Model Independent Greeks

Mathematical Finance Winter School
Lunteren
January 2014

Jesper Andreasen
Danske Markets, Copenhagen
kwant.daddy@danskebank.com

Outline

- Motivation: Wots me Δelδa?
- The short maturity arbitrage condition for implied volatility.
- The minimum variance delta.
- The ATM MV delta and its relation to the volatility skew.
- MV delta in stochastic versus pure local volatility models.
- The ATM MV gamma and its relation to the volatility smile.
- The ATM MV theta and the term structure of implied volatility.

- The MV delta when historical and implied parameters are different.
- Risk and return in option trading.
- Empirical examples.
- Conclusion.

References

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Motivation: Wots me Δelδa?

- It's the most pressing question for any option trader.
- The quant's standard answer is: "That depends on the model – I have ten of them, you pick one..."
- The ambiguity comes about because different models specify different dynamics for the implied volatility as the underlying moves.
- Let $g = g(s,v)$ be the (normal) option pricing formula and v the corresponding implied volatility. We have

$$c_s = g_s + g_v v_s$$

- In a Levy type (jump) model we have $v(s,k)=v(k-s)$ from which we get

$$c_s = g_s - g_v v_k$$

- In a local volatility model

$$v(s,k)=\frac{s-k}{\int_k^s \sigma(a)^{-1}da}+O(\tau) \;\Rightarrow\; v_s(s=k)\approx v_k(s=k)$$

- ... which leads to the exact opposite of the jump model

$$c_s = g_s + g_v v_k \quad ,k=s$$

- Hence, the truly scientific quant would say:

$$c_s = g_s \pm g_v v_k \quad ,k = s$$

- Hagan et al (2002) argues that the delta can be fine tuned in stochastic local volatility models by changing the correlation versus the local volatility component – when the volatility smile is kept the same.

- Dupire (2006), however, counters that and argues that close to ATM, the *minimum variance delta* is virtually independent of the choice of correlation versus local volatility – when the smile is kept the same.

- The minimum variance delta is the position in the underlying stock that (locally) hedges as much variance of the option, i.e. including volatility risk, as possible:

$$\delta = c_s + c_v \frac{\mathrm{cov}[dv, ds]}{\mathrm{var}[ds]}$$

- In this talk we provide a general proof of the Dupire statement in the context of short maturity expansions.

- We further show that the MV delta is uniformly higher for low strikes and lower for high strikes in stochastic volatility models than in pure local volatility models.

- ... and we produce a model free ATM MV gamma, and consider the link between ATM theta and the term structure of implied volatility.

- We investigate results empirically and find that there are significant differences between “realised” and implied MV delta.

- Ie historical and implied parameters of stochastic volatility models differ.

- We show how one can create trading strategies that attempt to benefit from this.

Important Note

- This is what I knew few months ago.

- It turns out, however, that many of our results are well-known to Lorenzo Bergomi, who states the Delta result in his (2004) paper.

- Further, our trading strategy ideas are spiritually related to investigations in Bergomi (2009).

- It also appears that some of our results can be dug out of Durrleman (2004).

Short Maturity Expansion

- Consider the following general class of stochastic volatility models

$$
\begin{aligned}
ds &= \sigma(s,z)dW \\
dz &= \mu(s,z)dt + \varepsilon(s,z)dZ \\
dW \cdot dZ &= \rho(s,z)dt
\end{aligned}
\tag{1}
$$

- Clearly the family of models (1) is rich enough to include several models that fit the same smile.

- As an example, one can think of the smile generated by a Heston model but fitted by a pure local volatility model.

- Let $c(t)$ be the time t price of a European option on $s(T)$:

$$c(t) = E_t[(s(T)-k)^+] \quad (2)$$

- Suppose we write the option price as

$$c(t) = g(t, s(t), v(t)) \quad (3)$$

- ... where $g(\cdot)$ is Bachelier's option price formula and v is the implied normal volatility. I.e.

$$g(t,s,v) = (s-k)\Phi\left(\frac{x}{\sqrt{\tau}}\right) + v\sqrt{\tau}\phi\left(\frac{x}{\sqrt{\tau}}\right) \quad ,x = \frac{s-k}{v} \quad ,\tau = T-t \quad (4)$$

- We think of v as a stochastic process and we want to identify the conditions v has to satisfy for the option prices to be consistent with absence of arbitrage.

- Ito expansion of the option price yields

$$dc = g_t + g_s ds + g_v dv + \frac{1}{2} g_{ss} ds^2 + g_{sv} ds \cdot dv + \frac{1}{2} g_{vv} dv^2 \quad (5)$$

- Using properties of g we obtain

$$dc = g_s ds + \frac{1}{2} g_{ss} [v^2 (dx^2 - dt) + 2\tau v dv] \quad , x = \frac{s-k}{v} \quad (6)$$

- Using that c must be a martingale and therefore $E_t[dc] = 0$ leads to

$$0 = (dx^2 - dt) + 2\frac{\tau}{v} E_t[dv] \quad , x = \frac{s-k}{v} \quad (7)$$

- As $\tau \to 0$ we get the condition that x needs to be of unit diffusion, i.e.

$$\frac{dx^2}{dt}=\sigma^2 x_s^2+2\sigma\rho\varepsilon x_s x_z+\varepsilon^2 x_z^2=1 \quad ,x(s=k)=0 \tag{8}$$

- This is the short maturity arbitrage condition on the implied volatility

$$v=\frac{s-k}{x} \qquad or \qquad v_{BS}=\frac{\ln(s/k)}{x}$$

- Equation (8) is the so-called *Eikonal* equation.

