

# Algorithmic Counterparty Credit Exposure for Multi-Asset Bermudan Options

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Winter School on Mathematical Finance, January 20-22, 2014, Netherlands

January 10, 2014

# Outline of the talk

- 1 Introduction
- 2 Credit Exposure of Bermudan Options
- 3 An Efficient Algorithm for Multi-Asset Case
- 4 Numerical Experiments
- 5 Conclusion

# Introduction

“During the financial crisis, however, roughly **two-thirds** of losses attributed to counterparty credit risk were due to CVA losses and only about **one-third** were due to actual defaults.” -Bank for International settlements

- According to Basel II and Basel III, counterparty credit risk (CCR) is the risk that a counterparty in an **over-the-counter (OTC)** derivatives transaction will default before the expiration of the instrument and will not make the current and future payments required by the contract. Exchange-traded derivatives are **not** subject to CCR because the payments promised by the derivatives are guaranteed by the exchange.
- Quantification of counterparty credit risk: potential future exposure (PFE), expected exposure (EE), and credit value adjustment (CVA). An efficient computation method for counterparty credit exposure is required for exotic instruments.

# Introduction

CVA is the difference between the risk free portfolio value,  $V$ , and the true portfolio value that takes into account the counterparty's default,  $\tilde{V}$ , i.e.,  $CVA = V - \tilde{V}$

Assuming **no correlation** between the exposure and default risk, an approximation of CVA in Gregory (2009, p. 194):

$$CVA \approx (1 - \delta) \sum_{m=1}^M D(0, t_m) EE_{t_m} (S_{ur}(t_{m-1}) - S_{ur}(t_m))$$

with  $D(0, t_m)$  the discounted factor,  $S_{ur}(t_m) = \mathbb{Q}[\tau > t_m]$  the risk-neutral survival function, and the expected exposure  $EE_{t_m}$ .

- **Multi-asset** portfolios: need efficient calculation of  $EE_{t_m}$ .
- Here we consider the efficient computation of  $EE_{t_m}$  of multi-asset **Bermudan equity options**; the default risk part will not be discussed. **All of the discussion is from the contract holder's point of view.**

# Definition

- Define a stochastic process  $V(t)$ ,  $0 \leq t \leq T$ , as the value of a derivative security at time  $t$ , under the risk-neutral measure  $\mathbb{Q}$ .
- $V(t)$  is driven by risk factors  $X(t)$ ,  $0 \leq t \leq T$ , such as stock prices, foreign exchange rates, and interest rates[4].
- We call  $(t, X(t))$  the state of the economy at time  $t$ .
- $V(t) = \mathbb{E}^{\mathbb{Q}}[C_{ASHFLOWS}(t, T) | \mathcal{F}_t]$ , with  $C_{ASHFLOWS}(t, T)$  as the derivative's discounted net cashflow between  $t$  and  $T$ .

## Definition

The credit exposure,  $E_t$ , of a derivative security at time  $t$  to a counterparty is defined as the non-negative value of the risk-neutral expected discounted value of future cashflows, i.e.,

$$E_t = \max(V(t), 0) = V(t)^+, 0 \leq t \leq T \quad (1)$$

## Definition

The potential future exposure (PFE) at time  $t$  as seen from time zero is defined as

$$PFE_{\alpha,t} = \inf \{x : \mathbb{P}(E_t \leq x) \geq \alpha\}, 0 \leq t \leq T \quad (2)$$

where  $\alpha$  is the given confidence level, and  $\mathbb{P}$  is the real-world measure.

## Definition

The expected exposure (EE) at time  $t$  as seen from time zero, is given by:

$$EE_t = \mathbb{E}^{\mathbb{P}} [E_t], 0 \leq t \leq T \quad (3)$$

here the expectation is taken under the real-world measure  $\mathbb{P}$ ; For CVA calculation, the expectation should be taken under the risk-neutral measure  $\mathbb{Q}$ .

## Definition

The exposure profile of counterparty credit exposure is defined as the the graph of  $PFE_{\alpha,t}$  or  $EE_t$ , as a function of  $t$ .

- Computation of  $PFE_{\alpha,t}$  and  $EE_t$  is equivalent to the computation of the **probability distribution** of  $E_t$  (or  $V(t)$ ) under measure  $\mathbb{P}$  or  $\mathbb{Q}$ .
- Empirical distribution of the sample results of  $E_t$  (or  $V(t)$ ) on each simulated state  $(t, X(t))$ .

