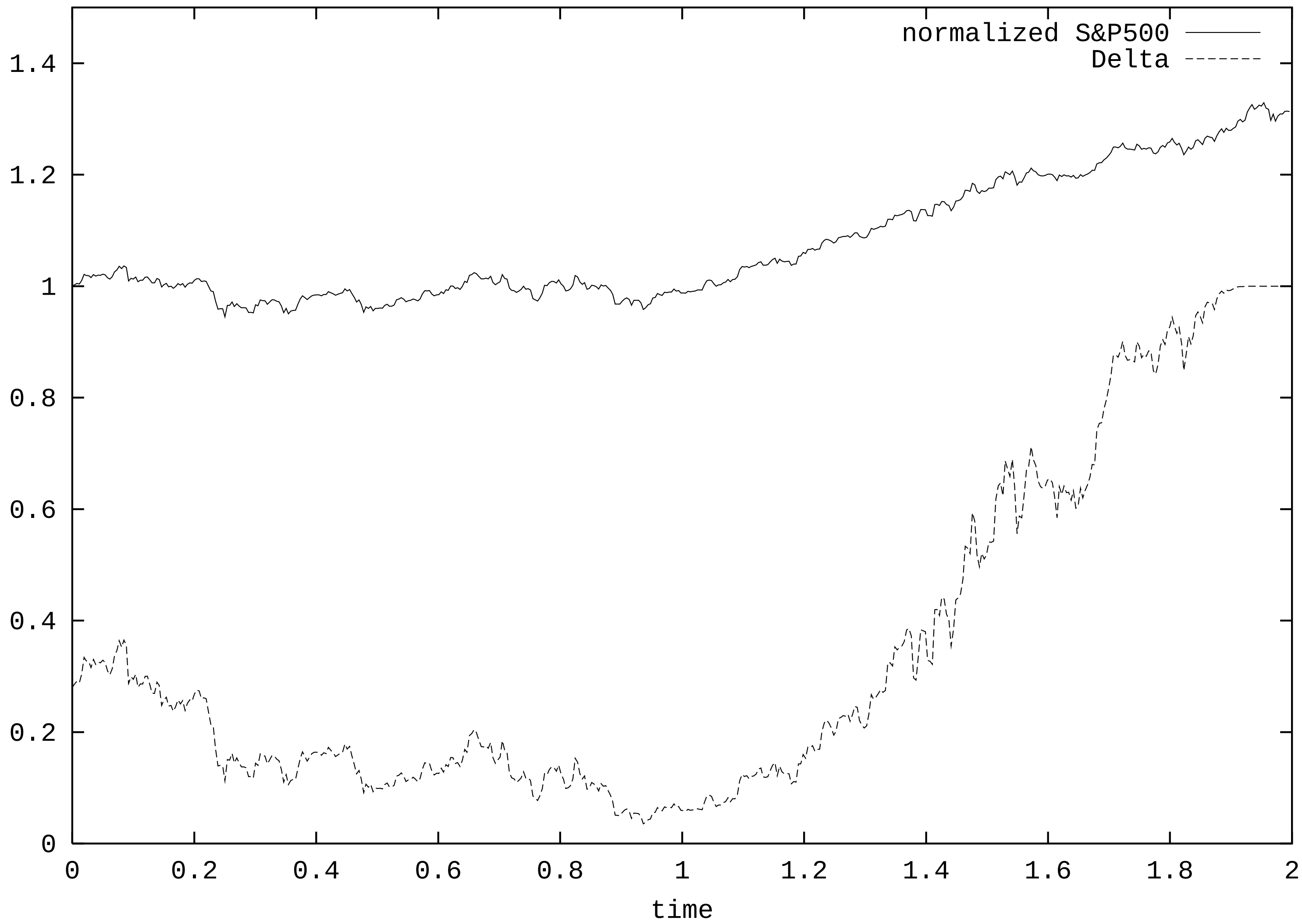


Underlying security and hedge ratio for out-of-the-money call.

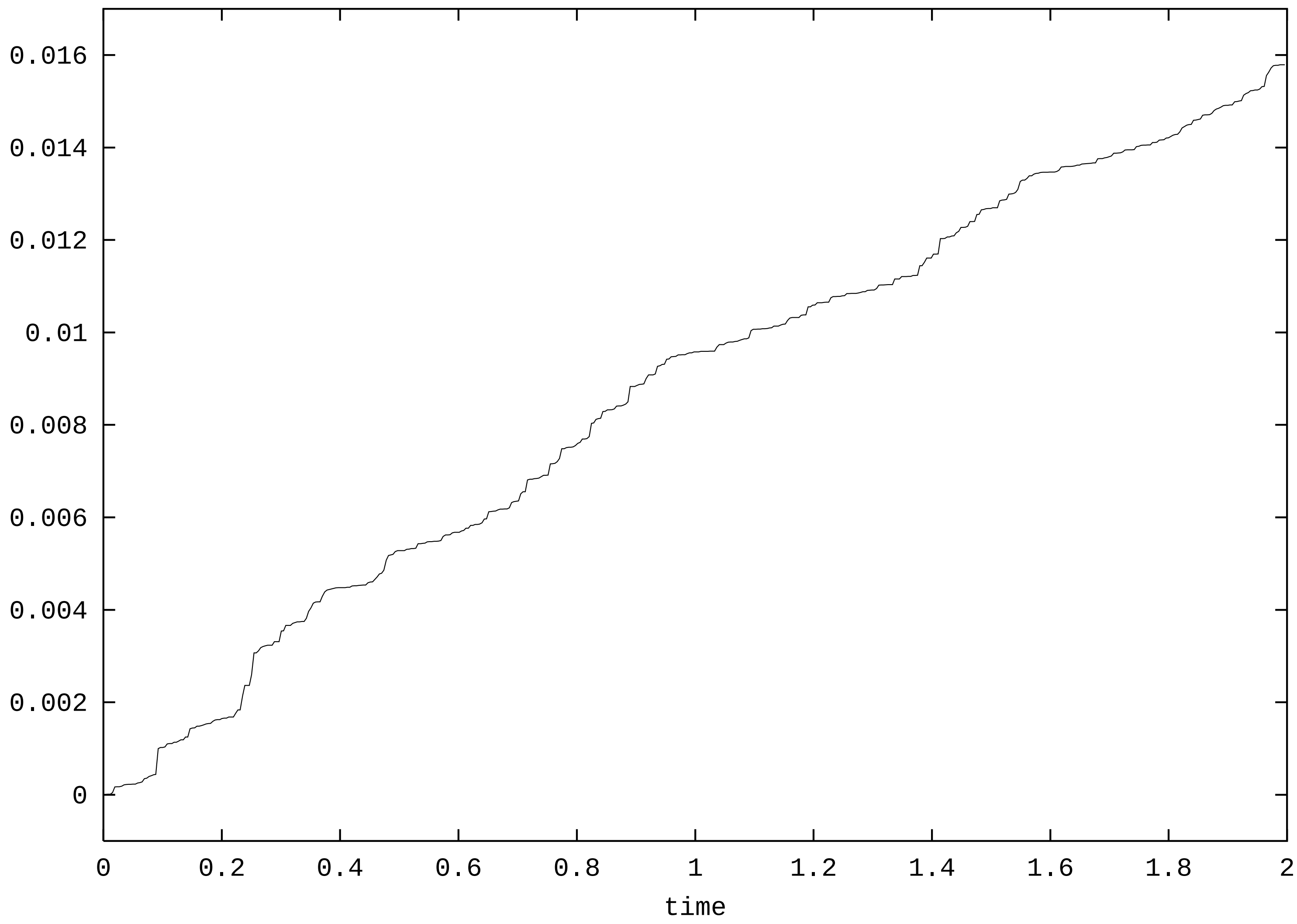


Price, intrinsic value and P&L for out-of-the-money call.

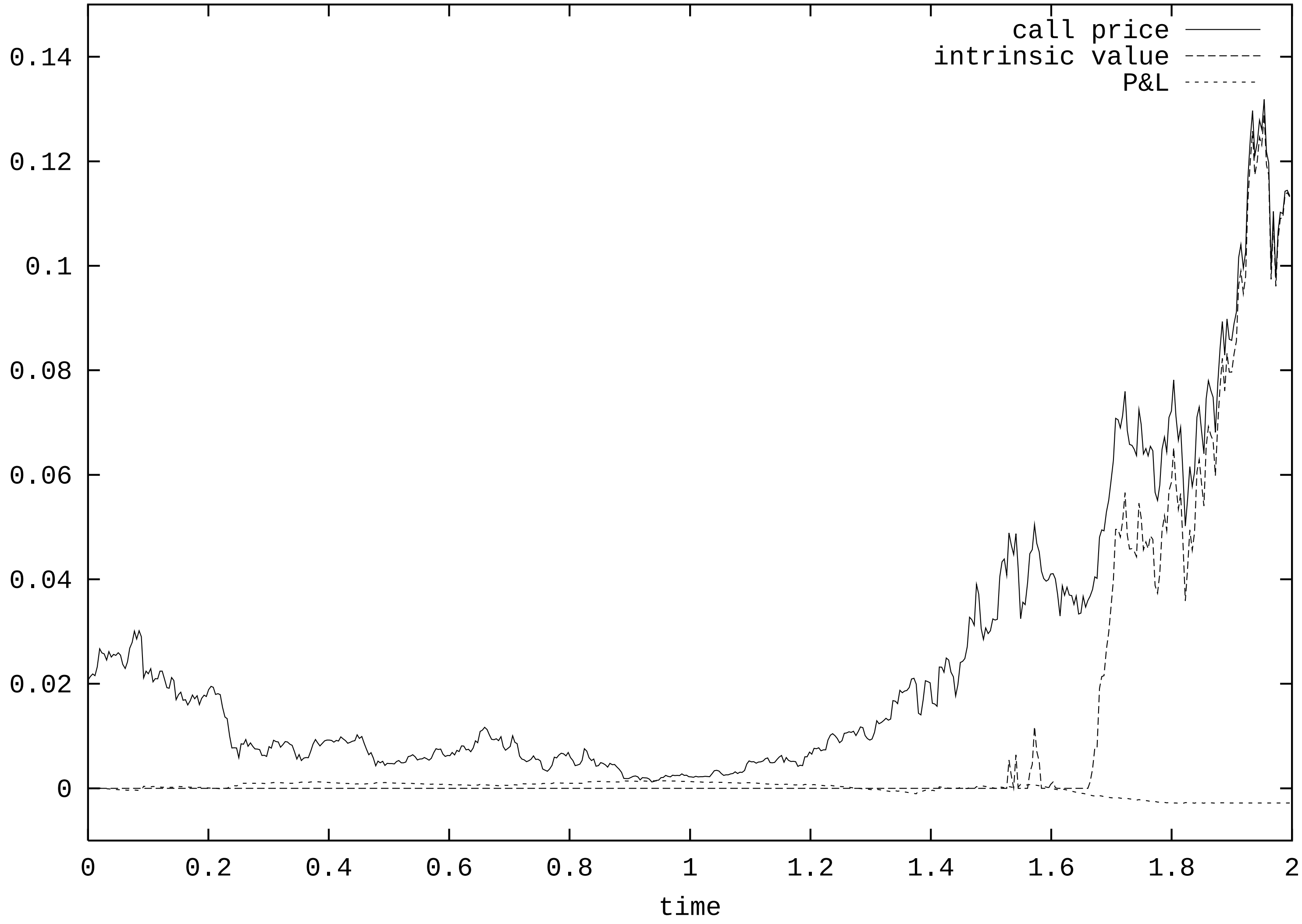
Hedging a European Call Option on the S&P500



Normalized S&P500 and hedge ratio for $K = 1.2$.



Quadratic variation of log-S&P500.



Call price on S&P500, intrinsic value and P&L.

Benchmarked Risk Minimization

Risk Minimization

- Föllmer and Sondermann (1986)
- Föllmer and Schweizer (1989)
- Schweizer (1991, 2001)
- Biagini et al. (1999)
- Biagini and Cretarola (2009)
- Du & Pl. (2012)

Financial Market

- d primary security accounts
- $S_t^{i,j}$ - jth primary security account in ith security denomination

- locally riskless savings account

$$S_t^{i,i} = S_0^{i,i} \exp\left\{\int_0^t r_s^i ds\right\}$$

- portfolio

$$S_t^{i,\delta} = \sum_{j=1}^{d} \delta_t^j S_t^{i,j}$$

- self-financing portfolio

$$S_t^{i,\delta} = S_0^{i,\delta} + \sum_{j=1}^{d} \int_0^t \delta_s^j dS_s^{i,j}$$

Numéraire Portfolio

- $S_t^{i,\delta*}$ numéraire portfolio

$$\hat{S}_t^{\delta} = \frac{S_t^{i,\delta}}{S_t^{i,\delta*}} \geq E_t(\hat{S}_s^{\delta})$$

$$0 \leq t \leq s < \infty$$

- supermartingale property

- benchmarked $\boldsymbol{j}$th primary security account

$$\hat{S}_t^j = \frac{S_t^{i,j}}{S_t^{i,\delta*}}$$

local martingale; supermartingale

- matrix valued optional covariation process

$$[\hat{\mathbf{S}}] = \{[\hat{\mathbf{S}}]_t = ([\hat{S}^i, \hat{S}^j]_t)_{i,j=1}^d, t \in [0, \infty)\}$$

- benchmarked primary security accounts

$$\hat{\mathbf{S}} = \{\hat{\mathbf{S}}_t = (\hat{S}_t^1, \ldots, \hat{S}_t^d)^\top, t \in [0, \infty)\}$$

Definition *A* dynamic trading strategy v, *initiated at time* $t = 0$, $v = \{v_t = (\eta_t, \vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0, \infty)\}$, *where* $\vartheta = \{\vartheta_t = (\vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0, \infty)\}$ *forms benchmarked self-financing part* $\vartheta_t^\top \hat{\mathbf{S}}_t$ *of benchmarked price process*

$$\hat{V}_t^v = \vartheta_t^\top \hat{\mathbf{S}}_t + \eta_t$$

ϑ *is predictable*

$$\int_0^t \vartheta_{\mathbf{u}}^\top \mathbf{d}[\hat{\mathbf{S}}]_{\mathbf{u}} \vartheta_{\mathbf{u}} < \infty$$

$\boldsymbol{\eta} = \{\boldsymbol{\eta}_t, t \in [0,\infty)\}$ *adapted with* $\boldsymbol{\eta}_0 = \mathbf{0}$ *monitors the benchmarked non-self-financing part of*

$$\hat{V}_t^v = \hat{V}_0^v + \int_0^t \boldsymbol{\vartheta}_{\mathbf{s}}^\top \mathbf{d}\hat{\mathbf{S}}_{\mathbf{s}} + \boldsymbol{\eta}_{\mathbf{t}}$$

$\hat{V}_t^v$ *forms a supermartingale.*

In general, capital has to be added or removed to match desired price

$$\hat{V}_t^v = \hat{S}_t^\delta = \sum_{j=1}^{d} \delta_t^j \hat{S}_t^j$$

with

$$\delta_t^j = \vartheta_t^j + \eta_t \delta_{*,t}^j \ .$$

- benchmarked contingent claim $\hat{\boldsymbol{H}}_{\boldsymbol{T}}$
  dynamic trading strategy $\boldsymbol{v}$ *delivers* $\hat{\boldsymbol{H}}_{\boldsymbol{T}}$ if

$$\hat{\boldsymbol{V}}_{\boldsymbol{T}}^{\boldsymbol{v}} = \hat{\boldsymbol{S}}_{\boldsymbol{T}}^{\boldsymbol{\delta}} = \hat{\boldsymbol{H}}_{\boldsymbol{T}}$$

  $\Longrightarrow$ *replicable* if self-financing

Proposition *If for* $\hat{H}_T$ *a self-financing benchmarked portfolio* $\hat{S}^{\delta_{\hat{H}_T}}$ *exists, satisfying*

$$\hat{S}_t^{\delta_{\hat{H}_T}} = E(\hat{H}_T | \mathcal{F}_t)$$

for all $t \in [0, T]$ P*-a.s., then this provides least expensive hedge for* $\hat{H}_T$.

Real World Pricing

- least expensive replication by a self-financing benchmarked portfolio

  $\Longrightarrow$

  real world pricing formula

  $$\hat{S}_t^{\delta_{\hat{H}_T}} = E_t(\hat{H}_T)$$

  minimal possible price process

Definition *For a dynamic trading strategy* $\boldsymbol{v}$ *which delivers*

$$\hat{S}_t^{\delta} = \sum_{j=1}^{d} \delta_t^j \hat{S}_t^j$$

benchmarked P&L

$$\hat{C}_t^{\delta} = \hat{S}_t^{\delta} - \sum_{j=1}^{d} \int_0^t \vartheta_u^j d\hat{S}_u^j - \hat{S}_0^{\delta}$$

Corollary $\hat{C}_t^\delta = \eta_t$ *for* $t \in [0, \infty)$.

- usually fluctuating benchmarked P&L
- intrinsic risk

What criterion would be most natural?

- symmetric view with respect to all primary security accounts, including the domestic savings account
- pooling in large trading book
  $\Longrightarrow$ vanishing total hedge error

Proposition $\hat{H}_{T,l}$, $\hat{V}^{v_l}$ *with*
$\hat{C}^{v_l}$ *independent square integrable martingales with*
$E\left(\left(\frac{\hat{C}_t^{v_l}}{\hat{V}_0^{v_l}}\right)^2\right) \leq K_t < \infty$ *for* $l \in \{1, 2, \ldots\}$, $t \in [0, T]$, $T \in [0, \infty)$.
at initial time well diversified trading book holds equal fractions

- *total benchmarked wealth* $\hat{U}_t = \frac{\hat{U}_0}{m} \sum_{l=1}^{m} \frac{\hat{V}_t^{v_l}}{\hat{V}_0^{v_l}}$
- *total benchmarked P&L*

$$\hat{R}_m(t) = \frac{\hat{U}_0}{m} \sum_{l=1}^{m} \frac{\hat{C}_t^{v_l}}{\hat{V}_0^{v_l}} \, ,$$

$\Rightarrow$

$$\lim_{m \to \infty} \hat{R}_m(t) = 0$$

P-*a.s.*

Definition *A dynamic trading strategy* $\boldsymbol{v}$*, initiated at time zero, is called* locally real-world mean-self-financing *if its adapted process* $\boldsymbol{\eta_t = \hat{C}_t^{\delta}}$ *forms an local martingale.*

- **classical risk minimization**

  Föllmer and Sondermann (1986)

  Föllmer and Schweizer (1989)

  Schweizer (1991, 1995)

  Schweizer (1999)

  Föllmer-Schweizer decomposition assuming a risk neutral probability measure

  Buckdahn (1993) Schweizer (1994) Stricker (1996)

  Delbaen et al. (1997) Pham et al. (1998)

- **local risk minimization**

  Bouleau and Lamberton (1989)

  Duffie and Richardson (1991)

  Schweizer (1994)

  Biagini et al. (1999)

  Schweizer (2001)

- Benchmarked P&L is *orthogonal* if

$$\boldsymbol{\eta_t} \hat{\mathbf{S}}_{\boldsymbol{t}}$$

is vector local martingale.

- above product has no trend

- benchmarked hedge error orthogonal to any benchmarked traded wealth

- $\mathcal{V}_{\hat{H}_T}$ set of locally real-world mean-self-financing dynamic trading strategies v initiated at time zero with

$$\hat{V}_t^v = \hat{S}_t^\delta, \quad t \geq 0$$

which deliver $\hat{H}_T$

with orthogonal benchmarked P&L

- market participants prefer **more for less**

$\Rightarrow$ **Benchmarked Risk Minimization**

For $\hat{H}_T$ strategy $\widetilde{v} \in \mathcal{V}_{\hat{H}_T}$
benchmarked risk minimizing (BRM) if for all $v \in \mathcal{V}_{\hat{H}_T}$ with $\hat{V}_t^v = \hat{S}_t^\delta$

price $\hat{S}_t^{\widetilde{\delta}}$ is minimal

$$\hat{S}_t^{\widetilde{\delta}} \leq \hat{S}_t^{\delta}$$

P-a.s. for all $t \in [0, T]$.

Regular Benchmarked Contingent Claims

$\hat{H}_T$ is called *regular* if

$$\hat{H}_T = E_t(\hat{H}_T) + \sum_{j=1}^{d} \int_t^T \vartheta^j_{\hat{H}_T}(s) d\hat{S}^j_s + \eta_{\hat{H}_T}(T) - \eta_{\hat{H}_T}(t)$$

$\vartheta_{\hat{\mathbf{H}}_\mathbf{T}}$ - predictable

$\eta_{\hat{H}_T}$ - local martingale, orthogonal to $\hat{S}$

Theorem *A regular* $\hat{H}_T$ *has a BRM strategy* $v = \{v_t = (\eta_t, \vartheta_t^1, \ldots, \vartheta_t^d)^\top, t \in [0, T]\} \in \mathcal{V}_{\hat{H}_T}$ *with*

$$\hat{S}_t^\delta = \hat{V}_t^v = E_t(\hat{H}_T) \,,$$

$$\hat{S}_T^\delta = \hat{H}_T \,,$$

and orthogonal benchmarked P&L: $\hat{C}_t^\delta = \eta_{\hat{H}_T}(t)$

$\Longrightarrow$

$\hat{C}_t^\delta$ - *local martingale*

$\hat{C}_t^\delta \hat{\mathbf{S}}_t$ - *vector local martingale*

orthogonal

hedgeable part:

$$E_t(\hat{H}_T) - \hat{C}_t^{\delta} = E_0(\hat{H}_T) + \int_0^t \vartheta_{\hat{H}_T}^{\top}(s) d\hat{S}_s \ .$$

local martingale

- benchmarked price $\hat{V}_t^v = E_t(\hat{H}_T)$ minimal
- do not request square integrability
- no equivalent risk neutral probability measure required
- need only to check zero drift of $\eta_{\hat{H}_T}$
- and $\eta_{\hat{H}_T}(t), \hat{\mathbf{S}}_t$ orthogonality

Hedging a Regular Claim

- **benchmarked contingent claim** $\hat{H}_T$ driven by continuous local martingales $W^1, W^2, \ldots, W^d$ orthogonal to each other

- **benchmarked primary security account** $\hat{S}_t^j$, $j \in \{1, \ldots, d\}$

$$d\hat{S}_t^j = -\hat{S}_t^j \sum_{k=1}^{d-1} \theta_t^{j,k} dW_t^k$$

- $d \times d$ matrix $\Phi_t = [\Phi_t^{i,k}]_{i,k=1}^d$

$$\Phi_t^{i,k} = \begin{cases} \theta_t^{i,k} & \text{for } k \in \{1, \ldots, d-1\} \\ 1 & \text{for } k = d \end{cases}$$

for $t \in [0, T]$.

Proposition *assume Φ_t invertible and $\hat{V}_t = E(\hat{H}_T|\mathcal{F}_t)$ has representation*

$$\hat{V}_t = \hat{V}_0 + \sum_{k=1}^{d} \int_0^t x_s^k dW_s^k + \int_0^t x_s^d dW_s^d \ .$$

Then $\hat{H}_T$ is a regular benchmarked contingent claim

$$\begin{aligned} \hat{V}_t^{v_{\hat{H}_T}} &= \hat{V}_t \quad for\ all\ t \in [0,T] \\ \vartheta_{\hat{H}_T}(t) &= diag(\hat{S}_t)^{-1}(\Phi_t^\top)^{-1}\xi_t \\ \eta_{\hat{H}_T}(t) &= \int_0^t x_s^d dW_s^d \end{aligned}$$

with $\xi_t = (x_t^1, \ldots, x_t^{d-1}, \hat{V}_0 + \sum_{k=1}^{d-1} \int_0^t x_s^k dW_s^k)^\top$.

$$\bar{S}_t^{i,j} = \frac{\hat{S}_t^j}{\hat{S}_t^i}$$

$$d\bar{S}_t^{i,j} = \bar{S}_t^{i,j} \sum_{k=1}^{d-1} (\theta_t^{i,k} - \theta_t^{j,k})(\theta_t^{i,k} dt + dW_t^k)$$

$$b_t^{i,j,k} = (\theta_t^{i,k} - \theta_t^{j,k}) \ .$$

$$b_t^d = [b_t^{d,j,k}]_{j,k=1}^{d-1,d-1}$$

Proposition *The matrix* $\boldsymbol{\Phi_t}$ *is invertible if and only if* $\mathrm{b}_{\boldsymbol{t}}^{\boldsymbol{d}}$ *is invertible.*

Proof: elementary transform of $\mathrm{b}_{\boldsymbol{t}}^{\boldsymbol{d}}$

$$\boldsymbol{\Phi_t} \longleftrightarrow \left(\begin{array}{ccc|c} & \mathrm{b}_{\boldsymbol{t}}^{\boldsymbol{d}} & & \mathbf{0} \\ \hline \boldsymbol{\theta}_{\boldsymbol{t}}^{\boldsymbol{d},\mathbf{1}} & \ldots & \boldsymbol{\theta}_{\boldsymbol{t}}^{\boldsymbol{d},\boldsymbol{d}-\mathbf{1}} & \mathbf{1} \end{array}\right)$$

$\boldsymbol{\Phi_t}$ has full rank if and only if $\boldsymbol{b}_{\boldsymbol{t}}^{\boldsymbol{d}}$ has full rank

Comparison with Quadratic Criterion

- $\int_0^t \vartheta_{\hat{H}_T}^\top(s) d\hat{S}_s$ and $\eta_{\hat{H}_T}(t)$ independent, square integrable martingales
- $\hat{S}_t^1, \ldots, \hat{S}_t^d$ and $\eta_{\hat{H}_T}(t)$ independent, square integrable martingales
- $\eta_{\hat{H}_T}$ orthogonal to benchmarked primary security accounts
- $\hat{H}_T$ square integrable
- $\hat{V}_t^v = \hat{S}_t^\delta$ delivers $\hat{H}_T$

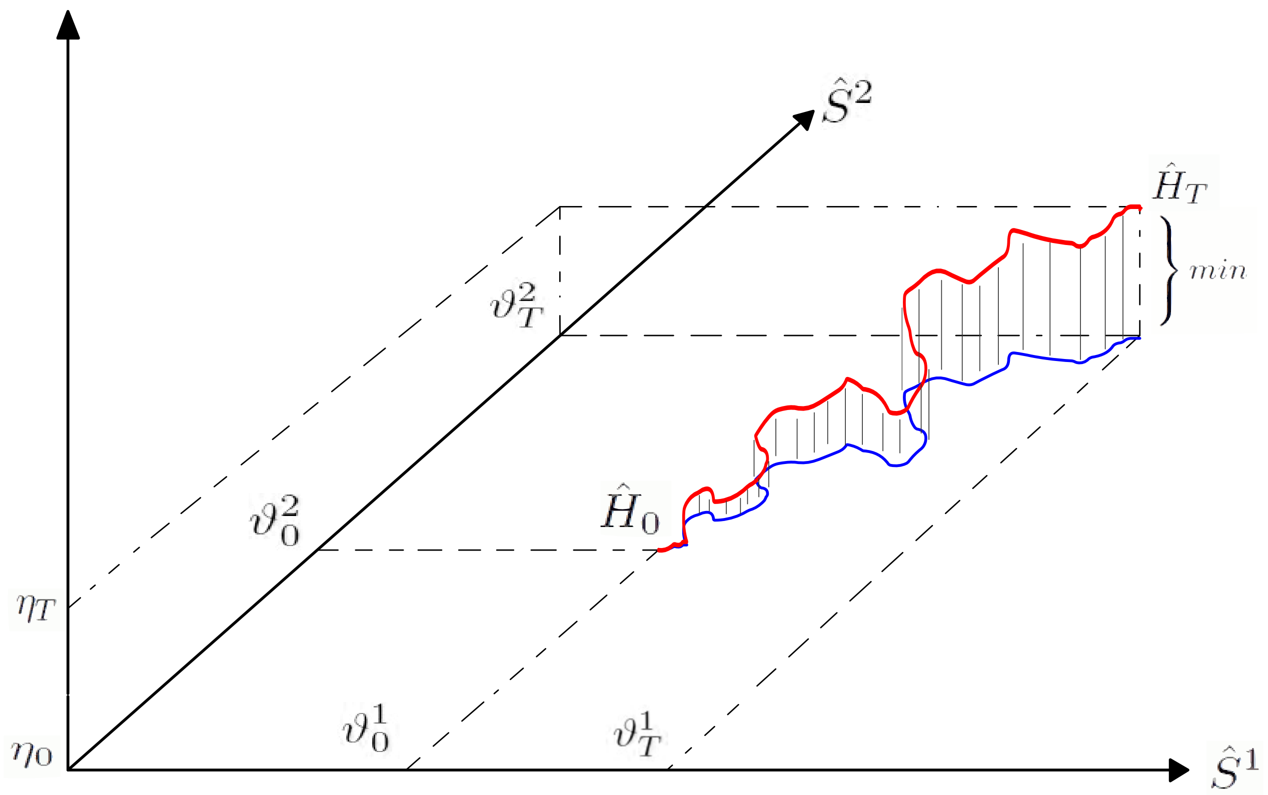
Quadratic Criterion

$$E\left((\hat{C}_T^{\delta})^2\right) = E\left((\hat{H}_T - \int_0^T \vartheta_s^{\top} d\hat{S}_s - \hat{S}_0^{\delta})^2\right) \quad \Rightarrow min$$

$$
\begin{aligned}
&E\left((\hat{C}_T^{\delta})^2\right)\\
&= E\left(\left(E(\hat{H}_T) - \hat{S}_0^{\delta} + \int_0^T (\vartheta_{\hat{H}_T}^{\top}(s) - \vartheta_s^{\top}) d\hat{S}_s + \eta_{\hat{H}_T}(T)\right)^2\right)\\
&= E\left(E(\hat{H}_T) - \hat{S}_0^{\delta}\right)^2\\
&\qquad + E\left(\int_0^T (\vartheta_{\hat{H}_T}^{\top}(s) - \vartheta_s^{\top})^2 d[\hat{S}]_s\right) + E((\eta_{\hat{H}_T}(T))^2)
\end{aligned}
$$

$\Longrightarrow$

- $\hat{S}_0^\delta = E(\hat{H}_T)$
- $\vartheta_t = \vartheta_{\hat{H}_T}(t), t \in [0, T]$
- $\eta_{\hat{H}_T}(T) = \hat{C}_T^\delta$
- second moment of benchmarked P&L becomes minimal



$\hat{S}^2$
$\hat{H}_T$
min
ϑ_T^2
ϑ_0^2
$\hat{H}_0$
η_T
ϑ_0^1
ϑ_T^1
η_0
$\hat{S}^1$

Nonhedgeable Contingent Claim

- savings account $S_t^{1,1} = 1$
- numéraire portfolio $S_t^{1,\delta*}$
- $H_t = E_t(H_T), t \in [0, T]$
  - independent from $S_t^{1,\delta*}$
  - continuous

Benchmarked risk minimization

- real world pricing formula

$$\begin{aligned}\hat{S}_t^{\delta_{\hat{H}_T}} &= E_t(\hat{H}_T) = E_t(H_T)E_t(\hat{S}_T^1) = H_t\hat{P}(t,T)\\ \hat{P}(t,T) &= E_t(\hat{S}_T^1)\\ d\hat{P}(t,T) &= \frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1} d\hat{S}_t^1\end{aligned}$$

- benchmarked NP $\hat{S}_t^{\delta *} = \frac{S_t^{1,\delta *}}{S_t^{1,\delta *}} = 1$
- benchmarked P&L

$$\begin{aligned}d\hat{C}_t^{\delta_{\hat{H}_T}} &= d\hat{S}_t^{\delta_{\hat{H}_T}} - \vartheta^1_{\hat{H}_T}(t)d\hat{S}_t^1\\ &= \left(H_t \frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1} - \vartheta^1_{\hat{H}_T}(t)\right) d\hat{S}_t^1 + \hat{P}(t,T)dH_t\end{aligned}$$

- Assume orthogonal benchmarked primary security accounts
- product $\hat{C}^{\delta_{\hat{H}_T}} \hat{S}$ has zero drift if

$$0 = \frac{d[\hat{S}, \hat{C}^{\delta_{\hat{H}_T}}]_t}{dt} = \left(H_t \frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1} - \vartheta^1_{\hat{H}_T}(t) \right) \frac{d[\hat{S}^1]_t}{dt} .$$

$\Longrightarrow$

$$\vartheta^1_{\hat{H}_T}(t) = H_t \frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1}$$

Depends on H_t !

- Benchmarked P&L

$$\eta_{\hat{H}_T}(t) = \hat{C}_t^{\delta_{\hat{H}_T}} = \int_0^t \hat{P}(s,T)\,dH_s$$

local martingale, orthogonal

- self-financing benchmarked hedgeable part

$$\hat{S}_t^{\delta_{\hat{H}_T}} - \eta_{\hat{H}_T}(t) = \hat{S}_0^{\delta_{\hat{H}_T}} + \int_0^t \vartheta^1_{\hat{H}_T}(s)\,d\hat{S}^1_s$$

- $\vartheta^1_{\hat{H}_T}(t)$ units in $\hat{S}^1_t$
- remaining wealth in NP
- minimizing $\frac{d}{dt}[\hat{C}^{\delta_{\hat{H}_T}}]_t$

- Regular benchmarked contingent claim

$$
\begin{aligned}
\hat{H}_T &= \frac{H_T}{S_T^{1,\delta*}} \\
&= \hat{S}_0^{\delta_{\hat{H}_T}} + \int_0^T H_t \frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1} d\hat{S}_t^1 + \int_0^T \hat{P}(t,T) dH_t \\
\vartheta^1_{\hat{H}_T}(t) &= H_t \frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1} \\
\eta_{\hat{H}_T}(t) &= \int_0^t \hat{P}(s,T) dH_s
\end{aligned}
$$

Evolving Information

- consider special case when $\hat{S}^1$ true martingale
- classical risk minimization can be applied
- real world pricing formula yields

$$\bar{S}_t^{\delta_{\hat{H}_T}} = \frac{\hat{S}_t^{\delta_{\hat{H}_T}}}{\hat{S}_t^1} = E_t\left(\frac{\hat{S}_T^1}{\hat{S}_t^1}\frac{H_T}{S_T^{1,1}}\right) = E_t\left(\frac{\Lambda_T}{\Lambda_t}\frac{H_T}{S_T^{1,1}}\right) = E_t^Q\left(\frac{H_T}{S_T^{1,1}}\right)$$

- equivalent minimal martingale measure Q

$$\Lambda_T = \frac{dQ}{dP}\Big|_{\mathcal{F}_T} = \frac{\hat{S}_T^1}{\hat{S}_t^1}$$

- same initial price as benchmarked risk minimization

$$\bar{S}_0^{\delta_{\hat{H}_T}} = H_0 = \hat{S}_0^{\delta_{\hat{H}_T}}(\hat{S}_0^1)^{-1}$$

- classical risk minimization invests total initial value in the savings account

$$\vartheta^1_{CRM}(t) = H_0$$

- BRM strategy different

- $\hat{P}(t,T) = \hat{S}^1_t$, $\frac{\partial \hat{P}(t,T)}{\partial \hat{S}^1_t} = 1$
  $\Longrightarrow$

$$\vartheta^1_{\hat{H}_T}(t) = H_t = E_t(H_T)$$

- intuitive and practically appealing

- under classical risk minimization, evolving information ignored

- provides less fluctuations for benchmarked P&L

- easier diversified

Modeling the Dynamics of Diversified Indices

Pl. & Rendek (2012)

- conjecture for normalized aggregate wealth dynamics
  time transformed square root process

- Naive Diversification Theorem $\Rightarrow$ equity index $=$ proxy of numéraire portfolio

- empirical stylized facts $\Rightarrow$ falsify models

- $\Rightarrow$ propose realistic one factor, two component long term index model

- realistic model outside classical theory $\Rightarrow$ benchmark approach

- exact, almost exact simulation $\Rightarrow$ verify empirical facts, effects of estimation techniques etc.