- The Eikonal equation is a non-linear first order partial differential equation on the diffusion rather than the linear second order partial differential equations on the drift that we are used to in finance.

Minimum Variance Delta

- The MV delta is the position in the underlying stock that minimises the noise of the portfolio of option and stock:

$$\min_{\delta} \operatorname{var}_t[dc - \delta ds] \qquad (9)$$

- The idea first appeared in a paper by Föllmer and Sondermann (1986) under the name of *locally risk minimizing strategies*.

- The solution can be found almost directly by rewriting the Brownian motion driver of the volatility process as $dZ = \rho dW + (1-\rho^2)^{1/2} dB$ where $dW \cdot dB = 0$, i.e.

$$
\begin{aligned}
dc - \delta ds &= c_s ds + c_z dz - \delta ds + O(dt) \\
&= [c_s + c_z \frac{\rho\varepsilon}{\sigma} - \delta] ds + c_z (1-\rho^2)^{1/2} dB + O(dt)
\end{aligned}
\tag{10}
$$

- This leads to the following break-down of the MV delta:

$$
\delta = \underset{\substack{naive \\ delta}}{c_s} + \underbrace{\frac{\rho\varepsilon}{\sigma} c_z}_{\substack{correction \\ for\, corr}} = \underbrace{\underset{\substack{sticky \\ strike \\ delta}}{g_s} + \underset{vega}{g_v}}_{\substack{only\, depend\, on \\ smile}} \underbrace{[v_s + \frac{\rho\varepsilon}{\sigma} v_z]}_{\substack{\min\mathrm{var}\, delta\, of \\ the\, implied\, volatility}}
\tag{11}
$$

- So for a given smile, the minimum variance delta of the option price is given from the minimum variance delta of the implied volatility.

- ...because g and all its derivatives in (s,v) only depend on the current smile.

- Here, we will identify the ATM MV delta of the implied volatility from the volatility smile.

- ... and consider what can be said for higher order derivatives.

- For later use we define the MV operator

$$Df = \left[\frac{\partial}{\partial s} + \eta \frac{\partial}{\partial z}\right] f \quad , \eta = \frac{\rho \varepsilon}{\sigma} \tag{12}$$

Rewriting the PDE

- Using the relation $x=(s-k)/v$ and the MV operator D we can rewrite equation (7) as an equation (directly) in implied volatility

$$
\begin{aligned}
0=&[\sigma^2(Dv)^2+\varepsilon^2(1-\rho^2)(v_z)^2](k-s)^2+[2\sigma^2 v(Dv)](k-s)+[\sigma^2 v^2-v^4] \\
&+2\tau v^3[v_t+\frac{1}{2}\sigma^2 D^2 v+\frac{1}{2}\varepsilon^2(1-\rho^2)v_{zz}+(\mu-\frac{1}{2}\sigma^2(D\eta))v_z]
\end{aligned}
\tag{13}
$$

- Equation (13) and differentials of this equation is what we will use for derivation of all results.

ATM Minimum Variance Delta

- Differentiating (13) wrt k and evaluating at $\tau = 0, k = s$ yields

$$v_k = v_s + \frac{\rho\varepsilon}{\sigma} v_z = Dv \tag{14}$$

- *For at-the-money the minimum variance delta of the implied volatility is equal to the slope of the smile.*

- ... for *any* model without jumps.

- This is the statement of Bergomi (2004) and Dupire (2006).

Away from ATM

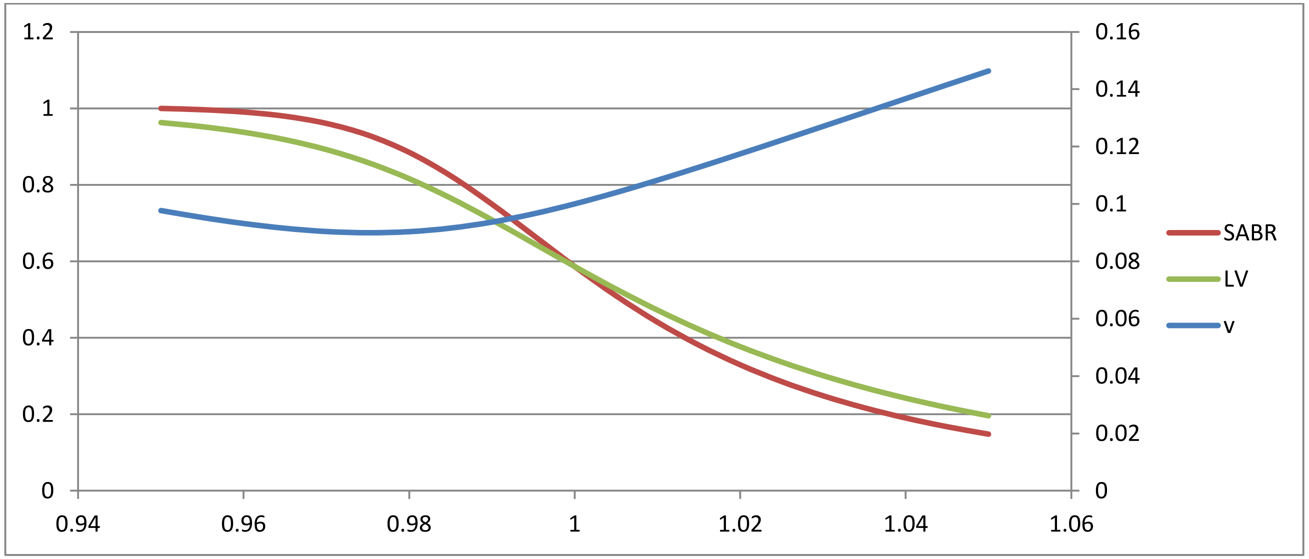
- Using the notation $v(k) = v(s,z;k)$, we can re-arrange equation (13) as

$$Dv(k) = \frac{v(k)}{v(s)} \frac{1}{s-k} \{ v(s) - v(k) [1 - \underbrace{\frac{\varepsilon^2 (1-\rho^2) v_z(k)^2 (k-s)^2}{v(k)^4}}_{\geq 0}]^{1/2} \} \quad , \tau = 0 \qquad (15)$$

