Basket option pricing and implied correlation in a Lévy copula model

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Abstract

In this paper we employ the Lévy copula model to determine basket option prices. More precisely, basket option prices are determined by replacing the distribution of the real basket with an appropriate approximation. For the approximate basket we determine the underlying characteristic function and hence we can derive the related basket option prices by using the Carr-Madan formula. Two approaches are considered. In the first approach, we replace the arithmetic sum by an appropriate geometric average, whereas the second approach can be considered as a three-moments-matching method. Numerical examples illustrate the accuracy of our approximations; several Lévy models are calibrated to market data and basket option prices are determined.

In a last part we show how our newly designed basket option pricing formula can be used to define implied Lévy correlation by matching model and market prices for basket options. Our main finding is that the implied Lévy correlation smile is flatter than its Gaussian counterpart. Furthermore, if (near) at-the-money option prices are used, the corresponding implied Gaussian correlation estimate is a good proxy for the implied Lévy correlation.

Keywords: basket options, characteristic function, implied correlation, Lévy market, Variance-Gamma.

1 Introduction

Nowadays, an increased volume of multi-asset derivatives is traded. An example of such a derivative is a *basket option*. The basic version of such a multivariate product has the same characteristics as a vanilla option, but now the underlying is a basket of stocks instead of a single stock. The pricing of these derivatives is not a trivial task because it requires a model that jointly describes the stock prices involved.

Stock price models based on the lognormal model proposed in Black and Scholes (1973) are popular choices from a computational point of view, however, they are not capable of capturing

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the skewness and kurtosis observed for log returns of stocks and indices. The class of Lévy processes provides a much better fit of the observed log returns and, consequently, the pricing of options and other derivatives in a Lévy setting is much more reliable. In this paper we consider the problem of pricing multi-asset derivatives in a multivariate Lévy model.

The most straightforward extension of the univariate Black & Scholes model is the *Gaussian copula model*, also called the multivariate Black & Scholes model. In this framework, the stocks composing the basket are assumed to be lognormal distributed and a Gaussian copula connects the marginals. Even in this simple setting, the price of a basket option is not given in closed form and has to be approximated; see e.g. Hull and White (1993), Brooks et al. (1994), Milevsky and Posner (1998), Rubinstein (1994), Deelstra et al. (2004), Carmona and Durrleman (2006) and Linders (2013), among others. However, the normality assumption for the marginals used in this pricing framework is too restrictive. Indeed, in Linders and Schoutens (2014) it is shown that calibrating the Gaussian copula model to market data can lead to non-meaningful parameter values. This dysfunctioning of the Gaussian copula model is typical observed in distressed periods. In this paper we extend the classical Gaussian pricing framework in order to overcome this problem.

Several extensions of the Gaussian copula model are proposed in the literature. For example, Luciano and Schoutens (2006) introduce a multivariate Variance Gamma model where dependence is modeled through a common jump component. This model was generalized in Semeraro (2008), Luciano and Semeraro (2010) and Guillaume (2013). A stochastic correlation model was considered in Fonseca et al. (2007). A framework for modeling dependence in finance using copulas was described in Cherubini et al. (2004). However, the pricing of basket options in these advanced multivariate stock price models is not a straightforward task. There are several attempts to derive closed form approximations for the price of a basket option in a non-Gaussian world. In Linders and Stassen (2014), approximate basket option prices in a multivariate Variance Gamma model are derived, whereas Xu and Zheng (2010, 2014) consider a local volatility jump diffusion model. McWilliams (2011) derives approximations for the basket option price in a stochastic delay model. Upper and lower bounds for basket option prices in a general class of stock price models with known joint characteristic function of the logreturns are derived in Caldana et al. (2014). This paper also provides an extensive literature overview of the research on basket option pricing.

In this paper we start from the one-factor Lévy model introduced in Albrecher et al. (2007) to build a multivariate stock price model with correlated Lévy marginals. Stock prices are assumed to be driven by an idiosyncratic and a systematic factor. The idea of using a common market factor is not new in the literature and goes back to Vasicek (1987). Conditional on the common (or market) factor, the stock prices are independent. We show that our model generalizes the Gaussian model (with single correlation). Indeed, the idiosyncratic and systematic component are constructed from a Lévy process. Employing in that construction a Brownian motion delivers the Gaussian copula model, but other Lévy copulas arise by employing different Lévy processes like VG, NIG, Meixner, ... As a result, this new *Lévy copula model* is more flexible and can capture other types of dependence. This model has also some deficiencies. The correlation is by construction always positive and, moreover, we assume a single correlation. From a tractability point of view, however, reporting a single correlation number is often preferred over $n(n-1)$ pairwise correlations.

In a first part of this paper, we consider the problem of finding accurate approximations for the price of a basket option in the Lévy copula model. In order to value a basket option, the distribution of this basket has to be determined. However, the basket is a weighted sum of dependent stock prices and its distribution function is in general unknown or too complex to work with. Therefore, we replace the random variable describing the basket price at maturity by a random variable with a more simple structure. Moreover, the characteristic function of the log transformation of this approximate random variable is given in closed form, such that the Carr-Madan formula can be used to determine approximate basket option prices. We propose two different approximations. Both methods are already applied for deriving approximate basket option prices in the Gaussian copula model. In this paper we show how to generalize the methodologies to the Lévy case.

A basket is an arithmetic sum of dependent random variables. However, stock prices are modeled as exponentials of stochastic processes and therefore, a geometric average has a lot of computational advantages. The first methodology, proposed in Korn and Zeytun (2013), consists of constructing an approximate basket by replacing the arithmetic sum by a geometric average. The second valuation formula is based on a moment-matching approximation. To be more precise, the distribution of the basket is replaced by a shifted random variable having the same first three moments than the original basket. This idea was first proposed in Brigo et al. (2004). Note that the approximations proposed in Korn and Zeytun (2013) and Brigo et al. (2004) were only worked out in the Gaussian copula model, whereas the approximations introduced in this paper allow for Lévy marginals and a Lévy copula. Furthermore, we determine the approximate basket option price using the Carr-Madan formula, whereas a closed form expression is available in the Gaussian situation. Numerical examples illustrating the accuracy and the sensitivity of the approximations are provided.

In a second part of the paper we show how the well-established notions of implied volatility and implied correlation can be generalized in our Lévy copula model; see also Corcuera et al. (2009). We assume that a finite number of options, written on the basket and the components, are traded. The prices of these derivatives are observable and will be used to calibrate the parameters of our stock price model. An advantage of our modeling framework is that each stock is described by a volatility parameter and that the marginal parameters can be calibrated separately from the correlation parameter. We give numerical examples to show how to use the vanilla option curves to determine an implied Lévy volatility for each stock based on a Normal, VG, NIG and Meixner process and determine basket option prices for different choices of the correlation parameter.

An *implied Lévy correlation* estimate arises when we tune the correlation parameter such that the model price exactly hits the market price of a basket option for a given strike. We determine implied correlation levels for the stocks composing the Dow Jones Industrial Average in a Gaussian and a Variance Gamma copula model. We observe that implied correlation depends on the strike and in the VG copula model, this implied Lévy correlation *smile* is flatter than in the Gaussian copula model. The standard technique to price non-traded basket options (or other multi-asset derivatives), is by interpolating on the implied correlation curve. It is shown in Linders and Schoutens (2014) that in the Gaussian copula model, this can sometimes lead to non-meaningful correlation values. We show that the Lévy version of the implied correlation solves (at least to some extent) this problem.

Several papers consider the problem of measuring implied correlation between stock prices; see e.g. Fonseca et al. (2007), Tavin (2013), Ballotta et al. (2014)) and Austing (2014). Our approach is different in that we determine implied correlation estimates in the Lévy copula model using multi-asset derivatives consisting of many assets (30 assets for the Dow Jones). When considering multi-asset derivatives with a low dimension, determining the model prices of these multi-asset derivatives becomes much more tractable.

This paper is organized as follows. In Section 2 we introduce the Lévy copula model as an extension of the classical Gaussian copula model. In Section 3 and Section 4 we propose two different approximate random variables and show how the Carr-Madan formula, discussed in Section 5, can be used to determine approximate basket option prices. We give numerical illustrations in Section 6. Implied Lévy volatility and correlation are defined and investigated in Section 7.

2 Lévy copula model

We consider a market where n stocks are traded. The price level of stock j at some future time $t,\ 0 \leq t \leq T$ is denoted by $S_j(t)$[1]. Dividends are assumed to be paid continuously and the dividend yield of stock j is constant and deterministic over time. We denote this dividend yield by q_j. The current time is $t = 0$. We fix a future time T and we always consider the random variables $S_j(T)$ denoting the time-T prices of the different stocks involved. The price level at time T of a basket of stocks is denoted by $S(T)$ and given by

$$S(T) = \sum_{j=1}^{n} w_j S_j(T),$$

where $w_j > 0$ are weights which are fixed upfront. The pay-off of a basket option with strike K and maturity T is given by $(S(T) - K)_+$, where $(x)_+ = \max(x, 0)$. The price of this basket option is denoted by $C[K, T]$. We assume that the market is arbitrage-free and that there exists a risk-neutral pricing measure $\mathbb{Q}$ such that the basket option price $C[K, T]$ can be expressed as the discounted risk-neutral expected value. In this pricing formula, discounting is performed using the risk-free interest rate r, which is, for simplicity, assumed to be deterministic and constant over time. Throughout the paper, we always assume that all expectations we encounter are well-defined and finite.