The Affine Nature of Diversified Wealth Dynamics

Object: normalized units of wealth

Total wealth: $Y_{\tau_i}^{\Delta}$

Time steps: $\tau_i = i\Delta$

Wealth unit value: $\sqrt{\Delta}$

Economic activity: during $[\tau_i, \tau_{i+1})$ "projects"
generate each independent wealth with variance increment $v^2\Delta^{3/2}$;
consume each $\eta\Delta$ fraction of wealth;
generate together on average $\beta\sqrt{\Delta}$ new units.

Mean for increment of aggregate wealth: $\left(\boldsymbol{\beta} - \boldsymbol{\eta} \boldsymbol{Y}_{\boldsymbol{\tau}_i}^{\boldsymbol{\Delta}}\right) \boldsymbol{\Delta}$

Variance for increment aggregate wealth: $\triangleq$ sum of variances

$\Longrightarrow$ proportional to number of wealth units: $\frac{Y_{\tau_i}^{\Delta}}{\sqrt{\Delta}}$

$\Longrightarrow$ proportional to aggregate wealth

$\Longrightarrow$ variance of increment of aggregate wealth is $v^2 Y_{\tau_i}^{\Delta} \Delta$

for $\boldsymbol{\Delta \to 0}$

$$\boldsymbol{Y^{\Delta}_{\tau_{i+1}} - Y^{\Delta}_{\tau_i} = \left(\beta - \eta Y^{\Delta}_{\tau_i}\right)\Delta + \upsilon\sqrt{Y^{\Delta}_{\tau_i}}\Delta W_{\tau_i}}$$

$$\boldsymbol{E(\Delta W_{\tau_i}) = 0, \quad E((\Delta W_{\tau_i})^2) = \Delta}$$

conjectures drift and diffusion terms

$\Longrightarrow$ **Conjecture via**

Weak convergence

Kleoden & Pl. (1992), Alfonsi (2005), Diop (2003)

for parameters: $\boldsymbol{\beta = \eta = \upsilon = 1}$

square root process:

$$\boldsymbol{dY_{\tau_t} = (1 - Y_{\tau_t})\, d\tau_t + \sqrt{Y_{\tau_t}} dW_{\tau_t}}$$

- **Quadratic variation:**

$$[Y_{\tau_\cdot}]_t = \int_0^t Y_{\tau_s} d\tau_s = \int_0^t Y_{\tau_s} M_s ds$$

- **Market activity:**

$$M_t = \frac{d\tau_t}{dt}$$

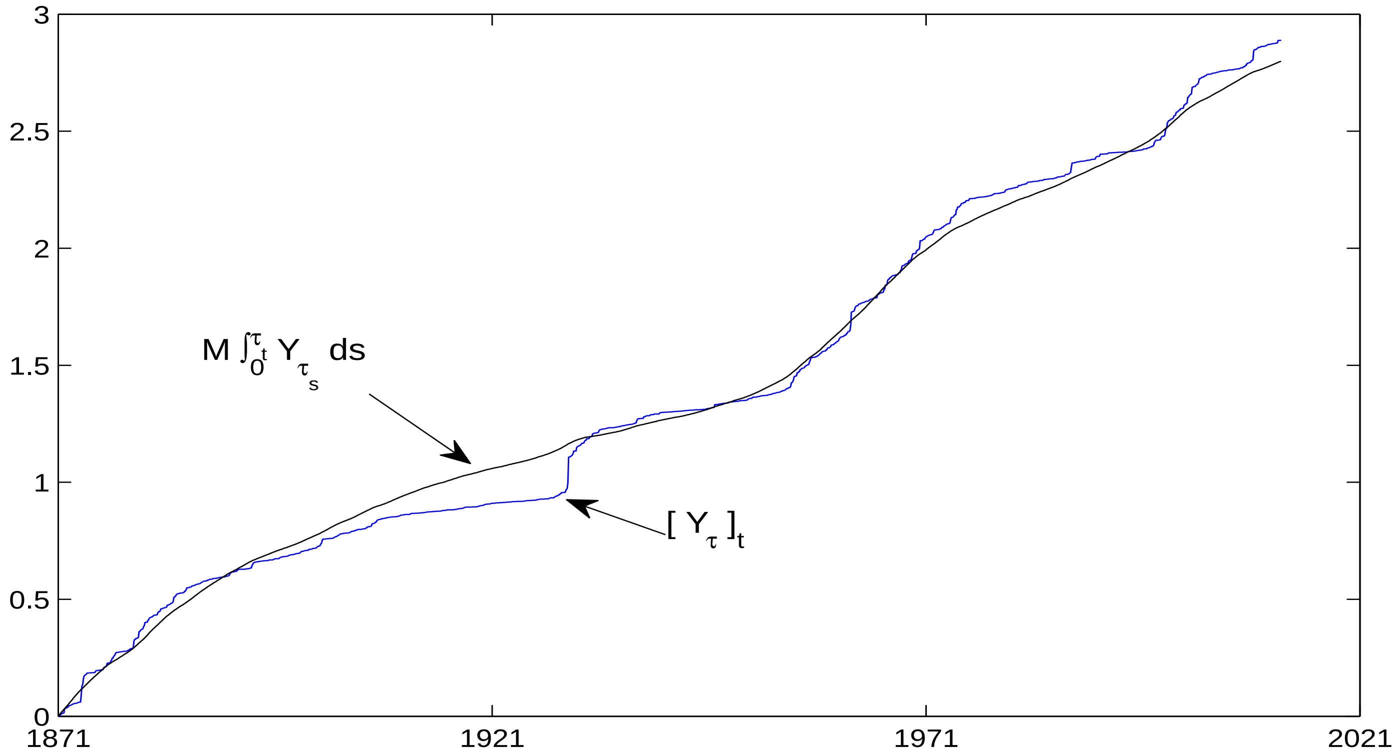
- **integrate normalized index:**

  can we find M =const. s.t.

$$M \int_0^t Y_{\tau_s} ds \approx [Y_{\tau_\cdot}]_t \quad ?$$

Quadratic variation and integrated normalized S&P500

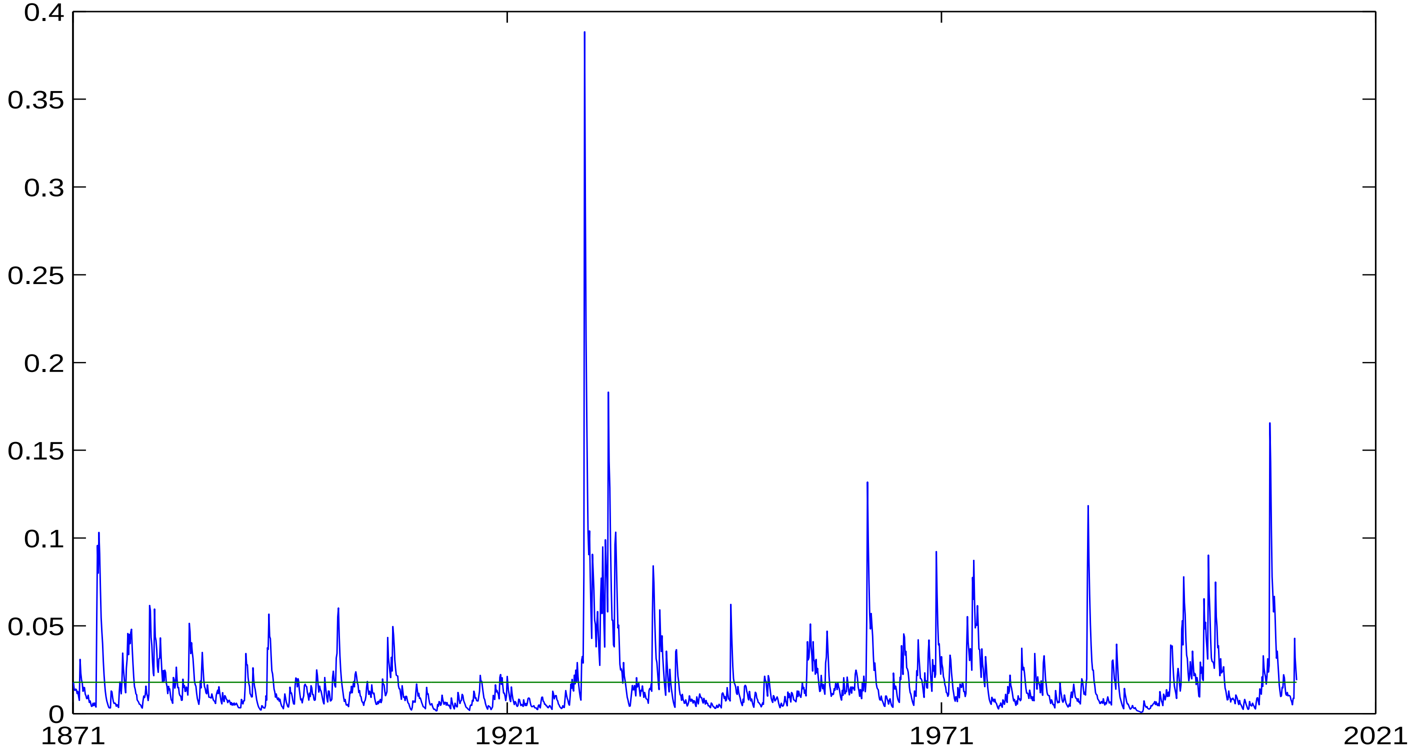
monthly data, calendar time, Shiller data



$M \approx 0.0178$

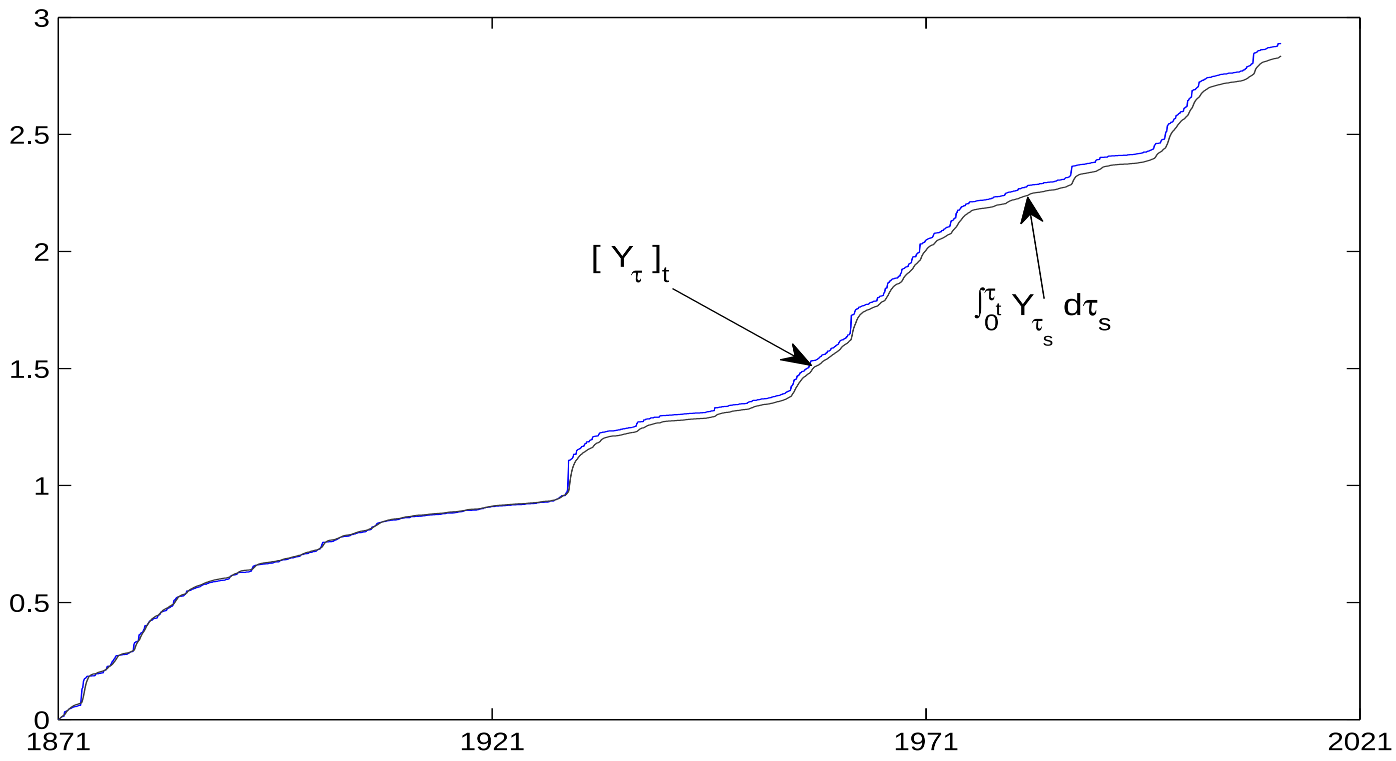
average long term fit

Market Activity: $M_t = \frac{d\tau_t}{dt}$ from model



Quadratic variation and integrated normalized S&P500

monthly data, $\boldsymbol{\tau}$-time

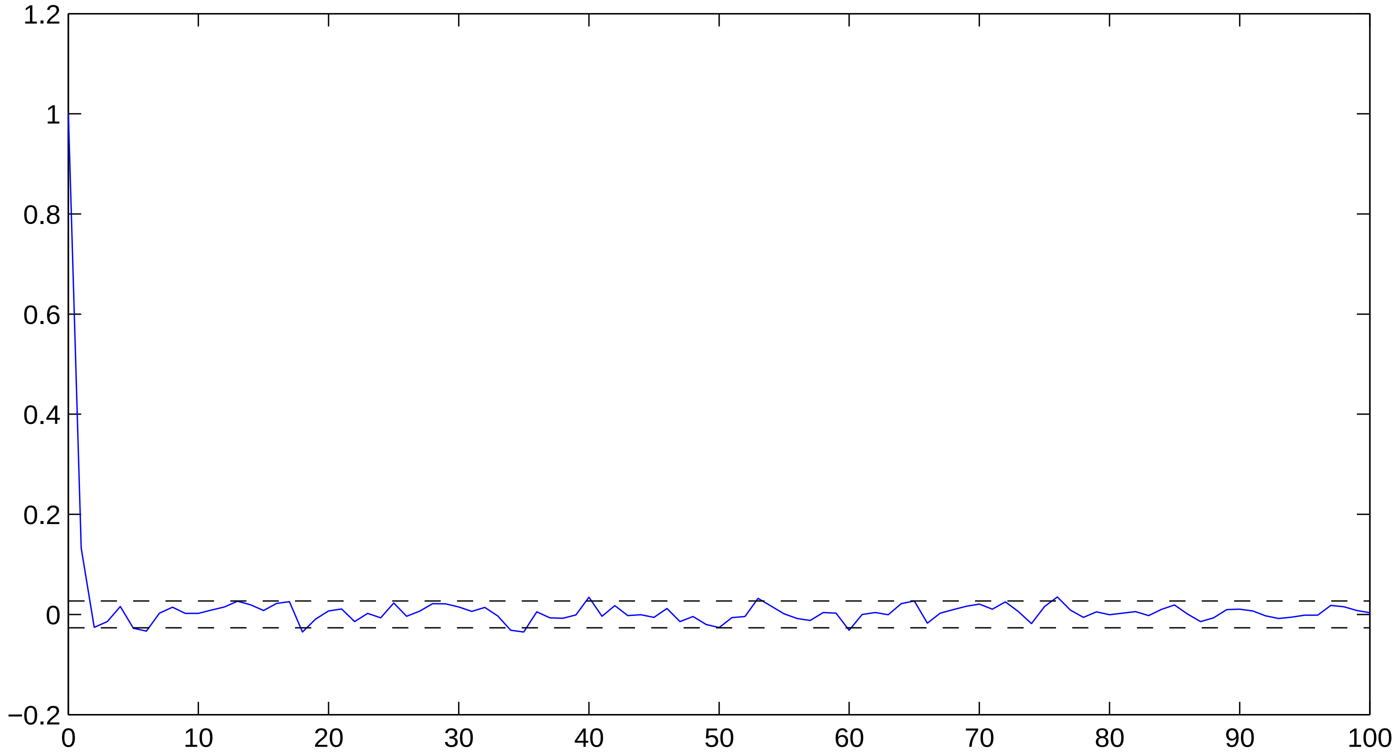


Eight Stylized Empirical Facts

- falsify potential models, Popper (1959)
- TOTMKWD in 26 currency denominations

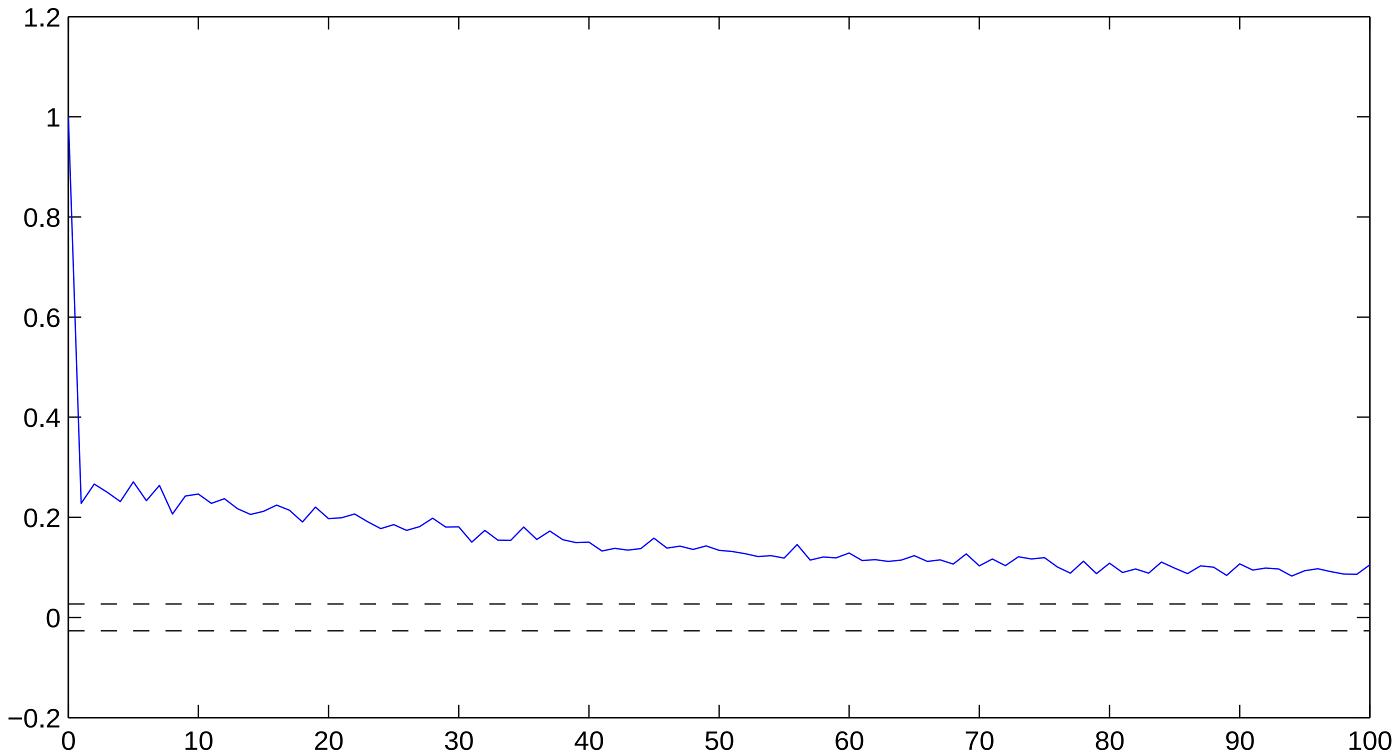
about 1000 years of daily data

(i) uncorrelated log-returns



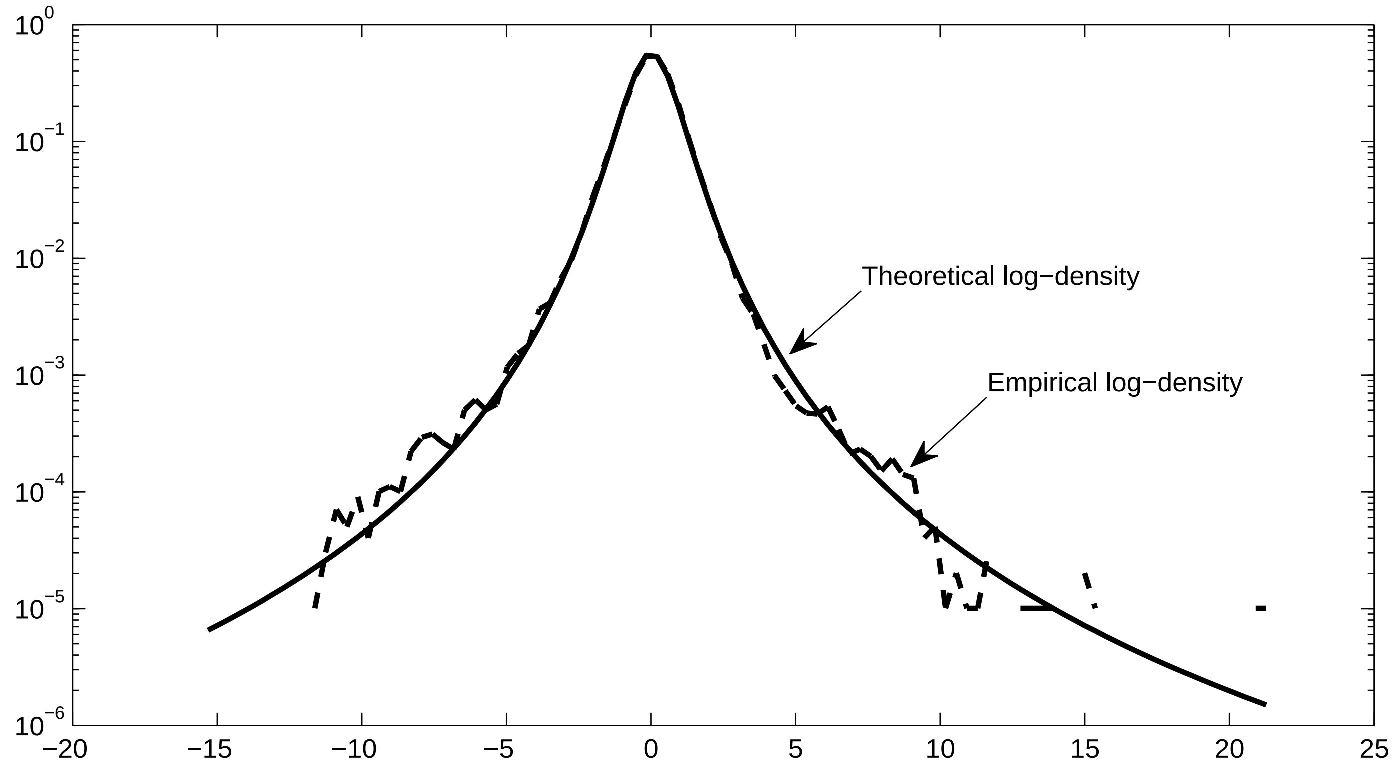
Average autocorrelation function for log-returns

(ii) correlated absolute log-returns



Average autocorrelation function for absolute log-returns

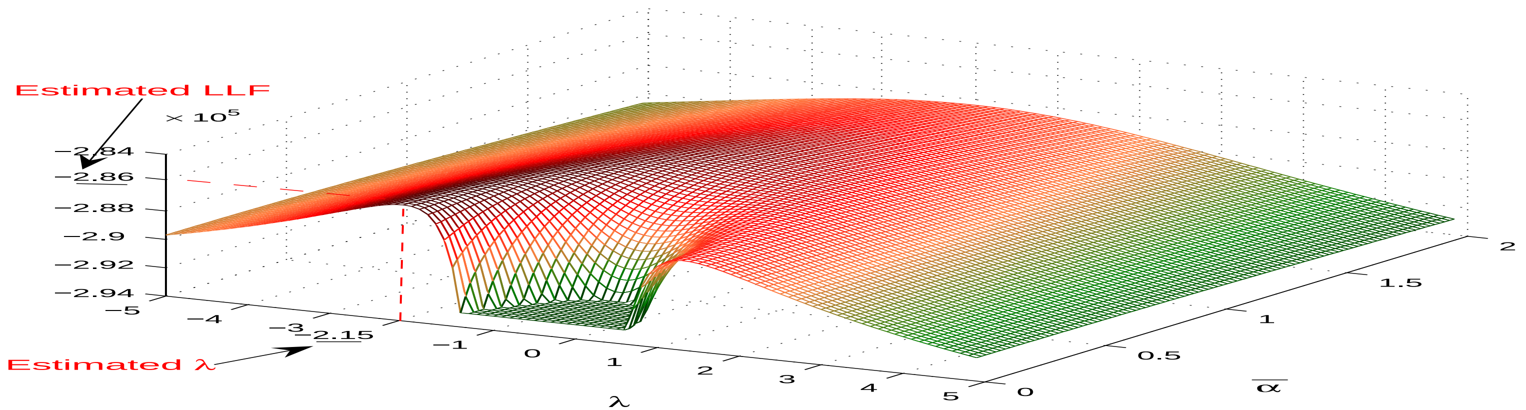
(iii) Student-t distributed log-returns



Logarithm of empirical density of normalized log-returns with Student-t density

Results for log-returns of the EWI104s

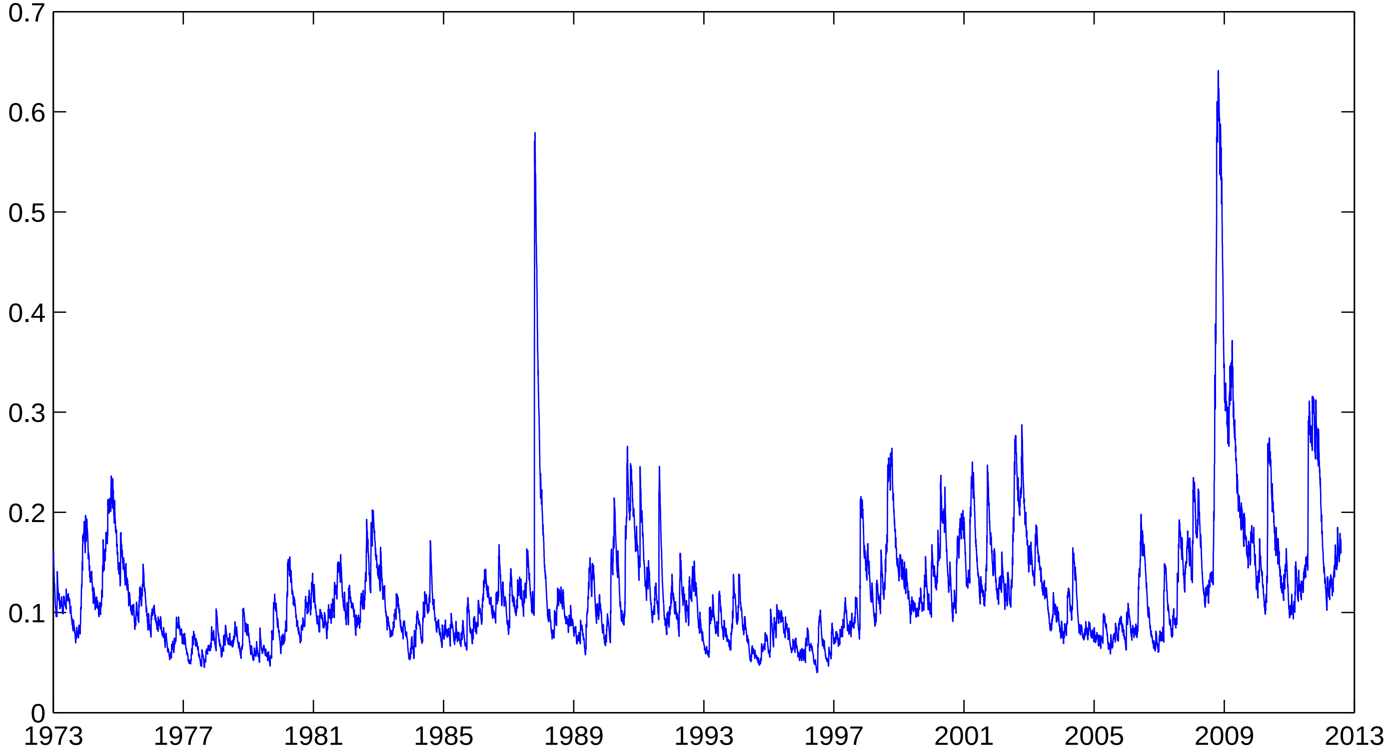
Pl. & Rendek (2008)



	SGH	Student-t	NIG	Hyperbolic	VG
σ	0.98	0.72	0.97	0.96	0.96
$\bar{\alpha}$	0.00		0.97	0.72	
λ	-2.16				1.49
ν		4.33			
$\ln(\mathcal{L}^*)$	-285796.39	-285796.39	-286448.94	-287152.08	-287499.83
L_n		**0.0000004**	1305.10	2711.38	3406.88

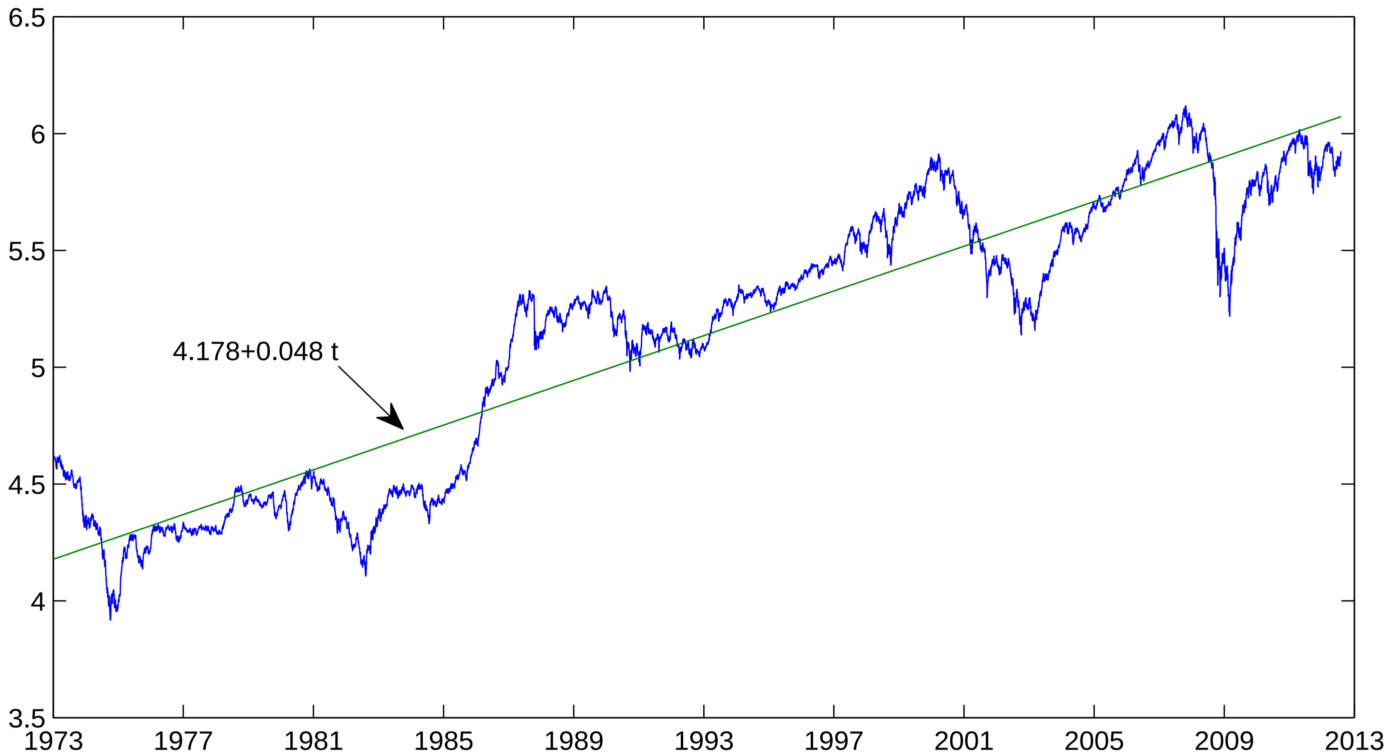
$L_n = 0.0000004 < \chi^2_{0.001,1} \approx 0.000002$