Assuming underlying risk factors  $X(t)$ ,  $0 \leq t \leq T$ , two basic steps needed in **the modelling framework**, see Antonov (2012) [1], Cesari (2010) [2]:

- 1 Scenario simulation of  $X(t)$  under the real-world measure  $\mathbb{Q}$ .
- 2 Calculation of derivative's value (under the risk-neutral measure  $\mathbb{Q}$ ) for each simulated state  $(t, X(t))$ , or the mark-to-market value.



# The modelling framework

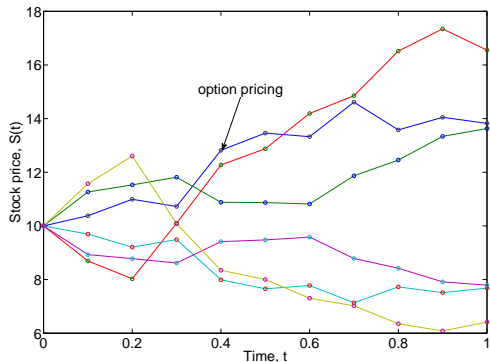


Figure: The modeling framework.

# Credit Exposure of Bermudan Option

Dynamic programming recursion for Bermudan option pricing, initial stock price  $S_0$ ,  $\mathcal{T} = \{t_1, \dots, t_M\}$ ,  $0 = t_0 \leq t_1 < \dots < t_M = T$ , the exercise dates are equally spaced with interval  $\Delta t$ , see Fang & Oosterlee (2009) [3]:

$$\begin{aligned}V_M(S_M) &= \max(h(t_M, S_M), 0) \\c(t_{m-1}, S_{m-1}) &= \exp(-r\Delta t)\mathbb{E}^{\mathbb{Q}}[V_m(S_m)|\mathcal{F}_{t_{m-1}}], m = M, M-1, \dots, 1 \\V_{m-1}(S_{m-1}) &= \max\{h(t_{m-1}, S_{m-1}), c(t_{m-1}, S_{m-1})\} \\V_0(S_0) &= c(t_0, S_0)\end{aligned}$$

On each exercise date, the credit exposure,  $E_{t_m} = \max(V_m(S_m), 0)$ , can be calculated as a **by-product** of the option pricing procedure.

- Let  $s_{m-1}(p)$  be the stock price at possible exercise time  $t_{m-1}$ , on sample path  $p$ ,  $m = 1, \dots, M$ ,  $p = 1, \dots, P$ . For each path  $p$ , the earliest exercise time  $\tau^P$  can be written as,

$$\tau^P = \min\{k \in \{1, \dots, M\} | h(t_k, s_k(p)) \geq c(t_k, s_k(p))\}.$$

- On each path  $p$ , the option will be exercised at  $\tau^P$ , and the option will not exist after exercise event, the exposure of Bermudan option on path  $p$  can be written as,

$$E_{t_m}^P = \begin{cases} \max(V_m(s_m(p)), 0) & t_m \leq \tau^P \\ 0 & t_m > \tau^P \end{cases}$$

## An efficient algorithm for multi-asset case

The continuation value for multi-asset case:

$$c(t_{m-1}, \mathbf{s}_{m-1}(p)) = \exp(-r\Delta t) \mathbb{E}^{\mathbb{Q}} [V_m(\mathbf{S}_m) | \mathbf{S}_{m-1} = \mathbf{s}_{m-1}(p)],$$

with

$$\mathbf{s}_{m-1}(p) = (s_{m-1}^1(p), \dots, s_{m-1}^d(p))$$

Difficulty in Longstaff-Schwartz regression:

- 1 As the dimension increases, the number of basis functions necessary for obtaining a given accuracy will increase. It becomes more difficult to choose a tractable set of basic functions;
- 2 Regression to almost the whole data set may generate bigger approximation error, comparing to more sophisticated regression methods, such as localized regression.

More sophisticated regression methods, such as,

- Cesari (2010): Longstaff-Schwartz regression, enhanced by bundling method.
- The Stochastic Grid Bundling method (SGBM) by Jain & Oosterlee (2013) [5]: A different regression method, enhanced by bundling method.