Since the seminal papers of Bachelier (1900) and Black and Scholes (1973), various models are proposed to capture the dynamics of the stocks and their dependence relations. We first revisit the Gaussian copula model. Next, we generalize this popular multidimensional model by allowing more flexibility when fitting the marginals.

[1]We use the common approach to describe the financial market via a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}\right)$. Furthermore, $(\mathcal{F}_t)_{0\leq t\leq T}$ is the natural filtration and we assume that $\mathcal{F}_T = \mathcal{F}$.

2.1 Gaussian copula model

We show how the Gaussian copula model can be constructed using standard Brownian motions. Consider the independent standard Brownian motions $W = \{W(t)|t \geq 0\}$ and $W_j = \{W_j(t)|t \geq 0\}$, for $j = 1, 2, \ldots, n$. Let $\rho \in [0, 1]$. The log returns of each stock j are driven by the r.v. Z_j. We assume that these log returns consist of a systematic component and a stock-specific, idiosyncratic, component. Therefore, the r.v. Z_j is given by:

$$Z_j = W(\rho) + W_j(1-\rho), \quad j = 1, 2, \ldots, n. \tag{1}$$

Because the Brownian motions W and W_j are independent, we find that $Z_j \stackrel{\mathrm{d}}{=} N(0,1)$. Furthermore, for $i \neq j$, the correlation between Z_i and Z_j is equal to ρ. Indeed, we have that

$$\begin{aligned} \text{Corr}\left[Z_i, Z_j\right] &= \text{Var}\left[W(\rho)\right] \\ &= \rho. \end{aligned}$$

The log returns of the different stocks are assumed to be described by the correlated r.v.'s Z_j, $j = 1, 2, \ldots, n$. Each r.v. Z_j is standard normal distributed. In order to adjust the mean and the variance of the time-T stock price $S_j(T)$, we add a stock specific drift parameter $\mu_j \in \mathbb{R}$ and a volatility parameter $\sigma_j > 0$. The stock prices $S_j(T)$, $j = 1, 2, \ldots, n$ at time T are then given by

$$S_j(T) = S_j(0)\mathrm{e}^{\mu_j T + \sigma_j \sqrt{T} Z_j}, \quad j = 1, 2, \ldots n. \tag{2}$$

The stock price model (2) is also called the multivariate Black & Scholes model or Gaussian copula model; the marginal log returns are modeled by Normal distributions and a Gaussian copula connects these marginals. For a detailed description of the Black & Scholes model and its extensions, we refer to Black and Scholes (1973), Björk (1998), Carmona and Durrleman (2006) and Dhaene et al. (2013).

2.2 Generalization: the Lévy copula model

A crucial (and simplifying) assumption in the Gaussian copula model (2) is the normality assumption for the risk factors Z_j. It is well-known that log returns do not pass the test for normality. Indeed, log returns exhibit a skewed and leptokurtic distribution which cannot be captured by a normal distribution; see e.g. Schoutens (2003).

We generalize the Gaussian copula model outlined in the previous section, by allowing the risk factors to be distributed according to any infinitely divisible distribution with known characteristic function. This larger class of distributions increases the flexibility to find a more realistic distribution for the log returns. In Albrecher et al. (2007) a similar framework was considered for pricing CDO tranches; related references are Baxter (2007) and Itkin and Lipton (2014) and the references therein. The Variance Gamma case was considered in Moosbrucker (2006*a*,*b*), whereas Guillaume et al. (2009) consider the pricing of CDO-squared tranches in this one-factor Lévy model.

Consider an infinitely divisible distribution for which the characteristic function is denoted by ϕ. A stochastic process X can be build using this distribution. Such a process is called a

Lévy process with mother distribution having characteristic function ϕ. The Lévy process $X = \{X(t)|t \geq 0\}$ based on this infinitely divisible distribution starts at zero and has independent and stationary increments. Furthermore, for $s, t \geq 0$ the characteristic function of the increment $X(t+s)-X(t)$ is ϕ^s. For more details on Lévy processes, we refer to Sato (1999) and Schoutens (2003).

Assume that the random variable L has an infinitely divisible distribution and denote its characteristic function by ϕ_L. Consider the Lévy process $X = \{X(t)|t \in [0,1]\}$ based on the distribution L. We assume that the process is standardized, i.e. $\mathbb{E}[X(1)] = 0$ and $\text{Var}[X(1)] = 1$. One can then show that $\text{Var}[X(t)] = t$, for $t \geq 0$. Define also a series of independent and standardized processes $X_j = \{X_j(t)|t \in [0,1]\}$, for $j = 1, 2, \ldots, n$. The process X_j is based on an infinitely divisible distribution L_j with characteristic function ϕ_{L_j}. Furthermore, the processes $X_1, X_2, \ldots, X_n$ are independent from X. Take $\rho \in [0,1]$. The r.v. A_j is defined by

$$A_j = X(\rho) + X_j(1-\rho), \quad j = 1, 2, \ldots n. \tag{3}$$

In this construction, $X(\rho)$ and $X_j(1-\rho)$ are random variables having characteristic function ϕ_L^ρ and $\phi_{L_j}^{1-\rho}$, respectively. Denote the characteristic function of A_j by ϕ_{A_j}. Because the processes X and X_j are independent and standardized, we immediately find that

$$\mathbb{E}[A_j] = 0, \quad \text{Var}[A_j] = 1 \text{ and } \quad \phi_{A_j}(t) = \phi_L^\rho(t)\phi_{L_j}^{1-\rho}(t), \text{ for } j = 1, 2, \ldots, n. \tag{4}$$

Note that if X and X_j are both Lévy processes based on the same mother distribution L we obtain the equality $A_j \stackrel{\text{d}}{=} L$.

The parameter ρ describes the correlation between A_i and A_j, if $i \neq j$. Indeed, it was proven in Albrecher et al. (2007) that in case A_j , $j = 1, 2, \ldots, n$ is defined by (3), we have that

$$\text{Corr}\,[A_i, A_j] = \rho. \tag{5}$$

We model the stock price levels $S_j(T)$ at time T for $j = 1, 2, \ldots, n$ as follows

$$S_j(T) = S_j(0)\mathrm{e}^{\mu_j T + \sigma_j \sqrt{T} A_j}, \quad j = 1, 2, \ldots, n, \tag{6}$$

where $\mu_j \in \mathbb{R}$ and $\sigma_j > 0$. Note that in this setting, each time-T stock price is modeled as the exponential of a Lévy process. Furthermore, a drift μ_j and a volatility parameter σ_j are added to match the characteristics of stock j. Our model, which we will call the *Lévy copula model*, can be considered as a generalization of the Gaussian copula model (2). Indeed, instead of a normal distribution, we allow for a Lévy distribution, while the Gaussian copula is generalized to a so-called Lévy copula. This Lévy copula model can also, at least to some extent, be considered as a generalization to the multidimensional case of the model proposed in Corcuera et al. (2009) and the parameter σ_j in (6) can then be interpreted as the Lévy space (implied) volatility of stock j. Another related model was proposed in Kawai (2009), where the dynamics of each stock price are modeled by a linear combination of independent Lévy processes. The idea of building a multivariate asset model by taking a linear combination of a systematic and an idiosyncratic process can also be found in Ballotta and Bonfiglioli (2014) and Ballotta et al. (2014).

2.3 The risk-neutral stock price processes

If we take

$$\mu_j = (r - q_j) - \frac{1}{T} \log \phi_L \left(-\mathrm{i}\sigma_j \sqrt{T} \right), \tag{7}$$

we find that

$$\mathbb{E}[S_j(T)] = \mathrm{e}^{(r-q_j)T} S_j(0), \quad j = 1, 2, \ldots, n.$$

From expression (7) we conclude that the risk-neutral dynamics of the stocks in the Lévy copula model are given by

$$S_j(T) = S_j(0)\mathrm{e}^{(r-q_j-\omega_j)T+\sigma_j\sqrt{T}A_j}, \quad j = 1, 2, \ldots, n, \tag{8}$$

where $\omega_j = \frac{1}{T} \log \phi_L \left(-\mathrm{i}\sigma_j \sqrt{T} \right)$. We always assume that ω_j is finite. The first three moments of $S_j(T)$ can be expressed in function of the characteristic function ϕ_L. By the martingale property, we have that $\mathbb{E}\left[S_j(T)\right] = S_j(0)\mathrm{e}^{(r-q_j)T}$. The risk-neutral variance $\mathrm{Var}\left[S_j(T)\right]$ can be written as follows

$$\mathrm{Var}\left[S_j(T)\right] = S_j(0)^2 \mathrm{e}^{2(r-q_j)T} \left(\mathrm{e}^{-2\omega_j T} \phi_{A_j} \left(-\mathrm{i}2\sigma_j\sqrt{T} \right) - 1 \right).$$

The second and third moment of $S_j(T)$ can be expressed in terms of the characteristic function ϕ_{A_j}:

$$\begin{aligned}
\mathbb{E}\left[S_j(T)^2\right] &= \mathbb{E}[S_j(T)]^2 \frac{\phi_{A_j}\left(-\mathrm{i}2\sigma_j\sqrt{T}\right)}{\phi_{A_j}\left(-\mathrm{i}\sigma_j\sqrt{T}\right)^2}, \\
\mathbb{E}\left[S_j(T)^3\right] &= \mathbb{E}[S_j(T)]^3 \frac{\phi_{A_j}\left(-\mathrm{i}3\sigma_j\sqrt{T}\right)}{\phi_{A_j}\left(-\mathrm{i}\sigma_j\sqrt{T}\right)^3}.
\end{aligned}$$

If the process X_j has mother distribution L, we can replace ϕ_{A_j} by ϕ_L in expression (7) and in the formulas for $\mathbb{E}\left[S_j(T)^2\right]$ and $\mathbb{E}\left[S_j(T)^3\right]$. From here on, we always assume that all Lévy processes are built on the same mother distribution. However, all results remain to hold in the more general case.