- As the term inside the square-root is positive for all stochastic volatility models and zero for pure local volatility models we can conclude that...

- *For a stochastic volatility model relative to a pure local volatility model, the MV Delta is **uniformly** higher (lower) for $k < s$ ($k > s$).*

- An example of MV Delta as function of strike in LV and SLV models:



- SABR parameters: $\sigma(s,z)=0.1z$, $\rho=0.5$, $\varepsilon(s,z)=3z$. $s=z=1, \tau=1/12$.

ATM Minimum Variance Gamma

- Hardcore (Huge) manipulations of (13) lead to

$$D^2 v = [\frac{\partial}{\partial s} + \frac{\rho\varepsilon}{\sigma}\frac{\partial}{\partial z}]^2 v = v_{kk} \quad ,\tau = 0, k = s \tag{16}$$

- *The ATM MV gamma is determined by the ATM curvature of the smile*.

ATM MV Theta

- Differentiating (13) with respect to t and T and combining yield

$$v_t = \sigma_t - v_T \quad , \tau = 0, k = s \tag{17}$$

- *The ATM theta is determined by the slope of the implied volatility in the maturity dimension.*

Relation to Variance Contracts

- Using even Huger manipulations of (13) we derive the following relation between the slope of ATM volatility and the variance forward

$$\frac{1}{4}\underbrace{\frac{\partial E_t[\sigma(T)^2]}{\partial T}}_{\approx slope\,of\,\mathrm{var}\,contract} = v v_T + \frac{1}{2} v^3 v_{kk} + v^2 v_k^2 + \frac{1}{2} v^2 v_k \quad ,\tau = 0, k = s \tag{19}$$

- To apply to VIX, equations need to be converted to BS vols.

The Difference Between P and Q

- We often see clear discrepancies between implied and historical parameters.

- Before we venture into a discussion of whether this is due to market inefficiency or model misspecification, let us first investigate what it implies for delta hedging.

- Suppose the realised dynamics are given by the model as in (1):

$$
\begin{aligned}
ds &= \sigma(s,z)dW \\
dz &= \varepsilon(s,z)dZ \\
dW \cdot dZ &= \rho(s,z)dt
\end{aligned}
\tag{21}
$$

- But suppose that options are priced as if they come from the model

$$
\begin{aligned}
ds &= \bar{\sigma}(s,z)dW \\
dz &= \bar{\varepsilon}(s,z)dZ \\
dW \cdot dZ &= \bar{\rho}(s,z)dt
\end{aligned}
\tag{22}
$$

- Consider a portfolio consisting of one option and $-\delta$ stocks. The portfolio evolves according to

$$
\begin{aligned}
dc - \delta ds = {} & [\bar{c}_t + \frac{1}{2}\sigma^2 \bar{c}_{ss} + \sigma\rho\varepsilon\bar{c}_{sz} + \frac{1}{2}\varepsilon^2 \bar{c}_{zz}]dt \\
& + [\bar{c}_s \sigma + \bar{c}_z \rho\varepsilon - \delta\sigma]dW + \bar{c}_z \sqrt{1-\rho^2}\varepsilon dB
\end{aligned}
\tag{23}
$$

- Because the pricing model is inconsistent with realised dynamics the portfolio value will not be a martingale, i.e. the hedge strategy will not be mean self-financing.

- However, the choice of δ only affects the risk, not the expected return.

- The delta that minimizes the risk is

$$\delta = \bar{c}_s + \frac{\rho\varepsilon}{\sigma}\bar{c}_z = g_s + g_v[\bar{v}_s + \eta\bar{v}_z] \tag{24}$$

- *So the minimum variance delta should be based on the historical (realised) parameter* $\eta = \rho\varepsilon/\sigma$ *in combination with derivatives* $\bar{v}_s, \bar{v}_z$ *computed on implied parameters*.