(iv) volatility clustering



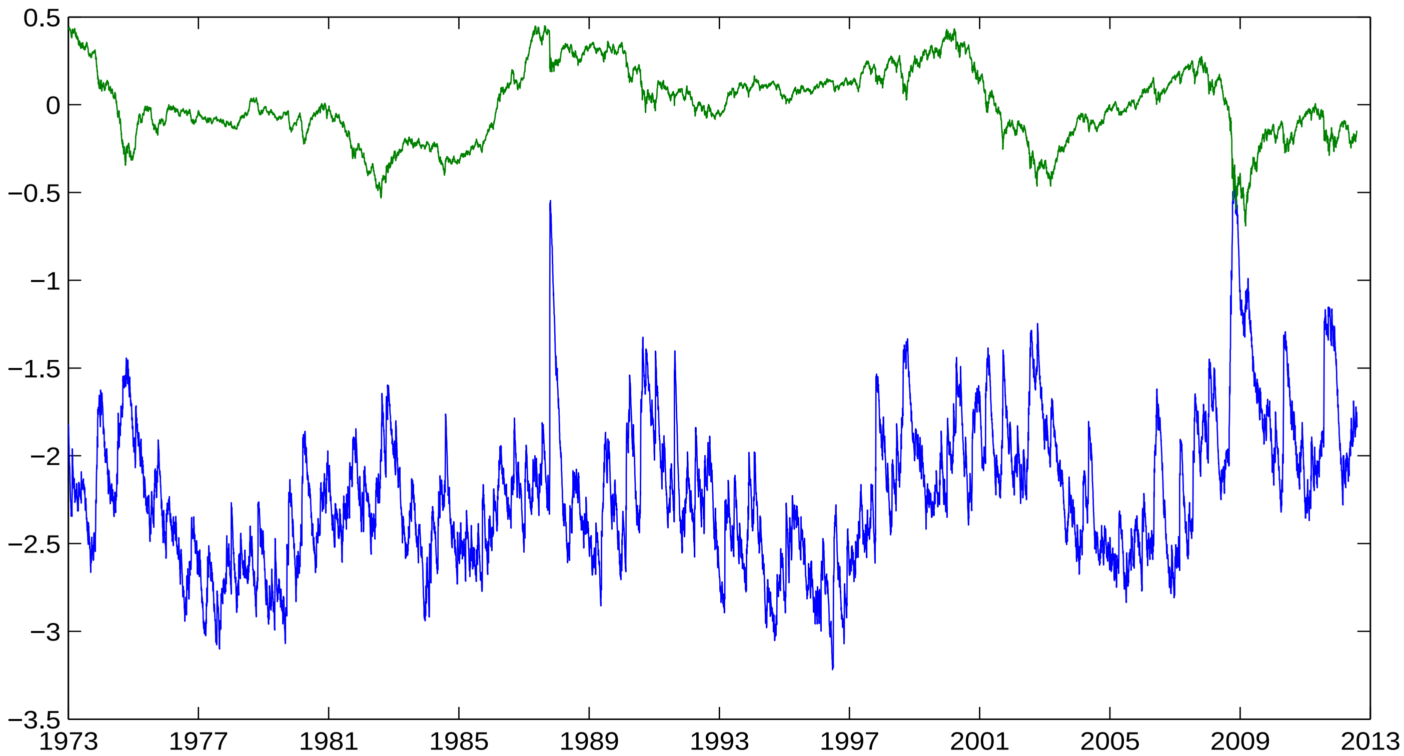
Estimated volatility

(v) long term exponential growth



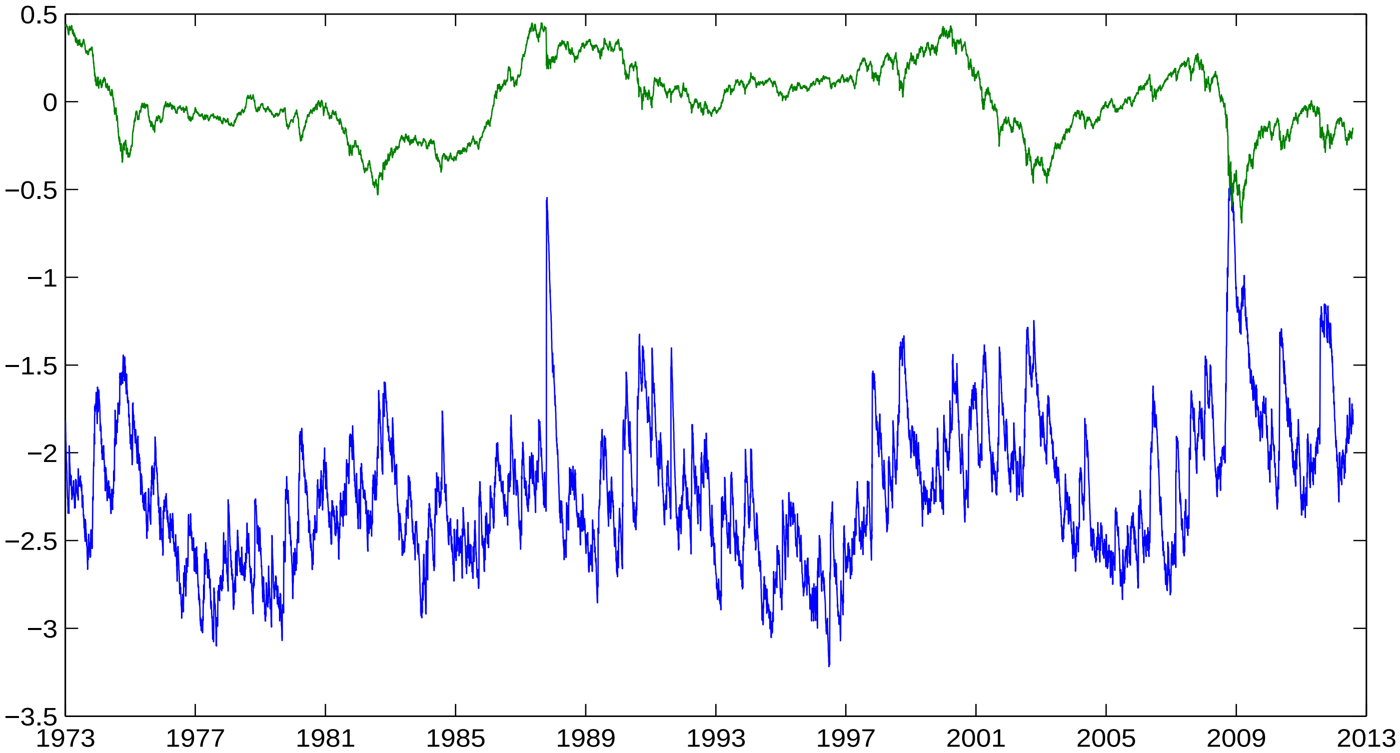
Logarithm of index with trend line

(vi) leverage effect



Logarithms of normalized index and its volatility

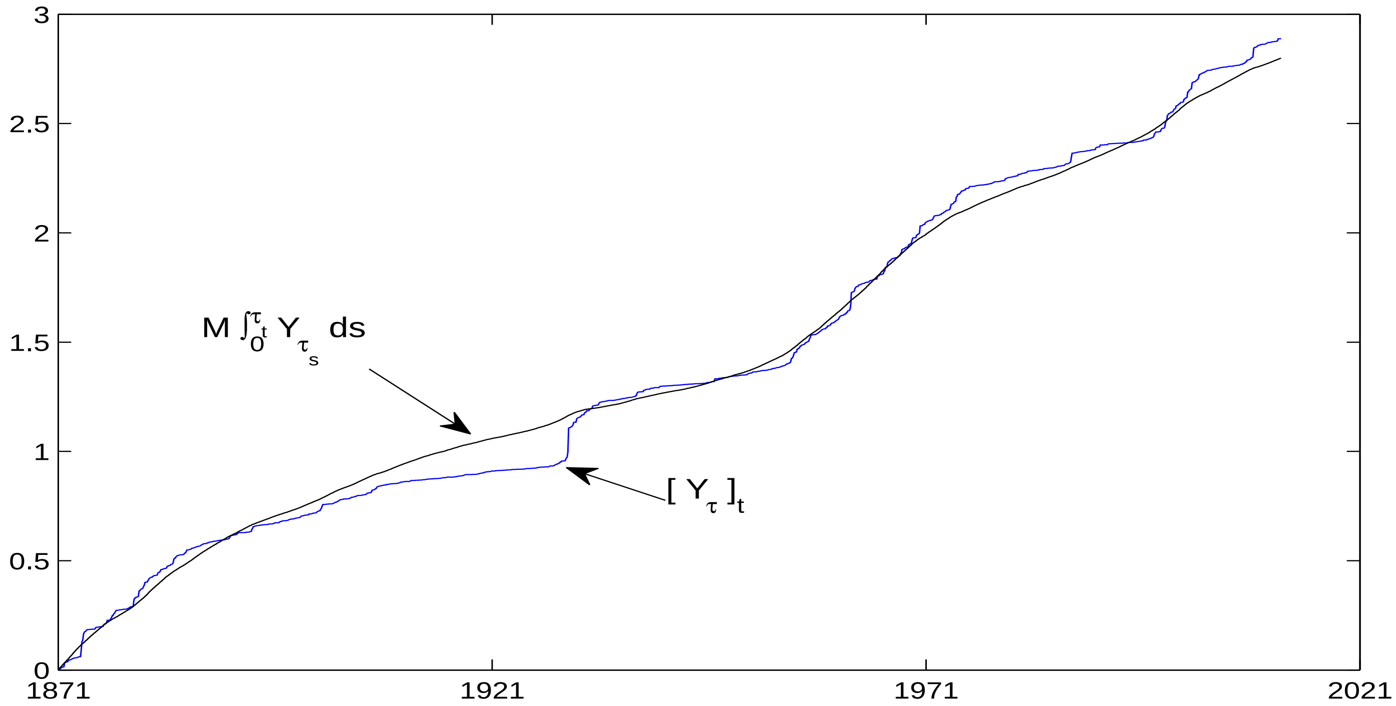
(vii) extreme volatility at major downturns



Logarithms of normalized index and its volatility

(viii) Integrated Normalized Index and its Quadratic Variation

monthly data, calendar time



$\boldsymbol{M \approx 0.0178}$

average long term fit

⇒ **Discounted Index Model generalizing conjecture**

$$S_t = A_{\tau_t}(Y_{\tau_t})^q,$$

$$A_{\tau_t} = A \exp\{a\tau_t\}$$

$q > 0$, $a > 0$, $A > 0$

conjecture expects $q = 1$

Normalized index: $(Y_{\tau_t})^q = \frac{S_t}{A_{\tau_t}}$

$$dY_\tau = \left(\frac{\delta}{4} - \frac{1}{2}\left(\frac{\Gamma\left(\frac{\delta}{2}+q\right)}{\Gamma\left(\frac{\delta}{2}\right)}\right)^{\frac{1}{q}} Y_\tau\right) d\tau + \sqrt{Y_\tau}\, dW(\tau)$$

Long term mean: $\lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau (Y_s)^q\, ds = 1$ P-a.s

Market activity time: $d\tau_t = M_t dt$

Inverse of market activity:

$$d\left(\frac{1}{M_t}\right) = \left(\frac{\nu}{4}\gamma - \epsilon\frac{1}{M_t}\right) dt + \sqrt{\frac{\gamma}{M_t}} dW_t,$$

where

$$dW(\tau_t) = \sqrt{\frac{d\tau_t}{dt}} dW_t = \sqrt{M_t} dW_t$$

◇ only one W_t

◇ two component model

Discounted index SDE:

$$dS_t = S_t \left(\mu_t dt + \sigma_t dW_t\right)$$

Expected rate of return:

$$\mu_t = \left(\frac{a}{M_t} - \frac{q}{2} \left(\frac{\Gamma\left(\frac{\delta}{2} + q\right)}{\Gamma\left(\frac{\delta}{2}\right)} \right)^{\frac{1}{q}} + \left(\frac{\delta}{4} q + \frac{1}{2} q(q-1) \right) \frac{1}{M_t Y_{\tau_t}} \right) M_t$$

Volatility:

$$\sigma_t = q \sqrt{\frac{M_t}{Y_{\tau_t}}}$$

Pl. & Rendek (2012c)

Benchmark Approach

$\hat{B}_t$ – benchmark savings account

$$d\hat{B}_t = \hat{B}_t \left(\left(-\mu_t + \sigma_t^2 \right) dt - \sigma_t dW_t \right)$$

$$\sigma_t^2 \leq \mu_t \quad \Rightarrow \quad \hat{B}_t \text{ is an } (\underline{\mathcal{A}}, P)\text{-supermartingale}$$

$$\Rightarrow \quad \text{no strong arbitrage; Pl. (2011)}$$

Assumptions:

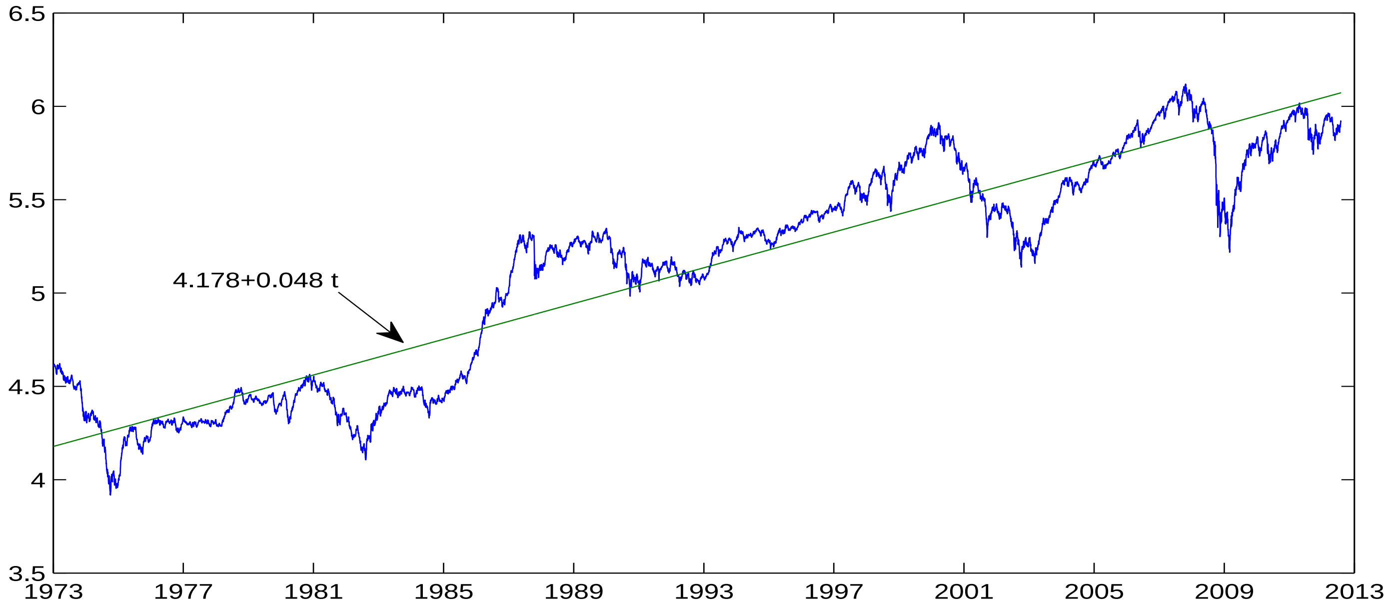
A1. $\delta = 2(q+1)$ **A2**. $\frac{q}{2}\left(\frac{\Gamma(2q+1)}{\Gamma(q+1)}\right)^{\frac{1}{q}} \leq a$

$$\Longrightarrow \quad \sigma_t^2 \leq \mu_t$$

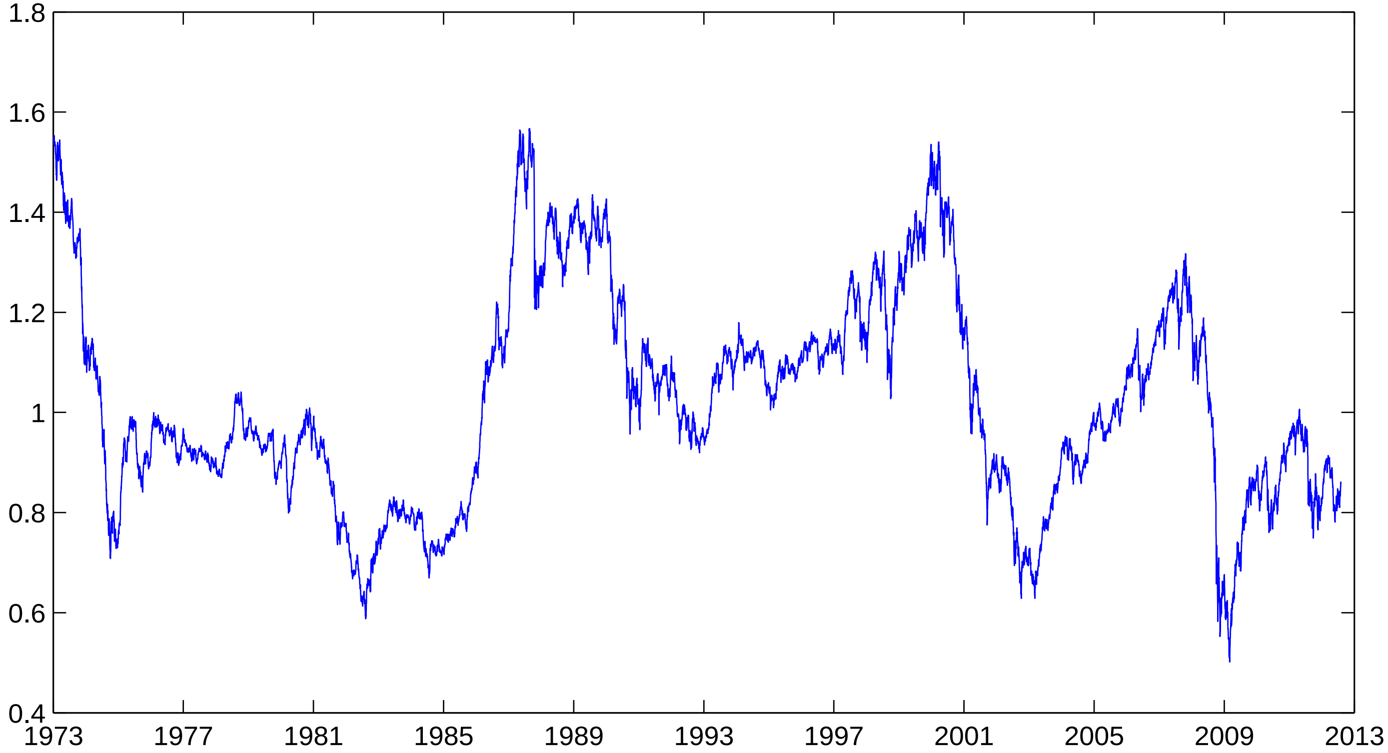
Fitting the model to TOTMKWD

Step 1: Normalization of Index

$$A_{\tau_t} \approx A \exp\{\frac{4a\epsilon}{\gamma(\nu-2)} t\} \Rightarrow A = 65.21, \frac{4a\epsilon}{\gamma(\nu-2)} \approx 0.048$$



Normalized TOTMKWD



Step 2: Power q:

Affine nature $\Longrightarrow$ conjecture: $\boldsymbol{q = 1 = \frac{\delta}{2} - 1} \Longrightarrow \boldsymbol{\delta = 4}$

- student-t with about four degree of freedom
  is considered and cannot be falsified,

$\Longrightarrow$ We set $\boldsymbol{q = 1}$ and $\boldsymbol{\delta = 4}$

Step 3: Observing Market Activity:

$$\frac{d[\sqrt{Y}]_{\tau_t}}{dt} = \frac{1}{4}\frac{d\tau_t}{dt} = \frac{M_t}{4}$$

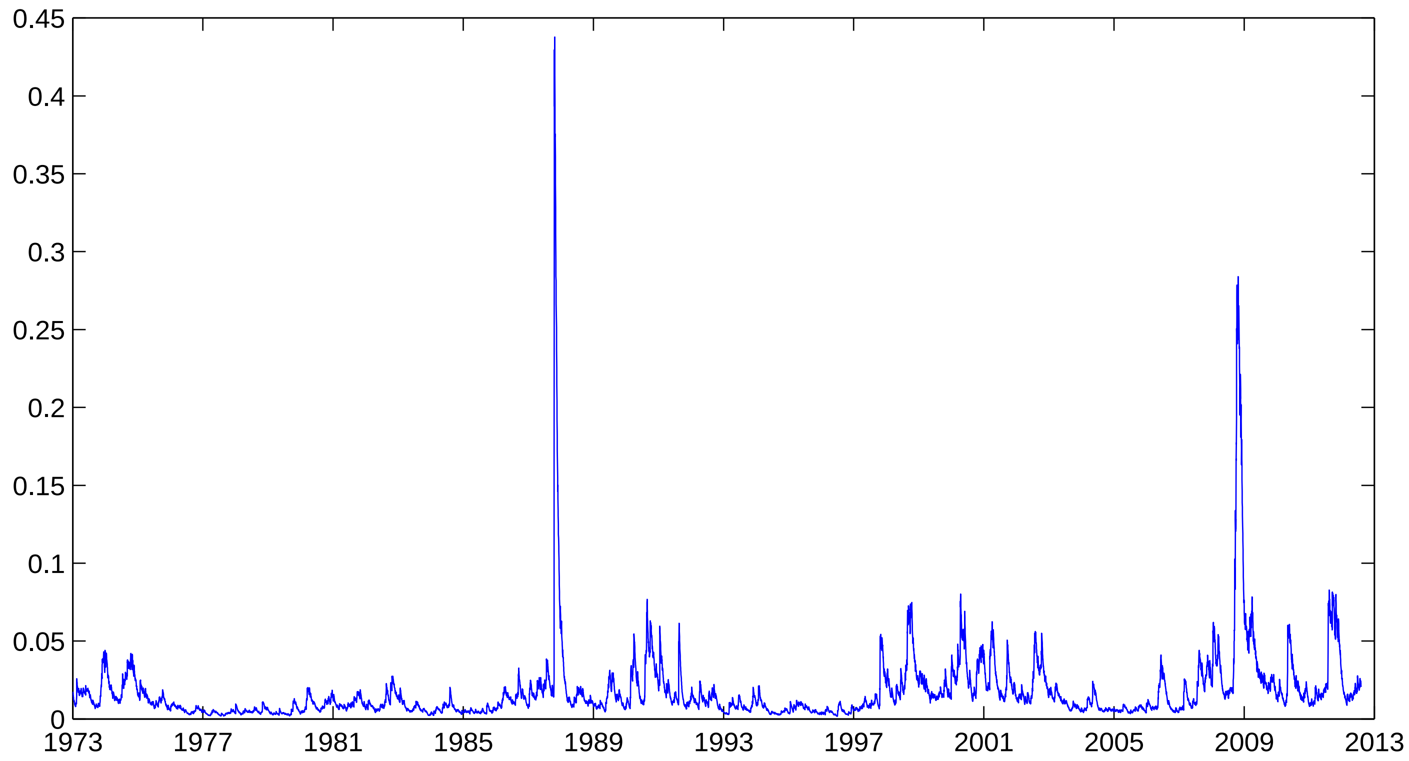
$$\hat{Q}_{t_i} \approx \frac{[\sqrt{Y}]_{\tau_{t_{i+1}}} - [\sqrt{Y}]_{\tau_{t_i}}}{t_{i+1} - t_i}$$

$$\tilde{Q}_{t_{i+1}} = \alpha\sqrt{t_{i+1} - t_i}\hat{Q}_{t_i} + (1 - \alpha\sqrt{t_{i+1} - t_i})\tilde{Q}_{t_i}, \alpha = 0.92$$

exponential smooth

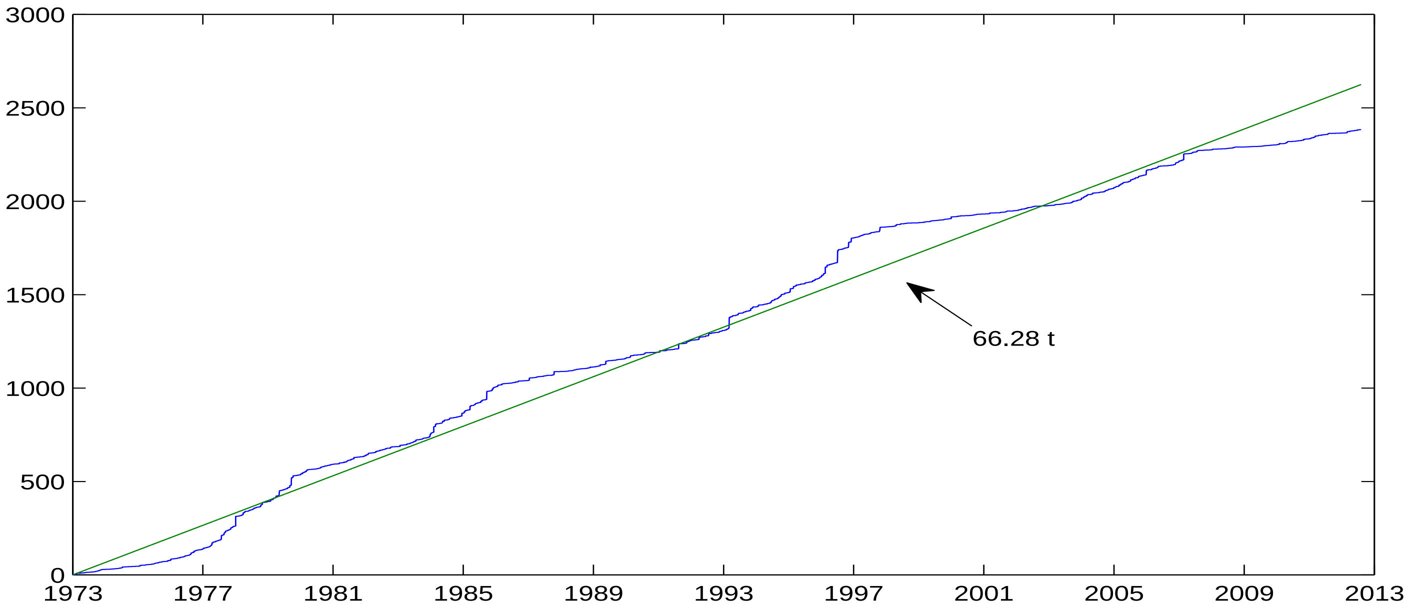
robust

Market activity: $M_t \approx 4\tilde{Q}_t$



$$M_0 = 0.0175$$

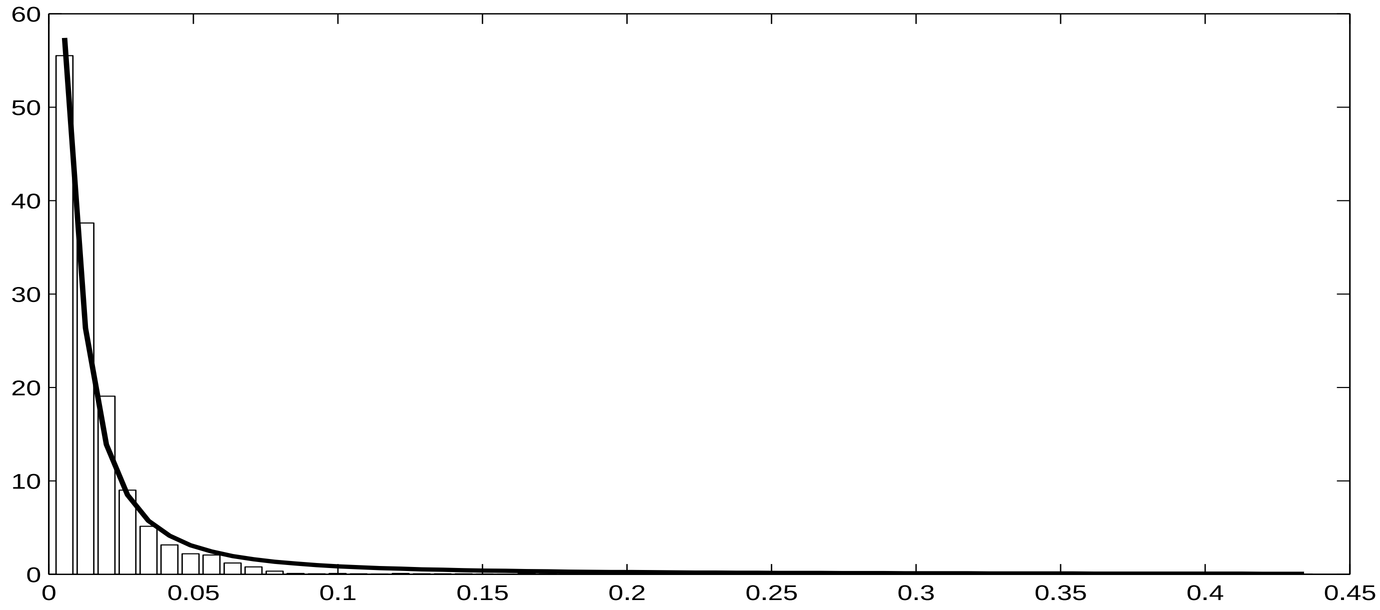
Step 4: Parameter γ:



$$\gamma = 265.12$$

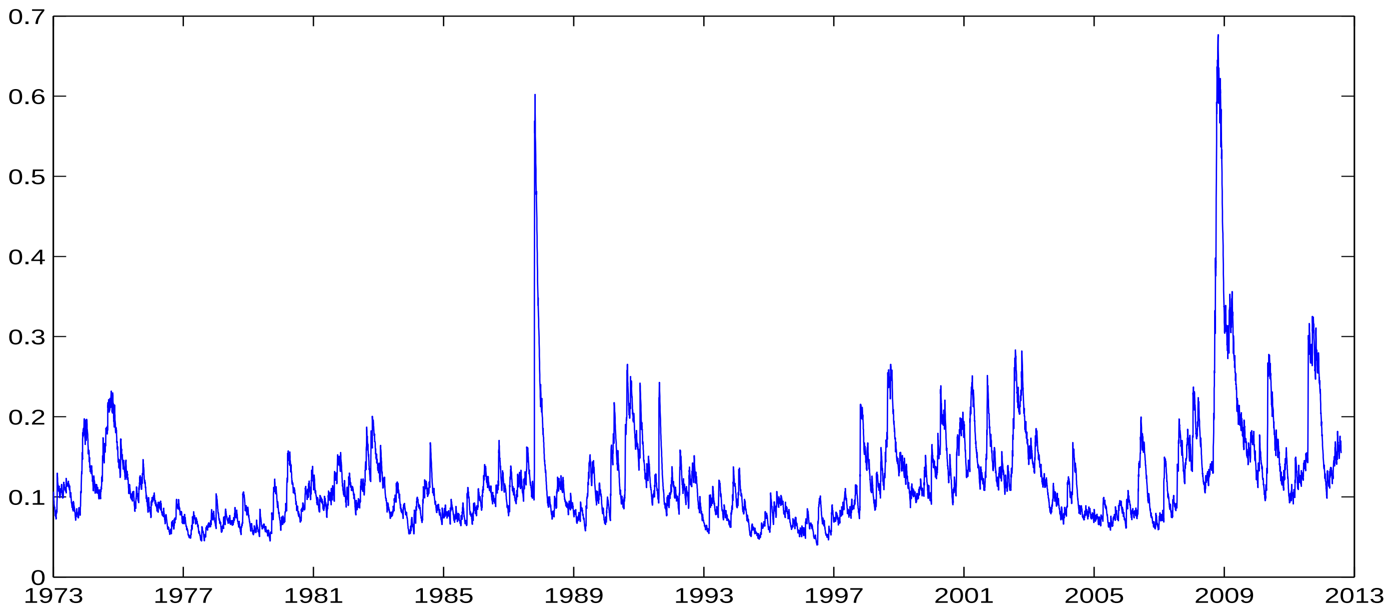
Step 5: Parameters ν and ϵ and

Step 6: Long Term Average Net Growth Rate a:



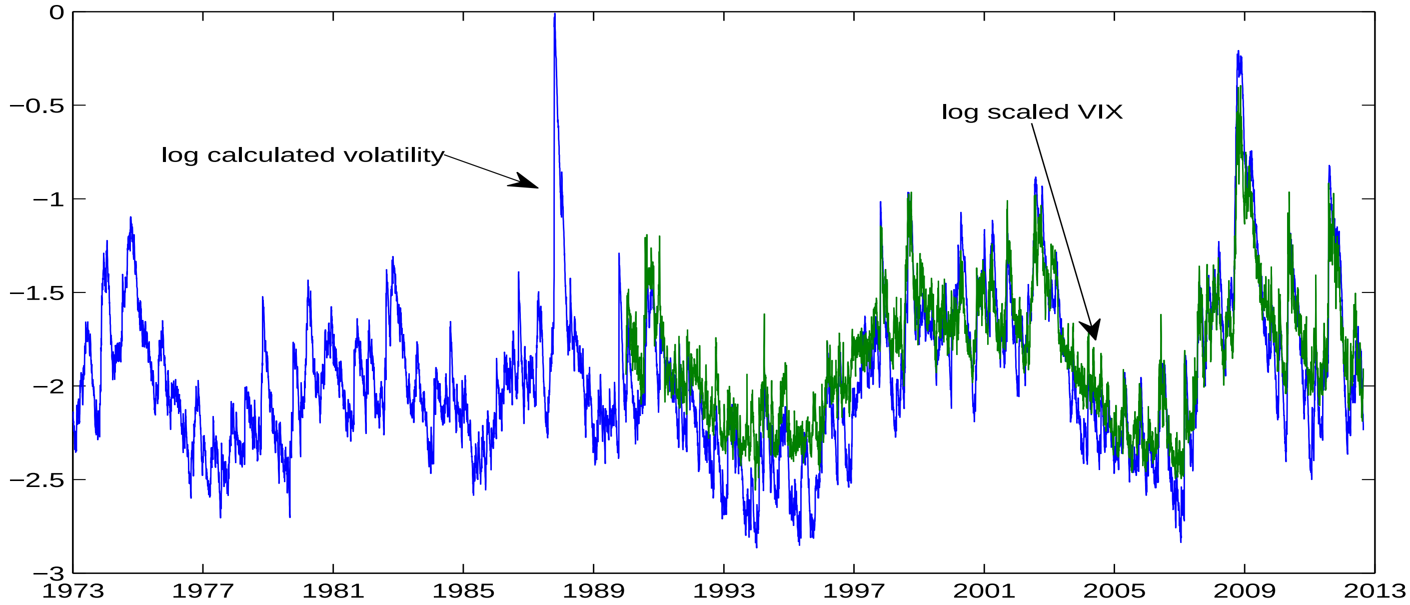
$\boldsymbol{\nu \approx 4}$, $\boldsymbol{\epsilon \approx 2.18 \Rightarrow a = 2.55} \Rightarrow$ no strong arbitrage

Fitted model allows visualizing
Calculated Volatility



$\boldsymbol{\sigma_t} \approx \sqrt{\frac{4\tilde{Q}_t}{Y_{\tau_t}}}$, average volatility: $\mathbf{11.9\%}$

Model applies also to proxies of numéraire portfolio
S&P500 and VIX

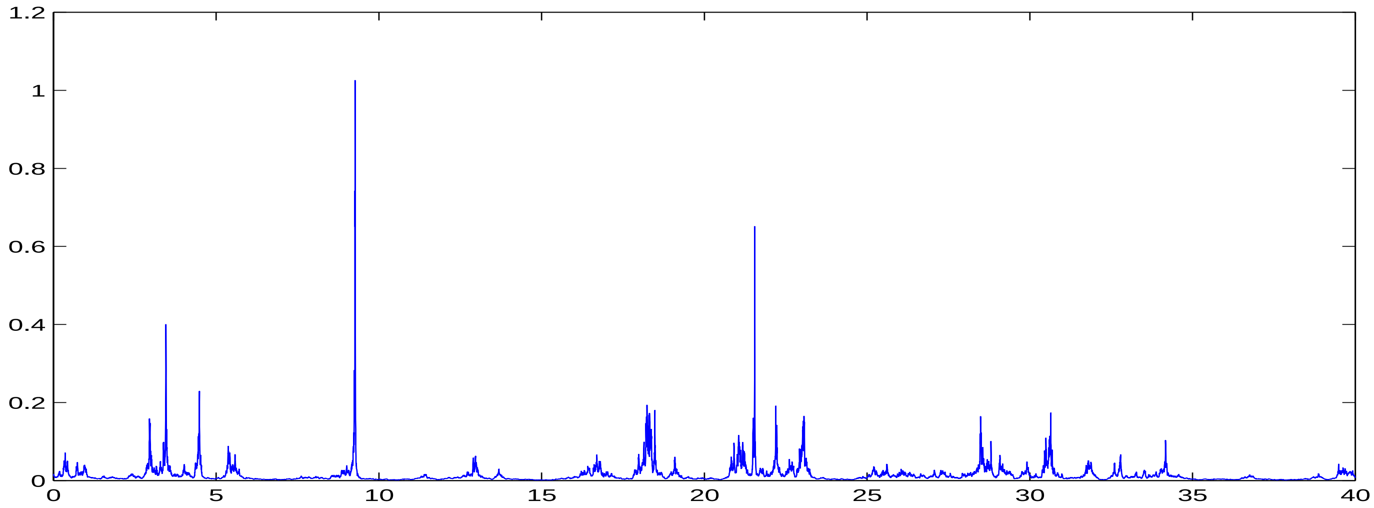


$$A = 52.09, \epsilon = 2.15, \gamma = 172.3, a = 1.5$$

Simulation Study

Step 1: Market activity:

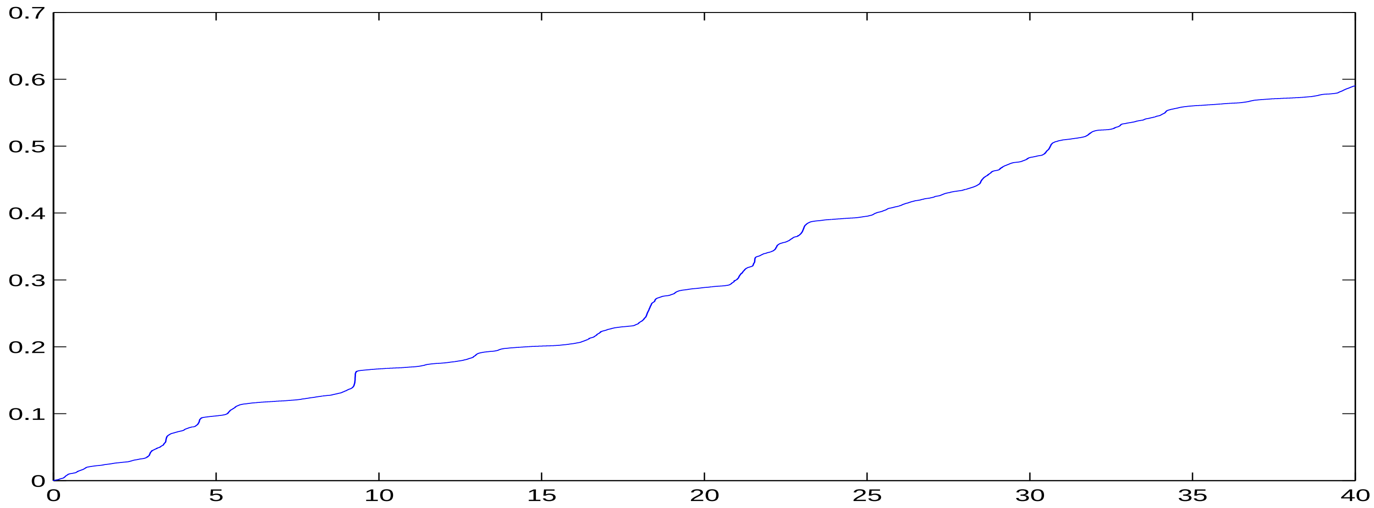
$$\frac{1}{M_{t_{i+1}}} = \frac{\gamma(1 - e^{-\epsilon(t_{i+1}-t_i)})}{4\epsilon}\left(\chi^2_{3,i} + \left(\sqrt{\frac{4\epsilon e^{-\epsilon(t_{i+1}-t_i)}}{\gamma(1 - e^{-\epsilon(t_{i+1}-t_i)})}\frac{1}{M_{t_i}}} + Z_i\right)^2\right)$$



exact simulation

Step 2: τ-time:

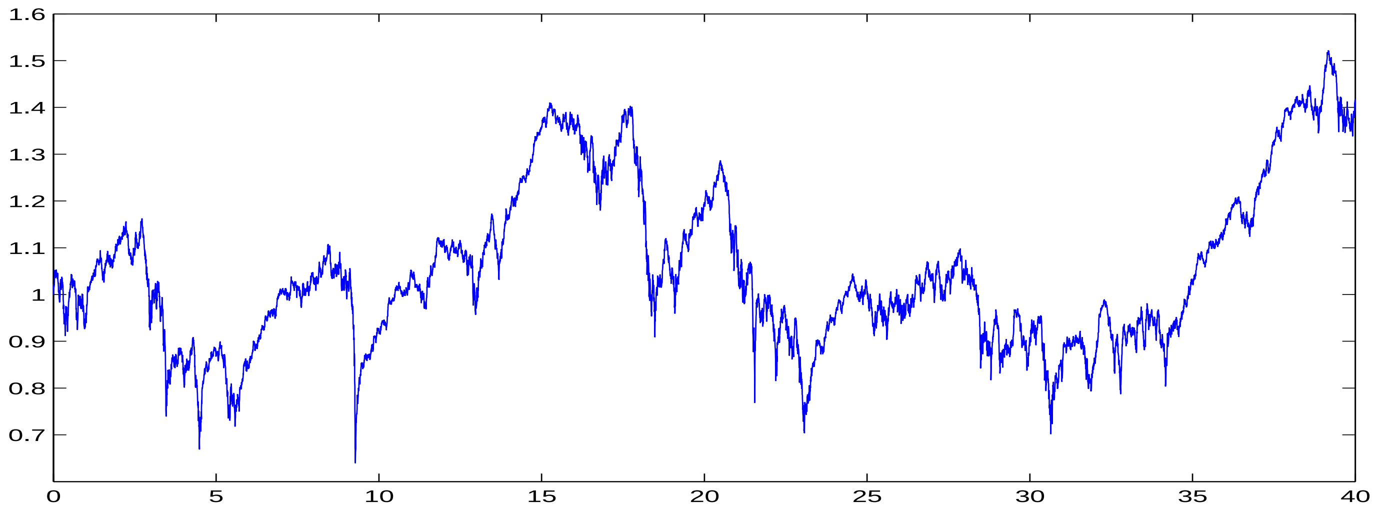
$$\tau_{t_{i+1}} - \tau_{t_i} = \int_{t_i}^{t_{i+1}} M_s ds \approx M_{t_i}(t_{i+1} - t_i)$$



almost exact simulation

Step 3: Normalized index:

$$Y_{\tau_{t_{i+1}}} = \frac{1 - e^{-(\tau_{t_{i+1}} - \tau_{t_i})}}{4} \left(\chi^2_{3,i} + \left(\sqrt{\frac{4e^{-(\tau_{t_{i+1}} - \tau_{t_i})}}{1 - e^{-(\tau_{t_{i+1}} - \tau_{t_i})}} Y_{\tau_{t_i}}} + Z_i \right)^2 \right)$$

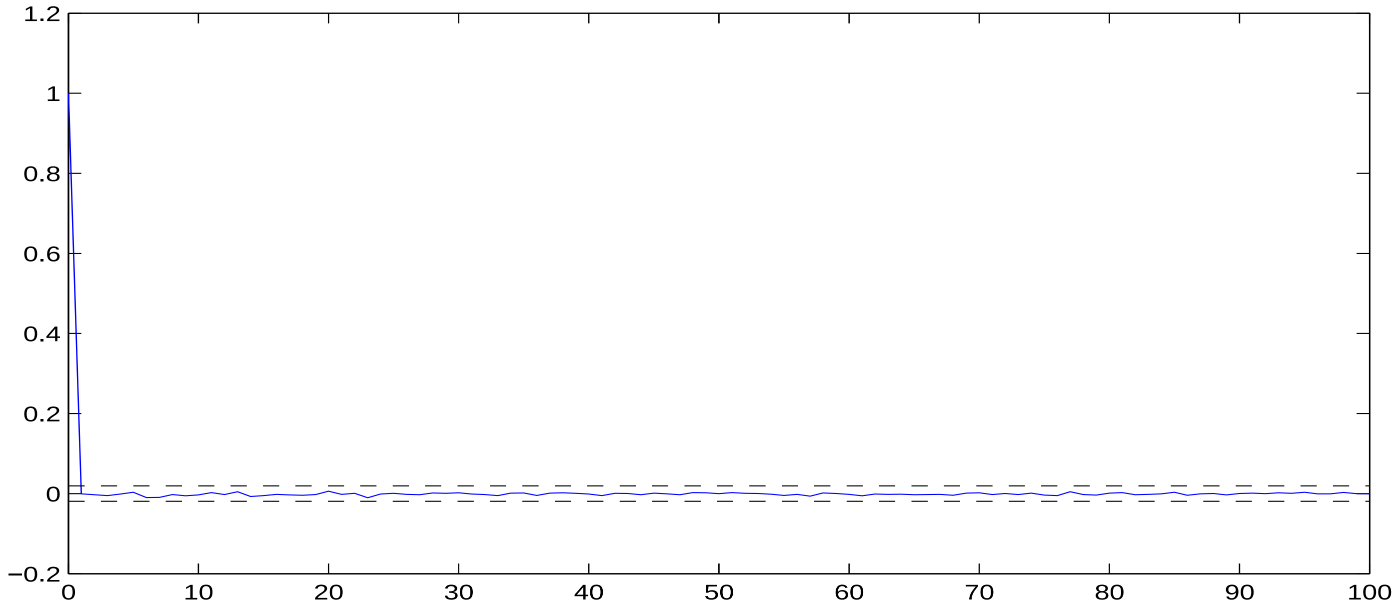


almost exact simulation

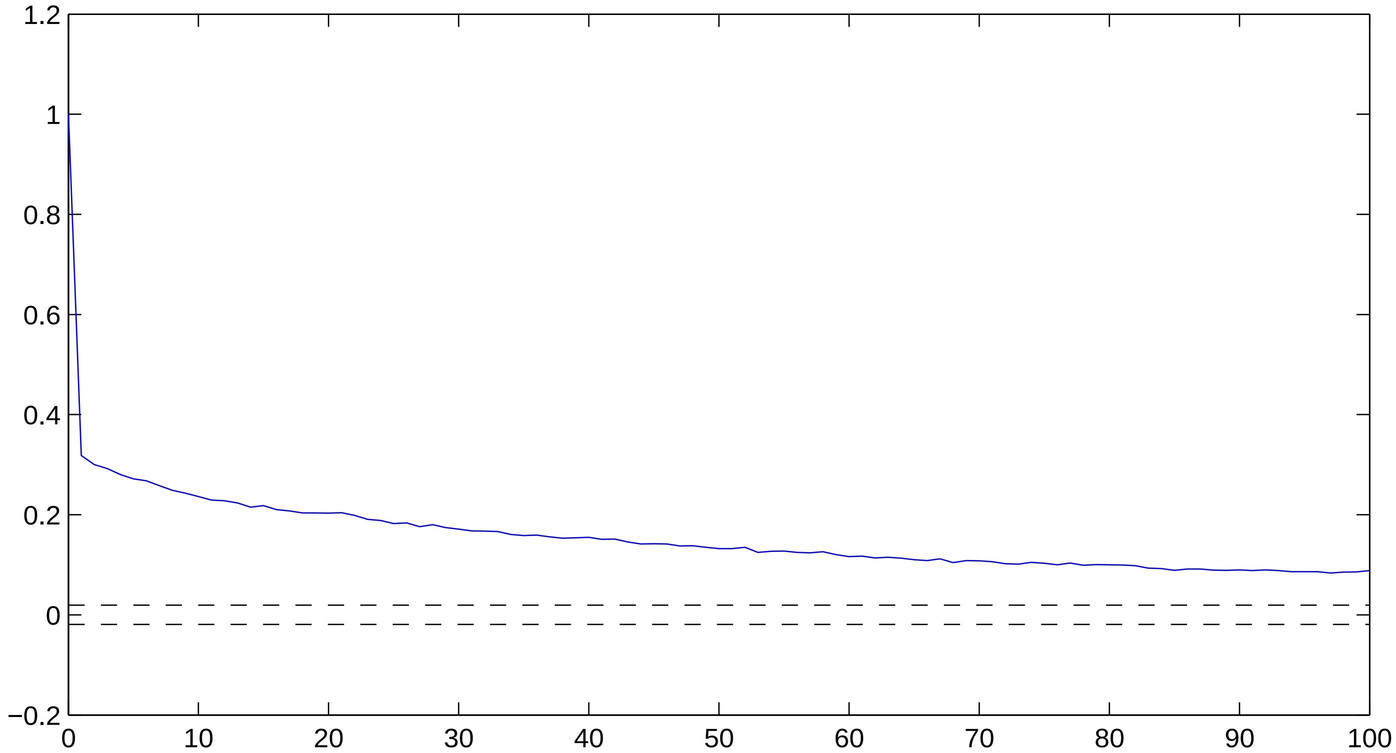
Model recovers stylized empirical facts:

Model is difficult to falsify: Popper (1934)

1. Uncorrelated returns



2. Correlated absolute returns

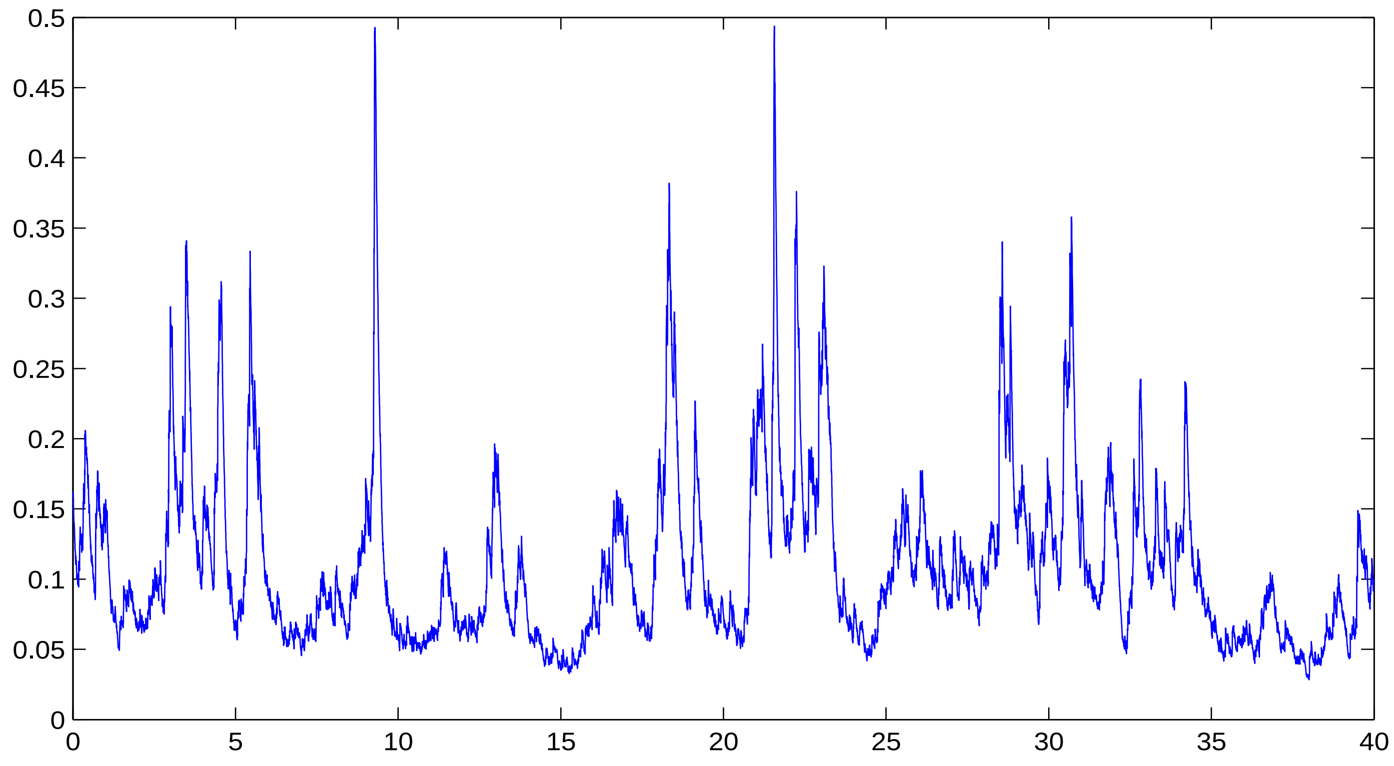


3. Student-t distributed returns

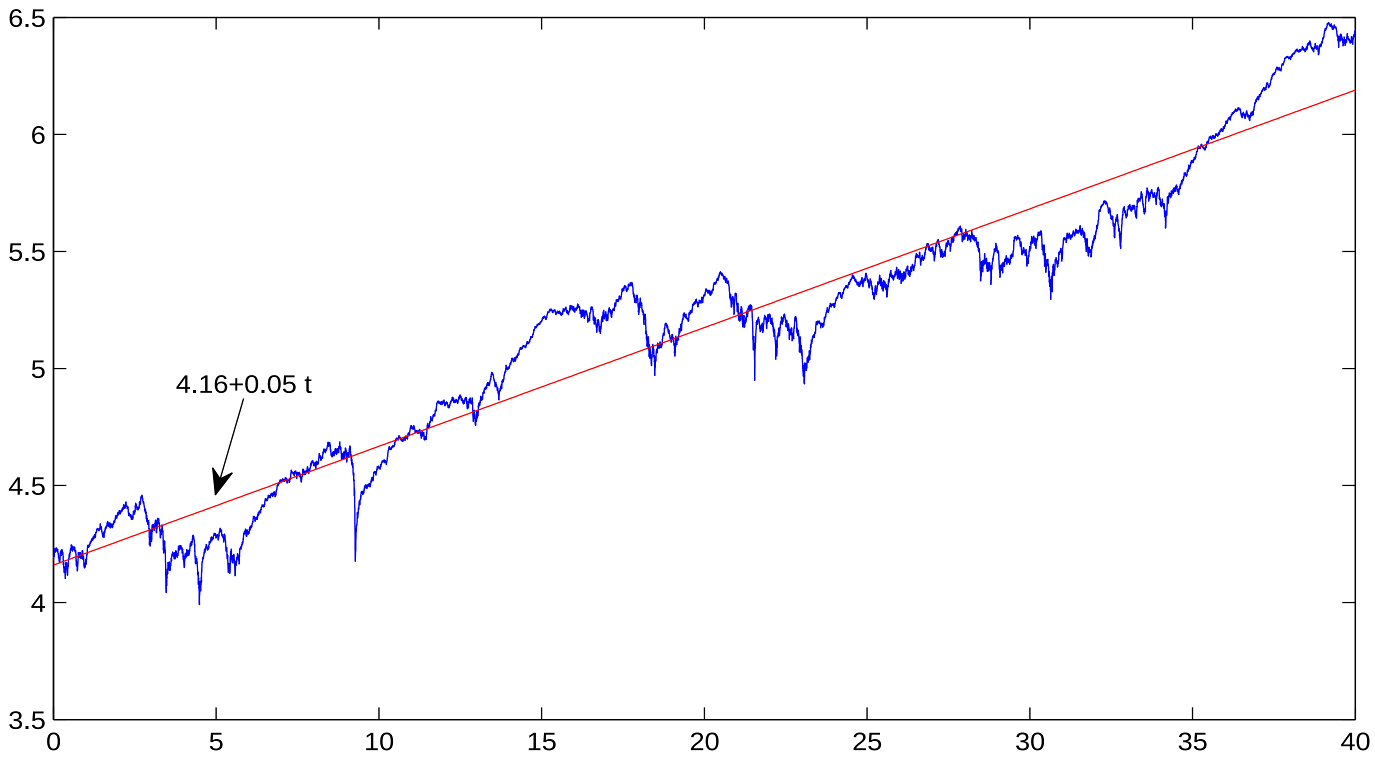
estimated statistics

Simulation	Student-t	NIG	Hyperbolic	VG
1	0.008934	37.474149	102.719638	131.240780
2	11.485226	11.175028	96.457136	132.916256
3	0.000000	100.928524	244.190151	294.719960
4	9.002421	35.759464	347.060676	331.014904
5	8.767003	11.551178	121.190482	144.084964
6	0.401429	60.570898	205.788160	252.591737
7	12.239056	4.354888	46.411554	78.273485
8	1.693411	23.910523	94.408789	130.623174
9	1.232454	47.830407	202.073144	237.168411
10	0.000000	43.037206	128.807757	162.582353
11	0.433645	47.782681	172.736397	208.847632
12	0.000000	56.019354	146.077121	185.624888
13	7.137154	48.219756	579.922931	477.383441
14	5.873948	16.515390	107.770531	135.508299
15	0.000000	54.718046	184.112794	217.304105
16	6.982560	3.991610	29.192198	47.105125
17	2.966916	22.914863	108.513143	138.044416
18	0.000000	52.066364	129.790856	160.373085
19	0.006909	39.568695	111.398645	143.914350
20	0.000001	56.845664	169.915512	211.260626
21	1.674578	17.681088	61.710576	90.679738
22	14.010840	3.279722	47.433693	73.789313
23	11.198940	12.074044	114.888817	143.800553
24	0.455557	27.676102	86.841704	114.452947

4. Volatility clustering

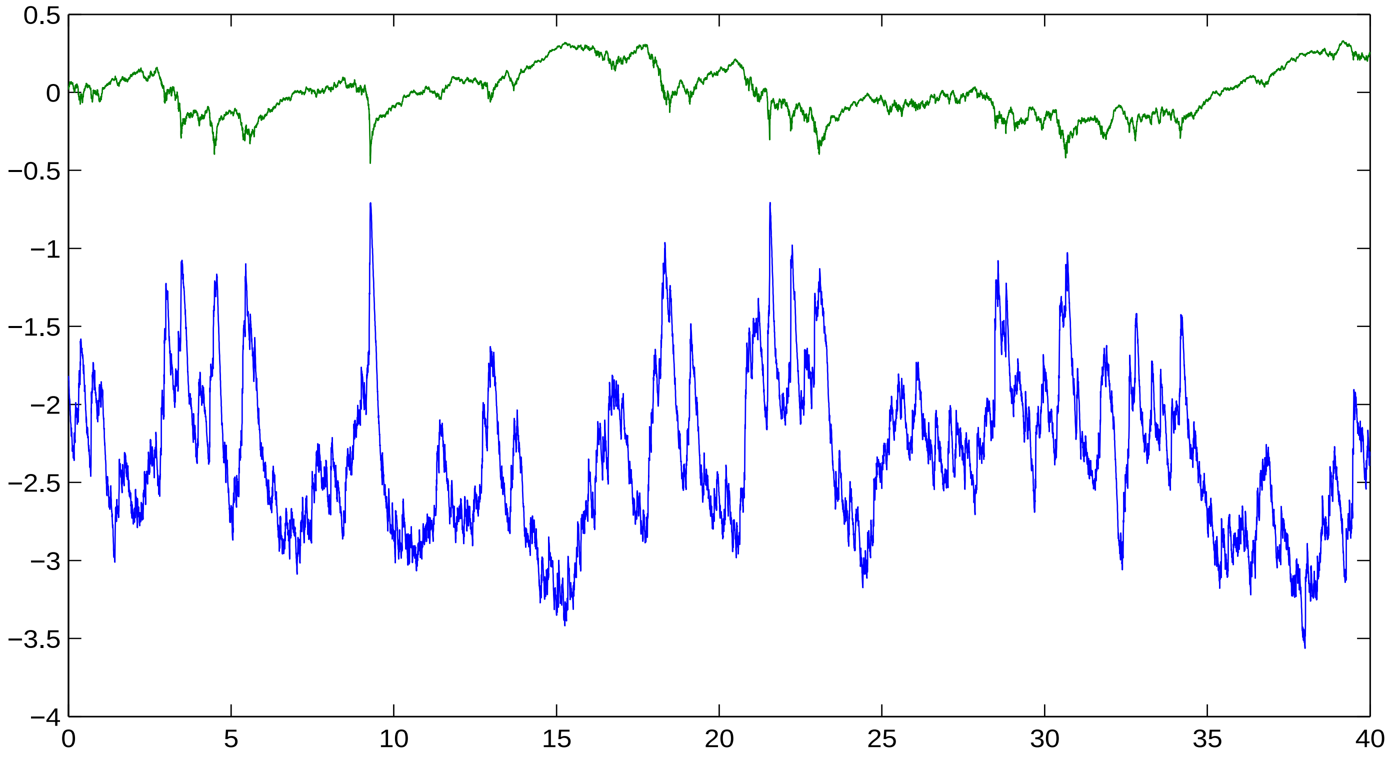


5. Long term exponential growth



6. Leverage effect and

7. Extreme volatility at major market downturns



Conclusions:

$\diamond$ equity index model: $\mathbf{3}$ initial parameters, $\mathbf{3}$ structural parameters and $\mathbf{1}$ Wiener process (nondiversifiable uncertainty)

$\diamond$ model recovers $\mathbf{7}$ stylized empirical facts

$\diamond$ allows long dated derivative pricing under benchmark approach

$\diamond$ leads outside classical theory

$\diamond$ if benchmarked savings account local martingale
$\Longrightarrow$ only $\mathbf{2}$ structural parameters

Squared Bessel Processes (*)

Revuz & Yor (1999)

- **squared Bessel process** BESQ$_{x}^{\delta}$

  dimension $\delta \geq 0$

  $$dX_{\varphi} = \delta \, d\varphi + 2 \sqrt{|X_{\varphi}|} \, dW_{\varphi}$$

  $\varphi \in [0, \infty)$ with $X_0 = x \geq 0$

- **scaling property**

  $$Z_{\varphi} = \frac{1}{a} X_{a\varphi}$$

  BESQ$_{\frac{x}{a}}^{\delta}$, $a > 0$

- **sum of squares**

$$X_\varphi = \sum_{k=1}^{\delta} (w^k + W_\varphi^k)^2$$

W_φ^k Wiener process

$\Longrightarrow$

$$dX_\varphi = \delta\, d\varphi + 2 \sum_{k=1}^{\delta} \left(w^k + W_\varphi^k \right) dW_\varphi^k$$

$$X_0 = \sum_{k=1}^{\delta} (w^k)^2 = x$$

- setting

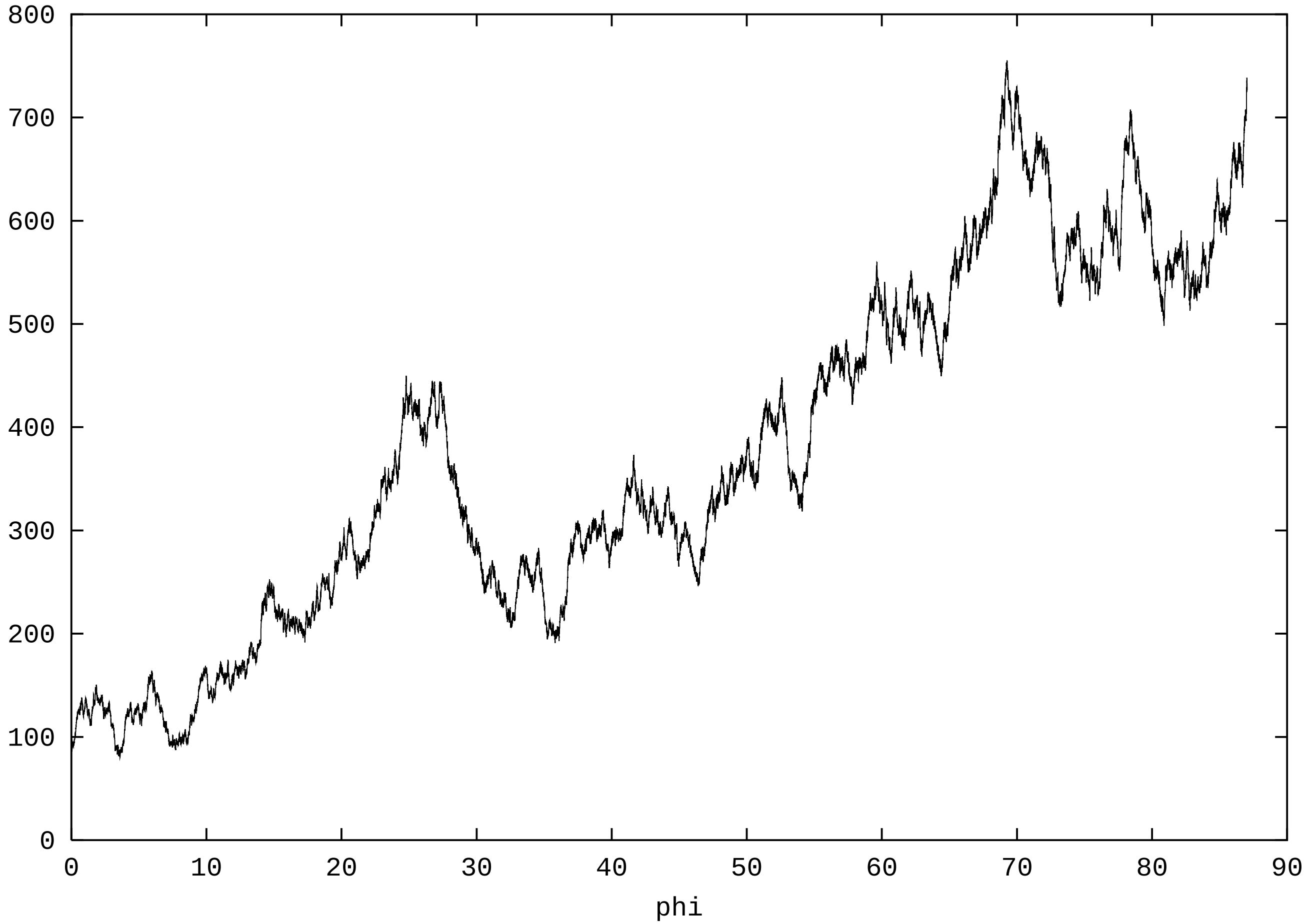
$$dW_\varphi = |X_\varphi|^{-\frac{1}{2}} \sum_{k=1}^{\delta} \left(w^k + W_\varphi^k\right) dW_\varphi^k$$

- quadratic variation

$$[W]_\varphi = \int_0^\varphi \frac{1}{X_s} \sum_{k=1}^{\delta} \left(w^k + W_s^k\right)^2 ds = \varphi$$

Lévy's theorem

$\Longrightarrow$ W Wiener process in φ time



Squared Bessel process of dimension $\delta = 4$ in φ-time

- **additivity property**

  Shiga & Watanabe (1973)

  $\boldsymbol{X} = \{\boldsymbol{X}_\varphi,\ \varphi \in [0,\infty)\}$ BESQ$_{\boldsymbol{x}}^{\boldsymbol{\delta}}$

  $\boldsymbol{Y} = \{\boldsymbol{Y}_\varphi,\ \varphi \in [0,\infty)\}$ independent BESQ$_{\boldsymbol{y}}^{\boldsymbol{\delta'}}$

  $\Longrightarrow \quad \boldsymbol{X}_\varphi + \boldsymbol{Y}_\varphi$ forms BESQ$_{\boldsymbol{x}+\boldsymbol{y}}^{\boldsymbol{\delta}+\boldsymbol{\delta'}}$

  $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{\delta}, \boldsymbol{\delta'} \geq \mathbf{0}$

- $\boldsymbol{\delta > 2}$, strict positivity

  with $\boldsymbol{X_0 = x > 0}$

$$\boldsymbol{P\left(\inf_{0\le\varphi<\infty} X_\varphi > 0\right) = 1}$$

- $\boldsymbol{\delta = 2}$, no strict positivity

$$\boldsymbol{P\left(\inf_{0\le\varphi<\infty} X_\varphi > 0\right) = 0}$$

- $\boldsymbol{\delta \in [0, 2)}$, no strict positivity

  $\boldsymbol{X_0 = x > 0}$

$$\boldsymbol{P\left(\inf_{0\le\varphi\le\varphi'} X_\varphi = 0\right) > 0}$$

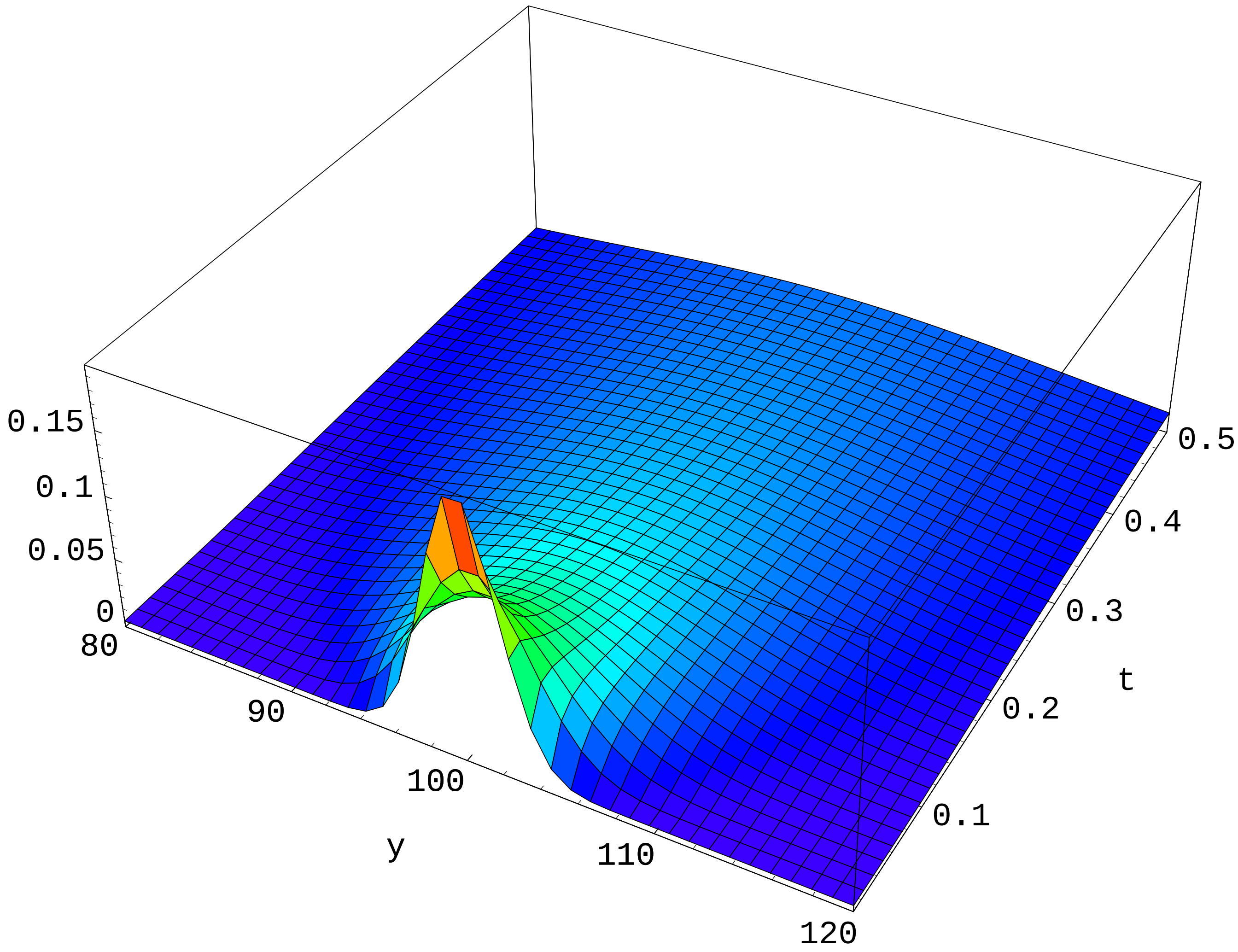
- **transition density**

  $\delta > 0\,,\ x > 0$

$$p_\delta(\varrho, x; \varphi, y) = \frac{1}{2(\varphi - \varrho)} \left(\frac{y}{x}\right)^{\frac{\nu}{2}} \exp\left\{-\frac{x+y}{2(\varphi - \varrho)}\right\} I_\nu\left(\frac{\sqrt{x\,y}}{\varphi - \varrho}\right)$$

I_ν modified Bessel function of the first kind

index $\nu = \frac{\delta}{2} - 1$



Transition density of squared Bessel process for $\delta = 4$

- $Y = \frac{X_\varphi}{\varphi}$ non-central chi-square distributed

  dimension δ

  non-centrality parameter $\ell = \frac{x}{\varphi}$

$$P\left(\frac{X_\varphi}{\varphi} < u\right) = \sum_{k=0}^{\infty} \frac{\exp\left\{-\frac{\ell}{2}\right\}\left(\frac{\ell}{2}\right)^k}{k\,!}\left(1 - \frac{\Gamma\left(\frac{u}{2};\frac{\delta+2k}{2}\right)}{\Gamma\left(\frac{\delta+2k}{2}\right)}\right)$$

$\Gamma(\cdot;\cdot)$ incomplete gamma function

- **Moments** for $\alpha > -\frac{\delta}{2},\ \varphi \in (0, \infty)$ and $\delta > 2$

$$E\left(X_\varphi^\alpha \,\middle|\, \mathcal{A}_0\right) = \begin{cases} (2\,\varphi)^\alpha \ \exp\left\{\frac{-X_0}{2\,\varphi}\right\} \sum_{k=0}^\infty \left(\frac{X_0}{2\,\varphi}\right)^k \frac{\Gamma\left(\alpha+k+\frac{\delta}{2}\right)}{k\,!\,\Gamma\left(k+\frac{\delta}{2}\right)} & \\ \qquad\qquad \text{for} \quad \alpha > -\frac{\delta}{2} & \\ \infty \qquad \text{for} \quad \alpha \le -\frac{\delta}{2} & \end{cases}$$

$\Longrightarrow$

$$E\left(X_\varphi \,\middle|\, \mathcal{A}_0\right) = X_0 + \delta\,\varphi$$

- by monotonicity of the gamma function

$$\boldsymbol{E}\left(\boldsymbol{X}_{\varphi}^{\alpha} \,\middle|\, \boldsymbol{\mathcal{A}}_0\right) \;\leq\; (2\varphi)^{\alpha} \exp\left\{\frac{-\boldsymbol{X}_0}{2\varphi}\right\}\left(\frac{\Gamma\left(\alpha+\frac{\delta}{2}\right)}{\Gamma\left(\frac{\delta}{2}\right)} + \exp\left\{\frac{\boldsymbol{X}_0}{2\varphi}\right\}\right)$$

$$< \;\infty$$

- for $\delta = 4$

$$\boldsymbol{E}\left(\boldsymbol{X}_{\varphi}^{-1} \,\middle|\, \boldsymbol{\mathcal{A}}_0\right) = \boldsymbol{X}_0^{-1}\left(1 - \exp\left\{\frac{-\boldsymbol{X}_0}{2\,\varphi}\right\}\right)$$

A Strict Local Martingale

$Z_\varphi = X_\varphi^{-1}$ inverse of a squared Bessel process X_φ of dimension 4