In the following sections, we propose two methodologies to approximate the risk-neutral distribution of the sum $S(T)$. In Section 3 we replace the arithmetic sum $S(T)$ by a geometric average, whereas a three-moments matching approach is considered in Section 4. In both situations, the r.v. $S(T)$ is replaced by an approximate r.v. S^* for which the characteristic function $\phi_{\log S^*}$ is known. Approximate basket option prices can then be derived by using the Carr-Madan formula.

3 Approximating the arithmetic sum by a geometric average

3.1 The geometric average

The random variable $S(T)$ is a weighted sum of the dependent random variables $S_j(T)$, $j = 1, 2, \ldots, n$. Its distribution function cannot be determined in an analytical form, which makes the calculation of $C[K,T]$ a difficult task. In this section we approximate the r.v. $S(T)$ by a more tractable r.v. S^* such that the characteristic function $\phi_{\log S^*}$ is known. A suitable approximating random variable is based on the following lemma by Korn and Zeytun (2013).

Lemma 1 *Let $s_1, s_2, \ldots, s_n$ be a set of non-negative numbers. Then the following asymptotic relations hold:*

$$\lim_{\kappa\to\infty} \frac{\left(\frac{1}{n}\sum_{j=1}^{n}(s_j+\kappa)\right)^n}{\prod_{j=1}^{n}(s_j+\kappa)} = 1, \tag{9}$$

$$\frac{\left(\frac{1}{n}\sum_{j=1}^{n}(s_j+\kappa)\right)^n}{\prod_{j=1}^{n}(s_j+\kappa)} = 1 + \frac{1}{\kappa}\sum_{j=1}^{n} s_j + \mathcal{O}\left(\frac{1}{\kappa^2}\right). \tag{10}$$

A geometric average of stock prices is more tractable than an arithmetic sum, because stock prices are modeled as exponentials of correlated Lévy processes. The lemma stated above provides a way to move from an arithmetic sum to a geometric average. Indeed, Lemma 1 shows that the geometric average $\prod_{j=1}^{n}(s_j+\kappa)^{1/n}$ is a good approximation for the arithmetic average $\frac{1}{n}\sum_{j=1}^{n}(s_j+\kappa)$, provided κ is sufficiently large.

The pay-off of the basket option $C[K,T]$ can be rewritten as follows

$$\left(\sum_{j=1}^{n} w_j S_j(T) - K\right)_+ = \left(\frac{1}{n}\sum_{i=1}^{n}(n w_j S_j(T)+\kappa) - (K+\kappa)\right)_+,$$

where $\kappa \in \mathbb{R}$. If κ is large, we can use Lemma 1 and derive that the pay-off can be approximated as follows

$$\left(\sum_{j=1}^{n} w_j S_j(T) - K\right)_+ \approx \left(\prod_{j=1}^{n}(n w_j S_j(T)+\kappa)^{1/n} - (K+\kappa)\right)_+ \tag{11}$$

$$= \left(\tilde{S}(T) - \tilde{K}\right)_+, \tag{12}$$

where

$$\tilde{S}(T) = \prod_{j=1}^{n}(n w_j S_j(T)+\kappa)^{1/n}, \tag{13}$$

and $\tilde{K} = K+\kappa$. By adding a sufficiently large number κ to the basket, one can approximate the arithmetic sum by a geometric average of r.v.'s, where each r.v. is now shifted by the constant κ. However, we would like to decompose $\log\tilde{S}(T)$ in terms of the logarithms $\log S_j(T)$, but the shift κ throws a spanner in the works. In the next section, we show how to deal with this problem.

3.2 Approximating the shifted log infinitely divisible distribution

In order to determine the basket option price $C[K,T]$, we replace the arithmetic sum $S(T)$ by the geometric average $\tilde{S}(T)$, defined in (13). The terms of the geometric average $\tilde{S}(T)$ are denoted by Y_j, $j = 1, 2, \ldots, n$ and we have that $Y_j = nw_jS_j(T) + \kappa$. The key ingredient for applying the Carr-Madan option pricing formula is an analytical expression for the characteristic function of $\log \tilde{S}(T)$. Unfortunately, the characteristic function of $\log Y_j$, and hence also the characteristic function of $\log \tilde{S}(T)$, is unknown. Therefore, we replace the r.v. Y_j by Y_j^* which is defined by

$$Y_j^* = \mathrm{e}^{\mu_j^* + \sigma_j^* A_j}. \tag{14}$$

Remember that the characteristic function of A_j is denoted by ϕ_L. If the parameters μ^*, σ^* and the characteristic function ϕ_L are known, also the characteristic function of $\log Y_j^*$ is given in an analytical form.

We can now show how to determine the parameters μ^* and σ^*. It can easily be obtained that

$$\mathbb{E}\left[Y_j^*\right] = \mathrm{e}^{\mu_j^*}\phi_L\left(-\mathrm{i}\sigma_j^*\right), \qquad \mathrm{Var}\left[Y_j^*\right] = \mathrm{e}^{2\mu_j^*}\left(\phi_L\left(-\mathrm{i}2\sigma_j^*\right) - \phi_L\left(-\mathrm{i}\sigma_j^*\right)^2\right). \tag{15}$$

The parameters μ_j^* and σ_j^* are determined such that the first and second moment of Y_j and Y_j^* coincide. This results in:

$$\mathbb{E}\left[Y_j^*\right] = \mathbb{E}\left[Y_j\right], \tag{16}$$

$$\mathrm{Var}\left[Y_j^*\right] = \mathrm{Var}\left[Y_j\right]. \tag{17}$$

Using the expression (15) for the mean of Y_j^*, we find that equation (16) can be rewritten as follows:

$$\mathrm{e}^{\mu_j^*}\phi_L\left(-\mathrm{i}\sigma_j^*\right) = \mathbb{E}\left[Y_j\right],$$

from which we derive that

$$\mu_j^* = \log\left(\frac{\mathbb{E}[Y_j]}{\phi_L\left(-\mathrm{i}\sigma_j^*\right)}\right). \tag{18}$$

Plugging expression (18) for μ_j^* in the equation for the variance $\mathrm{Var}[Y_j^*]$ yields

$$\mathrm{Var}\left[Y_j^*\right] = \mathbb{E}[Y_j]^2\left(\frac{\phi_L\left(-\mathrm{i}2\sigma_j^*\right)}{\phi_L\left(-\mathrm{i}\sigma_j^*\right)^2} - 1\right).$$

The equation (17) then becomes

$$\mathbb{E}[Y_j]^2\left(\frac{\phi_L\left(-\mathrm{i}2\sigma_j^*\right)}{\phi_L\left(-\mathrm{i}\sigma_j^*\right)^2} - 1\right) = \mathrm{Var}\left[Y_j\right].$$

We find that σ_j^* can be determined from the following implicit relation

$$\frac{\phi_L\left(-\mathrm{i}2\sigma_j^*\right)}{\phi_L\left(-\mathrm{i}\sigma_j^*\right)^2} = \frac{\mathrm{Var}[Y_j]}{\mathbb{E}[Y_j]^2} + 1. \tag{19}$$

We always silently assume that Pr $[L>0]>0$, which implies that $\lim_{a\to+\infty}\phi_L(-\mathrm{i}a)=+\infty$. Define the function f as $f(a)=\frac{\phi_L(-\mathrm{i}2a)}{\phi_L(-\mathrm{i}a)^2}$. Then f is a continuous function and $\lim_{a\to 0} f(a)=1$. By the Cauchy-Schwarz inequality, we also find that $\lim_{a\to+\infty} f(a)=+\infty$. Moreover, the right-hand-side of equation (19) is always larger than one. As a result we find that (19) always has a solution σ_j^*, for $j=1,2,\ldots,n$. We have that the mean $\mathbb{E}[Y_j]$ is equal to $nw_j\mathbb{E}[S_j(T)]+\kappa$ and the variance $\mathrm{Var}[Y_j]$ is equal to $n^2w_j^2\mathrm{Var}[S_j(T)]$. These moments can be determined using the expressions derived in Section 2.3 for $\mathbb{E}[S_j(T)]$ and $\mathrm{Var}[S_j(T)]$.