We are interested in the advantage of SGBM for exposure calculation of multi-asset portfolios, which includes two basic steps:

- 1 Regression along the payoff
- 2 Bundling, i.e., partition the state space into several non-overlapping groups.

## Regression Along the Payoff

Suppose  $V_m(\mathbf{S}_m)$  is a continuous function of  $\mathbf{S}_m$ , then  $V_m(\mathbf{S}_m)$  can be approximated by regressing on a set of basis function of  $f_i(\mathbf{S}_m)$ ,

$$\begin{aligned} V_m(\mathbf{S}_m) &\approx \widehat{V}_m(\mathbf{S}_m) \\ &= \text{Reg}(V_m(\mathbf{S}_m); \Psi_{k,l}(\mathbf{S}_m)), \end{aligned}$$

with

$$\Psi_{k,l}(\mathbf{S}_m) = (f_l(\mathbf{S}_m))^k, l = 1, \dots, L, k = 0, 1, \dots, K$$

and  $f_i : \mathbb{R}^d \rightarrow \mathbb{R}, i = 1, \dots, L$  are continuous functions that map the multi-dimensional process into a scalar. 'Reg' is the regression operator.

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}} [V_m(\mathbf{S}_m) | \mathbf{S}_{m-1} = \mathbf{s}_{m-1}(p)] &\approx \\ \mathbb{E}^{\mathbb{Q}} [\widehat{V}_m(\mathbf{S}_m) | \mathbf{S}_{m-1} = \mathbf{s}_{m-1}(p)], \end{aligned}$$

Since  $\widehat{V}_m$  is a **polynomial function** of basis function  $f$ , **analytical formula or approximation** of  $\mathbb{E}^{\mathbb{Q}}[\widehat{V}_m(\mathbf{S}_m) | \mathbf{S}_{m-1} = \mathbf{s}_{m-1}(p)]$  exists for a large set of Markov process.

**Example:** call-on-max payoff, with GBM model for underlying stock price process

$$h(S_t) = \max(S_t^1, \dots, S_t^d) - K,$$

For 2 dimension case, the choice of basis functions could be:

$$\begin{aligned} f_1(S_t) &= \log(\max(S_t^1, S_t^2)) \\ (f_1(S_t))^2 \\ (f_1(S_t))^3 \end{aligned}$$

$$f_2(S_t) = (S_t^1 S_t^2)^{\frac{1}{2}}$$

$$f_3(S_t) = S_t^1$$

$$f_4(S_t) = S_t^2$$

## Enhance the accuracy

$$\mathbb{E}^{\mathbb{Q}}[\widehat{V}_m(\mathbf{S}_m) | \mathbf{S}_{m-1} = \mathbf{s}_{m-1}(p)] = \int_{\mathbb{R}^d} \widehat{V}_m(x) \mathbb{P}[\mathbf{S}_m \in dx | \mathbf{S}_{m-1} = \mathbf{s}_{m-1}(p)]$$

The regression approximation  $\widehat{V}_m$  should be as accurate as possible for the region where  $\mathbf{S}_m$  has the 'most' probability mass, originating from the source  $\mathbf{s}_{m-1}(p)$ .

- We do not have the grid points original from  $\mathbf{s}_{m-1}(p)$ , but we have the grid points original from the 'neighbourhood' of  $\mathbf{s}_{m-1}(p)$ .
- The probability distribution of grid points original from grid point  $\mathbf{s}_{m-1}(p)$  could be approximated by the probability distribution of grid points original from the 'neighbourhood' of  $\mathbf{s}_{m-1}(p)$



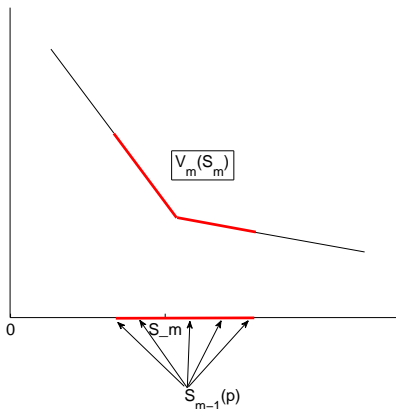
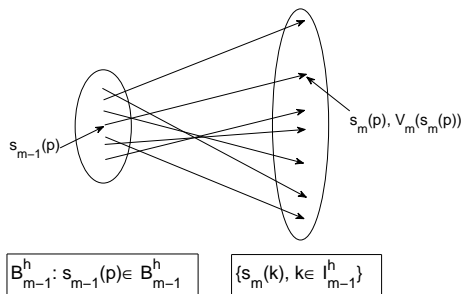


Figure: The regression approximation  $\hat{V}_m$  should be as accurate as possible for the region where  $\mathbf{S}_m$  has the 'most' probability mass, originating from the source  $\mathbf{s}_{m-1}(p)$ .

