Optimal Consumption and Investment During Retirement

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Motivation

Main question typically asked in consumption and portfolio choice literature:

how should an individual spend a given amount of (initial) wealth in order to maximize expected utility?

Preferences $\Longrightarrow$ Optimal Consumption (and Portfolio) Choice

Motivation (2)

We propose an **inverse** approach to optimal consumption and portfolio choice problems.

As a starting point, we assume a general specification of the consumption profile in retirement.

Questions:

- What is the optimal wealth profile?
  - The optimal wealth profile implies a market-consistent discount rate.
- What is the optimal investment strategy?
- Which preference models are consistent with our consumption profile?

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The Economy: The State Variables

We define an economy following Brennan and Xia (2002).

The state of the economy is characterized by the *real interest rate* r, the *expected rate of inflation* π, the *stock price* S and *the consumer price index* Π.

- $\mathrm{d}r_t = \kappa(\bar{r} - r_t)\mathrm{d}t + \sigma_r \mathrm{d}W_t^r$;
- $\mathrm{d}\pi_t = \theta(\bar{\pi} - \pi_t)\mathrm{d}t + \sigma_\pi \mathrm{d}W_t^\pi$;
- $\frac{\mathrm{d}S_t}{S_t} = \left(R_t^f + \lambda_S \sigma_S\right)\mathrm{d}t + \sigma_S \mathrm{d}W_t^S$;
- $\frac{\mathrm{d}\Pi_t}{\Pi_t} = \pi_t \mathrm{d}t + \sigma_\Pi \mathrm{d}W_t^\Pi = \pi_t \mathrm{d}t + \xi_r \mathrm{d}W_t^r + \xi_\pi \mathrm{d}W_t^\pi + \xi_S \mathrm{d}W_t^S + \xi_u \mathrm{d}W_t^u$.

Assumption: $\mathrm{d}W_t^u$ is orthogonal to $\mathrm{d}W_t^r$, $\mathrm{d}W_t^\pi$ and $\mathrm{d}W_t^S$.

The Economy: Bond Price Dynamics

The agent has the opportunity to invest in four risky assets: a real bond, two nominal bonds (with distinct maturities) and a stock.

- Nominal bond price $P_{t,h}$ satisfies:

$$\frac{\mathrm{d}P_{t,h}}{P_{t,h}} = \left(r_t + \pi_t - [\lambda^\top \xi + \lambda_u \xi_u] - B_h h \sigma_r \lambda_r - C_h h \sigma_\pi \lambda_\pi\right)\mathrm{d}t - B_h h \sigma_r \mathrm{d}W_t^r - C_h h \sigma_\pi \mathrm{d}W_t^\pi.$$

- Real bond price $p_{t,h}$ satisfies:

$$\frac{\mathrm{d}p_{t,h}}{p_{t,h}} = \left(r_t - B_h h \sigma_r \lambda_r\right)\mathrm{d}t - B_h h \sigma_r \mathrm{d}W_t^r + \sigma_\Pi \mathrm{d}W_t^\Pi.$$

Here, B_h, $C_h \in [0,1]$ are horizon-dependent constants; $\lambda \equiv (\lambda_r, \lambda_\pi, \lambda_S)$ and λ_u represent market prices of risk. We assume that λ is known.

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Consumption Profile in Retirement

The log nominal consumption profile $\log Y_s^N \equiv \log[\Pi_s^\alpha Y_s]$ is defined as follows:

$$\begin{aligned}\log Y_s^N &= \alpha \log \Pi_s + \log Y_s \\ &= \alpha \log \Pi_s + \log Y_0^s + \int_0^s \psi_{s-v}\left(r_v + [1-\alpha]\pi_v\right) \mathrm{d}v \\ &\quad \int_0^s q_{s-v}\left(w_r \mathrm{d}W_t^r + w_\pi \mathrm{d}W_t^\pi + w_S \mathrm{d}W_t^S + w_u \mathrm{d}W_t^u\right).\end{aligned}$$

Here, $\alpha, q_j \in [0,1]$. We assume that q_j is non-decreasing with the horizon j.