3.3 Approximating the basket option price

The basket option price $C[K,T]$ is approximated by replacing the arithmetic sum $S(T)$ by an appropriate geometric average and by changing the strike K; see (11). For each j, the random variable $nw_jS_j(T)+\kappa$ is then approximated by Y_j^*, defined in (14), where σ_j^* has to be determined using the relation (19) and μ_j^* then follows from (18). This results in the approximation $C^{GA}[K,T]$ for $C[K,T]$:

$$C^{GA}[K,T]=\mathrm{e}^{-rT}\mathbb{E}\left[\left(\prod_{j=1}^n\left(Y_j^*\right)^{1/n}-(K+\kappa)\right)_+\right]. \tag{20}$$

We define the random variable S^* as follows

$$S^*=\prod_{j=1}^n\left(Y_j^*\right)^{1/n}. \tag{21}$$

In the following theorem, we give an expression for the characteristic function of the random variable $\log S^*$ in terms of the characteristic function ϕ_L of the mother infinitely divisible distribution L.

Theorem 1 *Consider the Lévy copula model* (8) *with mother infinitely divisible distribution* L. *The characteristic function* $\phi_{\log S^*}$ *of the random variable* $\log S^*$ *is given by*

$$\phi_{\log S^*}(u)=\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u\log S^*}\right]=\mathrm{e}^{\frac{\mathrm{i}u}{n}\sum_{j=1}^n\mu_j^*}\phi_L\left(u\sum_{j=1}^n\frac{\sigma_j^*}{n}\right)^{\rho}\prod_{j=1}^n\phi_L\left(\frac{u\sigma_j^*}{n}\right)^{1-\rho}. \tag{22}$$

Proof. Combining expressions (14) and (21), we find that

$$\log S^* \;=\; \frac{1}{n}\left(\sum_{j=1}^n\mu_j^*+\sum_{j=1}^n\sigma_j^*A_j\right).$$

Then we can express $\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u\log S^*}\right]$ as follows

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u\log S^*}\right] \;=\; \mathrm{e}^{\frac{\mathrm{i}u}{n}\sum_{j=1}^n\mu_j^*}\mathbb{E}\left[\exp\left\{\frac{\mathrm{i}u}{n}\sum_{j=1}^n\sigma_j^*\left(X(\rho)+X_j(1-\rho)\right)\right\}\right].$$

We can now rewrite $\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u\log S^*}\right]$ using the independence between $X(\rho)$ and $\sum_{j=1}^n X_j(1-\rho)$

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u\log S^*}\right] = \mathrm{e}^{\frac{\mathrm{i}u}{n}\sum_{j=1}^n \mu_j^*}\mathbb{E}\left[\exp\left\{\frac{\mathrm{i}u}{n}\sum_{j=1}^n \sigma_j^* X(\rho)\right\}\right]\mathbb{E}\left[\exp\left\{\frac{\mathrm{i}u}{n}\sum_{j=1}^n \sigma_j^* X_j(1-\rho)\right\}\right].$$

Note also that X_j, $j=1,2,\ldots,n$ are independent processes, which results in the following expression:

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}u\log S^*}\right] = \mathrm{e}^{\frac{\mathrm{i}u}{n}\sum_{j=1}^n \mu_j^*}\mathbb{E}\left[\exp\left\{\frac{\mathrm{i}u}{n}\sum_{j=1}^n \sigma_j^* X(\rho)\right\}\right]\prod_{j=1}^n\mathbb{E}\left[\exp\left\{\frac{\mathrm{i}u}{n}\sigma_j^* X_j(1-\rho)\right\}\right].$$

The processes X and X_j, $j=1,2,\ldots,n$ have mother infinitely divisible distribution L. This implies that

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}uX(\rho)}\right] = \phi_L(u)^{\rho}, \tag{23}$$

$$\mathbb{E}\left[\mathrm{e}^{\mathrm{i}uX_j(1-\rho)}\right] = \phi_L(u)^{1-\rho}, \quad \text{for } j=1,2,\ldots,n, \tag{24}$$

from which we find that (22) holds. ■

In Section 5 we show that if $\phi_{\log S^*}$ is given in closed form, the approximate basket option prices $C^{GA}[K,T]$ can be determined using a robust and fast algorithm.

4 A three-moments-matching approximation

In this section we introduce a second approach for approximating $C[K,T]$ by replacing the sum $S(T)$ with an appropriate random variable $\widetilde{S}(T)$ which has a simpler structure, but for which the first three moments coincide with the first three moments of the original basket $S(T)$. This moment-matching approach was also considered in Brigo et al. (2004) for the multivariate Black & Scholes model. Consider the Lévy process $Y=\{Y(t)\mid 0\le t\le 1\}$ with infinitely divisible distribution L. Furthermore, we define the random variable A as

$$A = Y(1).$$

In this case, the characteristic function of A is given by ϕ_L. The sum $S(T)$ is a weighted sum of dependent random variables and its cdf is unknown. We approximate the sum $S(T)$ by $\widetilde{S}(T)$, defined by

$$\widetilde{S}(T) = \bar{S}(T) + \lambda, \tag{25}$$

where $\lambda \in \mathbb{R}$ and

$$\bar{S}(T) = S(0)\exp\left\{(\bar{\mu}-\bar{\omega})T + \bar{\sigma}\sqrt{T}A\right\}. \tag{26}$$

The parameter $\bar{\mu}\in\mathbb{R}$ determines the drift and $\bar{\sigma}>0$ is the volatility parameter. These parameters, as well as the shifting parameter λ are determined such that the first three moments of $\widetilde{S}(T)$ coincide with the corresponding moments of the real basket $S(T)$. The parameter $\bar{\omega}$, defined as follows

$$\bar{\omega} = \frac{1}{T}\log\phi_L\left(-\mathrm{i}\bar{\sigma}\sqrt{T}\right),$$

is assumed to be finite.

4.1 Matching the first three moments

The first three moments of the basket $S(T)$ are denoted by m_1, m_2 and m_3 respectively. In the following lemma, we express the moments m_1, m_2 and m_3 in terms of the characteristic function ϕ_L and the marginal parameters. A proof of this lemma is provided in the appendix.

Lemma 2 *Consider the Lévy copula model* (8) *with mother infinitely divisible distribution L. The first two moments m_1 and m_2 of the basket $S(T)$ can be expressed as follows*

$$m_1 = \sum_{j=1}^{n} w_j \mathbb{E}\left[S_j(T)\right], \tag{27}$$

$$m_2 = \sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k \mathbb{E}\left[S_j(T)\right]\mathbb{E}\left[S_k(T)\right]\left(\frac{\phi_L\left(-i(\sigma_j+\sigma_k)\sqrt{T}\right)}{\phi_L\left(-i\sigma_j\sqrt{T}\right)\phi_L\left(-i\sigma_k\sqrt{T}\right)}\right)^{\rho_{j,k}}, \tag{28}$$

where

$$\rho_{j,k} = \begin{cases} \rho, & \text{if } j \neq k; \\ 1, & \text{if } j = k. \end{cases}$$

The third moment m_3 of the basket $S(T)$ is given by

$$m_3 = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} w_j w_k w_l \mathbb{E}\left[S_j(T)\right]\mathbb{E}\left[S_k(T)\right]\mathbb{E}\left[S_l(T)\right] \times \frac{\phi_L\left(-i\left(\sigma_j+\sigma_k+\sigma_l\right)\sqrt{T}\right)^{\rho}}{\phi_L\left(-i\sigma_j\sqrt{T}\right)\phi_L\left(-i\sigma_k\sqrt{T}\right)\phi_L\left(-i\sigma_l\sqrt{T}\right)} A_{j,k,l}, \tag{29}$$

where

$$A_{j,k,l} = \begin{cases} \left(\phi_L\left(-i\sigma_j\sqrt{T}\right)\phi_L\left(-i\sigma_k\sqrt{T}\right)\phi_L\left(-i\sigma_l\sqrt{T}\right)\right)^{1-\rho}, & \text{if } j \neq k, k \neq l \text{ and } j \neq l; \\ \left(\phi_L\left(-i(\sigma_j+\sigma_k)\sqrt{T}\right)\phi_L\left(-i\sigma_l\sqrt{T}\right)\right)^{1-\rho}, & \text{if } j = k, k \neq l; \\ \left(\phi_L\left(-i(\sigma_k+\sigma_l)\sqrt{T}\right)\phi_L\left(-i\sigma_j\sqrt{T}\right)\right)^{1-\rho}, & \text{if } j \neq k, k = l; \\ \left(\phi_L\left(-i(\sigma_j+\sigma_l)\sqrt{T}\right)\phi_L\left(-i\sigma_k\sqrt{T}\right)\right)^{1-\rho}, & \text{if } j = l, k \neq l; \\ \phi_L\left(-i\left(\sigma_j+\sigma_k+\sigma_l\right)\sqrt{T}\right)^{1-\rho}, & \text{if } j = k = l. \end{cases}$$

In Section 2.3 we derived the first three moments for each stock j, $j = 1, 2, \ldots, n$. Taking into account the similarity between the price $S_j(T)$ defined in (8) and the approximate r.v. $\bar{S}(T)$, defined in (26), we can determine the first three moments of $\bar{S}(T)$:

$$\begin{aligned}
\mathbb{E}\left[\bar{S}(T)\right] &= S(0)\mathrm{e}^{\bar{\mu}T} =: \xi, \\
\mathbb{E}\left[\bar{S}(T)^2\right] &= \mathbb{E}\left[\bar{S}(T)\right]^2 \frac{\phi_L\left(-\mathrm{i}2\bar{\sigma}\sqrt{T}\right)}{\phi_L\left(-\mathrm{i}\bar{\sigma}\sqrt{T}\right)^2} =: \xi^2\alpha, \\
\mathbb{E}\left[\bar{S}(T)^3\right] &= \mathbb{E}\left[\bar{S}(T)\right]^3 \frac{\phi_L\left(-\mathrm{i}3\bar{\sigma}\sqrt{T}\right)}{\phi_L\left(-\mathrm{i}\bar{\sigma}\sqrt{T}\right)^3} =: \xi^3\beta.
\end{aligned}$$