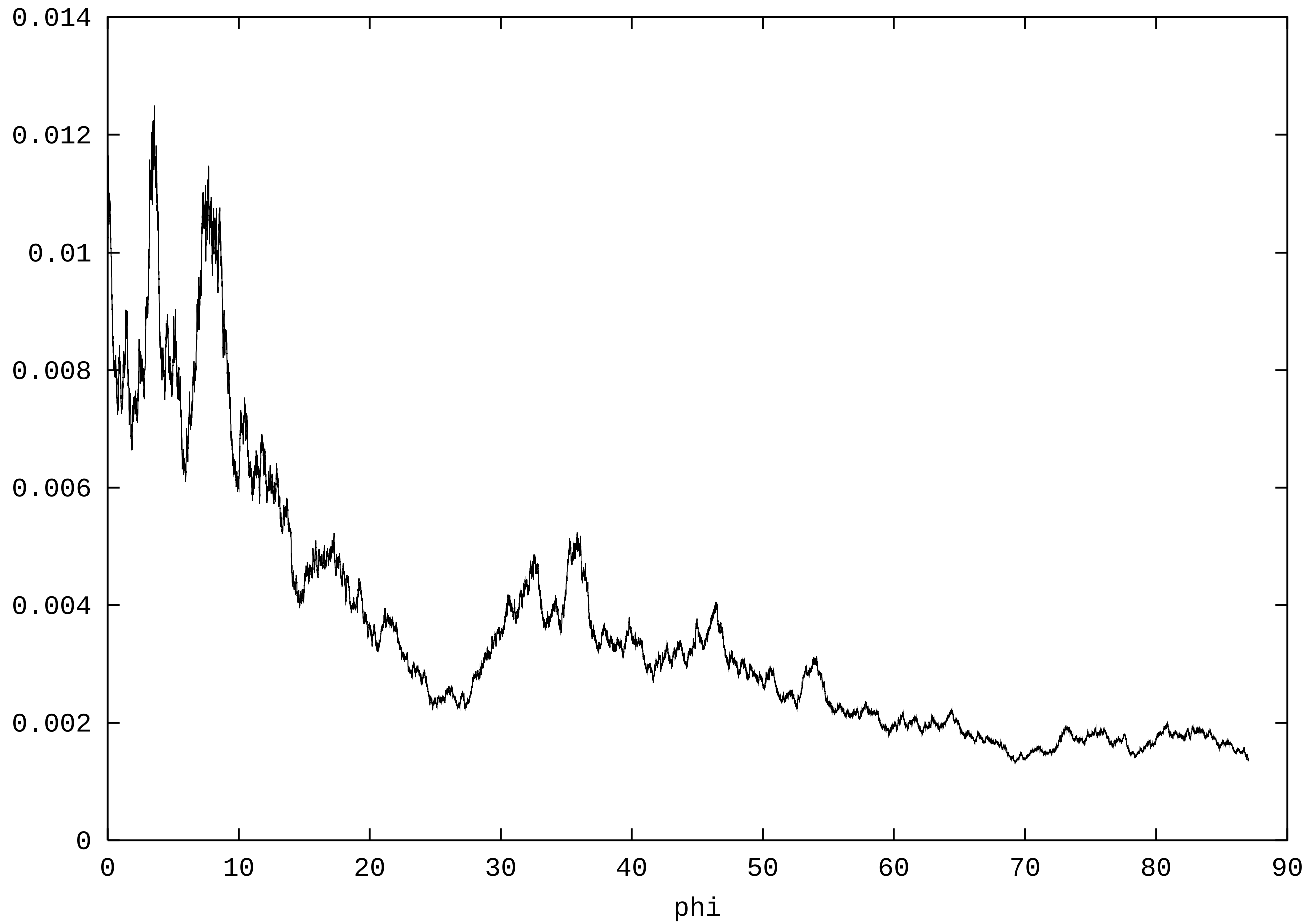
$$dZ_\varphi = -2\, Z_\varphi^{\frac{3}{2}}\, dW_\varphi$$

$\Longrightarrow$ local martingale

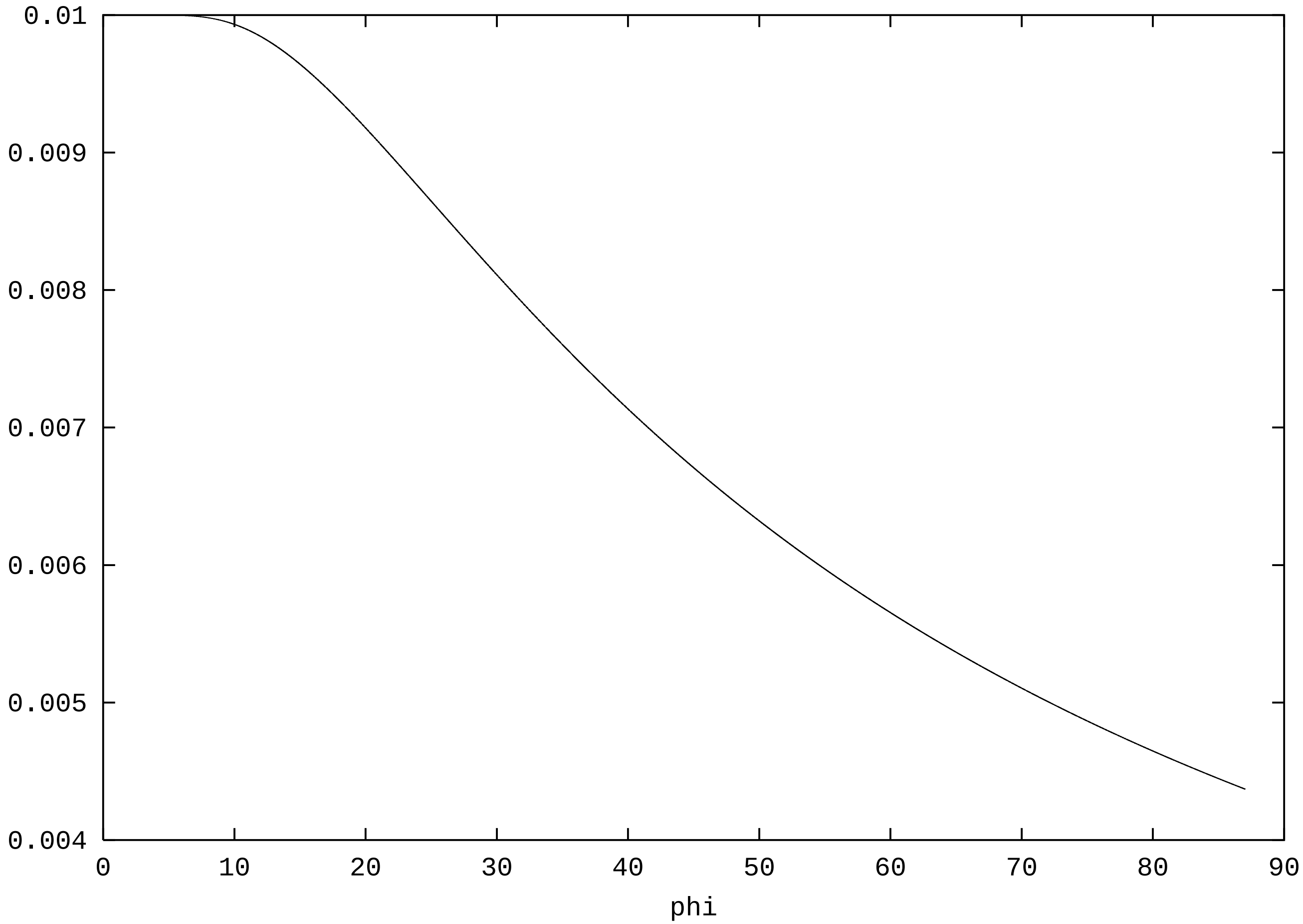
$$\begin{aligned} E\left(Z_\varphi \,\middle|\, \mathcal{A}_0\right) &= E\left(X_\varphi^{-1} \,\middle|\, \mathcal{A}_0\right) \\ &= Z_0 \left(1 - \exp\left\{\frac{-1}{2\, Z_0\, \varphi}\right\}\right) < Z_0 \end{aligned}$$

$\Longrightarrow$ strict local martingale

$\Longrightarrow$ **strict supermartingale**



Inverse of a squared Bessel process of dimension $\delta = 4$ in φ-time



Expectation of the inverse of the squared Bessel process for $\boldsymbol{\delta = 4}$ in φ-time

Time Transformation

- **φ-time**

$$\varphi(t) = \varphi(0) + \frac{1}{4} \int_0^t \frac{c_u^2}{s_u} \, du$$

$$t \in [0, \infty), \; s_0 > 0$$

with

$$s_t = s_0 \, \exp\left\{ \int_0^t b_u \, du \right\}$$

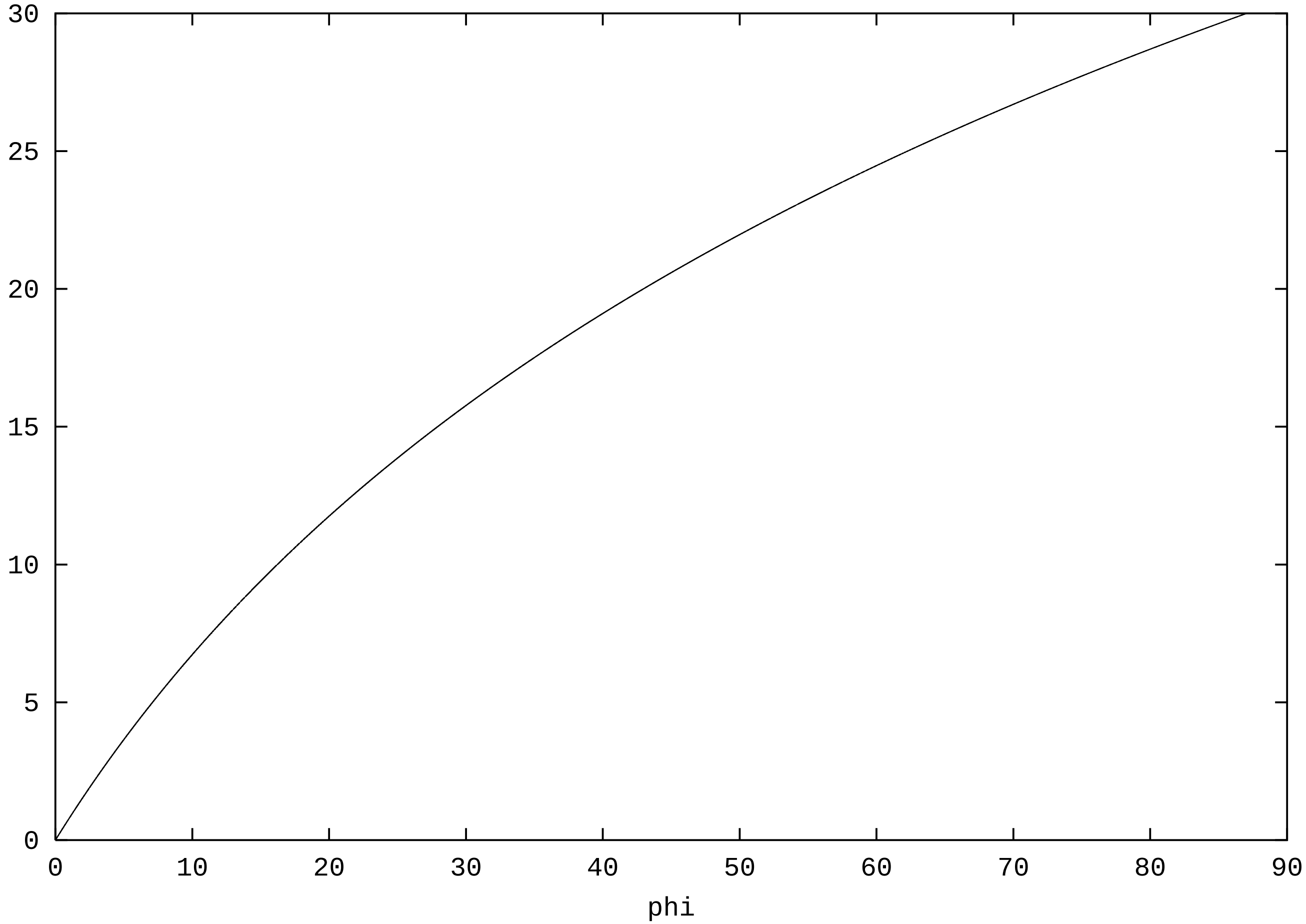
- **φ-time**

  for constant $b < 0$, $c \neq 0$

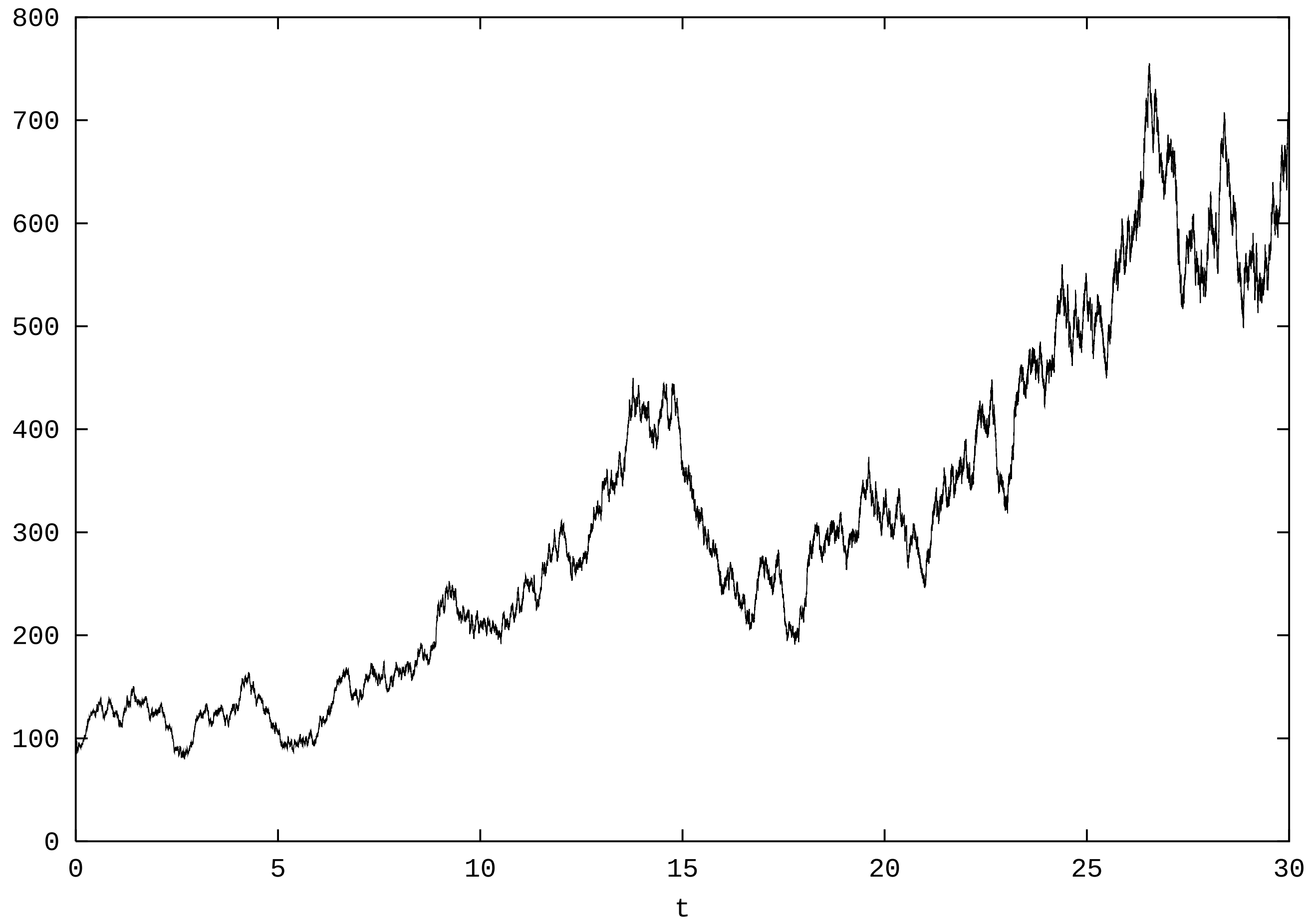
$$\varphi(t) = \varphi(0) + \frac{c^2}{4\,b\,s_0}\,(1 - \exp\{-b\,t\})$$

- time

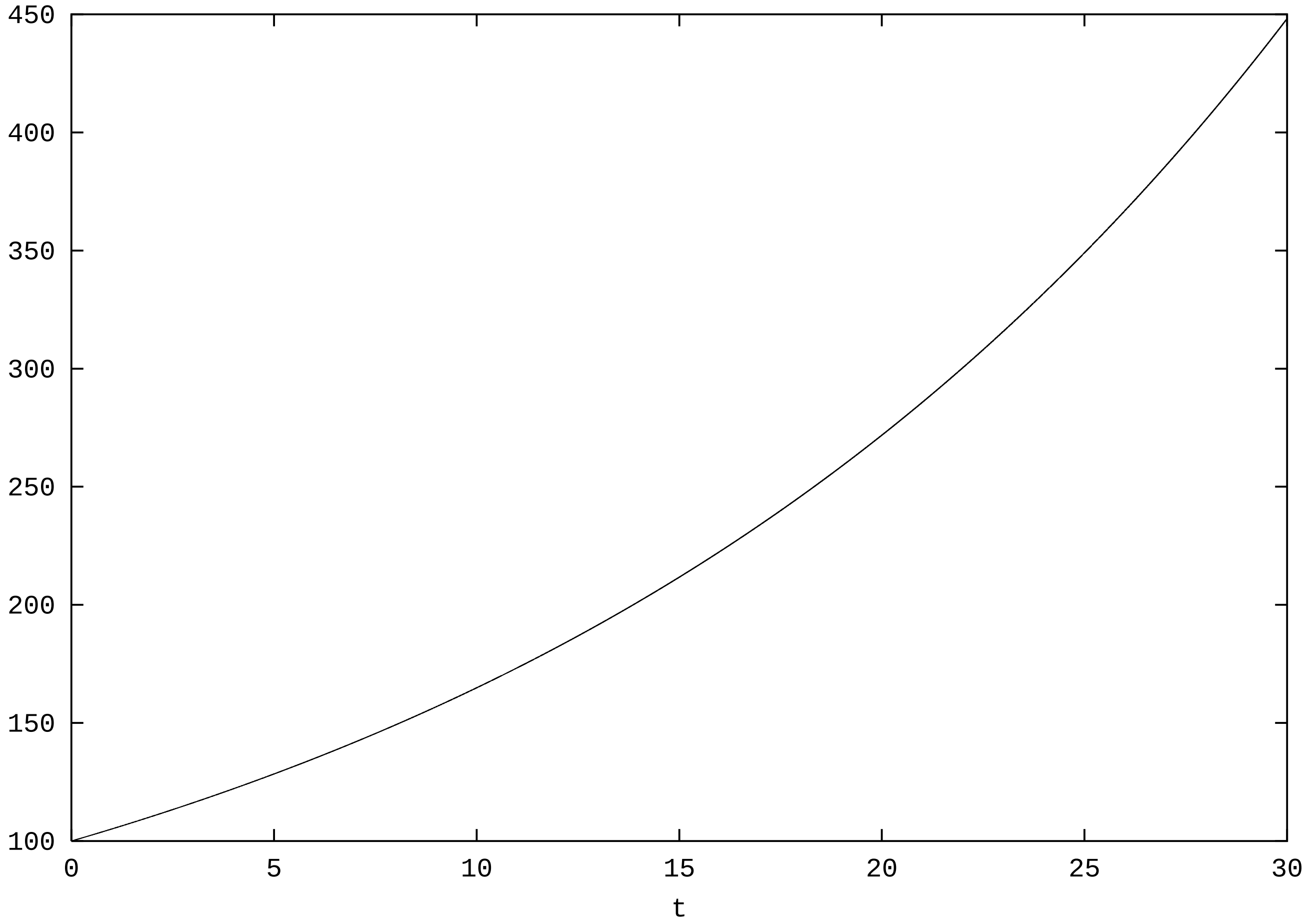
$$t(\varphi) = -\frac{1}{b}\,\ln\left(1 - \frac{4\,b\,s_0}{c^2}\,(\varphi - \varphi(0))\right)$$



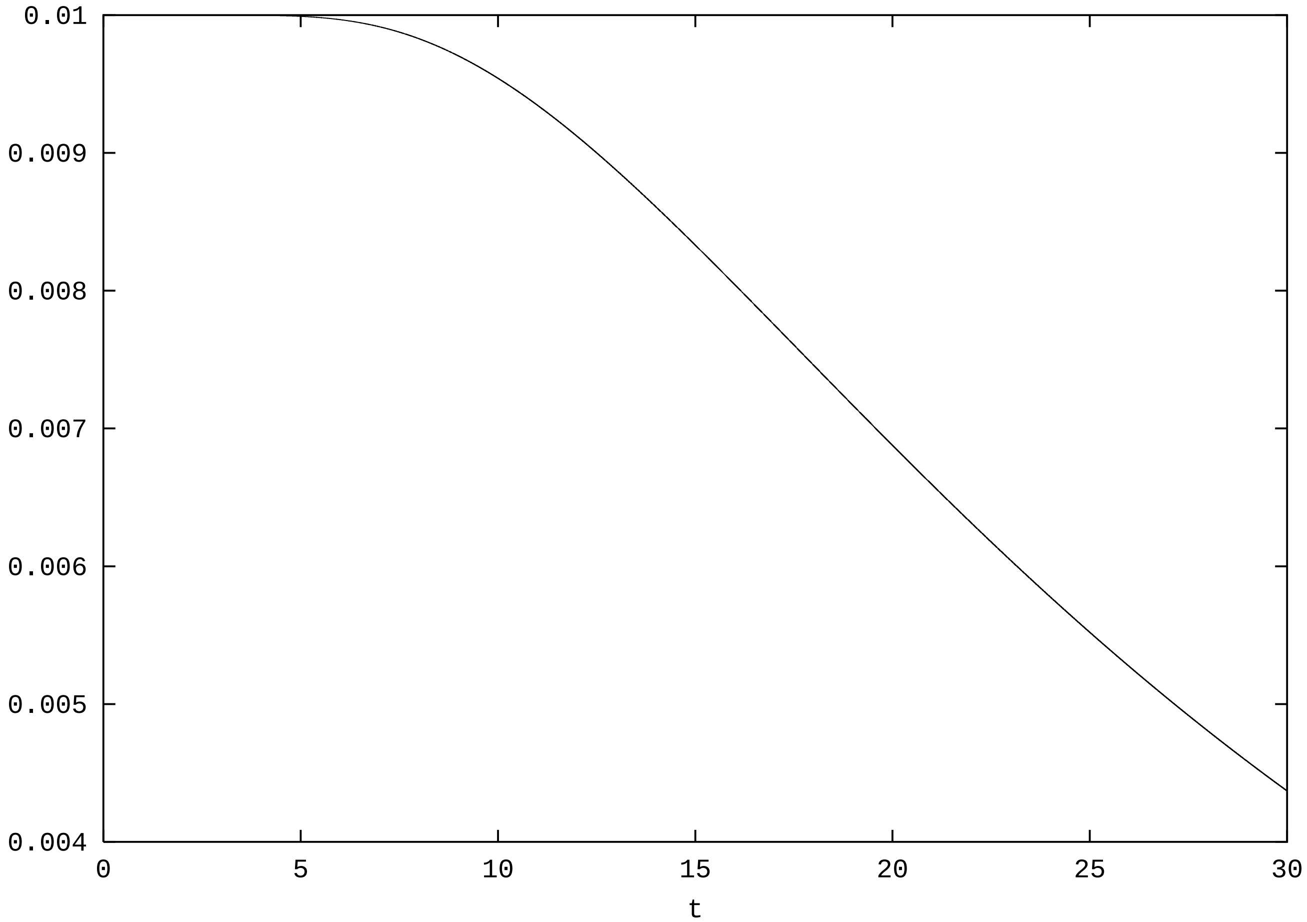
Time $t(\varphi)$ against φ-time



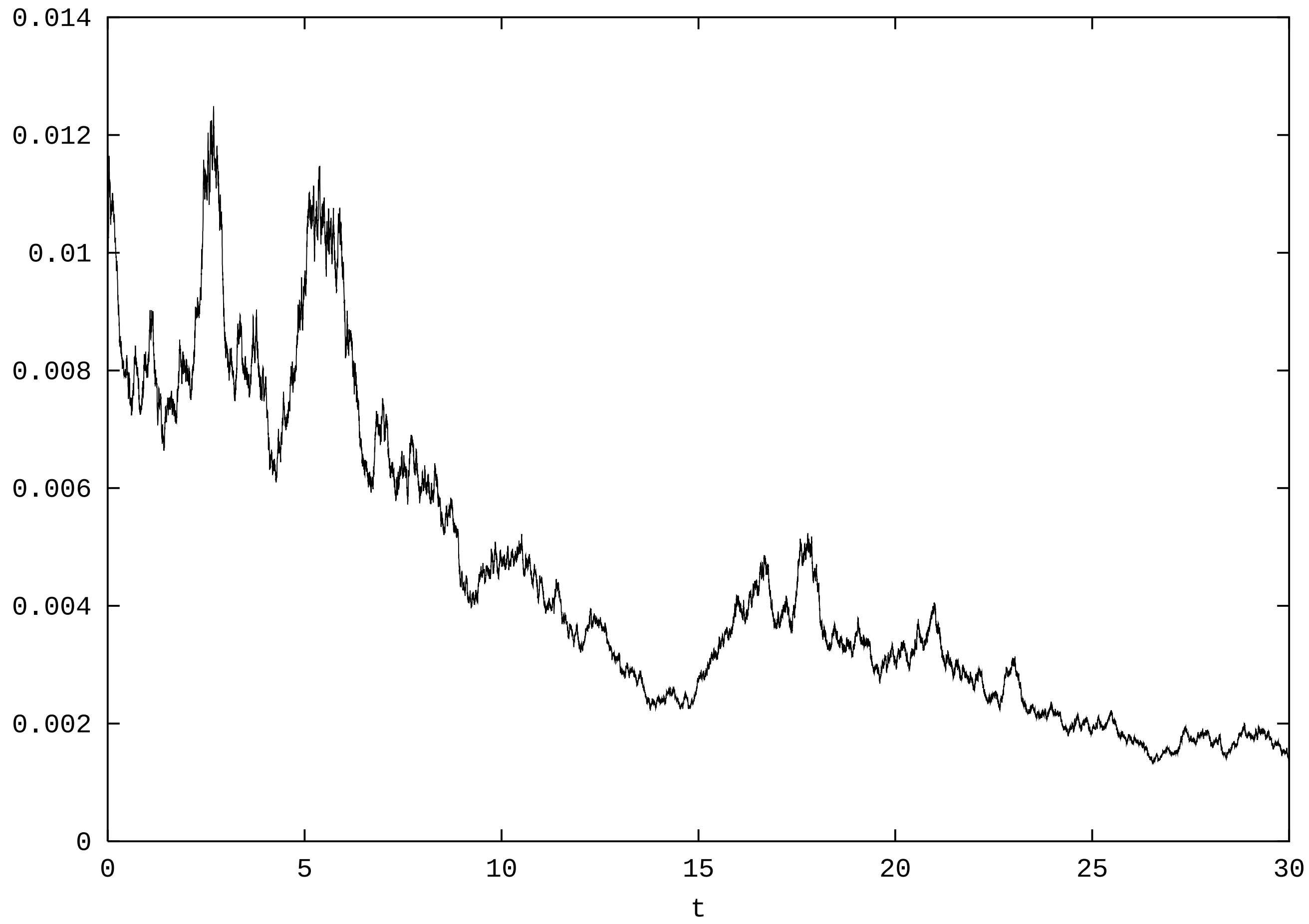
Squared Bessel process in dependence on time t



Expectation of a squared Bessel process in dependence on time t



Expectation of the inverse of a squared Bessel process in dependence on time t



Inverse of squared Bessel process in dependence on time t

Square Root Process

$$Y_t = s_t \, X_{\varphi(t)}$$

$$\begin{aligned} dY_t &= s_t \, dX_{\varphi(t)} + X_{\varphi(t)} \, ds_t \\ &= s_t \, \delta \, d\varphi(t) + s_t \, 2 \, \sqrt{X_{\varphi(t)}} \, dW_{\varphi(t)} + X_{\varphi(t)} \, s_t \, b_t \, dt \\ &= \left(\frac{\delta}{4} \, c_t^2 + b_t \, Y_t \right) dt + c_t \, \sqrt{Y_t} \, \sqrt{\frac{4 \, s_t}{c_t^2}} \, dW_{\varphi(t)} \end{aligned}$$

$$dU_t = \sqrt{\frac{4\,s_t}{c_t^2}}\,dW_{\varphi(t)}$$

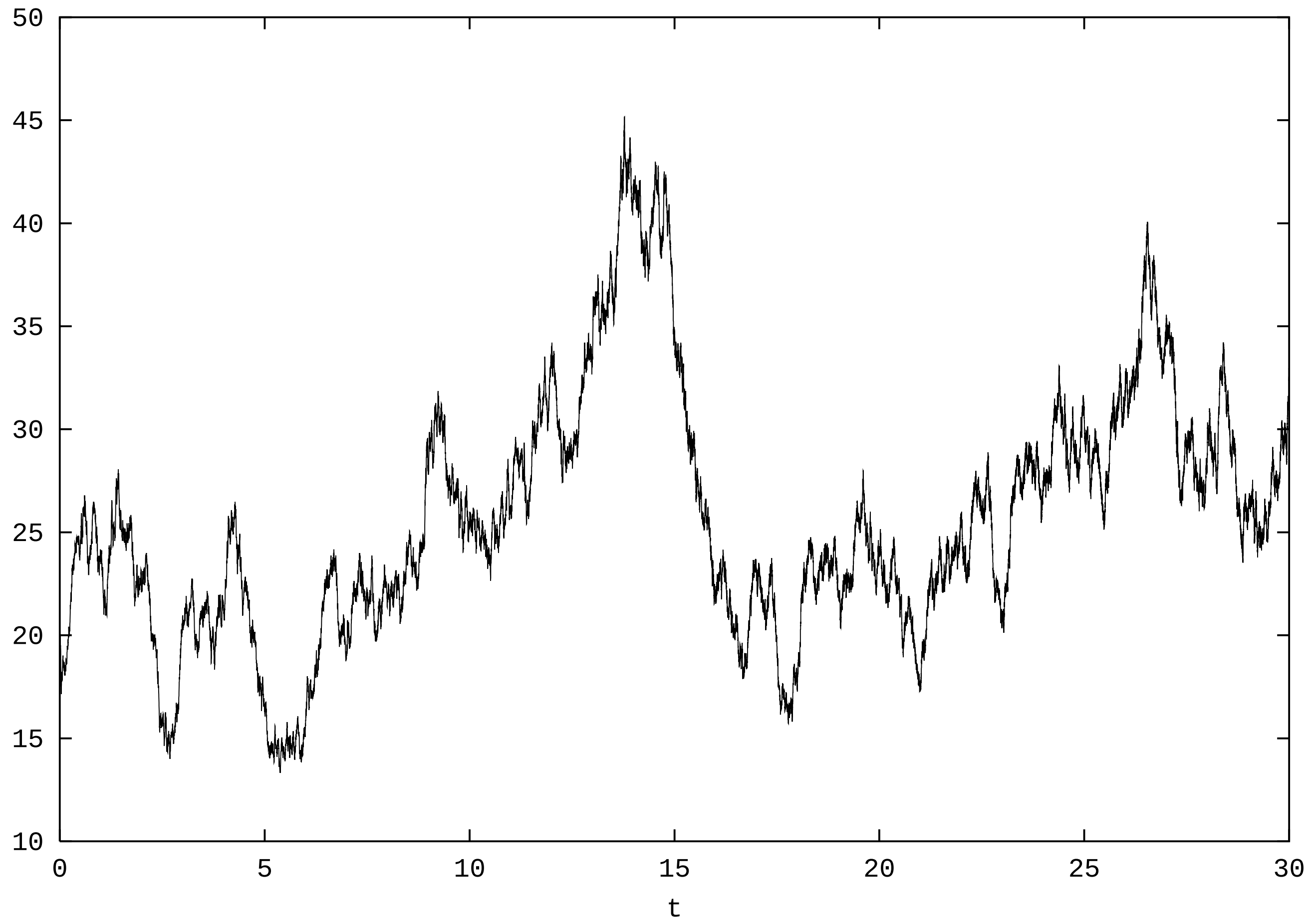
has quadratic variation

$$[U]_t = \int_0^t \frac{4\,s_z}{c_z^2}\,d\varphi(z) = t$$

U Wiener process

$\Longrightarrow$ SR process

$$dY_t = \left(\frac{\delta}{4}\,c_t^2 + b_t\,Y_t\right)dt + c_t\,\sqrt{Y_t}\,dU_t$$



Sample path of a square root process in dependence on time t

- **moment**

$$E\left(Y_t^{\alpha} \,\middle|\, \mathcal{A}_0\right) = (2\,\bar\varphi_t\,\bar s_t)^{\alpha}\exp\left\{-\frac{Y_0}{2\,\bar\varphi_t}\right\}\sum_{k=0}^{\infty}\left(\frac{Y_0}{2\,\bar\varphi_t}\right)^{k}\frac{\Gamma(\alpha+k+\frac{\delta}{2})}{k!\,\Gamma(k+\frac{\delta}{2})}$$

$$\delta > 2,\ \alpha > -\tfrac{\delta}{2}$$

- **moment estimate**

$$E\left(Y_t^{\alpha} \,\middle|\, \mathcal{A}_0\right) \le (2\,\bar\varphi_t\,\bar s_t)^{\alpha}\ \exp\left\{-\frac{Y_0}{2\,\bar\varphi_t}\right\}\left(\frac{\Gamma(\alpha+\frac{\delta}{2})}{\Gamma(\frac{\delta}{2})} + \exp\left\{\frac{Y_0}{2\,\bar\varphi_t}\right\}\right)$$

$$< \infty \qquad \text{for } \alpha \in (-\frac{\delta}{2}, 0)$$

$$\bar s_t = \frac{s_t}{s_0} = \exp\left\{\int_0^t b_u\,du\right\}$$

$$\bar\varphi_t = s_0\,(\varphi(t)-\varphi(0)) = \frac{1}{4}\int_0^t \frac{c_u^2}{\bar s_u}\,du$$

- **first moment of SR process**

$$E\left(Y_t \,\middle|\, \mathcal{A}_0\right) = E\left(Y_0 \,\middle|\, \mathcal{A}_0\right) \exp\left\{\int_0^t b_s\, ds\right\} + \int_0^t \frac{\delta}{4}\, c_s^2 \exp\left\{\int_s^t b_z\, dz\right\} ds$$

- case $\delta = 4$

  for $c_t^2 = c^2 > 0$ and $b_t = b < 0$

$$\lim_{t\to\infty} E\left(Y_t \,\middle|\, \mathcal{A}_0\right) = -\frac{c^2}{b}$$

$$\lim_{t\to\infty} E\left(Y_t^{-1} \,\middle|\, \mathcal{A}_0\right) = -2\,\frac{b}{c^2}$$

- **transition density**

$$p(s, Y_s; t, Y_t) = \frac{p_\delta\left(\varphi(s), \frac{Y_s}{s_s}; \varphi(t), \frac{Y_t}{s_t}\right)}{s_t}$$

$$p(0, x; t, y) = \frac{1}{2\bar{s}_t\,\bar{\varphi}_t}\left(\frac{y}{x\,\bar{s}_t}\right)^{\frac{\nu}{2}} \exp\left\{-\frac{x + \frac{y}{\bar{s}_t}}{2\,\bar{\varphi}_t}\right\} I_\nu\left(\frac{\sqrt{x\,\frac{y}{\bar{s}_t}}}{\bar{\varphi}_t}\right)$$

for $0 < t < \infty,\ x, y \in (0, \infty)$,

where $\nu = \frac{\delta}{2} - 1$, $\bar{s}_t = \exp\{b\,t\}$ and $\bar{\varphi}_t = \frac{c^2}{4\,b}(1 - \frac{1}{\bar{s}_t})$

- **stationary density**

  is gamma density

$$p_{Y_\infty}(y) = \frac{\left(\frac{-2\,b}{c^2}\right)^{\frac{\delta}{2}} \; y^{\frac{\delta}{2}-1}}{\Gamma(\frac{\delta}{2})} \; \exp\left\{\frac{2\,b}{c^2}\,y\right\}$$