Estimating MV Delta

- We have

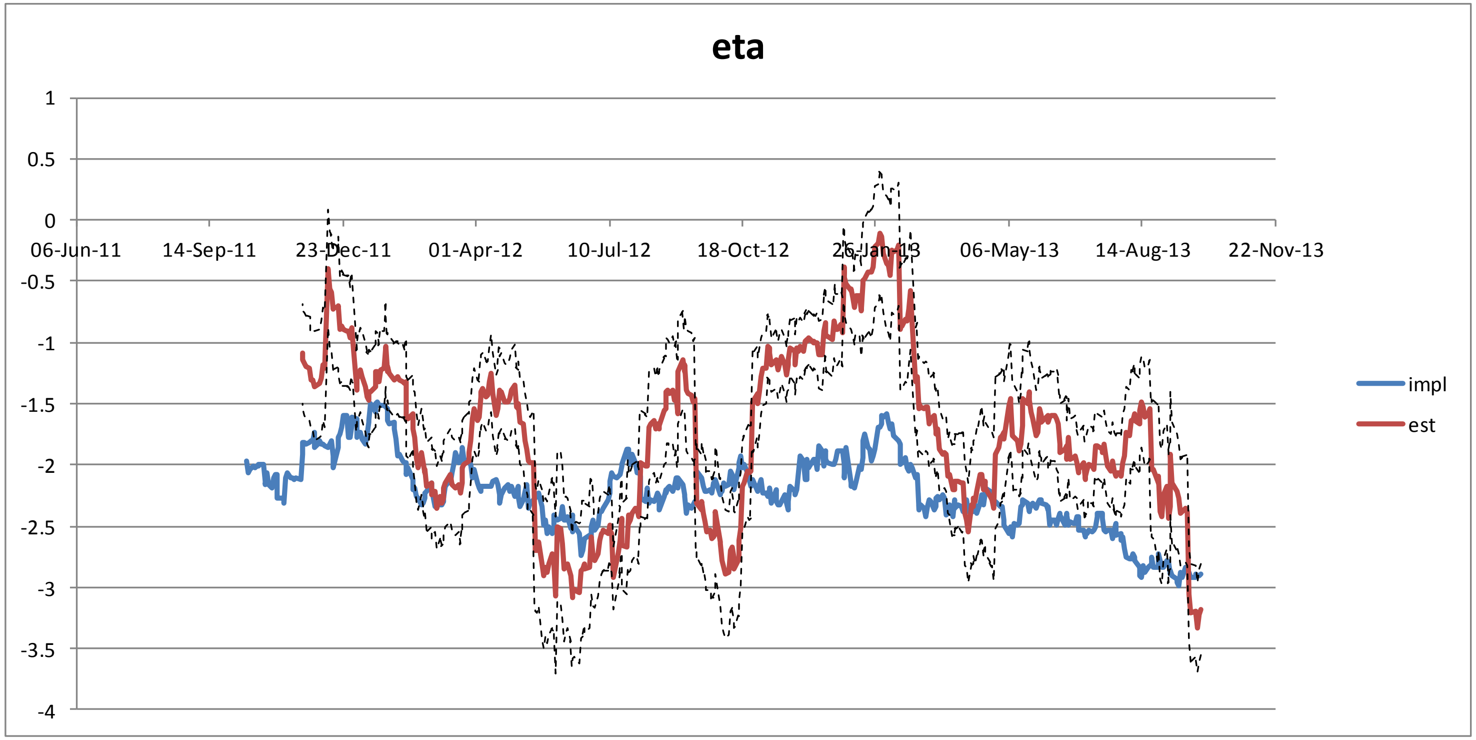
$$
\begin{aligned}
& dv = \bar{v}_s ds + \bar{v}_z dz + O(dt) = \bar{v}_s ds + \bar{v}_z \eta ds + \bar{v}_z \varepsilon \sqrt{1-\rho^2} dB + O(dt) \\
& \Downarrow \\
& \frac{dv - \bar{v}_s ds}{\bar{v}_z} = O(dt) + \eta ds + \varepsilon \sqrt{1-\rho^2} dB
\end{aligned} \tag{25}
$$

- This equation in combination with a model can be used for estimating the historical η and the expected noise of the Delta hedge which is proportional to