# Bundling



**Figure:** Grid points  $S_m(p)$  are original from bundle  $B_{m-1}^h$ ;  $I_{m-1}^h$  is the set of path indices in bundle  $B_{m-1}^h$

- **'localized regression':**  $\hat{V}_m^h$  is the regression approximation of  $\hat{V}_m$  that originates from  $B_{m-1}^h$  (the 'neighbourhood' of  $s_{m-1}(p)$ ).

Suppose the whole state space has been separated into  $H$  non-overlapping bundles.

- The continuation value of the grid points **in each bundle**  $\mathcal{B}_{m-1}^h$ ,  $h = 1, \dots, H$ ,

$$c^h(t_{m-1}, \mathbf{s}_{m-1}(p)) \approx \exp(-r\Delta t) \\ \times \mathbb{E}^{\mathbb{Q}}[\widehat{V}_m^h(\mathbf{S}_m) | \mathbf{S}_{m-1} = \mathbf{s}_{m-1}(p)]$$

where  $\widehat{V}_m^h(\mathbf{S}_m)$  is the regression approximation of function form of  $V_m(\mathbf{S}_m)$ , with  $\mathbf{S}_m$  original from bundle  $\mathcal{B}_{m-1}^h$ .

- The continuation value for **the all grid points** at  $t_{m-1}$  is denoted as  $c(t_{m-1}, \mathbf{s}_{m-1}(p))$ , which is a **combination** of  $c^h(t_{m-1}, \mathbf{s}_{m-1}(p))$ ,  $h = 1, \dots, H$ .

## Bundling the state space

To bundle  $P$  grid points  $\mathbf{s}_m(p) = (s_m^1(p), s_m^2(p))$ ,  $p = 1, 2, \dots, P$  at time step  $t_m$ ,  $m = 1, \dots, M$ , the following steps need to be performed recursively:

- 1 Estimate the mean value for each stock  $i$  at time  $t_m$ , i.e.,

$$\hat{\mu}_m^i = \frac{1}{P} \sum_{p=1}^P s_m^i(p), i = 1, 2$$

- 2 Define the following subsets of grid points:

$$G_m^i = \{\mathbf{s}_m(p) : s_m^i(p) > \hat{\mu}_m^i\}, \quad \overline{G}_m^i = \{\mathbf{s}_m(p) : \mathbf{s}_m(p) \notin G_m^i\}$$

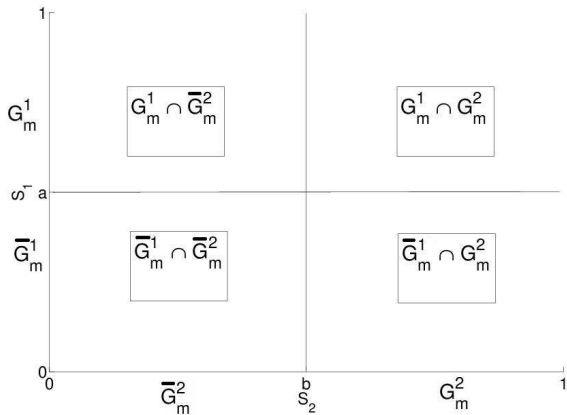


Figure: Combination of different subsets

- 3 The 4 (i.e.,  $2^2$ ) unique bundles are obtained through combinations of different subsets, i.e.,

$$\begin{aligned} \mathcal{B}_m^1 &= G_m^1 \cap G_m^2 & \mathcal{B}_m^2 &= \overline{G}_m^1 \cap G_m^2 \\ \mathcal{B}_m^3 &= G_m^1 \cap \overline{G}_m^2 & \mathcal{B}_m^4 &= \overline{G}_m^1 \cap \overline{G}_m^2 \end{aligned}$$

- 4 If more bundles are required, the same procedure, from (1) to (3), can be performed, either for each bundles,  $\mathcal{B}_m^1, \dots, \mathcal{B}_m^4$  or some of them.