These expressions can now be used to determine the first three moments of the approximate r.v. $\widetilde{S}(T)$:

$$\begin{aligned}
\mathbb{E}\left[\widetilde{S}(T)\right] &= \mathbb{E}\left[\bar{S}(T)\right] + \lambda, \\
\mathbb{E}\left[\widetilde{S}(T)^2\right] &= \mathbb{E}\left[\bar{S}(T)^2\right] + \lambda^2 + 2\lambda\mathbb{E}\left[\bar{S}(T)\right], \\
\mathbb{E}\left[\widetilde{S}(T)^3\right] &= \mathbb{E}\left[\bar{S}(T)^3\right] + \lambda^3 + 3\lambda^2\mathbb{E}\left[\bar{S}(T)\right] + 3\lambda\mathbb{E}\left[\bar{S}(T)^2\right].
\end{aligned}$$

Determining the parameters $\bar{\mu}$, $\bar{\sigma}$ and the shifting parameter λ by matching the first three moments results in the following set of equations

$$\begin{aligned}
m_1 &= \xi + \lambda, \\
m_2 &= \xi^2\alpha + \lambda^2 + 2\lambda\xi, \\
m_3 &= \xi^3\beta + \lambda^3 + 3\lambda^2\xi + 3\lambda\xi^2\alpha.
\end{aligned}$$

These equations can be recast in the following set of equations

$$\begin{aligned}
\lambda &= m_1 - \xi, \\
\xi^2 &= \frac{m_2 - m_1^2}{\alpha - 1}, \\
0 &= \left(\frac{m_2 - m_1^2}{\alpha - 1}\right)^{3/2} (\beta + 2 - 3\alpha) + 3m_1m_2 - 2m_1^3 - m_3.
\end{aligned}$$

Remember that α and β are defined by

$$\alpha = \frac{\phi_L\left(-\mathrm{i}2\bar{\sigma}\sqrt{T}\right)}{\phi_L\left(-\mathrm{i}\bar{\sigma}\sqrt{T}\right)^2} \quad \text{and} \quad \beta = \frac{\phi_L\left(-\mathrm{i}3\bar{\sigma}\sqrt{T}\right)}{\phi_L\left(-\mathrm{i}\bar{\sigma}\sqrt{T}\right)^3}.$$

Solving the third equation results in the parameter $\bar{\sigma}$. Note that this equation does not always have a solution. This issue was also discussed in Brigo et al. (2004) for the Gaussian copula case. However, in our numerical studies we did not encounter any numerical problems. If we know $\bar{\sigma}$, we can also determine ξ and λ from the first two equations. Next, the drift $\bar{\mu}$ can be determined from

$$\bar{\mu} = \frac{1}{T}\log\frac{\xi}{S(0)}.$$

4.2 Approximate basket option pricing

The price of a basket option with strike K and maturity T is denoted by $C[K,T]$. This unknown price is in this section approximated by $C^{MM}[K,T]$, which is defined as

$$C^{MM}[K,T] = \mathrm{e}^{-rT}\mathbb{E}\left[\left(\widetilde{S}(T) - K\right)_+\right].$$

Using expression (25) for $\widetilde{S}(T)$, the price $C^{MM}[K,T]$ can be expressed as

$$C^{MM}[K,T] = \mathrm{e}^{-rT}\mathbb{E}\left[\left(\bar{S}(T) - (K-\lambda)\right)_+\right].$$

Note that $\bar{S}(T)$ is also depending on the choice of λ. In order to determine the price $C^{MM}[K,T]$, we should be able to price an option written on $\bar{S}(T)$, with a shifted strike $K-\lambda$. Determining the approximation $C^{MM}[K,T]$ using the Carr-Madan formula requires knowledge about the characteristic function $\phi_{\log \bar{S}(T)}$ of $\log \bar{S}(T)$:

$$\phi_{\log \bar{S}(T)}(u) = \mathbb{E}\left[\mathrm{e}^{\mathrm{i}u \log \bar{S}(T)}\right].$$

Using expression (26) we find

$$\phi_{\log \bar{S}(T)}(u) = \mathbb{E}\left[\exp\left\{\mathrm{i}u\left(\log S(0) + (\bar{\mu} - \bar{\omega})T + \bar{\sigma}\sqrt{T}A\right)\right\}\right].$$

The characteristic function of A is ϕ_L, from which we find that

$$\phi_{\log \bar{S}(T)}(u) = \exp\left\{\mathrm{i}u\left(\log S(0) + (\bar{\mu} - \bar{\omega})T\right)\right\}\phi_L\left(u\bar{\sigma}\sqrt{T}\right).$$

Note that nowhere in this section, we used the assumption that the basket weights w_j are strictly positive. Therefore, the three-moments-matching approach proposed in this section can also be used to price e.g. spread options. However, for pricing spread options, alternative methods exist; see e.g. Carmona and Durrleman (2003), Hurd and Zhou (2010) and Caldana and Fusai (2013).

5 The FFT method and basket option pricing

Our two methodologies for pricing basket options consist both in approximating the basket $S(T)$ by a random variable with a more simple structure and for which the characteristic function is known in closed form. Denote this approximate random variable by X. In this section we show that in case the characteristic function $\phi_{\log X}$ of a random variable X is known, one can approximate

$$\mathrm{e}^{-rT}\mathbb{E}\left[(X-K)_+\right],$$

for any $K > 0$.

Let $\alpha > 0$ and assume that $\mathbb{E}\left[X^{\alpha+1}\right]$ exists and is finite. It was proven in Carr et al. (1999) that the price $\mathrm{e}^{-rT}\mathbb{E}\left[(X-K)_+\right]$ can be expressed as follows

$$\mathrm{e}^{-rT}\mathbb{E}\left[(X-K)_+\right] = \frac{\mathrm{e}^{-\alpha\log(K)}}{\pi}\int_0^{+\infty}\exp\left\{-\mathrm{i}v\log(K)\right\}g(v)\mathrm{d}v, \tag{30}$$

where

$$g(v) = \frac{\mathrm{e}^{-rT}\phi_{\log X}\left(v-(\alpha+1)\mathrm{i}\right)}{\alpha^2+\alpha-v^2+\mathrm{i}(2\alpha+1)v}. \tag{31}$$

In order to determine the approximation $C^{GA}[K,T]$ proposed in Section 3, the random variable X has to be understood as the geometric average S^*. Furthermore, $C[K,T]$ is approximated by determining option prices written on S^* but where the strike price K is shifted by a constant κ:

$$C^{GA}[K,T] = \mathrm{e}^{-rT}\mathbb{E}\left[(S^*-(K+\kappa))_+\right].$$

Note that the random variable S^* also depends on κ. According to Lemma 1, the shift κ has to be chosen sufficiently large; typical values are in the range $[10^4, 10^7]$. The approximation $C^{MM}[K,T]$ was introduced in Section 4 and the random variable X now denotes the moment-matching approximation $\widetilde{S}(T) = \bar{S}(T)+\lambda$. The approximation $C^{MM}[K,T]$ can then be determined as the option price written on $\bar{S}(T)$ and with shifted strike price $K-\lambda$.

6 Examples and numerical illustrations

The Gaussian copula model is a member of our class of Lévy copula models and is already described in Section 2. In this section we discuss how to build the Variance Gamma, Normal Inverse Gaussian and Meixner copula models. However, the reader is invited to construct Lévy copula models based on other Lévy based distributions; e.g. CGMY, Generalized hyperbolic, ... distributions. In each situation, the methodology for pricing basket options consists of replacing the original basket by an appropriate approximate basket. This idea was also considered in Dhaene et al. (2002a,b) for the Gaussian copula case and extended in Valdez et al. (2009) to the elliptical copula case.

Table 1 summarizes the Variance Gamma, Normal Inverse Gaussian and the Meixner distributions, which are all infinitely divisible. In the last row, it is shown how to construct for each of these distributions a standardized version. We assume that L is distributed according to one of these standardized distributions. Hence, L has zero mean and unit variance. Furthermore, the characteristic function ϕ_L of L is given in closed form. We can then define the Lévy processes X and X_j, $j=1,2,\ldots,n$ based on the mother distribution L. The random variables A_j, $j=1,2,\ldots,n$, are modeled as follows

$$A_j = X(\rho) + X_j(1-\rho), \quad j=1,2,\ldots,n,$$

where X and X_j, $j=1,2,\ldots,n$ are independent Lévy processes with mother infinitely divisible distribution L. More details can be found in Albrecher et al. (2007).

6.1 Variance Gamma

Although pricing basket options under a normality assumption is tractable from a computational point of view, it introduces a high degree of model risk; see e.g. Leoni and Schoutens (2008). The Variance Gamma distribution has already been proposed as a more flexible alternative to the Brownian setting; see e.g. Madan and Seneta (1990) and Madan et al. (1998).