Minimal Market Model

Pl. (2001), special case of Pl.-Rendek model

Volatility Parametrization

- **Diversification Theorem** $\Longrightarrow$

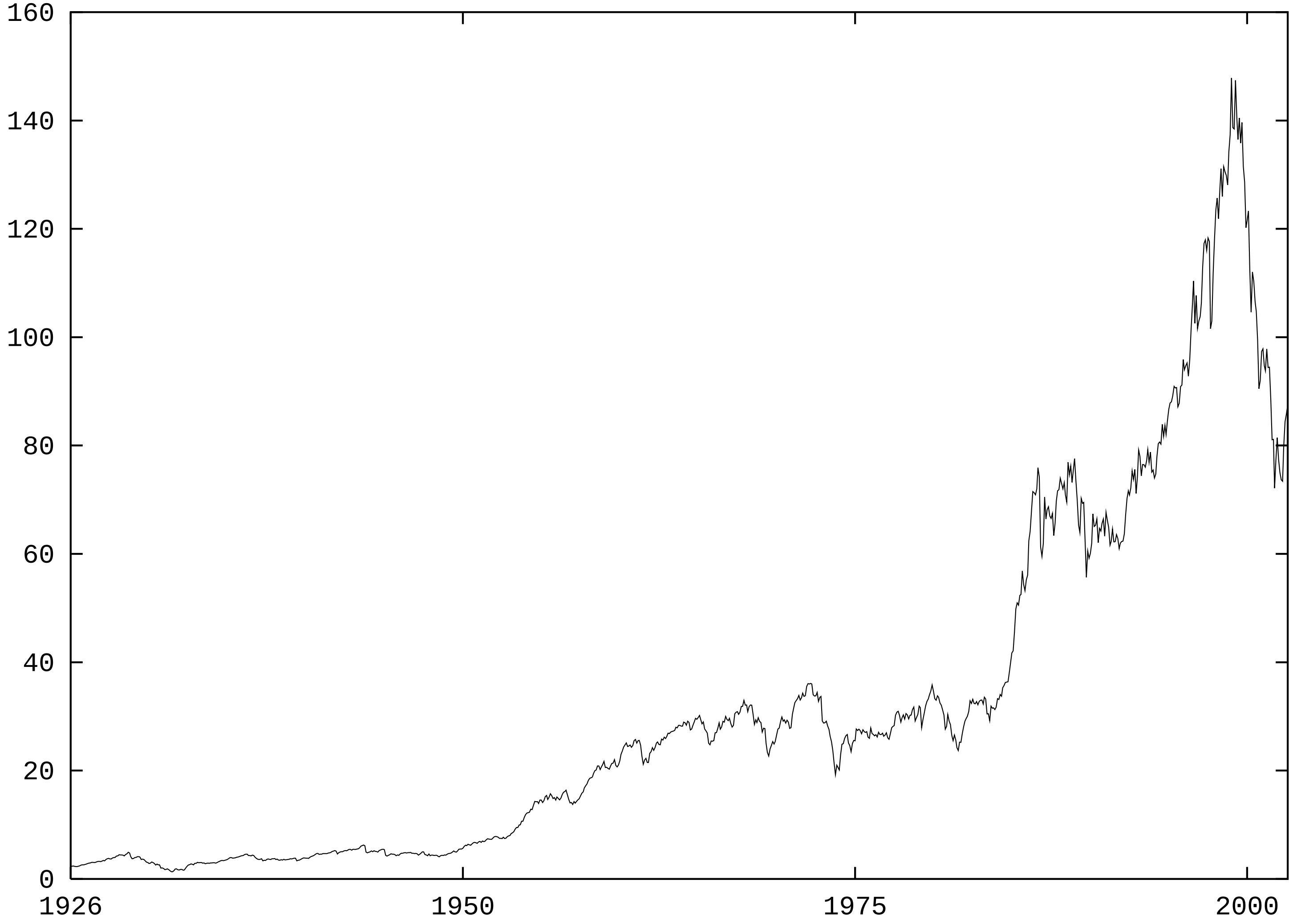
  well diversified stock market index approximates NP

- **discounted NP**

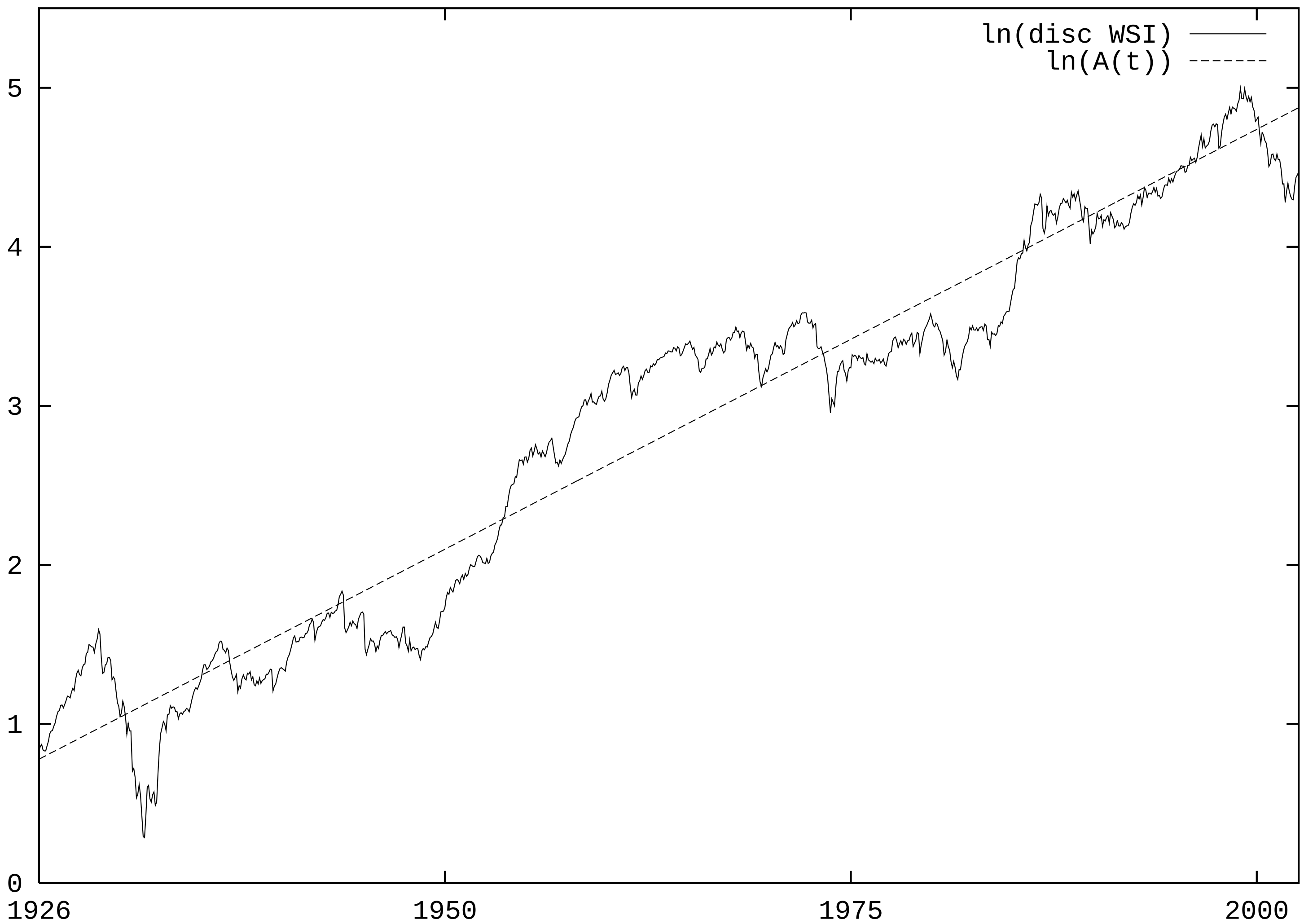
$$d\bar{S}_t^{\delta_*} = \bar{S}_t^{\delta_*} \, |\theta_t| \, (|\theta_t| \, dt + dW_t),$$

  where

$$dW_t = \frac{1}{|\theta_t|} \sum_{k=1}^{d} \theta_t^k \, dW_t^k$$



Discounted NP



Logarithm of discounted NP

Drift Parametrization

- **discounted NP drift**

$$\alpha_t = \bar{S}_t^{\delta_*} \, |\theta_t|^2$$

strictly positive, predictable

$\Longrightarrow$

- **volatility**

$$|\theta_t| = \sqrt{\frac{\alpha_t}{\bar{S}_t^{\delta_*}}}$$

leverage effect creates natural **feedback**
under reasonably independent drift

- **discounted NP**

$$d\bar{S}_t^{\delta_*} = \alpha_t \, dt + \sqrt{\bar{S}_t^{\delta_*} \, \alpha_t} \, dW_t$$

drift has economic meaning

- **transformed time**

$$\varphi_t = \frac{1}{4} \int_0^t \alpha_s \, ds$$

- **squared Bessel process of dimension four**

$$X_{\varphi_t} = \bar{S}_t^{\delta_*}$$

$$dW(\varphi_t) = \sqrt{\frac{\alpha_t}{4}}\, dW_t$$

$$dX_\varphi = 4\, d\varphi + 2\sqrt{X_\varphi}\, dW(\varphi)$$

Revuz & Yor (1999)

economically founded dynamics in φ-time

still no specific dynamics in t-time assumed

Time Transformed Bessel Process

$$d\sqrt{\bar{S}^{\delta_*}_t} = \frac{3\,\alpha_t}{8\,\sqrt{\bar{S}^{\delta_*}_t}}\,dt + \frac{1}{2}\,\sqrt{\alpha_t}\,dW_t$$

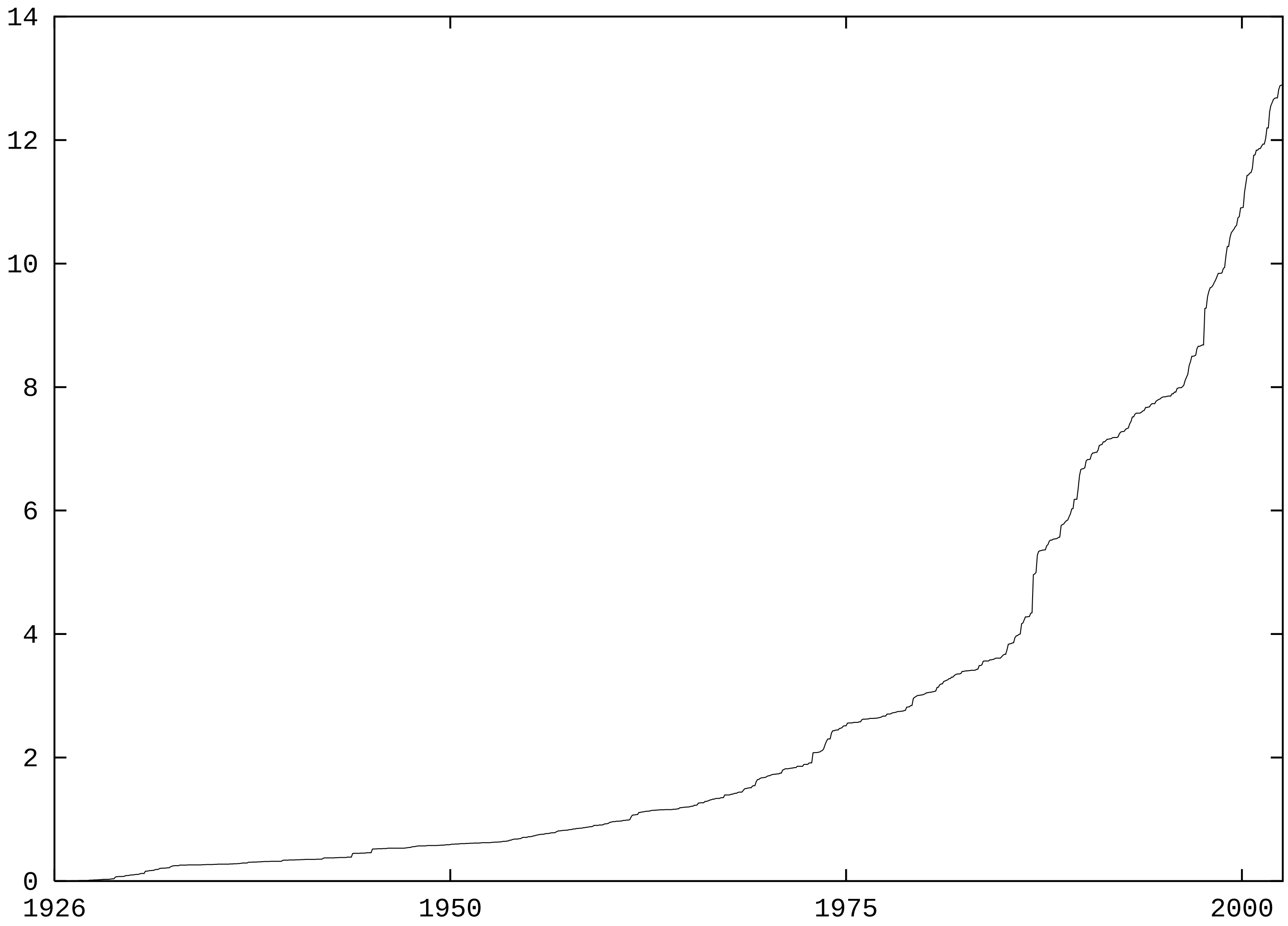
- **quadratic variation**

$$\left[\sqrt{\bar{S}^{\delta_*}}\right]_t = \frac{1}{4}\int_0^t \alpha_s\,ds$$

- **transformed time**

$$\varphi_t = \left[\sqrt{\bar{S}^{\delta_*}}\right]_t$$

observable



Empirical quadratic variation of the square root of the discounted S&P 500

Stylized Minimal Market Model

Pl. (2001, 2002, 2006), Pl. & Rendek (2012)

- assume **discounted NP drift** as

$$\alpha_t = \alpha \exp\{\eta\, t\}$$

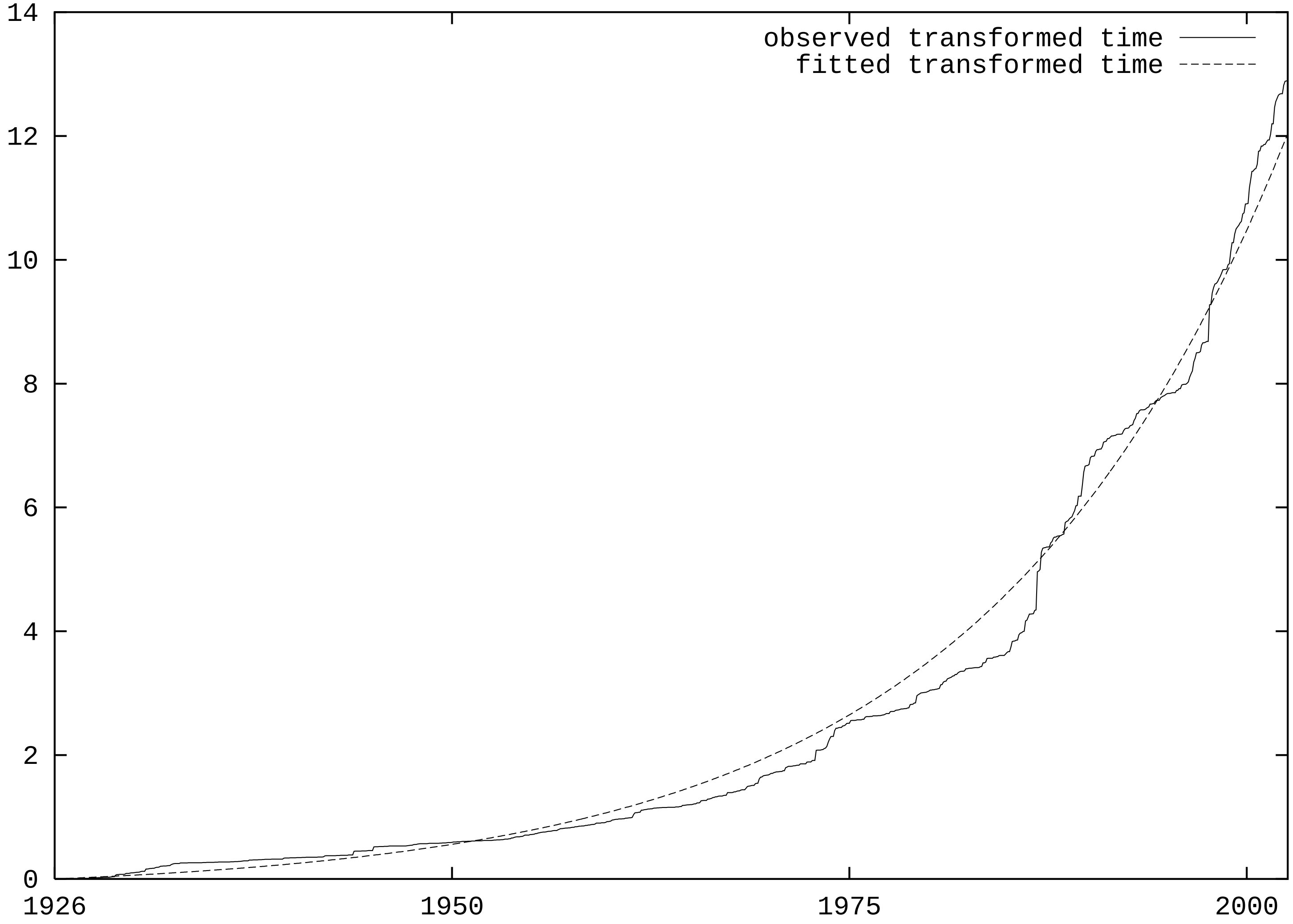
- initial parameter $\alpha > 0$

- net growth rate η

- **transformed time**

$$\varphi_t = \frac{\alpha}{4} \int_0^t \exp\{\eta z\}\, dz$$

$$= \frac{\alpha}{4\,\eta} \left(\exp\{\eta t\} - 1\right)$$



Fitted and observed transformed time

- **normalized NP**

$$Y_t = \frac{\bar{S}_t^{\delta_*}}{\alpha_t}$$

$$dY_t = (1 - \eta \, Y_t) \, dt + \sqrt{Y_t} \, dW_t$$

square root process of dimension four

$\Longrightarrow$ parsimonious model, long term viability

confirmation of stylized Pl.-Rendek model

Pl. & Rendek (2012)

$\Longrightarrow$

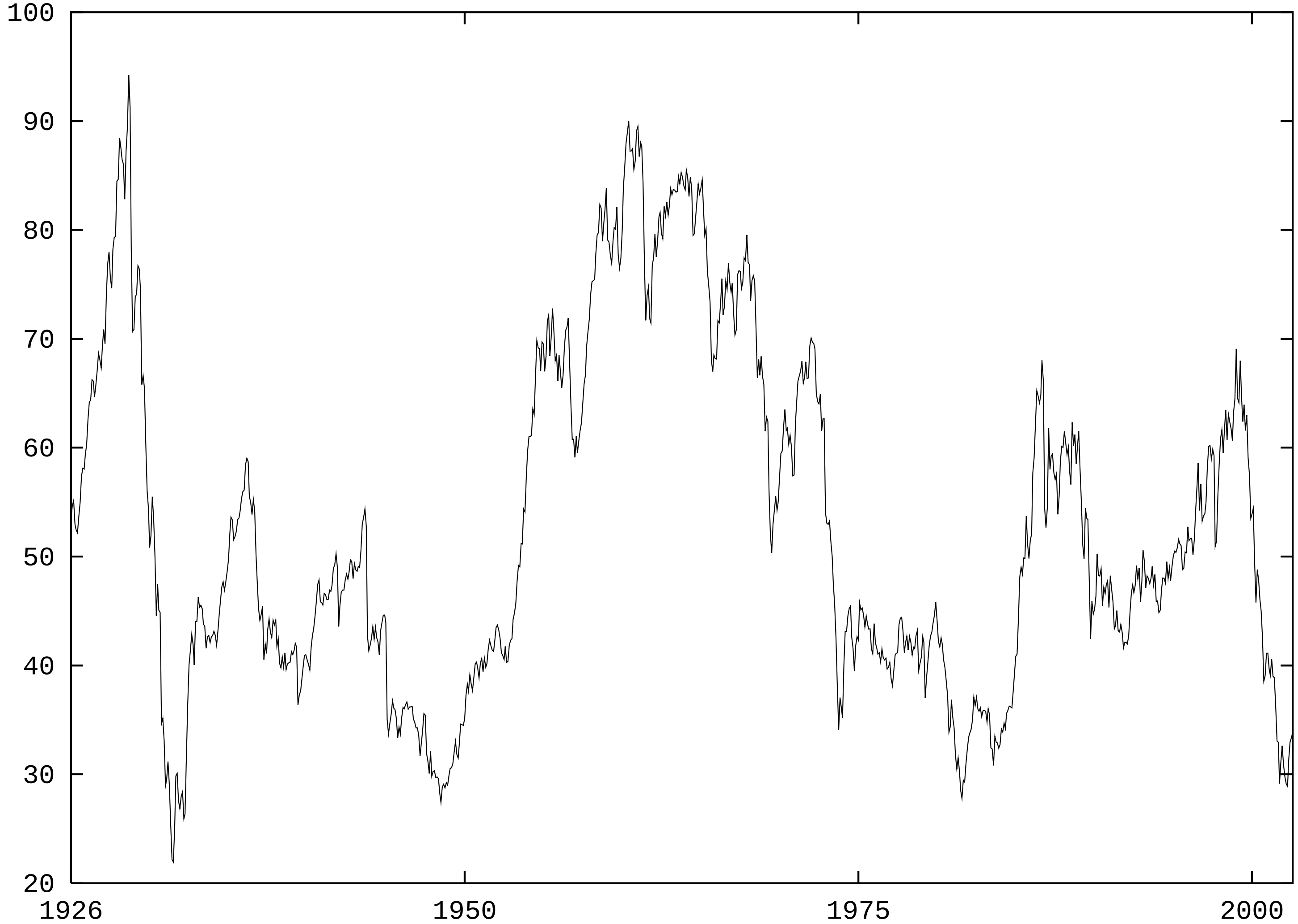
- **discounted NP**

$$\bar{S}_t^{\delta_*} = Y_t \, \alpha_t$$

- **NP**

$$S_t^{\delta_*} = S_t^0 \, \bar{S}_t^{\delta_*} = S_t^0 \, Y_t \, \alpha_t$$

- scaling parameter $\alpha = 0.043$
- net growth rate $\eta = 0.0528$
- reference level $\frac{1}{\eta} = 18.93939$
- speed of adjustment η
- half life time of major displacement $\frac{\ln(2)}{\eta} \approx 13$ years



Normalized NP

- **logarithm of discounted NP**

$$\ln(\bar{S}_t^{\delta_*}) = \ln(Y_t) + \ln(\alpha) + \eta\, t$$

- no need for extra volatility process in MMM

- realistic long term dynamics

Volatility of NP under the stylized MMM

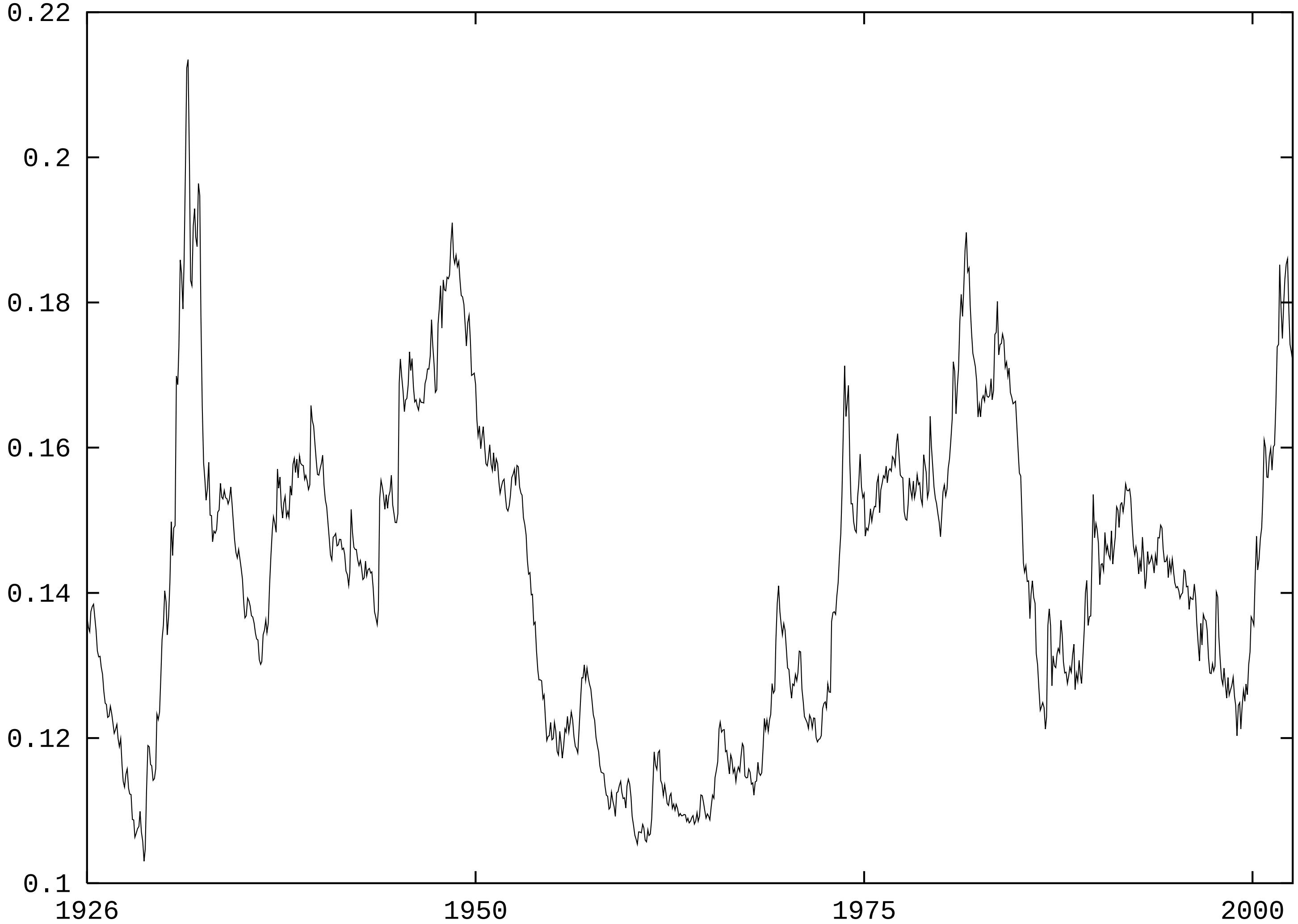
$$|\theta_t| = \frac{1}{\sqrt{Y_t}}$$

- **squared volatility**

$$d|\theta_t|^2 = d\left(\frac{1}{Y_t}\right) = |\theta_t|^2\,\eta\,dt - \left(|\theta_t|^2\right)^{\frac{3}{2}}\;dW_t$$

$3/2$ volatility model

Pl. (1997), Lewis (2000)



Volatility of the NP under the stylized MMM

Transition Density of Stylized MMM

- transition density of **discounted NP** $\bar{S}^{\delta_*}$

$$p(s,x;t,y) \;=\; \frac{1}{2\,(\varphi_t - \varphi_s)} \left(\frac{y}{x}\right)^{\frac{1}{2}} \exp\left\{-\frac{x+y}{2\,(\varphi_t - \varphi_s)}\right\} \times I_1\!\left(\frac{\sqrt{x\,y}}{\varphi_t - \varphi_s}\right)$$

$$\varphi_t = \frac{\alpha}{4\,\eta}\,(\exp\{\eta\,t\} - 1)$$

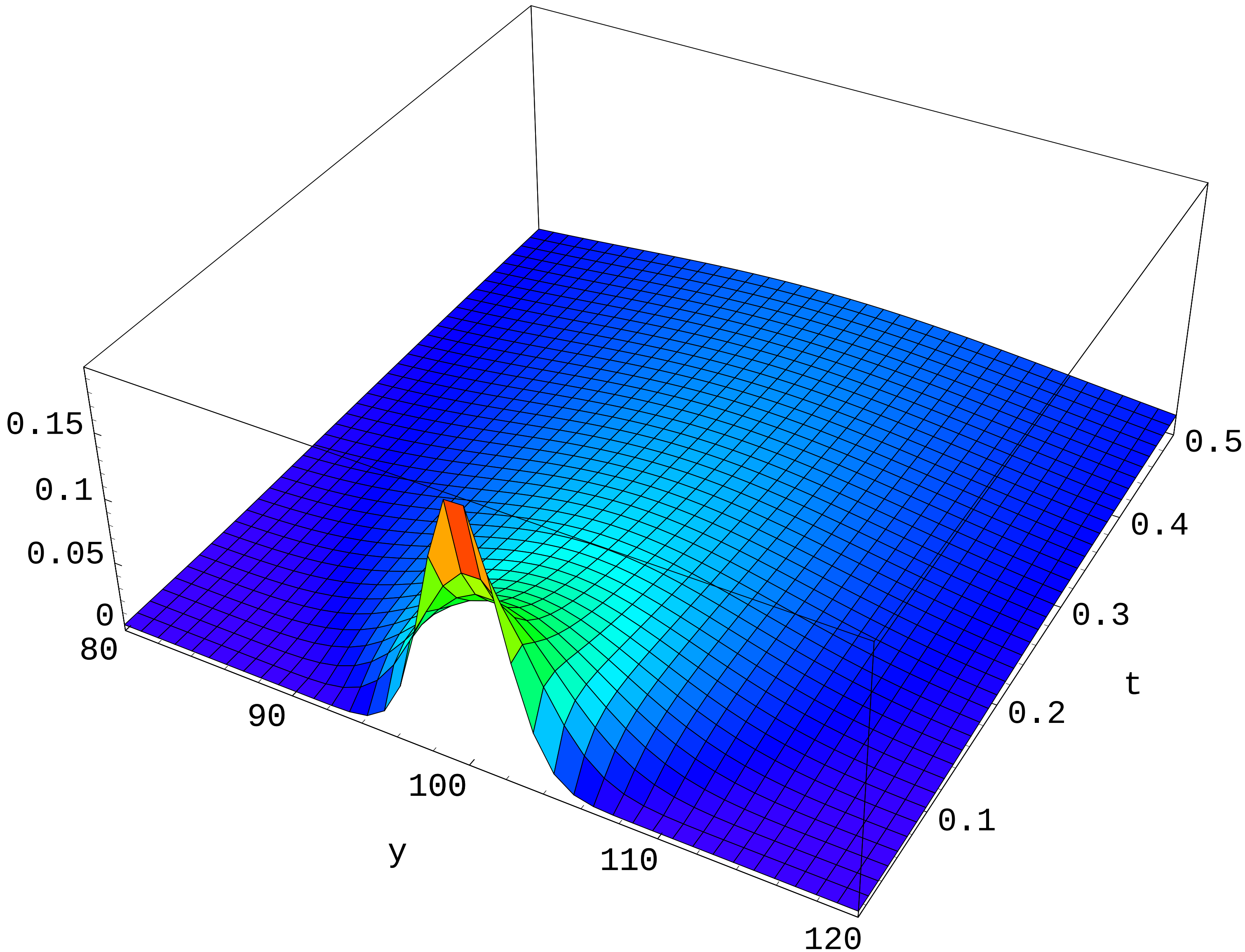
$I_1(\cdot)$ modified Bessel function of the first kind

- **non-central chi-square** distributed random variable:

$$\frac{y}{\varphi_t - \varphi_s} = \frac{\bar{S}_t^{\delta_*}}{\varphi_t - \varphi_s}$$

with $\delta = 4$ degrees of freedom and **non-centrality parameter:**

$$\frac{x}{\varphi_t - \varphi_s} = \frac{\bar{S}_s^{\delta_*}}{\varphi_t - \varphi_s}$$



Transition density of squared Bessel process for $\delta = 4$

Zero Coupon Bond under the stylized MMM

- **zero coupon bond**

$$\boldsymbol{P(t,T) = S_t^{\delta_*}\, E_t\left(\frac{1}{S_T^{\delta_*}}\right) = P_T^*(t)\, E_t\left(\frac{\bar{S}_t^{\delta_*}}{\bar{S}_T^{\delta_*}}\right)}$$

with **savings bond**

$$\boldsymbol{P_T^*(t) = \exp\left\{-\int_t^T r_s\, ds\right\}}$$

$$\delta = 4 \quad \Longrightarrow$$

$$P(t,T) = P_T^*(t)\left(1 - \exp\left\{-\frac{\bar{S}_t^{\delta_*}}{2\,(\varphi_T - \varphi_t)}\right\}\right) < P_T^*(t)$$

for $t \in [0, T)$, Pl. (2002)