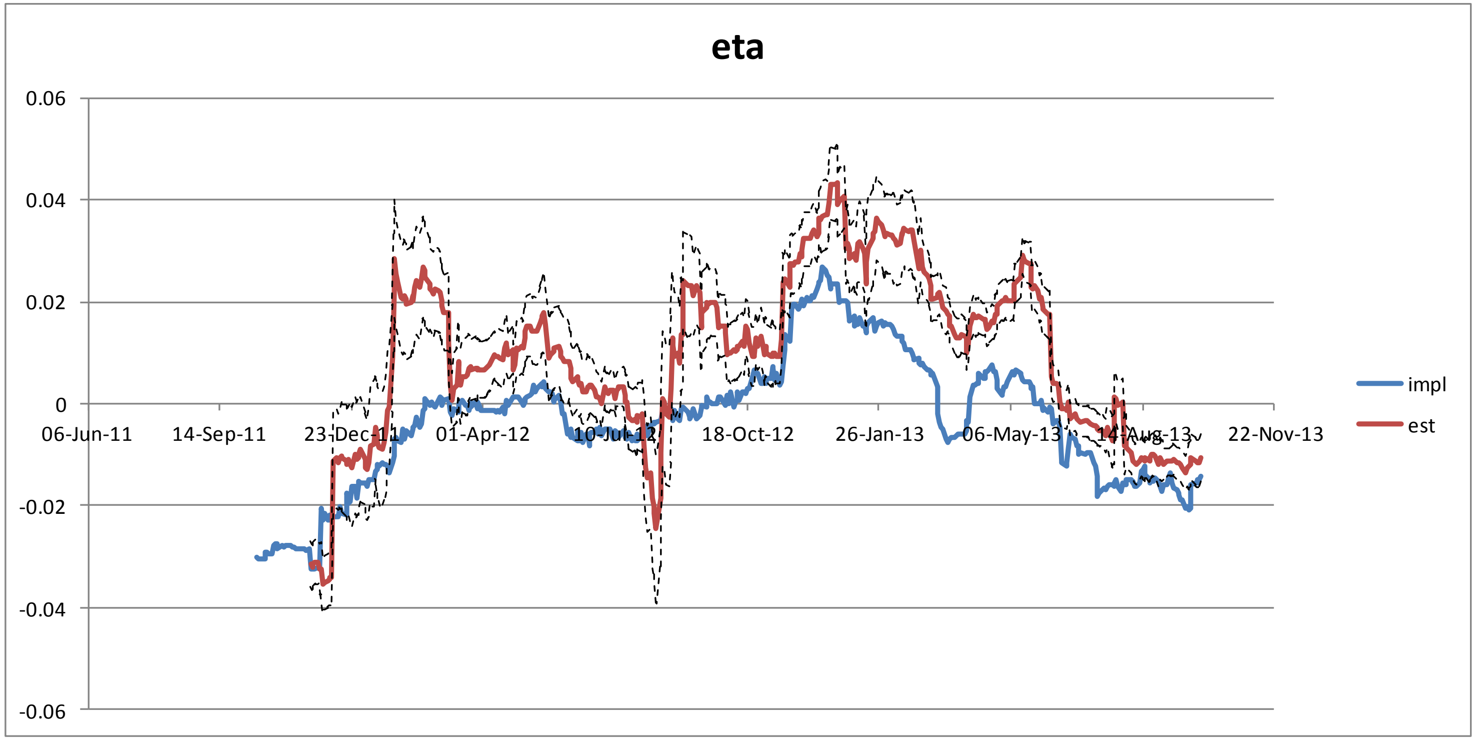
$$\varepsilon\sqrt{1-\rho^2}$$

Realised and Implied MV Delta

- EUR/USD 1y options over a time series of two years.



- USD/JPY 1y options over a time series of two years.



- Dotted lines are error bars of the statistical error.

- Significant differences between historical η and implied $\bar{\eta}$.
- … and a lot of variation and co-variation in both quantities.
- Which shows
    - Implied $\bar{\eta}$ will *not* minimize P&L noise.
    - Hedge η has to be frequently updated.

Risk and Return

- The MV delta hedged option position evolves according to (23).
- This equation can be re-written using the Eikonal equation (13).
- We have the ATM limits

$$\underbrace{dc - \delta ds}_{atm\ delta\ hedged} = g_{ss}\{\underbrace{\frac{1}{2}[\sigma^2 - \bar{\sigma}^2]dt}_{atm\ carry} + \underbrace{\tau v v_z \varepsilon\sqrt{1-\rho^2}dB}_{atm\ vega\ risk} + O(\tau)dt\}$$

$$\underbrace{dc_k - \delta_k ds}_{atm\ skew\ delta\ hedged} = g_{ss}\{\underbrace{v^{-1}[\sigma^2 Dv - \bar{\sigma}^2 \bar{D}v]dt}_{skew\ carry} + \underbrace{\tau(v_k v_z + v v_{kz})\varepsilon\sqrt{1-\rho^2}dB}_{skew\ vega\ risk} + O(\tau)dt\}$$

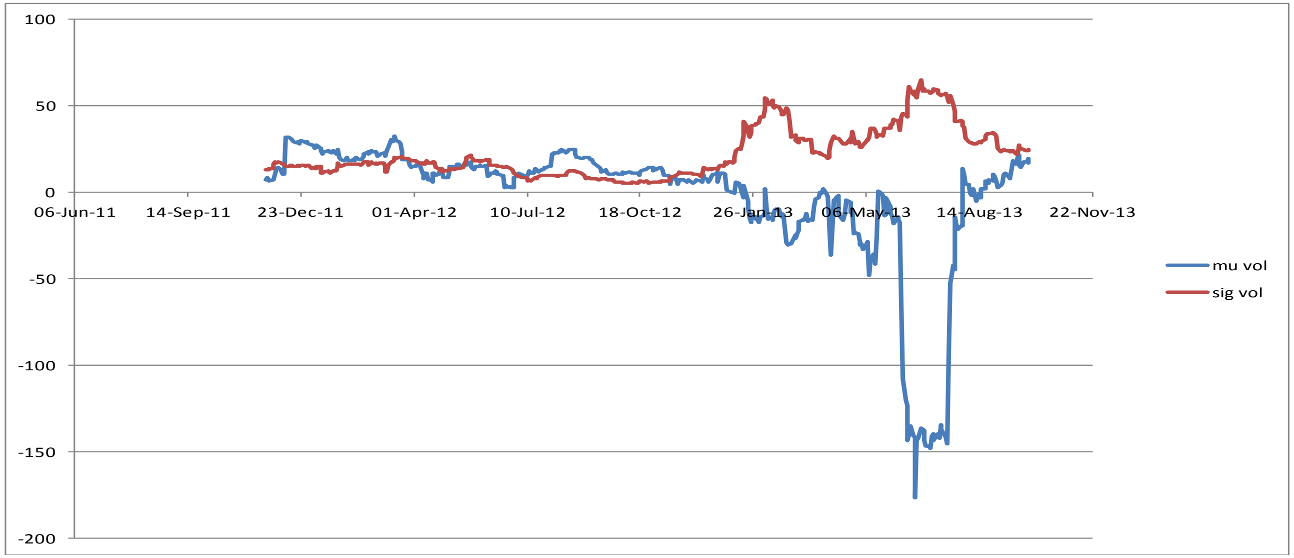
- Which shows that as long as the underlying evolves continuously, the risk of the delta hedged ATM and the skew positions vanish in the short maturity limit.

- Using our empirical estimates we can now compute the risk and return in option positions.

- So option prices combined with historical estimates actually give ***both risk and return information***.