We consider two numerical examples where L has a Variance Gamma distribution with parameters $\sigma = 0.5695, \nu = 0.75, \theta = -0.9492, \mu = 0.9492$. Table 2 contains the numerical values for the first illustration, where a four-basket option paying $\left(\frac{1}{4}\sum_{j=1}^{4} S_j(T) - K\right)_+$ at time T is considered. We use the following parameter values: $r = 6\%$, $T = 0.5$, $\rho = 0$ and $S_1(0) = 40,\ S_2(0) = 50,\ S_3(0) = 60,\ S_4(0) = 70$. These parameter values are also used in Section 5 of Korn and Zeytun (2013). We denote by $C^{mc}[K,T]$ the corresponding Monte Carlo estimate for the price $C[K,T]$. Here, 10^6 number of simulations are used. One can approximate the price $C[K,T]$ by replacing the arithmetic sum by a geometric average as explained in Section 3. This approximation is denoted by $C^{GA}[K,T]$. In order to determine $C^{GA}[K,T]$, we use $\kappa = 10^4$. Alternatively, the price $C[K,T]$ is approximated by using the moment-matching approach outlined in Section 4. This approximation is denoted by $C^{MM}[K,T]$.

When determining the basket option price using the geometric average $C^{GA}[K,T]$, the arithmetic sum $S(T)$ is replaced by the geometric average $\tilde{S}(T)$. In a second stage, each term of the geometric average $\tilde{S}(T)$ is approximated by the r.v. Y_j^*; see (14). The difference between the real basket option price $C[K,T]$ and its approximation is mainly caused in the second step. Indeed, consider the numerical illustration described above, where $\sigma_1 = 0.6, \sigma_2 = 1.2, \sigma_3 = 0.3$ and $\sigma_4 = 0.9$. The basket strike is $K = 60$. Using Monte Carlo simulation, we find that $\mathrm{e}^{-rT}\mathbb{E}\left[\tilde{S}(T) - (K+\kappa)\right] \approx 5.4903$, whereas $C^{GA}[K,T] = 5.4535$; see Table 2. Hence, the approximation $C^{GA}[K,T]$ can be improved if a more suitable r.v. Y_j^* can be found.

In a second example, we consider the basket $S(T) = w_1 X_1(T) + w_2 X_2(T)$. The interest rate r is set to 5%. We determine option prices for the maturities $T = 1$ and $T = 3$. Note that strike prices are expressed in terms of forward moneyness. A basket strike price K has forward moneyness equal to $\frac{K}{\mathbb{E}[S]}$. We assume that the current prices of the non-dividend paying stocks are given by $X_1(0) = X_2(0) = 100$ and the weights are equal, $w_1 = w_2 = 0.5$. These parameter values are also used in Section 7 of Deelstra et al. (2004). Table 3 gives numerical values for these basket options. We can conclude that the three-moments-matching approximation gives more accurate results than the geometric average approximation. Especially for far out-of-the money call options, the approximation based on the geometric average is not able to closely approximate the real basket option price, whereas the accuracy of the three-moments-matching approximation is better.

We also investigate the sensitivity with respect to the Variance Gamma parameters σ, ν and θ. We consider a basket option consisting of 2 stocks where $r = 0.05, \rho = 0.5, T = 1, \sigma_1 = \sigma_2 = 0.4, w_1 = w_2 = 0.5$. We determine the prices $C^{mc}[K,T]$, $C^{MM}[K,T]$ and $C^{GA}[K,T]$ where $K = 105.13$. The first panel of Figure 1 shows the relative error for varying σ. The other parameters are fixed: $\nu = 0.75, \theta = -\mu = -0.9492$. The second panel of Figure 1 shows the relative error in function of ν, where the other parameters are taken as follows:

Table 1: Overview of infinitely divisible distributions.

	Gaussian	Variance Gamma
Parameters	$\mu \in \mathbb{R}, \sigma > 0$	$\mu, \theta \in \mathbb{R}, \sigma, \nu > 0$
Notation	$N(\mu, \sigma^2)$	$VG(\sigma, \nu, \theta, \mu)$
$\phi(u)$	$\mathrm{e}^{\mathrm{i}u\mu + \frac{1}{2}\sigma^2 u}$	$\mathrm{e}^{\mathrm{i}u\mu}\left(1 - \mathrm{i}u\theta\nu + u^2\sigma^2\nu/2\right)^{-1/\nu}$
Mean	μ	$\mu + \theta$
Variance	σ^2	$\sigma^2 + \nu\theta^2$
Standardized version	$N(0,1)$	$VG(\kappa\sigma, \nu, \kappa\theta, -\kappa\theta)$ where $\kappa = \frac{1}{\sqrt{\sigma^2+\theta^2\nu}}$

	Normal Inverse Gaussian	Meixner
Parameters	$\alpha, \delta > 0, \beta \in (-\alpha, \alpha), \mu \in \mathbb{R}$	$\alpha, \delta > 0, \beta \in (-\pi, \pi), \mu \in \mathbb{R}$
Notation	$NIG(\alpha, \beta, \delta, \mu)$	$MX(\alpha, \beta, \delta, \mu)$
$\phi(u)$	$\mathrm{e}^{\mathrm{i}u\mu - \delta\left(\sqrt{\alpha^2-(\beta+\mathrm{i}u)^2} - \sqrt{\alpha^2-\beta^2}\right)}$	$\mathrm{e}^{\mathrm{i}u\mu}\left(\frac{\cos(\beta/2)}{\cosh((\alpha u - \mathrm{i}\beta)/2)}\right)^{2\delta}$
Mean	$\mu + \frac{\delta\beta}{\sqrt{\alpha^2-\beta^2}}$	$\mu + \alpha\delta\tan(\beta/2)$
Variance	$\alpha^2\delta\left(\alpha^2 - \beta^2\right)^{-3/2}$	$\cos^{-2}(\beta/2)\alpha^2\delta/2$
Standardized version	$NIG\left(\alpha, \beta, (\alpha^2-\beta^2)^{3/2}, \frac{-(\alpha^2-\beta^2)\beta}{\alpha^2}\right)$	$MX\left(\alpha, \beta, \frac{2\cos^2(\frac{\beta}{2})}{\alpha^2}, \frac{-\sin(\beta)}{\alpha}\right)$

$\sigma = 0.5695, \theta = -\mu = -0.9492$. Finally, the sensitivity with respect to θ is shown in the third panel of Figure 1. The other VG parameters are given by $\sigma = 0.5695, \nu = 0.75, \mu = -\theta$.

The numerical results show that the approximations do not always manage to closely approximate the true basket option price. Especially when some of the volatilities are big, compared to the other ones, the accuracy of the approximation deteriorates. The dysfunctioning of the moment-matching and the geometric average approximation in the Gaussian copula model was already reported in Korn and Zeytun (2013) and Brigo et al. (2004). However, in order to calibrate the Lévy copula model to available option data, the availability of a basket option pricing formula which can be evaluated in a fast way, is of crucial importance. Table 5 shows the CPU times[2] for the Variance Gamma copula model for different basket dimensions. The computation time for the moment-matching approximation is relatively fast, even when the basket dimension is large. For example, the calculation time of approximate basket option prices when 30 stocks are involved is 0.227 seconds. Therefore, the moment-matching approximation is a good candidate for calibrating the Lévy copula model.

[2]the numerical illustrations are performed on an Intel Core i7, 2.70 GHz.

Table 2: Basket option prices in the VG copula model with $S_1(0) = 40,\ S_2(0) = 50,\ S_3(0) = 60,\ S_4(0) = 70$ and $\rho = 0$.

K	$C^{mc}[K,T]$	$C^{GA}[K,T]$	$C^{MM}[K,T]$
$\sigma_1 = 0.2; \sigma_2 = 0.2; \sigma_3 = 0.2; \sigma_4 = 0.2$			
50	6.5757 (6.5736 6.5778)	6.5601	6.5676
55	2.4360 (2.4345 2.4375)	2.4357	2.4781
60	0.2652 (0.2647 0.2657)	0.2675	0.2280
$\sigma_1 = 0.5; \sigma_2 = 0.5; \sigma_3 = 0.5; \sigma_4 = 0.5$			
55	4.1056 (4.1024 4.1087)	4.0780	4.2089
60	1.7777 (1.7756 1.7798)	1.5862	1.7976
65	0.5476 (0.5466 0.5487)	0.3428	0.4637
$\sigma_1 = 0.8; \sigma_2 = 0.8; \sigma_3 = 0.8; \sigma_4 = 0.8$			
60	3.2429 (3.2393 3.2465)	2.9522	3.3371
65	1.6807 (1.6782 1.6832)	1.2463	1.6429
70	0.7580 (0.7563 0.7596)	0.3612	0.6375
$\sigma_1 = 0.6; \sigma_2 = 1.2; \sigma_3 = 0.3; \sigma_4 = 0.9$			
55	5.5030 (5.4983 5.5077)	5.4535	5.6766
60	3.2240 (3.2204 3.2277)	2.8540	3.1933
65	1.6955 (1.6929 1.6982)	1.1421	1.4524
70	0.7881 (0.7863 0.7898)	0.2933	0.4763

Table 3: Basket option prices in the VG copula model with $r = 0.05, w_1 = w_2 = 0.5, X_1(0) = X_2(0) = 100$.