**In summary**, we have the following SGBM algorithm for exposure calculation of multi-asset Bermudan options, see Shen, Anderluh, and Van Der Weide (2013) [7]:

- 1 Simulate sample paths for the stock price,  $\mathbf{s}_0, \mathbf{s}_1(p), \dots, \mathbf{s}_M(p)$ , at time steps  $0 = t_0, \dots, t_M = T$ , with indices of paths  $p = 1, \dots, P$ , under the risk-neutral measure  $\mathbb{Q}$ .
- 2 At terminal date  $t_M = T$ , set

$$V_M(\mathbf{s}_M(p)) = \max(h(t_M, \mathbf{s}_M(p)), 0)$$

for  $p = 1, \dots, P$ .

- ③ Apply backward induction, i.e.,  $m \rightarrow m - 1$  for  $m = M, \dots, 1$ .
- ① Bundling the grid points at  $t_{m-1}$ , into  $H$  distinct bundles
  - ② For each bundle  $\mathcal{B}_{m-1}^h$ ,  $h = 1, \dots, H$ , compute the regression approximation  $\widehat{V}_m^h(\mathbf{S}_m)$
  - ③ For every grid point  $\mathbf{s}_{m-1}(p) \in \mathcal{B}_{m-1}^h$ , by using the regression functions and moment calculation, compute the continuation value  $c^h(t_{m-1}, \mathbf{s}_{m-1}(p))$
- ② For each sample path  $p = 1, \dots, P$ , set

$$V_{m-1}(\mathbf{s}_{m-1}(p)) = \max(h(t_{m-1}, \mathbf{s}_{m-1}(p)), c(t_{m-1}, \mathbf{s}_{m-1}(p)));$$

if  $h(t_{m-1}, \mathbf{s}_{m-1}(p)) > c(t_{m-1}, \mathbf{s}_{m-1}(p))$ , set

$$V_m(\mathbf{s}_m(p)) = 0, V_{m+1}(\mathbf{s}_{m+1}(p)) = 0, \dots, V_M(\mathbf{s}_M(p)) = 0.$$



- 4 The initial option price,  $V_0(\mathbf{s}_0) = c(0, \mathbf{s}_0)$  (exercise is not allowed at  $t_0$ ).
- 5  $E_{t_m}^P = \max(V_m(\mathbf{s}_m(p)), 0)$  is the credit exposures, at time step  $t_m$ , on sample path  $p$ .

In the end, based on the results of credit exposures obtained from the algorithm, we can estimate the **risk measures**, such as expected exposure (EE) or potential future exposure (PFE), under the risk neutral measure  $\mathbb{Q}$ . And for the probability distribution under the real world measure  $\mathbb{P}$ , we can use **change of measure**.

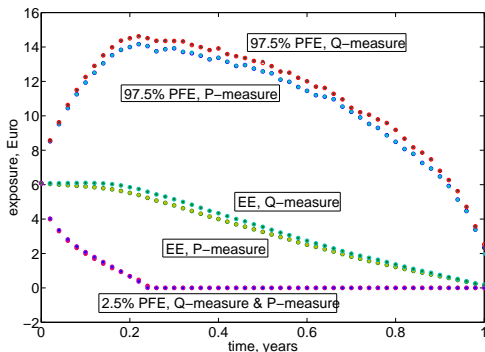
# Numerical Examples

Parameter setting of single-asset Bermudan option: GBM stock price process, initial price  $s_0 = 100$ , strike price  $K = 100$ , constant interest rate  $r = 0.05$ , real world drift  $\mu = 0.1$ , volatility  $\sigma = 0.2$ , possible early exercise dates  $M = 50$ , 18,000 simulation paths and 50 discrete time steps. We use SGBM exposure algorithm to provide:

- 1  $\mathbb{Q}$ -exposure profile, i.e., the probability distribution of exposure is under measure  $\mathbb{Q}$ .
- 2  $\mathbb{P}$ -exposure profile, i.e., the probability distribution of exposure is under measure  $\mathbb{P}$ .

The benchmark result for compare is provided by combination of Monte Carlo and Fourier COS option pricing method (MCCOS), see Shen, van der Weide and Anderluh (2013) [8]. For all the cases, at each time step, the results of PFE and EE from SGBM match almost perfectly with the results from MCCOS, which means SGBM has an excellent performance for exposure calculation.