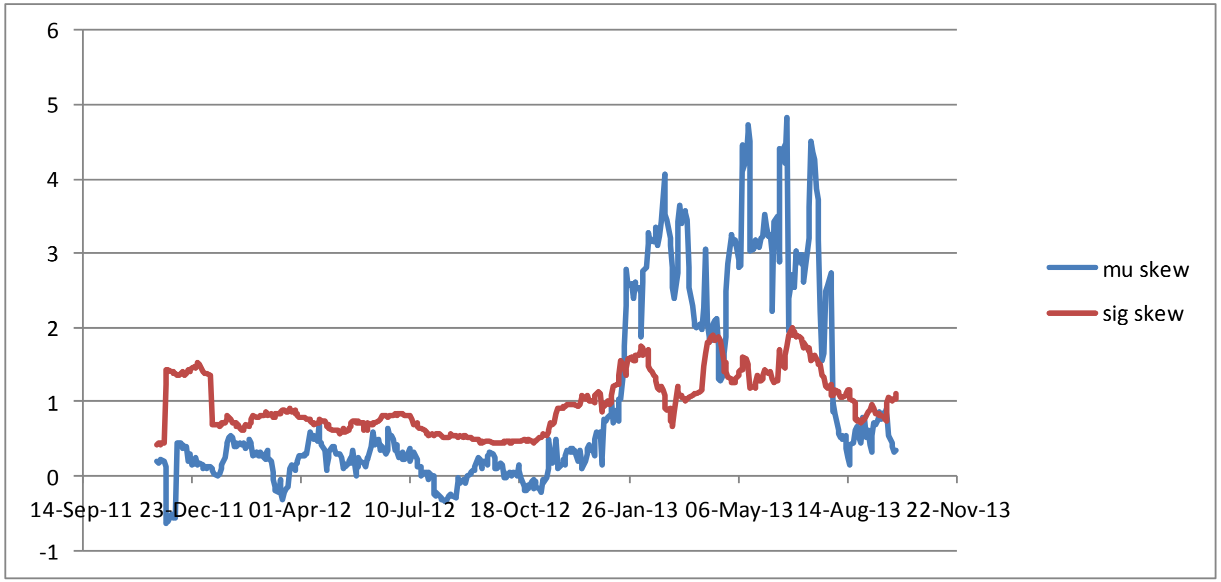
Historical Risk and Return

- Selling ATM 1y USD/JPY volatility (risk and return):



- So if your target Sharpe ratio is 1, then sell ATM volatility over first period, then go neutral and buy volatility towards the end of the last (Abe'nomic) period.

- Selling 1y USD/JPY skew (risk and return)



- There is no point in either buying or selling skew over the first period, whereas it appears profitable to sell over the last period.

- Note that all calculations exclude transaction costs.

Model Misspecification?

- All results here depend on no jumps.

- So the question is whether there are enough jumps to worry.

- A simple test for jumps is to consider

$$\underbrace{\sum \Delta \ln s}_{\log contract} = \underbrace{\sum \frac{\Delta s}{s}}_{delta\, hedge} - \frac{1}{2} \underbrace{\sum (\frac{\Delta s}{s})^2}_{\text{var}\, contract} + \frac{1}{3} \underbrace{\sum (\frac{\Delta s}{s})^3}_{skew\, contract} - \frac{1}{4} \underbrace{\sum (\frac{\Delta s}{s})^4}_{kurtosis\, contract} + \ldots$$

- Hence, can we hedge a log-contract with Delta hedge + variance swap?

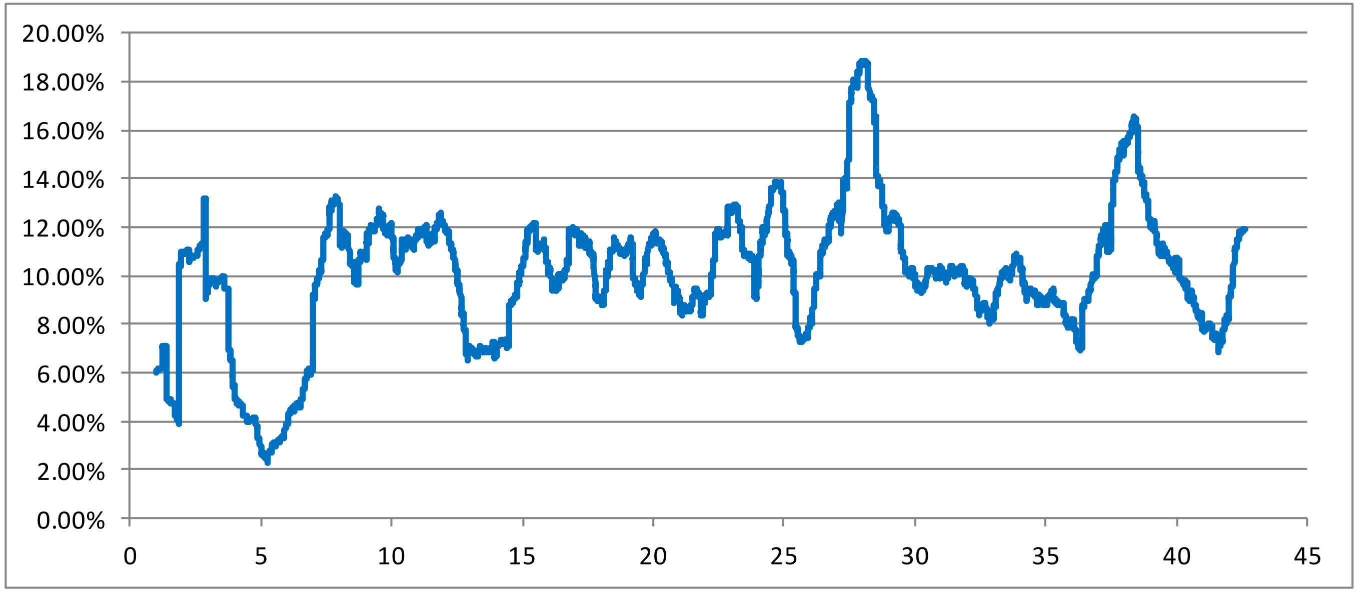
- Or do we need to include skew and kurtosis contracts...?