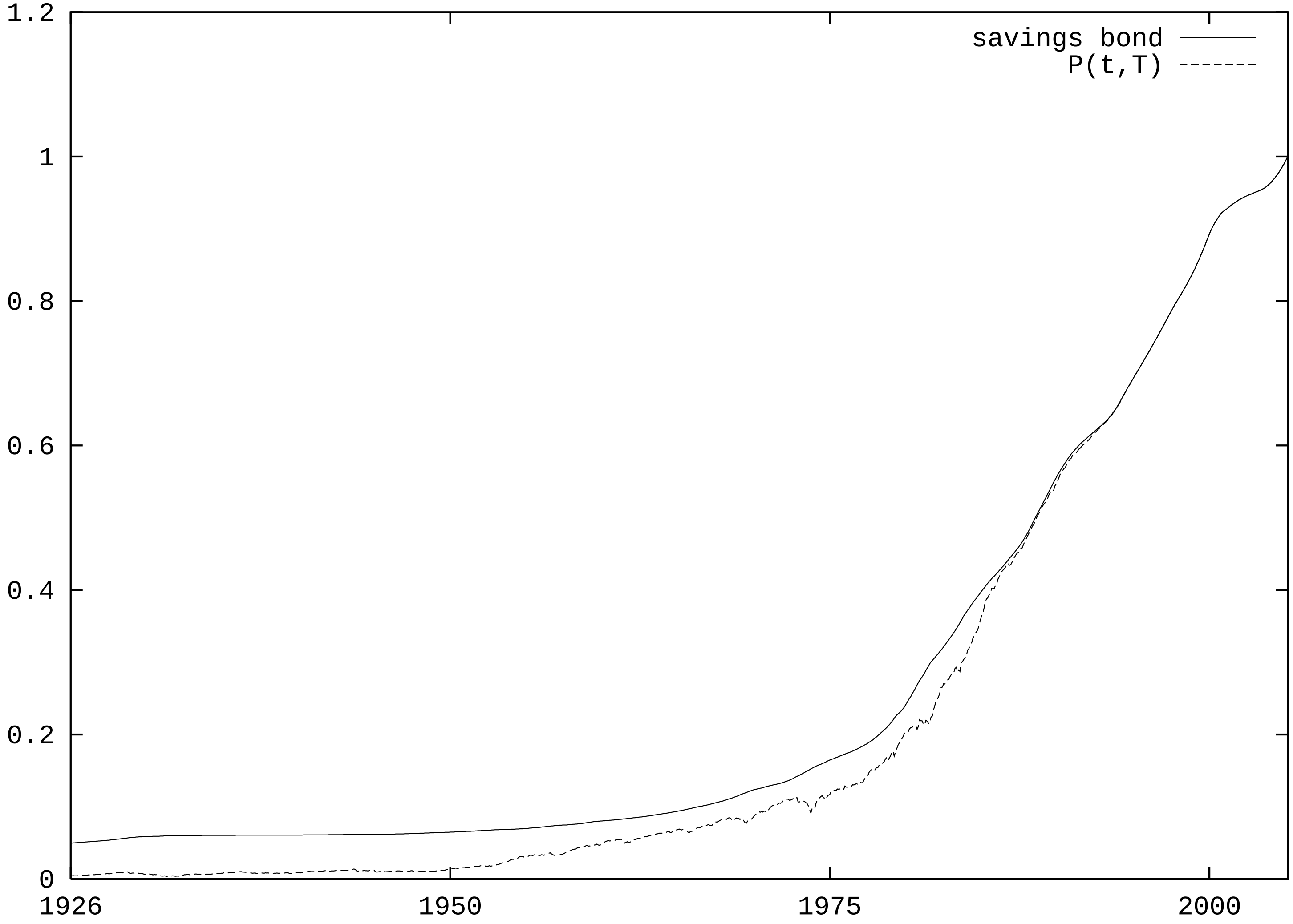
with time transform

$$\varphi_t = \frac{\alpha}{4\,\eta}\,(\exp\{\eta\,t\} - 1)$$

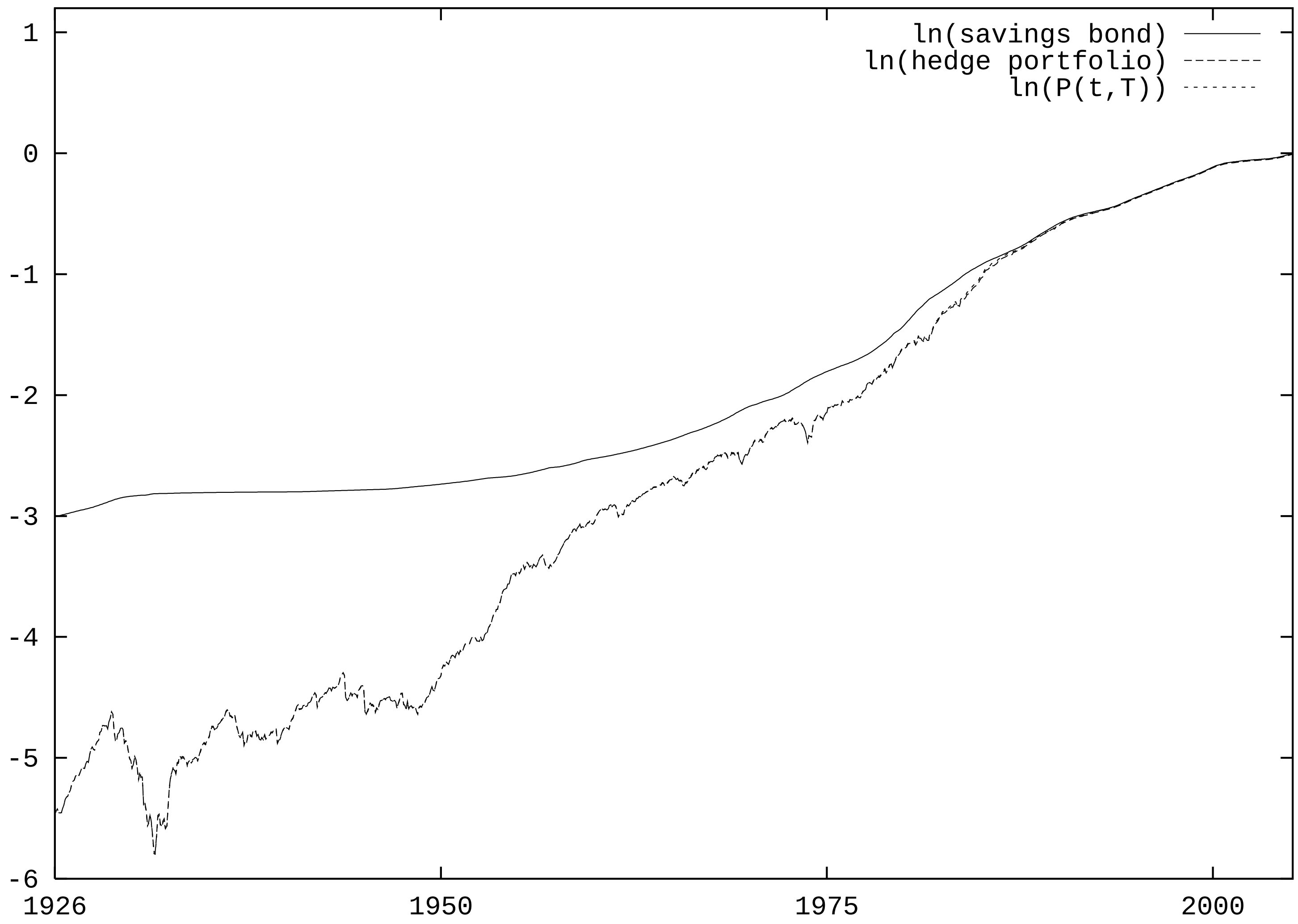
$$P(0,T) = 0.00423$$

$$P_T^*(0) = 0.04947$$

$$\frac{P(0,T)}{P_T^*(0)} \approx 0.0855$$



Zero coupon bond and savings bond



ln from Zero coupon bond, hedge portfolio and savings bond

Forward Rates under the MMM

- **forward rate**

$$
\begin{aligned}
f(t,T) &= -\frac{\partial}{\partial T}\,\ln(P(t,T)) \\
&= r_T + n(t,T)
\end{aligned}
$$

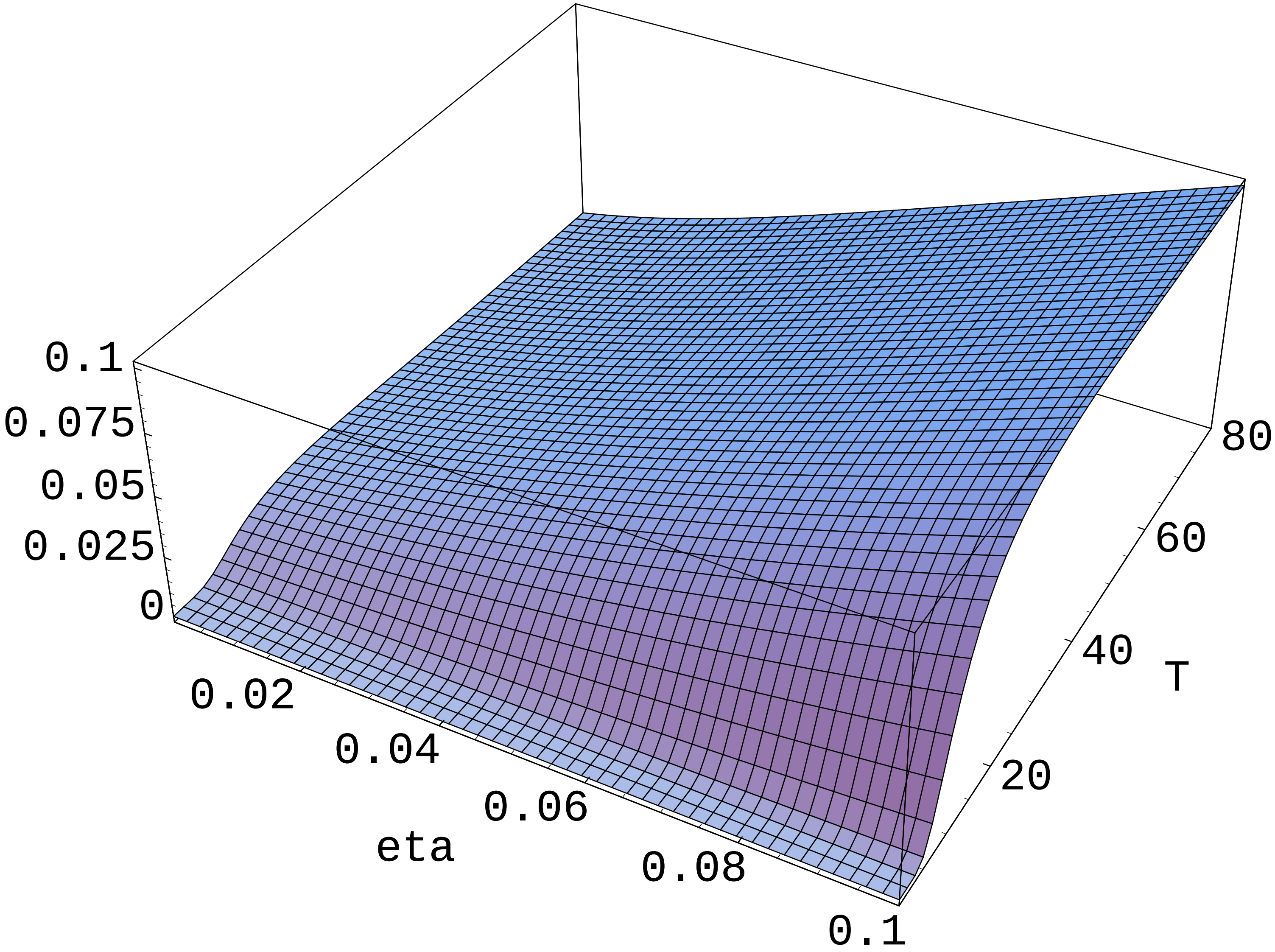
- **market price of risk contribution**

$$n(t,T) = -\frac{\partial}{\partial T} \ln\left(1 - \exp\left\{-\frac{\bar{S}_t^{\delta_*}}{2(\varphi_T - \varphi_t)}\right\}\right)$$

$$= \frac{1}{\left(\exp\left\{\frac{\bar{S}_t^{\delta_*}}{2(\varphi_T - \varphi_t)}\right\} - 1\right)} \frac{\bar{S}_t^{\delta_*}}{(\varphi_T - \varphi_t)^2} \frac{\alpha_T}{8}$$

$$\lim_{T\to\infty} n(t,T) = \eta$$

Pl. (2005), Miller & Pl. (2005)



Market price of risk contribution in dependence on η and T

Free Snack from Savings Bond

- potential existence of weak form of arbitrage?

  borrow amount $P(0,T)$ from savings account

$$S_t^{\delta} = P(t,T) - P(0,T)\,\exp\{r\,t\}$$

  such that $S_0^{\delta} = 0$

$$S_T^{\delta} = 1 - P(0,T)\,\exp\{r\,T\} > 0$$

  lower bond

$$S_t^{\delta} \geq -P(0,T)\,\exp\{r\,t\}$$

- since S_t^δ may become negative

  not strong arbitrage

Fundamental Theorem of Asset Pricing

Delbaen & Schachermayer (1998) $\Longrightarrow$ the MMM does not admit an equivalent risk neutral probability measure

$\Longrightarrow$

there is a **free lunch with vanishing risk**

Delbaen & Schachermayer (2006)

Absence of an Equivalent Risk Neutral Probability Measure

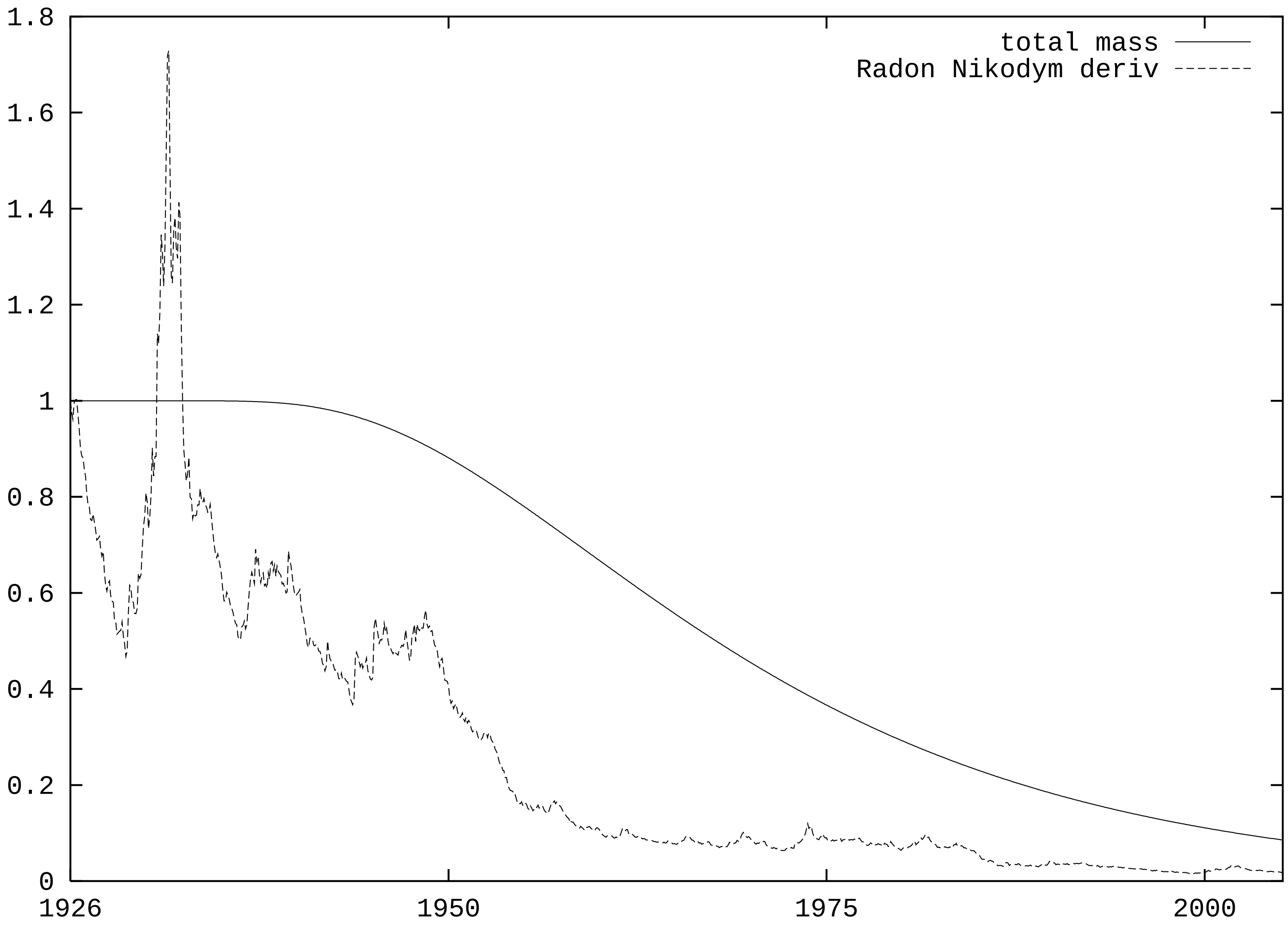
- fair zero coupon bond has a lower price than the savings bond

$$P(t,T) < P^*_T(t) = \exp\left\{-\int_t^T r_s\,ds\right\} = \frac{S^0_t}{S^0_T}$$

- **Radon-Nikodym derivative**

$$\Lambda_t = \frac{\hat{S}^0_t}{\hat{S}^0_0} = \frac{\bar{S}^{\delta_*}_0}{\bar{S}^{\delta_*}_t}$$

strict supermartingale under MMM



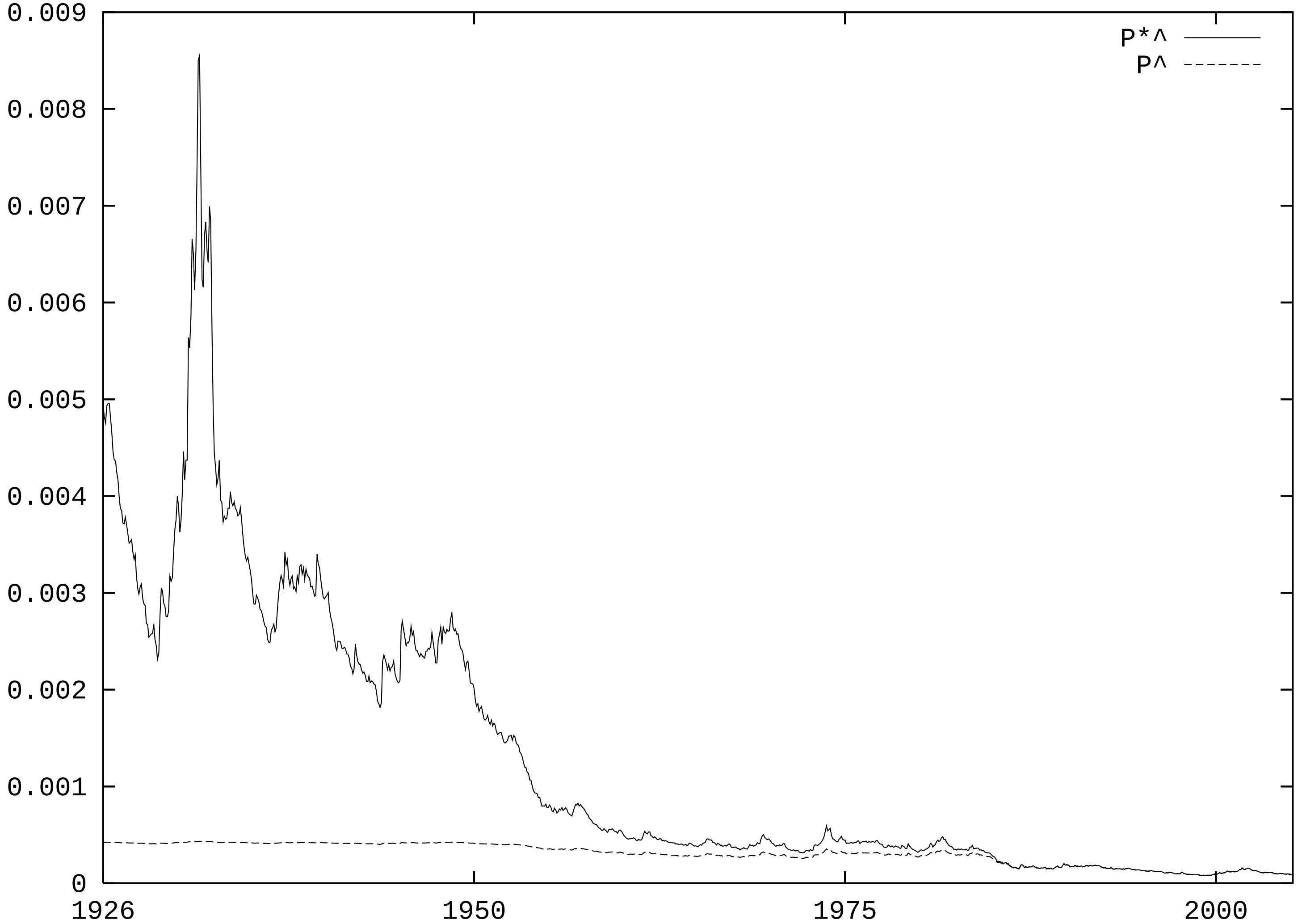
Radon-Nikodym derivative and total mass of candidate risk neutral measure

- **hypothetical risk neutral measure**

  total mass:

$$P_{\theta,T}(\Omega) = E\left(\Lambda_T\right) = 1 - \exp\left\{-\frac{\bar{S}_0^{\delta_*}}{2\,\varphi_T}\right\} < \Lambda_0 = 1$$

  P_θ is **not** a probability measure



Benchmarked zero coupon bond

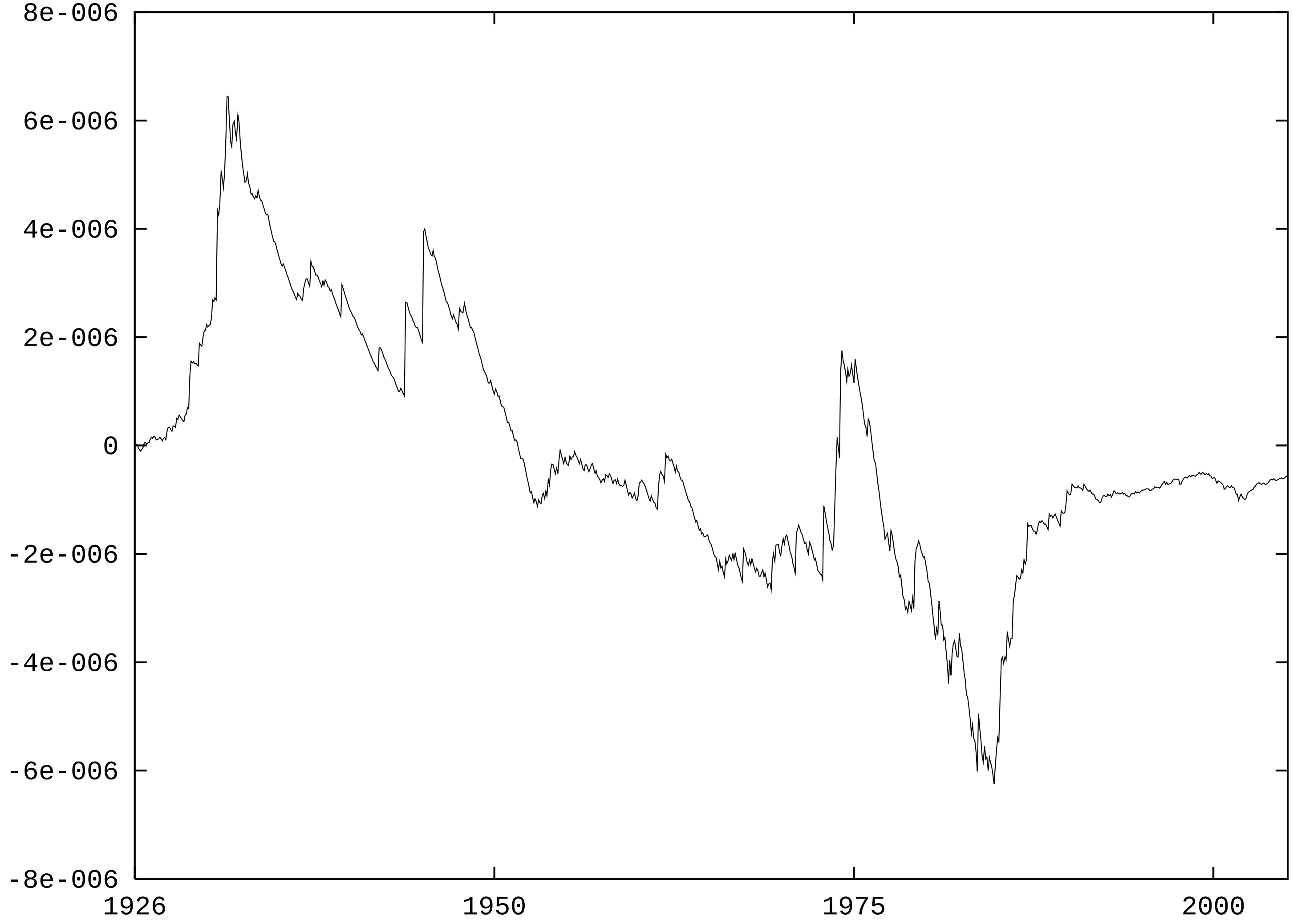
Hedge Simulation

- **delta** units in the NP

$$\delta_*(t) = \frac{\partial P(t,T)}{\partial S_t^{\delta_*}}$$

$$= \exp\left\{-\int_0^T r_s\, ds\right\} \exp\left\{-\frac{\bar{S}_t^{\delta_*}}{2(\varphi_T - \varphi_t)}\right\} \frac{1}{2(\varphi_T - \varphi_t)}$$

Ratio in the NP



Benchmarked P&L for hedge portfolio

European Call Options under the MMM

real world pricing formula $\Longrightarrow$

$$
\begin{aligned}
c_{T,K}(t, S_t^{\delta_*}) &= S_t^{\delta_*}\, E_t\left(\frac{(S_T^{\delta_*} - K)^+}{S_T^{\delta_*}}\right) \\
&= E_t\left(\left(S_t^{\delta_*} - \frac{K\, S_t^{\delta_*}}{S_T^{\delta_*}}\right)^+\right) \\
&= S_t^{\delta_*}\, \left(1 - \chi^2(d_1; 4, \ell_2)\right) \\
&\quad - K\, \exp\{-r\,(T-t)\}\,(1 - \chi^2(d_1; 0, \ell_2))
\end{aligned}
$$

$\chi^2(\cdot; \cdot, \cdot)$ non-central chi-square distribution with

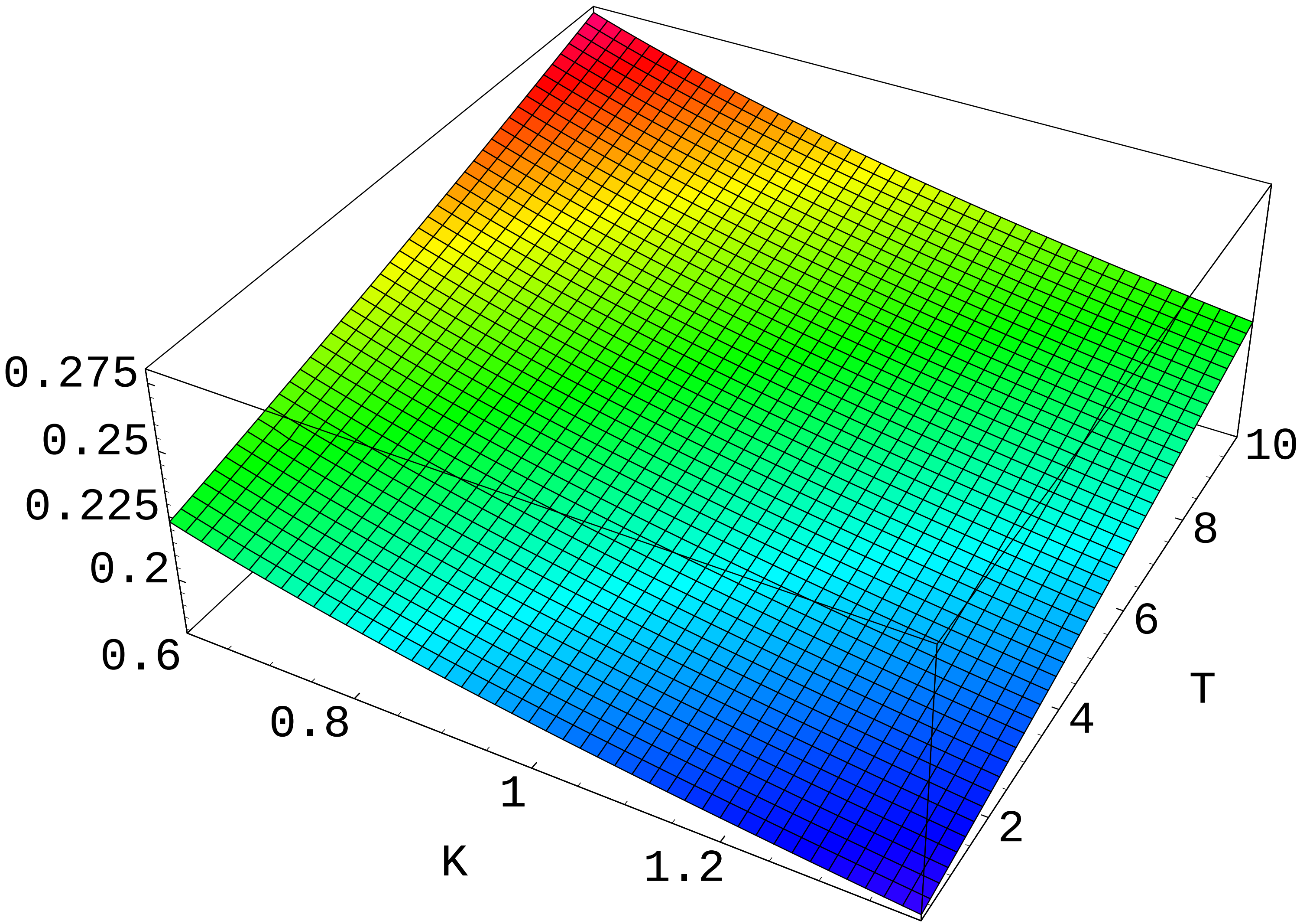
$$d_1 = \frac{4\,\eta\,K\ \exp\{-r\,(T-t)\}}{S_t^0\,\alpha_t\,(\exp\{\eta\,(T-t)\}-1)}$$

and

$$\ell_2 = \frac{2\,\eta\,S_t^{\delta_*}}{S_t^0\,\alpha_t\,(\exp\{\eta\,(T-t)\}-1)}$$

Hulley, Miller & Pl. (2005)

Miller & Pl. (2008)



Implied volatility surface for the stylized MMM

- **implied volatility**

  in BS-formula adjust short rate to

$$\hat{r} = -\frac{1}{T - t} \ln(P(t, T))$$

  otherwise put and call implied volatilities do not match

European Put Options under the MMM

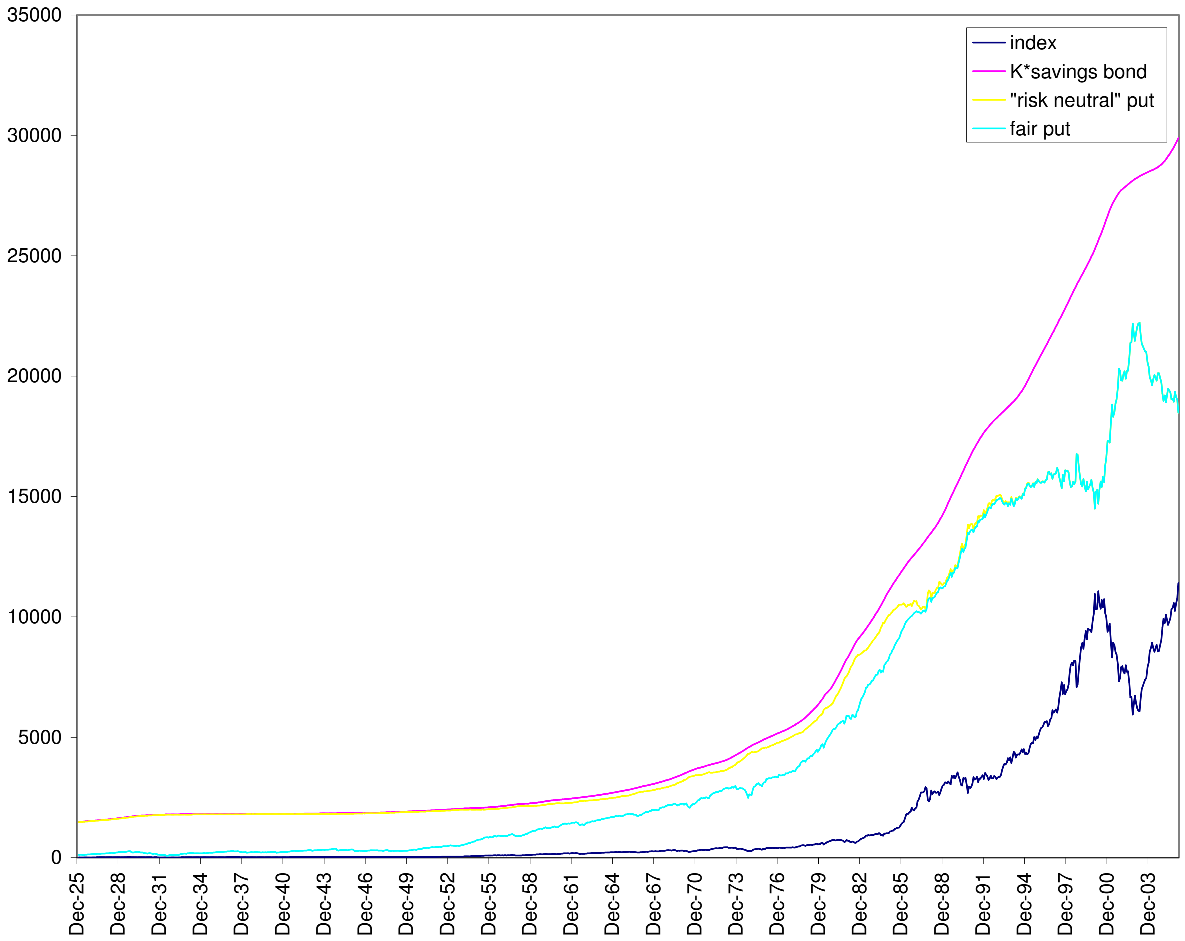
- **fair put-call parity relation**

$$p_{T,K}(t, \bar{S}_t^{\delta_*}) = c_{T,K}(t, \bar{S}_t^{\delta_*}) - S_t^{\delta_*} + K\,P(t,T)$$

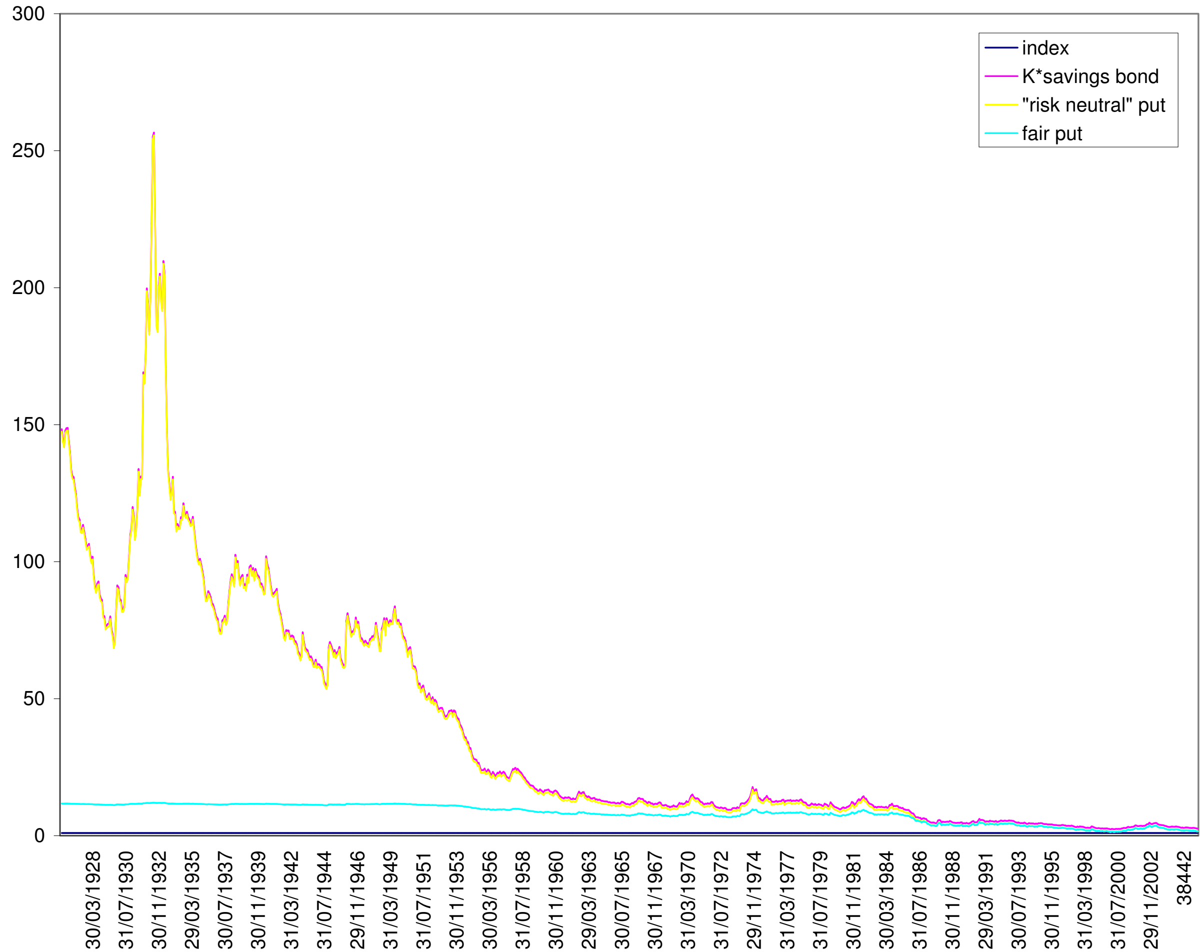
- **European put option formula**

$$p_{T,K}(t, S_t^{\delta_*}) \;=\; -S_t^{\delta_*}\,\big(\chi^2(d_1; 4, \ell_2)\big)$$
$$+\,K \exp\{-r\,(T-t)\}\big(\chi^2(d_1; 0, \ell_2) - \exp\{-\ell_2\}\big)$$

Hulley, Miller & Pl. (2005)