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Parameter Interpretations

- α represents the extent to which pension is linked to the price index.
  - $\alpha = 1$: real pension; $\alpha = 0$: nominal pension.
- ψ_j represents the sensitivity of pension to the interest rate.
  - $\psi_j > 0$: pension tends to increase as return on savings rises.
- q_j represents the exposure to (current) financial shocks.
  - Pension consumption in the distant future is more affected by current shocks than pension consumption in the near future.
- w_r, w_π, w_S and w_u represent long-term exposures.

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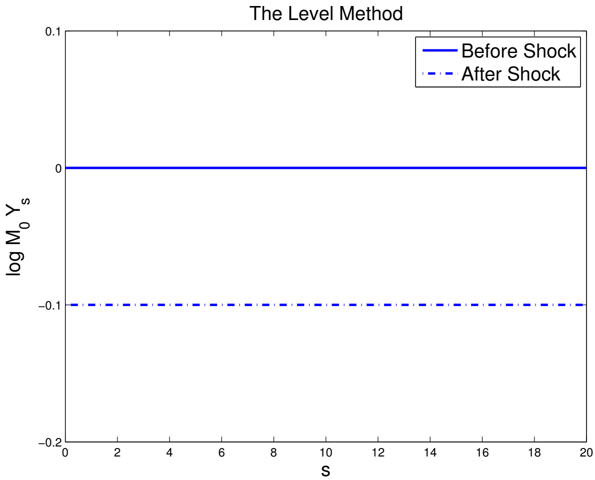
Consumption Profile: Special Cases

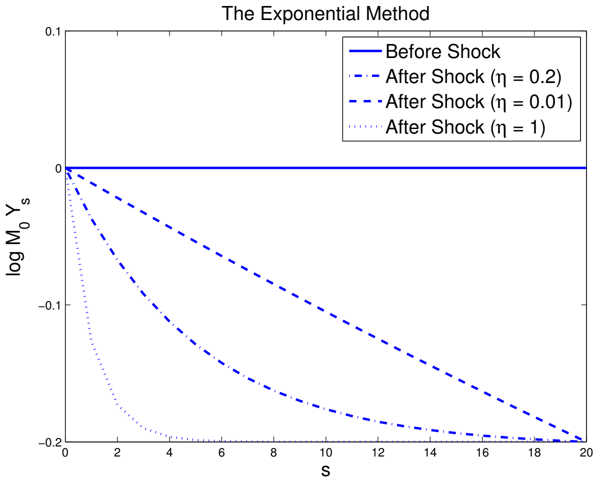
- **Nominal Defined Benefit Pension Scheme.**
  $\alpha = 0$, $\psi_j = 0$ and $q_j = 0$ for all j.
- **Real Defined Benefit Pension Scheme.**
  $\alpha = 1$, $\psi_j = 0$ and $q_j = 0$ for all j.

- **The Level Method.**
  $\psi_j = 0$ and $q_j = 1$ for all j.
- **The Exponential Method.**
  $\psi_j = 0$ and $q_j = 1 - e^{-\eta j}$ for all j.
- **The *N*-years Exponential Method.**
  $\psi_j = 0$ and $q_j = \left(1 - e^{-\eta j}\right) \mathbb{1}_{[j \leq N]} + \left(1 - e^{-\eta N}\right) \mathbb{1}_{[j > N]}$ for all j.

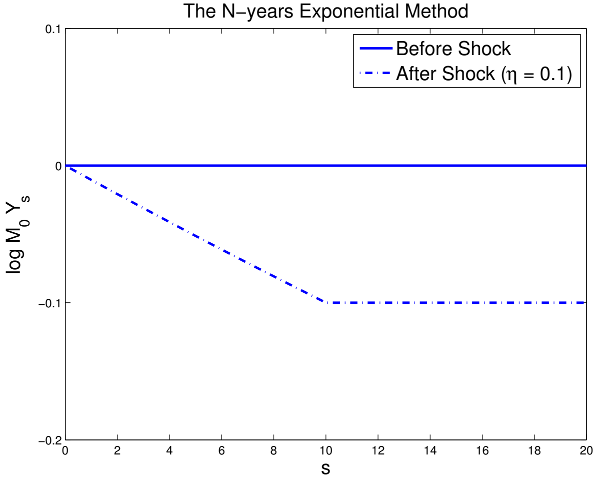
The Level Method



The Exponential Method



The N-years Exponential Method



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Market-Consistent Discount Rate