- This test can be done without any option and interest rate data or assumptions (!) *plus* we can translate the outcomes in terms of implied volatility, through

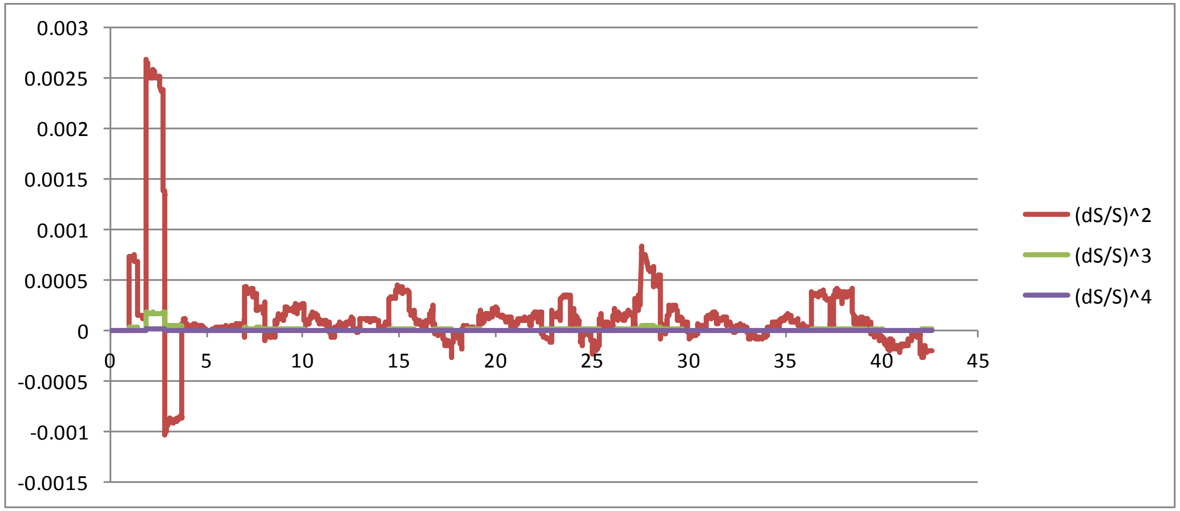
$$\ln S(T)/S(t) = -\frac{1}{2} v_{BS}^2 (T-t)$$

Jumps in FX?

- Rolling 1y log-contract on USD/JPY data (1971-2013) in implied vol:



- Hedging errors:

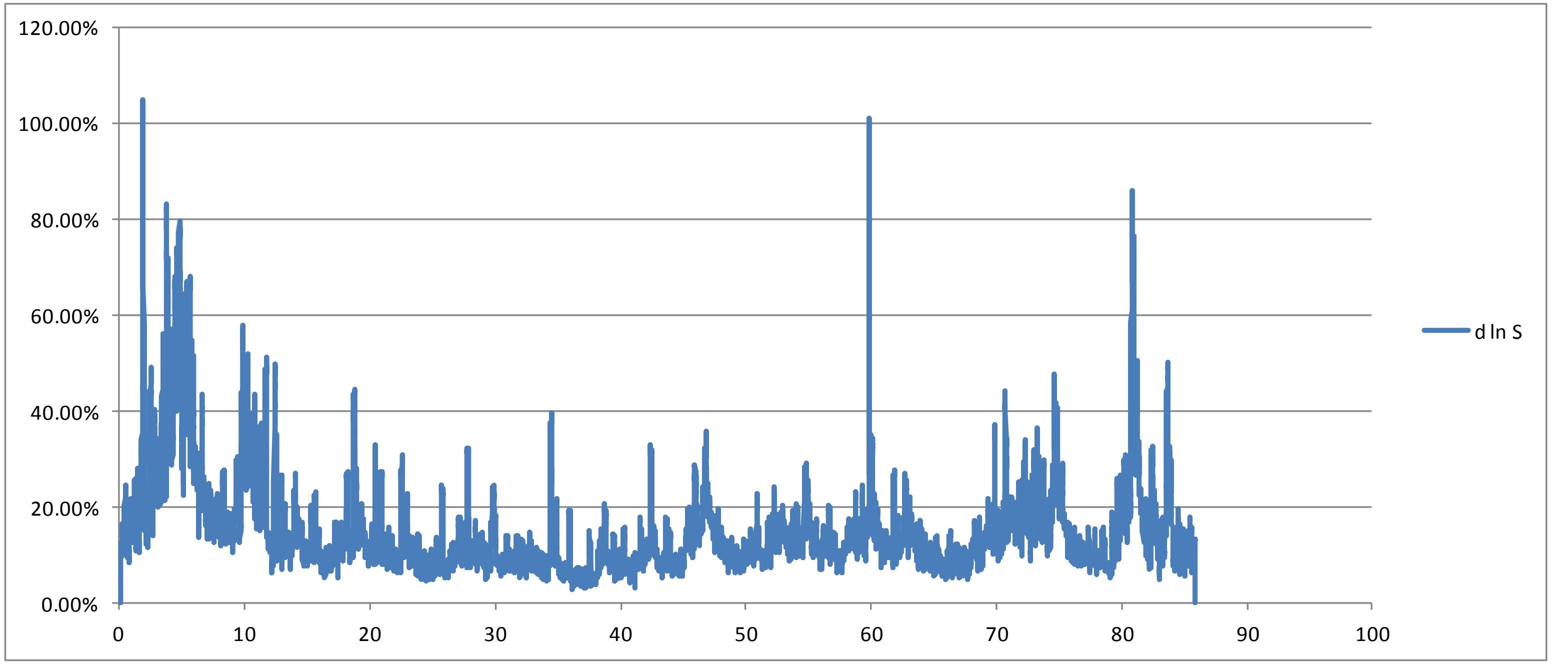


- So the maximum error of hedging log-contract with variance contract over a period of 40y is ~ 0.25% Black volatility!

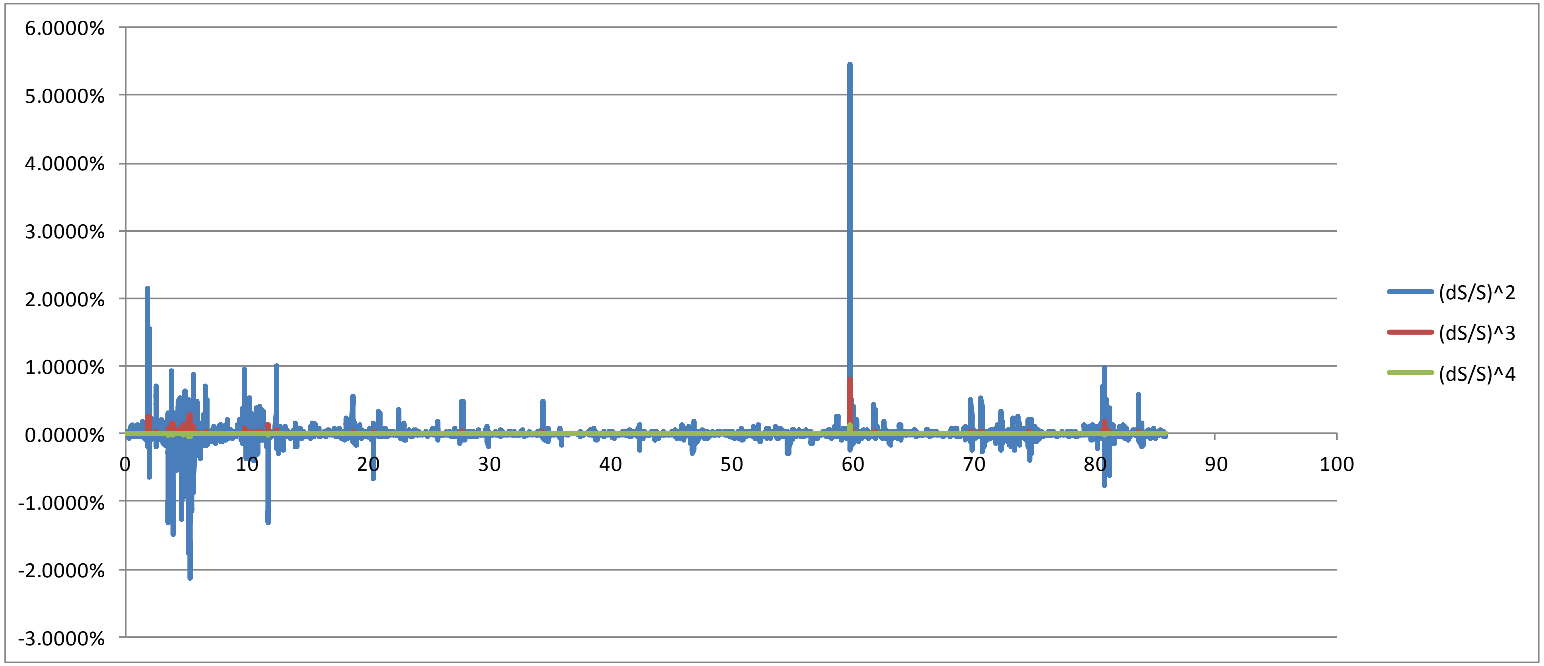
- In other words: the skewness contract never realised more than 0.25% BS vol!

Jumps in Equities?

- Rolling 1m log-contract on S&P500 (1927-2013) in implied volatility:



- Hedging errors in implied volatility terms.



- So very little jump risk in selling ATM options. Primary risk is Vega.

Conclusion

- We have produced model free short maturity limits of

  - ATM minimum variance delta.

  - ATM minimum variance gamma.

- We have shown that MV delta is uniformly higher for low strikes and lower for high strikes in stochastic volatility models relative to pure local volatility models.

- An ATM theta estimate can be produced from slope in the maturity direction of the implied volatility.

- Results hold for all models without jumps that match the observed smile.
- Empirical investigations suggest that historical and implied MV delta can differ significantly.
- ... and that realised parameters tend to fluctuate more than implied.
- MV delta should be computed using historical parameter estimates in combination with implied parameters for the option price derivatives.
- Differences in implied and historical parameters create non-zero expected returns on option books.
- Risk premiums for option positions can be estimated – as opposed to conventional investments.

- Methodologies around the latter need further development.
- ... for example in the direction of identifying the full efficient frontier of volatility trading.
- Qualitatively, our empirical results for FX options are very similar to results obtained for equity options by Bergomi (2004) and Bergomi (2008).