	T	ρ	σ_1	$C^{mc}[K,T]$	$C^{GA}[K,T]$	$C^{MM}[K,T]$
K=115.64	1	0.3	0.2	1.3966 (1.3945 1.3986)	1.0474	1.3113
			0.4	5.5900 (5.5836 5.5963)	4.4345	5.6267
		0.7	0.2	1.8974 (1.8949 1.8999)	1.4532	1.8706
			0.4	6.9576 (6.9502 6.9650)	5.5577	7.0095
K=127.80	3	0.3	0.2	4.4558 (4.4501 4.4615)	3.498	4.4565
			0.4	11.2968 (11.2830 11.3106)	9.371	11.5920
		0.7	0.2	5.6097 (5.6030 5.6164)	4.4588	5.6368
			0.4	13.7531 (13.7369 13.7692)	11.2444	13.9336
K=105.13	1	0.3	0.2	5.5259 (5.5215 5.5303)	5.2622	5.5965
			0.4	10.1619 (10.1533 10.1706)	9.5181	10.3515
		0.7	0.2	6.3340 (6.3292 6.3389)	5.958	6.3731
			0.4	11.7417 (11.7320 11.7514)	10.7703	11.8379

Table 4: Table 3 continued.

	T	ρ	σ_1	$C^{mc}[K,T]$	$C^{GA}[K,T]$	$C^{MM}[K,T]$
K=116.18	3	0.3	0.2	8.9945 (8.9862 9.0029)	8.4505	9.1489
			0.4	15.8681 (15.8517 15.8844)	14.8124	16.2498
		0.7	0.2	10.3662 (10.3569 10.3755)	9.5619	10.4528
			0.4	18.4113 (18.3927 18.4300)	16.7648	18.6214
K=94.61	1	0.3	0.2	12.3417 (12.3352 12.3482)	12.4384	12.4371
			0.4	16.2285 (16.2176 16.2394)	16.247	16.4493
		0.7	0.2	13.0869 (13.0799 13.0939)	13.1218	13.1269
			0.4	17.7742 (17.7622 17.7862)	17.5251	17.8690
K=104.57	3	0.3	0.2	15.1992 (15.1885 15.2100)	15.2468	15.3869
			0.4	21.3709 (21.3521 21.3897)	21.2958	21.7592
		0.7	0.2	16.5418 (16.5300 16.5536)	16.3782	16.6232
			0.4	23.8698 (23.8487 23.8910)	23.2868	24.0507

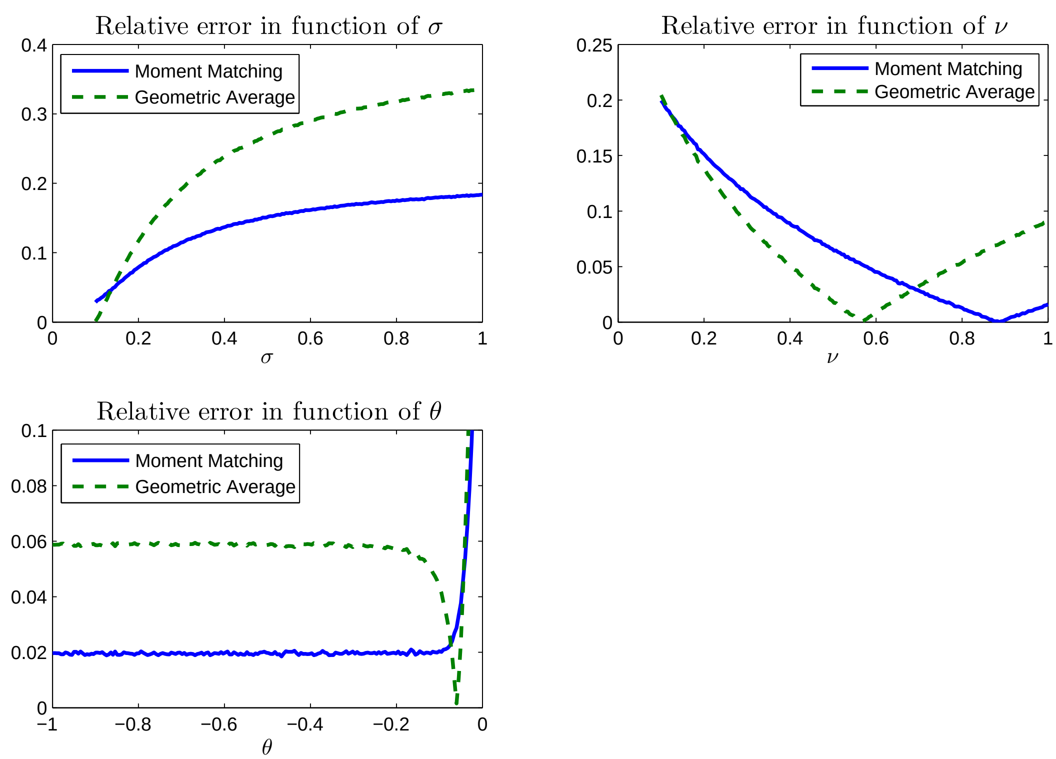

Figure 1: Relative error in the VG copula model for the moment matching approximation and the geometric average approximation. The basket option consists of 2 stocks and $r = 0.05, \rho = 0.5, T = 1, \sigma_1 = \sigma_2 = 0.4, w_1 = w_2 = 0.5$. The strike price is $K = 105.13$. In the benchmark model, the VG parameters are $\sigma = 0.5695, \nu = 0.75, \theta = -0.9492, \mu = 0.9492$

Table 5: The CPU time (in seconds) for the VG copula model for increasing basket dimension n. The following parameters are used: $r = 0.05, T = 1, \rho = 0.5, w_j = \frac{1}{n}, \sigma_j = 0.4, q_j = 0, S_j(0) = 100$, for $j = 1, 2, \ldots, n$. The basket strike is $K = 105.13$.

	CPU TIMES	
n	Moment Matching	Geometric Average
5	0.026	0.427
10	0.032	0.599
20	0.088	0.951
30	0.227	1.293
40	0.472	1.702
50	0.890	1.990
60	1.520	2.324
70	2.397	2.661
80	3.564	3.014
90	5.049	3.428
100	6.922	3.698

6.2 Pricing basket options

In this subsection we explain how to determine the price of a basket option in a realistic situation where option prices of the components of the basket are available and used to calibrate the marginal parameters. In our example, the basket under consideration consists of 2 major stock market indices, the S&P500 and the Nasdaq:

$$\text{Basket} = w_1 \text{S\&P 500} + w_2 \text{Nasdaq}.$$

The pricing date is February 19, 2009 and we determine prices for the Normal, VG, NIG and Meixner case. The details of the basket are listed in Table 6. The weights w_1 and w_2 are chosen such that the initial price $S(0)$ of the basket is equal to 100. The maturity of the basket option is equal to 30 days.

The S&P 500 and Nasdaq option curves are denoted by C_1 and C_2 respectively, and are shown in Figure 2. These option curves are only partially known. The traded strikes for curve C_j are denoted by $K_{i,j},\ i = 1, 2, \ldots, N_j$, where $N_j > 1$. If the volatilities σ_1 and σ_2 and the characteristic function ϕ_L of the mother distribution L are known, we can determine the model price of an option on asset j with strike K and maturity T. This price is denoted by $C_j^{model}[K, T; \Theta, \sigma_j]$, where Θ denotes the vector containing the model parameters of L. We use the observed option curves C_1 and C_2 to calibrate the model parameters as follows:

Algorithm 1 *[Determining the parameters Θ and σ_j of the Lévy copula model]*

Step 1: Choose a parameter vector Θ.

Step 2: For each stock $j = 1, 2, \ldots, n$, determine the implied Lévy volatility σ_j as follows:

$$\sigma_j = \arg\min_{\sigma} \frac{1}{N_j} \sum_{i=1}^{N_j} \frac{\left| C_j^{model}[K_{i,j}, T; \Theta, \sigma] - C_j[K_{i,j}] \right|}{C_j[K_{i,j}]},$$

Step 3: Determine the total error:

$$\text{error} = \sum_{j=1}^{2} \frac{1}{N_j} \sum_{i=1}^{N_j} \frac{\left| C_j^{model}[K_{i,j}, T; \Theta, \sigma_j] - C_j[K_{i,j}] \right|}{C_j[K_{i,j}]}.$$

Repeat these three steps until the parameter vector Θ is found for which the total error is minimal.

Only a limited number of option quotes is required to calibrate the Lévy copula model. Indeed, the parameter vector Θ can be determined using all available option quotes. Additional, one volatility parameter has to be determined for each stock. However, other methodologies for determining Θ exist. For example, one can fix the parameter Θ upfront, as is shown in Section 7.2. In such a situation, only one implied Lévy volatility has to be calibrated for each stock.