# Single-Asset Bermudan Option



**Figure:** Bermudan put option, single asset, by SGBM ('o') and MCCOS ('\*'). In Schöftner (2008)[6], the author showed a similar exposure profile shape for American options, **but without benchmark results**. Here we confirm the accuracy of SGBM by using an accurate result from MCCOS.

# Single-Asset European Option

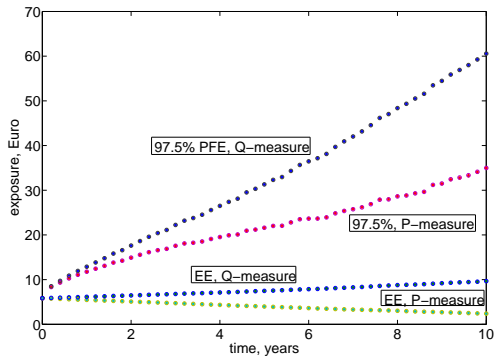


Figure: European put option, single asset, by SGBM ('o') and MCCOS ('\*').

## Expected Exposure

Bermudan option's EE (expected exposure) under measure  $\mathbb{P}$  and  $\mathbb{Q}$ , where  $\mathbb{Q}$ -EE is used for CVA calculation:

Time	0.1	0.2	0.3	0.4	0.5
$\mathbb{P}$	5.8983	5.5188	4.7929	4.0037	3.2563
$\mathbb{Q}$	6.1020	5.8501	5.1485	4.3417	3.5437

**Table:** Expected Exposure (EE) calculated under measure  $\mathbb{P}$  and  $\mathbb{Q}$ .

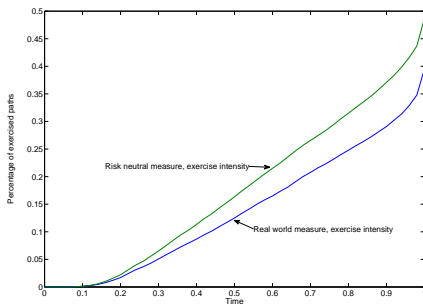
Time	0.6	0.7	0.8	0.9	1
$\mathbb{P}$	2.5100	1.8140	1.2148	0.6762	0.1654
$\mathbb{Q}$	2.7390	1.9942	1.3643	0.7519	0.1799

**Table:** Expected Exposure (EE) calculated under measure  $\mathbb{P}$  and  $\mathbb{Q}$ .

Comparing to the increasing of EE for European option, the **decreasing** of EE for Bermudan option is because of **the early exercise feature**.

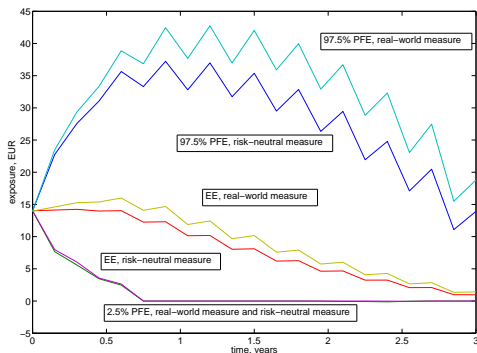
# Early Exercise

This figure shows the **percentage of exercised paths** at each time step  $t$  (**exercise intensity**).



**Figure:** The  $\mathbb{Q}$ -exercise intensity is higher (i.e., exercise more often) than  $\mathbb{P}$ -exercise intensity.

# Multi-asset Bermudan Option



**Figure:** Exposure profiles of **2-asset Bermudan call-on-max option** under different measures. Parameter setting: GBM model,  $K = 100$ ,  $r = 0.05$ ,  $q = 0.1$ ,  $\rho = 0$ ,  $T = 3$ ,  $\sigma = [0.2, 0.2]$ ,  $S_0 = [100, 100]$ , 10 equally spaced exercise opportunities, 30,000 simulation paths, 32 bundles, and 20 simulation steps.

# Conclusion

- We propose an efficient algorithm for the credit exposure calculation of multi-asset Bermudan options.
- Future work: CVA of multi-asset Bermudan option, wrong way risk for put option (i.e., correlation between default and exposure) and early exercise feature, etc.



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