Risk neutral and fair put on index



Benchmarked “risk neutral” and fair put on index

- put-call parity **breaks down** if one uses the savings bond $P_T^*(t)$

$$p_{T,K}(t, \bar{S}_t^{\delta_*}) < c_{T,K}(t, \bar{S}_t^{\delta_*}) - S_t^{\delta_*} + K \exp\{-r\,(T-t)\}$$

for $t \in [0, T)$

Comparison to Hypothetical Risk Neutral Prices

- **hypothetical risk neutral price** $c^*_{T,K}(t, S^{\delta_*}_t)$

  of a European call option on the GOP

- benchmarked hypothetical risk neutral call price

$$\hat{c}^*_{T,K}(t, S^{\delta_*}_t) = \frac{c^*_{T,K}(t, S^{\delta_*}_t)}{S^{\delta_*}_t}$$

  local martingale

  $\hat{c}^*_{T,K}(\cdot, \cdot)$ uniformly bounded

$\Longrightarrow \quad \hat{c}^*_{T,K}$ - martingale $\Longrightarrow$

$$\hat{c}^*_{T,K}(t, S_t^{\delta_*}) = \hat{c}_{T,K}(t, S_t^{\delta_*})$$

$\Longrightarrow$

$$c^*_{T,K}(t, S_t^{\delta_*}) = c_{T,K}(t, S_t^{\delta_*})$$

- **hypothetical risk neutral put-call parity**

$$p^*_{T,K}(t, S^{\delta_*}_t) = c^*_{T,K}(t, S^{\delta_*}_t) - S^{\delta_*}_t + K\, P^*_T(t)$$

since $P^*_T(t) > P(t,T) \quad \Longrightarrow$

$$p_{T,K}(t, S^{\delta_*}_t) < p^*_{T,K}(t, S^{\delta_*}_t)$$

$t \in [0, T)$

Difference in Asymptotic Put Prices

- hypothetical risk neutral prices can become extreme if NP value tends towards zero

- **asymptotic fair zero coupon bond**

$$\lim_{\bar{S}_t^{\delta_*} \to 0} P_T(t,T) \overset{\text{a.s.}}{=} 0$$

for $t \in [0,T)$

- **asymptotic fair European call**

$$\lim_{\bar{S}_t^{\delta_*} \to 0} c_{T,K}(t, S_t^{\delta_*}) \overset{\text{a.s.}}{=} \lim_{\bar{S}_t^{\delta_*} \to 0} S_t^{\delta_*} \, \hat{c}_{T,K}(t, S_t^{\delta_*}) = 0$$

- **asymptotic fair put**

  fair put-call parity $\Longrightarrow$

$$\lim_{\bar{S}_t^{\delta_*} \to 0} p_{T,K}(t, S_t^{\delta_*}) \overset{\text{a.s.}}{=} 0$$

- **asymptotic hypothetical risk neutral put**

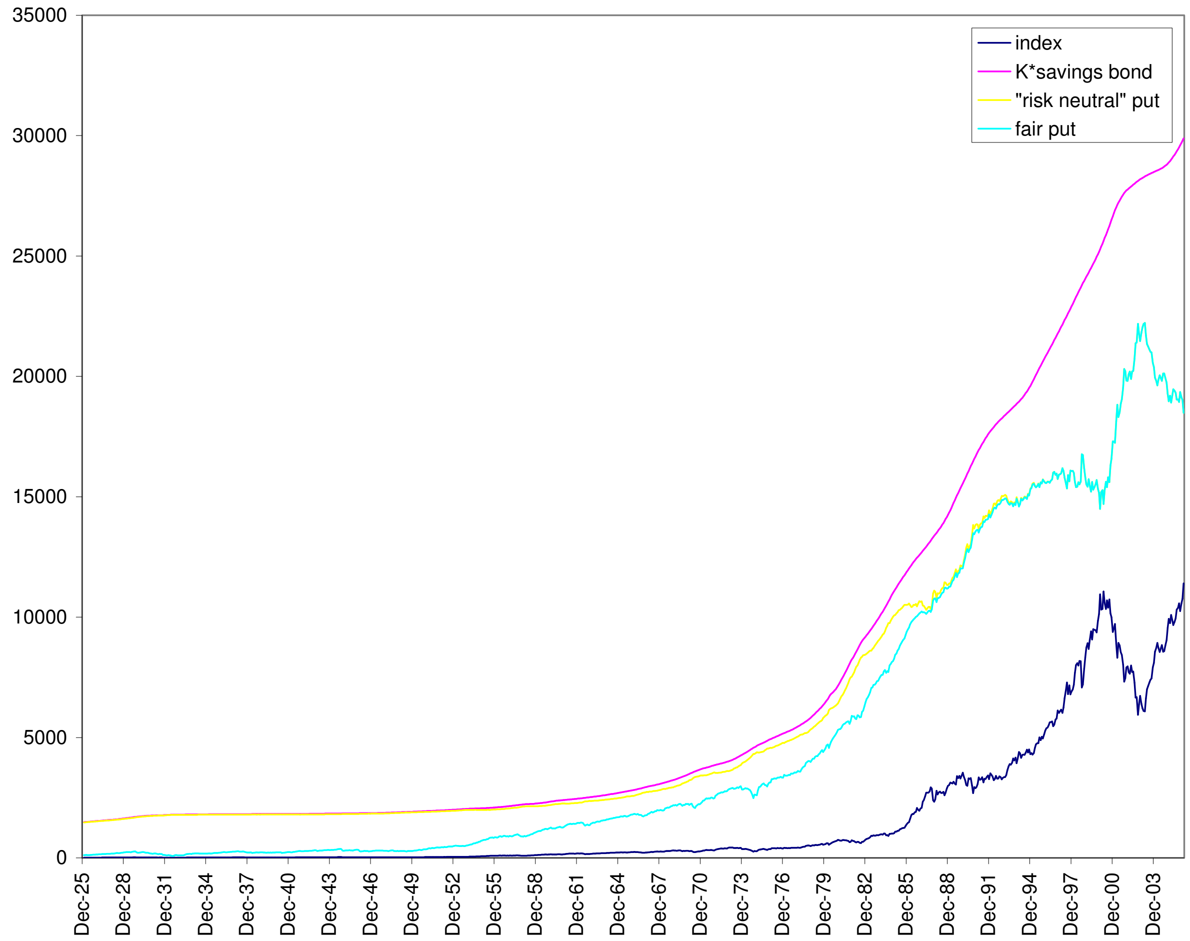
  hypothetical risk neutral put-call parity $\Longrightarrow$

$$\lim_{\bar{S}_t^{\delta_*} \to 0} p^*_{T,K}(t, S_t^{\delta_*})$$

$$\overset{\text{a.s.}}{=} \lim_{\bar{S}_t^{\delta_*} \to 0} \left(p_{T,K}(t, S_t^{\delta_*}) + K \, \frac{S_t^0}{S_T^0} \, \exp\left\{ -\frac{\bar{S}_t^{\delta_*}}{2(\varphi_T - \varphi_t)} \right\} \right)$$

$$\overset{\text{a.s.}}{=} K \, \frac{S_t^0}{S_T^0} > 0$$

dramatic differences can arise



“Risk neutral” and fair put on index

Pricing Annuities

Baldeaux and Pl. (2012)

Interest Indexed Payouts

$$\bar{Q}(t,T) = \bar{S}^*_t \, E_t\left(\frac{B_T}{S^*_T}\right)$$

$$\bar{S}^*_t = \frac{S^*_t}{B_t}$$

B_t - savings account

e.g.: Minimal Market Model (MMM), Pl. (2001)

$\Longrightarrow$

$$\bar{Q}(t,T) = 1 - \exp\left\{-\frac{2\,\eta\,\bar{S}_t^*}{\alpha\,(\exp\{\eta\,T\} - \exp\{\eta\,t\})}\right\}$$

Interest Indexed Life Annuity

θ-entitlement level

$\bar{T}$-retirement date

$T_0 < T_1 < T_2 < \cdots$

ξ_x - remaining life time of individual aged x at time 0

$$\bar{U}^{\theta}_{x,\bar{T}}(t) = \bar{S}^*_t \, E_t \left(\sum_{T_i \geq \bar{T}} \mathbb{I}_{\{\xi_x > T_i\}} \frac{\theta \, B_{T_i}}{S^*_{T_i}} \right)$$

$$= \theta \sum_{T_i \geq \bar{T}} P_t \left(\xi_x > T_i \right) \, \bar{Q}(t, T_i)$$

Threshold Life Table

$$P(\xi_x > T) = \begin{cases} \exp\left\{ -\frac{b}{\ln(C)}\left(C^{x+T} - 1\right)\right\} & \text{for} \quad T \leq N - x \\ q\left(1 + \lambda\left(\frac{x+T-N}{\omega}\right)\right)^{-\frac{1}{\lambda}} & \text{for} \quad T > N - x \end{cases}$$

?)

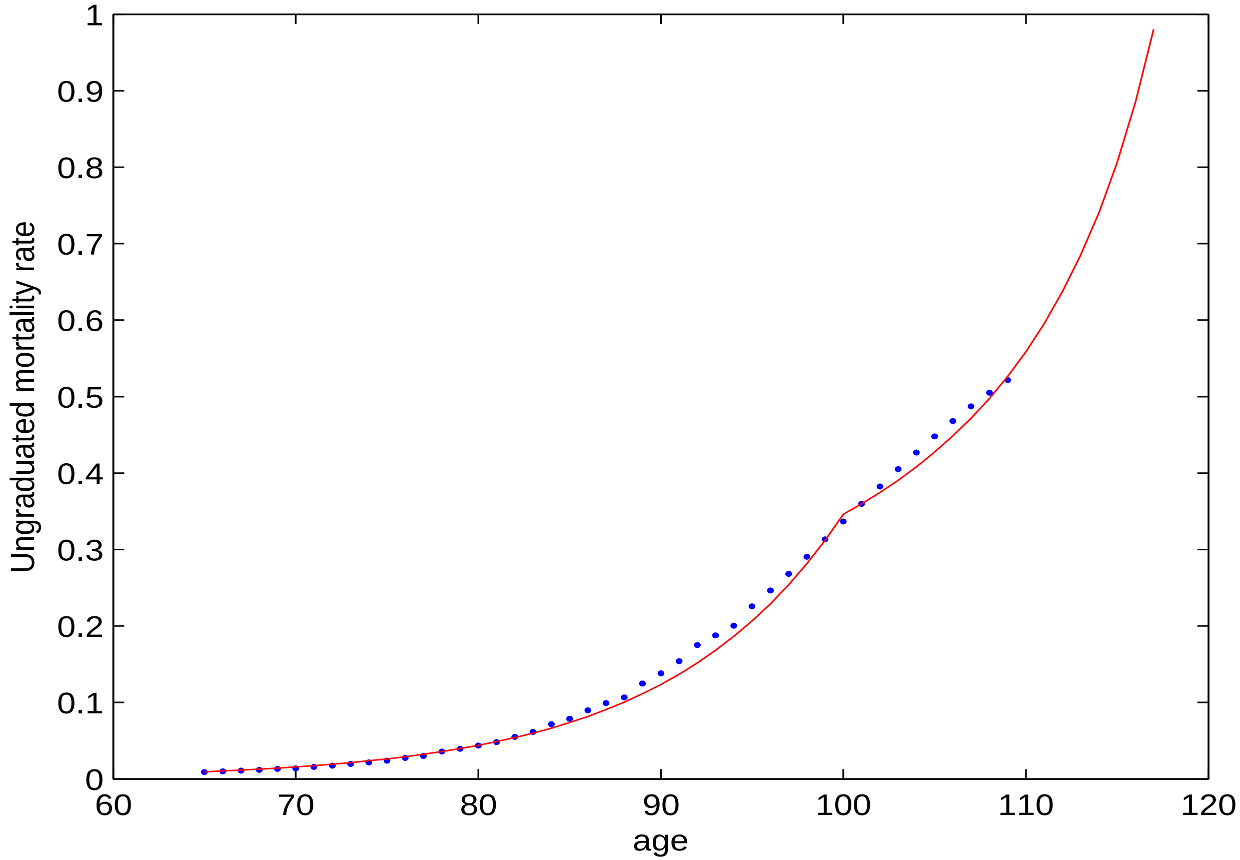


Figure 1: Mortality rates fitted to mortality data of the US population in 2007 with threshold at $\boldsymbol{N = 93}$.

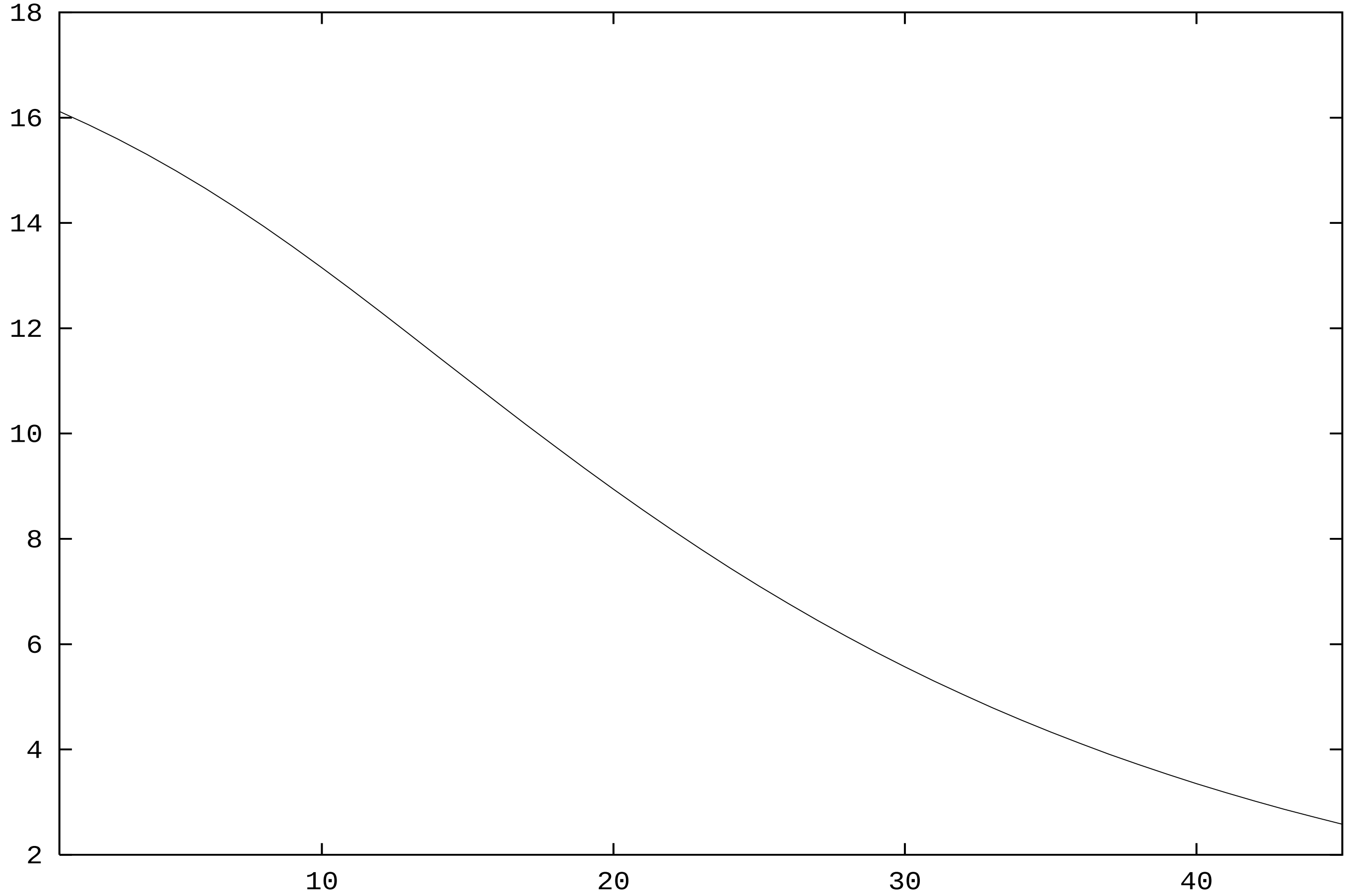


Figure 2: Discounted fair interest indexed life annuity as function of time to retirement.

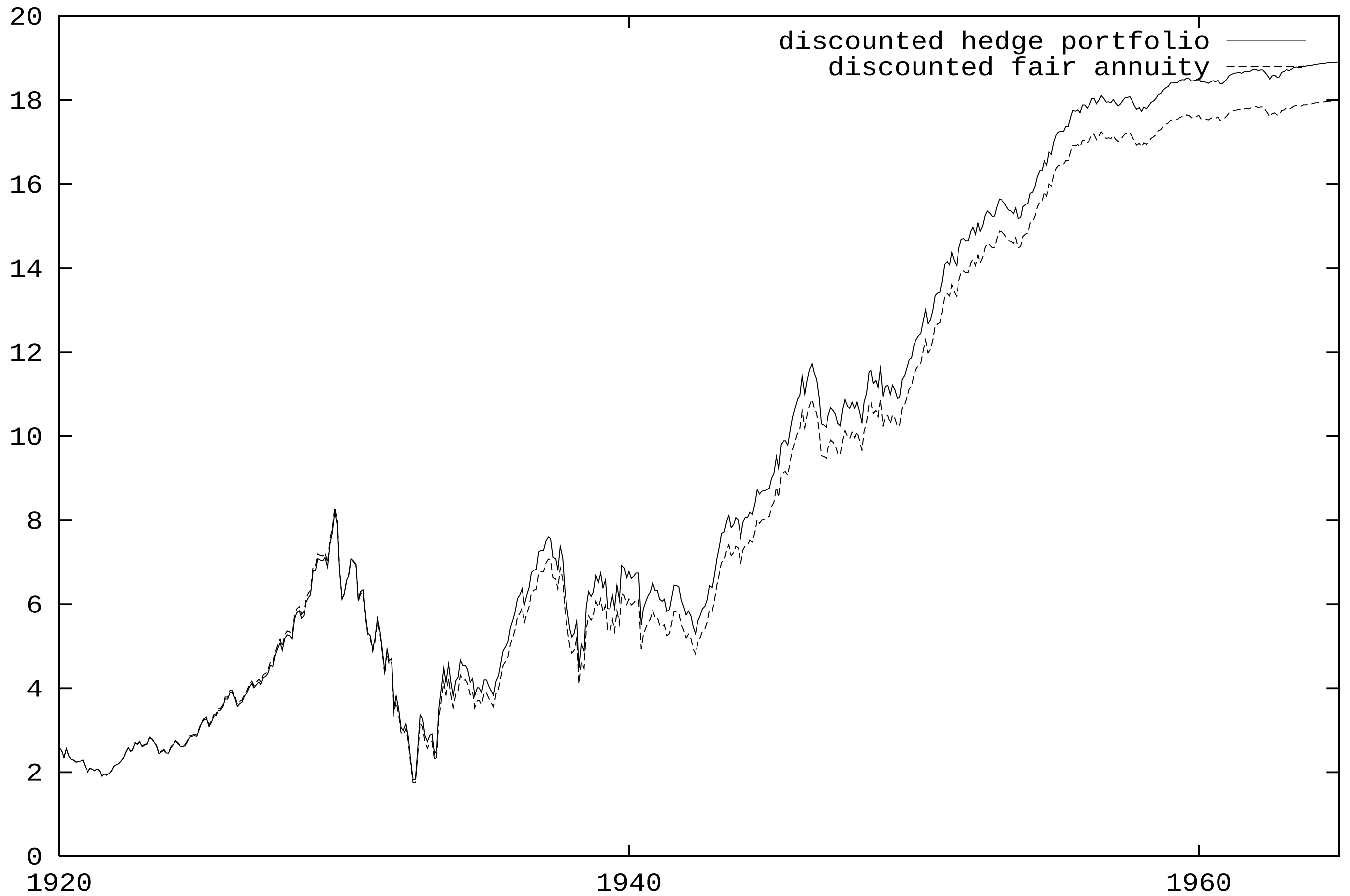


Figure 3: Discounted hedging portfolio and discounted fair interest indexed life annuity evolving over the years.

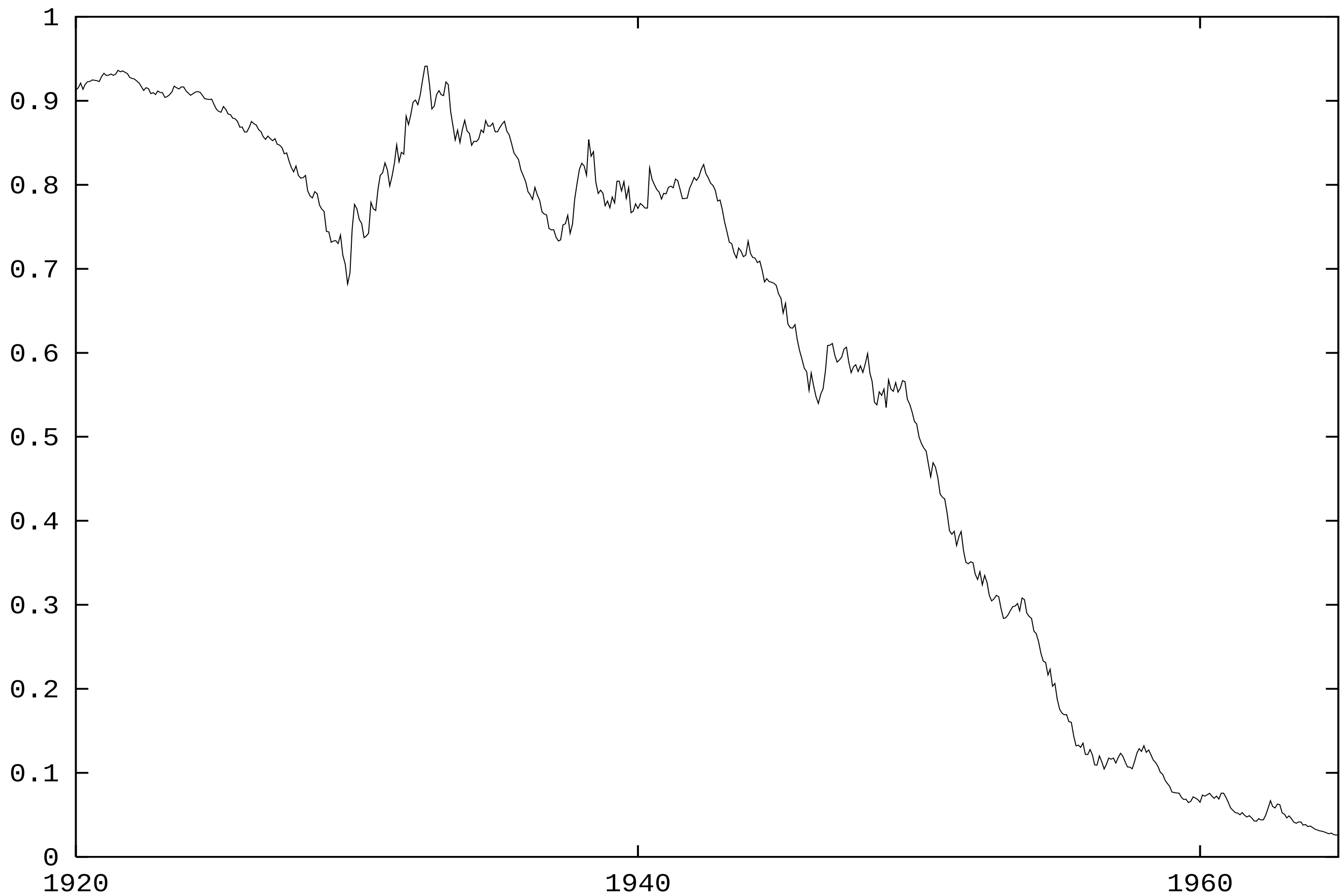


Figure 4: Fraction of the hedging portfolio invested in the benchmark portfolio.

Targeted Pension

$\Gamma_l(t)$-account of l th member in units of interest indexed life annuity

- **total discounted value**

$$\sum_{l=1}^{M_{t_i}} \Gamma_l(t_i)\bar{U}^{\theta_{t_i}}_{x_l,\bar{T}_l}(t_i) = \delta^0_{\text{total}}(t_i) + \delta^*_{\text{total}}(t_i)\,\bar{S}^*_{t_i}$$

$\Longrightarrow$ entitlement level

$$\theta_{t_i} = \frac{\delta^0_{\text{total}}(t_i) + \delta^*_{\text{total}}\,\bar{S}^*_{t_i}}{\sum_{l=1}^{M_i}\Gamma_l(t_i)\sum_{T_k\geq\bar{T}_l} P_{t_i}(\xi_{x_l} > T_k)\bar{Q}(t_i,T_k)}$$

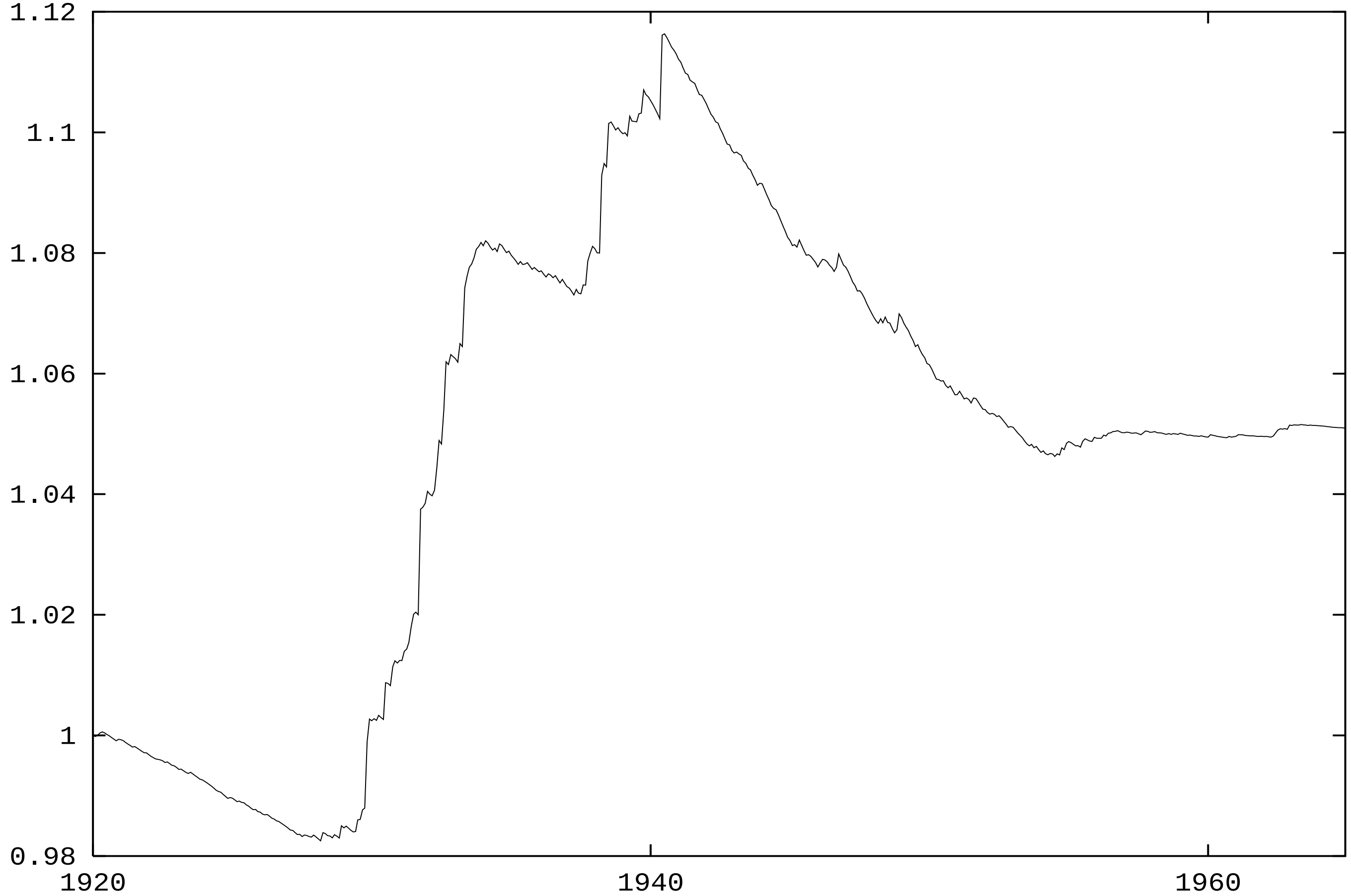


Figure 5: Entitlement level over the years.

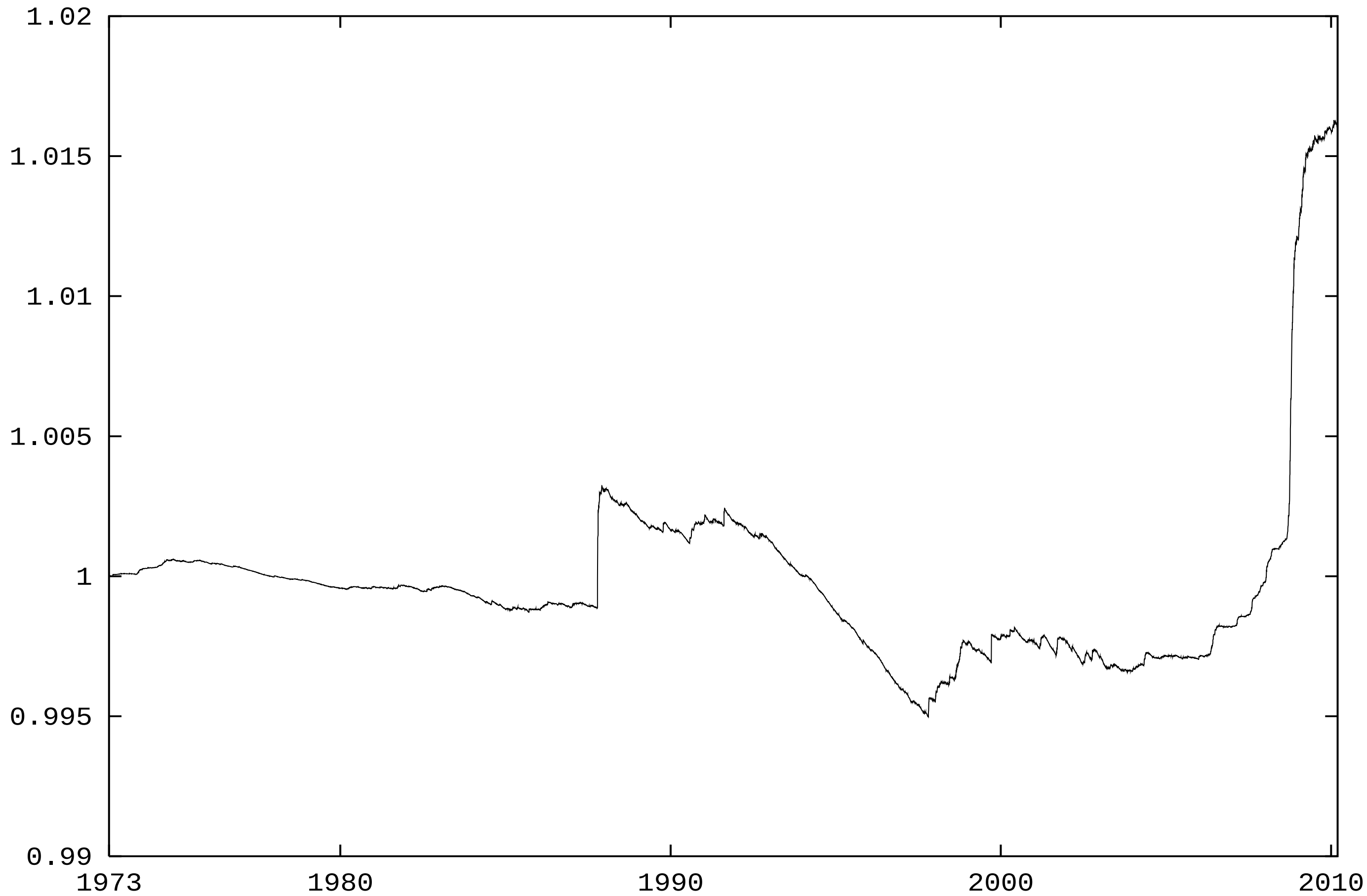


Figure 6: Entitlement level using fast growing portfolio and daily hedging.

Variable Annuity

Guaranteed Minimum Death Benefits

Marquardt, Pl. & Jaschke (2008)

- payout to the policyholder

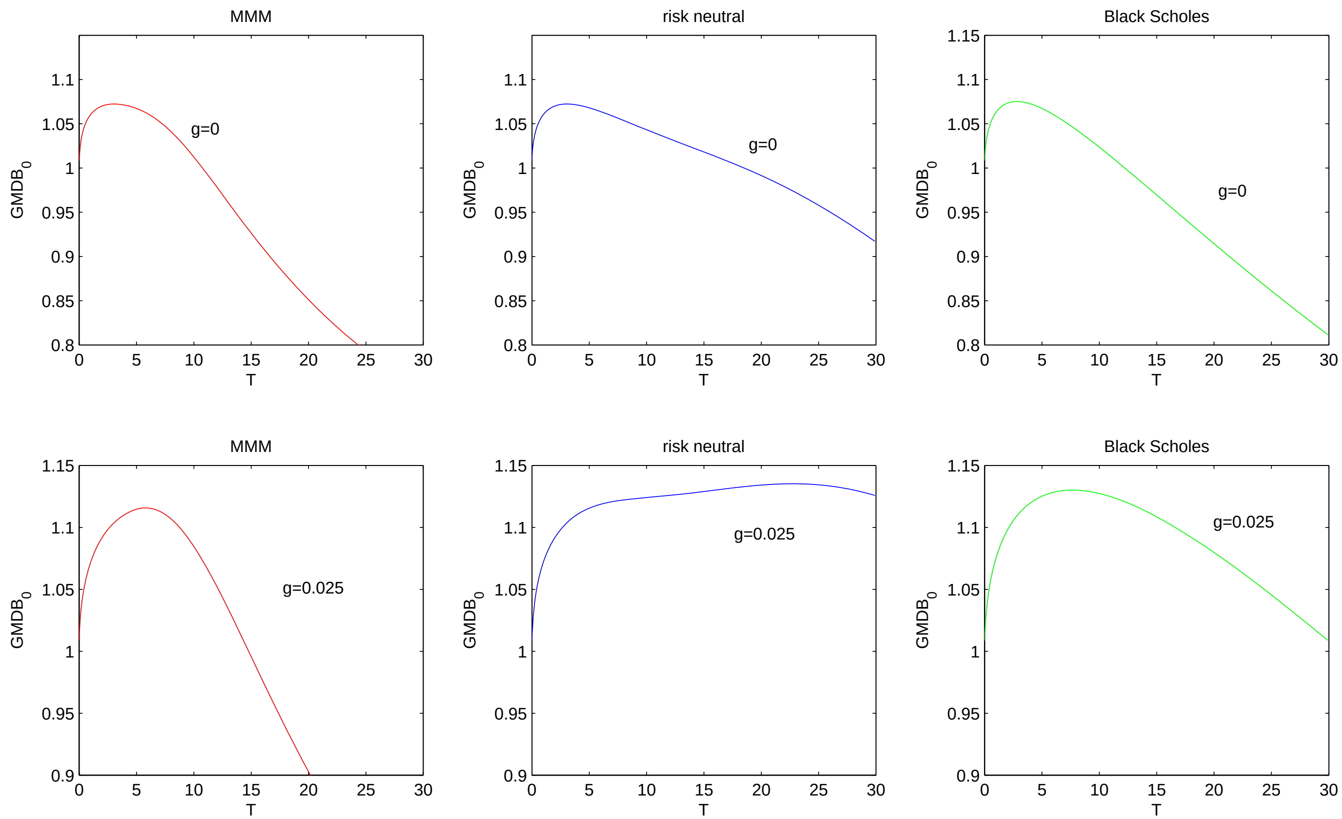
$$\max(e^{g\tau} V_0, V_\tau)$$

  τ time of death

  $g \geq 0$ is the guaranteed instantaneous growth rate

  V_0 is the initial account value

  V_τ is the unit value of the policyholder's account at time of death τ
  embedded put option



Present value of the GMDB under the real world pricing formula (left), the risk neutral pricing formula (middle) and the Black Scholes formula (right) for $\eta = 0.05$, $\alpha_0 = 0.05$, $r = 0.05$, $\xi = 0.01$ and $Y_0 = 20$.

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