The market-consistent discount rate can be determined as follows:

$$L_t = \mathbb{E}_t\left[\int_t^T \frac{M_s}{M_t} Y_s^N \mathrm{d}s\right] = \Pi_t^{\alpha} \int_t^T e^{-\mu_t^s (s-t)} \mathbb{M}_t Y_s \mathrm{d}s.$$

Here, M denotes the nominal pricing kernel and $\mathbb{M}_t$ stands for the conditional median.

Straightforward computations show that

$$\mu_t^s = R_t^s + \text{risk premium}$$

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Interpretation

- The first term R_t^s corresponds to the discount rate required to finance the log pension ambition $\log Y_0^s + \int_t^s \psi_{s-v}(r_v + [1-\alpha]\pi_v)\mathrm{d}v$.
- When $\alpha = 0$ and $\psi_j = 0$, R_t^s collapses to the nominal term structure of interest rates.
- When $\alpha = 1$ and $\psi_j = 0$, R_t^s boils down to the real term structure of interest rates.
- The last term represents a horizon-dependent risk premium. This term arises because we allow the agent to take speculative risk.
- The risk premium increases with q_j.

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Investment Strategy

The investment strategy is determined in such a way that the individual's wealth matches the market value of the liabilities.

Explicit analytical expressions for the optimal portfolio weights are provided in the paper.

Portfolio Strategy = Hedge Component + Speculative Component.

Observations

- Hedge component → **Liability-driven investment.** The real interest rate and the expected inflation rate duration largely determine the hedge component.
- *Expected* hedge component decreases as the individual ages. The liabilities of old individuals are less sensitive to interest rate and expected inflation rate shocks.
- ψ_j determines the hedge component, while q_j determines the speculative component.
- Speculative portfolio is proportional to q_t where

$$q_t \equiv \frac{1}{L_t} \int_t^T q_{s-t} \Phi_{t,s} \mathrm{d}s.$$

q_t typically decreases as the agent ages → life cycle strategy.

Efficient Investment Strategy

The efficient investment strategy can be obtained by maximizing the expected excess return subject to $w^\top \rho w + w_u^2 = w_1^2$. Specifically, the individual considers the following maximization problem:

$$\underset{w_r, w_\pi, w_S, w_u}{\text{maximize}} \quad q_t w^\top (\lambda - \alpha\rho\xi) + q_t w_u (\lambda_u - \alpha\xi_u)$$
$$\text{subject to} \quad w^\top \rho w + w_u^2 = w_1^2.$$

The optimal solution is given by

$$w_i = \frac{(1-\alpha)\xi_i - \phi_i}{\phi_1} w_1, \qquad \text{for } i = r, \pi, S, \Pi.$$

Here, $\phi = \xi - \rho^{-1}\lambda$. The efficient investment portfolio can be obtained by substituting the expressions for w_r, w_π, w_S and w_u into the expressions for the portfolio weights.

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Preferences

- Our consumption profile is consistent with CRRA utility.
  - $\psi_j = \frac{1}{\gamma}$ and $q_j = 1$ for all j. Here, γ denotes the coefficient of relative risk aversion.
- Under the assumption of linearizing the budget constraint, we can show that (multiplicative) habit formation is consistent with our consumption profile.
  - $q_j = c + (1 - c)\left[1 - e^{-\eta j}\right]$ for all j, for some c.

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Summary

- We have determined the market-consistent discount rate and the investment strategy for a general specification of the consumption profile.
- The consumption profile is controlled by four preference parameters.
  - α: nominal vs. real;
  - ψ_j: sensitivity to the interest rate;
  - q_j: relative exposure to shocks and
  - w_1: long-term exposure.
- All results are analytical!

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Thank you for your attention!