The calibrated parameters together with the calibration error are listed in Table 7. Note that the relative error in the VG, Meixner and NIG case is significantly smaller than in the normal case. Using the calibrated parameters for the mother distribution L together with the volatility parameters σ_1 and σ_2, we can determine basket option prices in the different model settings. Note that here and in the sequel of the paper, we always use the three-moments-matching approximation for determining basket option prices. We put $T = 30$ days and consider the cases where the correlation parameter ρ is given by 0.1, 0.5 and 0.8. The corresponding basket option prices are listed in Table 8. One can observe from the table that each model generates a different basket option price, i.e. there is model risk. However, the difference between the Gaussian and the non-Gaussian models is much more pronounced than the difference within the non-Gaussian models. We also find that using normally distributed log returns, one underestimates the in-the-money basket option prices. Indeed, the basket option price $C^{VG}[K,T], C^{Meixner}[K,T]$ and $C^{NIG}[K,T]$ are larger than $C^{BLS}[K,T]$ when K is below 105. Note, however, that for the strike $K = 110$, the price $C^{BLS}[K,T]$ is much closer to the other Lévy basket values. The reason for this behavior is that marginal log returns in the non-Gaussian situations are negatively skewed, whereas these distributions are symmetric in the Gaussian case. This skewness results in a lower probability of ending in the money for options with a sufficiently large strike. In the next section, we encounter situations where the Gaussian basket option price is larger than the corresponding VG price for out-of-the-money options.

Table 6: Input data for the basket option.

Date	Feb 19, 2009	
Maturity	March 21, 2009	
	S&P 500	Nasdaq
Forward	777.76	1116.72
Weights	0.06419	0.0428

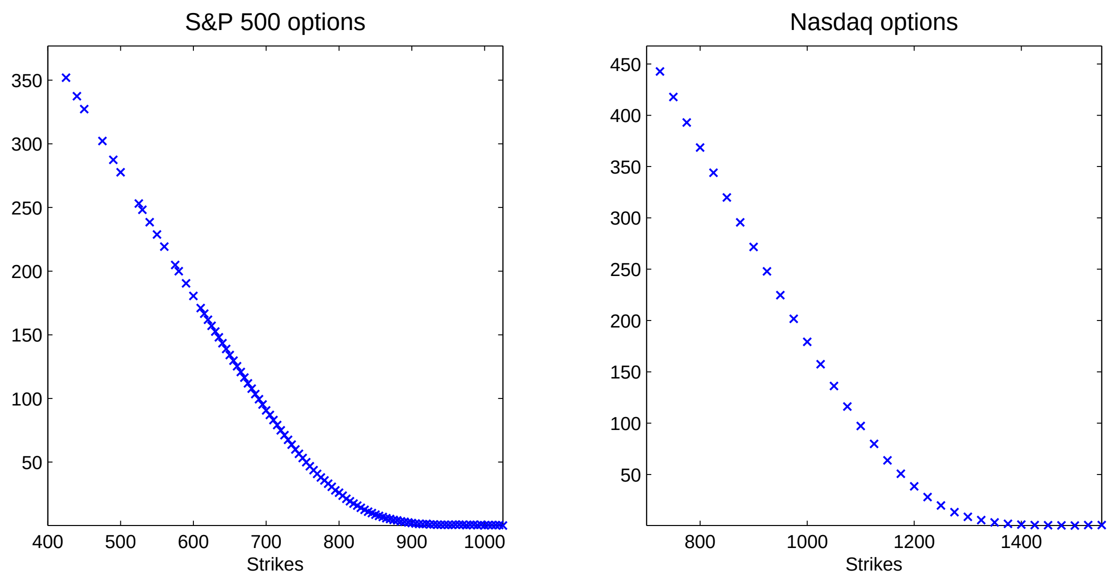

Figure 2: Available option data on February 19, 2009 for the S&P 500 (left) and the Nasdaq (right), with time to maturity 30 days.

Table 7: Lévy copula models: Calibrated model parameters

Model	Calibration error	Model Parameters			Volatilities	
		μ_{normal}	σ_{normal}		σ_1	σ_2
Normal	15.91%	0	1		0.2863	0.2762
		σ_{VG}	ν_{VG}	θ_{VG}		
VG	9.39 %	0.3640	0.7492	-0.3123	0.3876	0.3729
		$\alpha_{Meixner}$	$\beta_{Meixner}$			
Meixner	9.33%	1.5794	-1.6235		0.4015	0.3833
		α_{NIG}	β_{NIG}			
NIG	9.51 %	1.5651	-1.0063		0.4130	0.3941

Table 8: Basket option prices for the basket given in Table 6.

ρ	K	$C^{BLS}[K,T]$	$C^{VG}[K,T]$	$C^{Meixner}[K,T]$	$C^{NIG}[K,T]$
0.1	90	10.1906	10.8285	10.8957	10.9126
	95	5.9737	6.7858	6.8183	6.795
	100	2.8761	3.4893	3.4658	3.4237
	105	1.1009	1.2942	1.2802	1.2714
	110	0.3307	0.3721	0.3723	0.3764
0.5	90	10.3741	11.2496	11.3205	11.3242
	95	6.35	7.316	7.3414	7.3026
	100	3.3556	4.0513	4.017	3.9607
	105	1.5057	1.736	1.7024	1.6801
	110	0.5706	0.5768	0.5715	0.5737
0.8	90	10.5226	11.5343	11.6064	11.6023
	95	6.6127	7.6707	7.6923	7.6457
	100	3.6748	4.4373	4.3983	4.3352
	105	1.7868	2.0708	2.0276	1.9973
	110	0.758	0.7669	0.7586	0.7582

7 Implied Lévy correlation

In Section 6.2 we showed how the basket option formulas can be used to obtain basket option prices in the Lévy copula model. The parameter vector Θ describing the mother distribution L and the implied Lévy volatility parameters σ_j can be calibrated using the observed vanilla option curves $C_j[K,T]$ of the stocks composing the basket $S(T)$. In this section we show how an implied Lévy correlation estimate ρ can be obtained if on top of the vanilla options, also market prices for the basket option are available.

We assume that $S(T)$ represents the time-T price of a stock market index. Examples of such stock market indices are the Dow Jones, S&P 500, EUROSTOXX 50, ... Furthermore, options on $S(T)$ are traded and their prices are observable for a finite number of strikes. In this situation, pricing these index options is not a real issue; we denote the market price of an index option with maturity T and strike K by $C[K,T]$. Assume now that the stocks composing the index can be described by the Lévy copula model (8). If the parameter vector Θ and the marginal volatility vector $\underline{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ are determined, the model price $C^{model}[K,T;\underline{\sigma},\Theta,\rho]$ for the basket option only depends on the correlation ρ. An *implied correlation* estimate for ρ arises when we match the model price with the observed index option price.

Definition 1 (implied Lévy correlation) *Consider the Lévy copula model defined in* (8). *The* implied Lévy correlation *of the index $S(T)$ with moneyness $\pi = \frac{S(T)}{S(0)}$, denoted by $\rho[\pi]$, is defined by the following equation:*

$$C^{model}\left[K,T;\underline{\sigma},\Theta,\rho\left[\pi\right]\right] = C[K,T], \tag{32}$$

where $\underline{\sigma}$ contains the marginal implied volatilities and Θ is the parameter vector of L.

Determining an implied correlation estimate $\rho\left[\frac{K}{S(0)}\right]$ requires an inversion of the pricing formula $\rho \to C^{model}[K,T;\underline{\sigma},\Theta,\rho]$. However, the basket option price is not given in closed form and determining this price using Monte Carlo simulation is not an option, because the calibration would be too slow. Therefore, from here on, $C^{model}[K,T;\underline{\sigma},\Theta,\rho]$ has to be interpreted as the three-moments-matching approximation. In this case, implied correlations can be determined in a fast and efficient way. The idea of determining implied correlation estimates based on an approximate basket option pricing formula was already proposed in Chicago Board Options Exchange (2009) and Linders and Schoutens (2014).

Note that in case we take L to be the standard normal distribution, $\rho[\pi]$ is an implied Gaussian correlation; see e.g. Chicago Board Options Exchange (2009) and Skintzi and Refenes (2005). Equation (32) can be considered as a generalization of the implied Gaussian correlation. Indeed, instead of determining the single correlation parameter in a multivariate model with normal log returns and a Gaussian copula, we can now extend the model to the situation where the log returns follow a Lévy distribution and a Lévy copula connects the marginals. A similar idea was proposed in Garcia et al. (2009) and further studied in Masol and Schoutens (2011). In these papers, Lévy base correlation is defined using CDS and CDO prices.

The proposed methodology for determining implied correlation estimates can be generalized to other multi-asset derivatives. For example, implied correlation estimates can be extracted from traded spread options (Tavin (2013)), best-of basket options (Fonseca et al. (2007)) and quanto options (Ballotta et al. (2014)). Implied correlation estimates based on various multi-asset products are discussed in Austing (2014).

7.1 Variance Gamma

In order to illustrate the proposed methodology for determining implied Lévy correlation estimates, we use the Dow Jones Industrial Average (DJ). The DJ is composed of 30 underlying stocks and for each underlying we have a finite number of option prices to which we can calibrate the parameter vector Θ and the Lévy volatility parameters σ_j. Using the available vanilla option data for June 20, 2008 we will work out the Gaussian and the Variance Gamma case. Note that options on components of the Dow Jones are of American type. In the sequel, we assume that the American option price is a good proxy for the corresponding European option price. This assumption is justified because we use short term and out-of-the-money options.

The single volatility parameter σ_j is determined for stock j by minimizing the relative error between the model and the market vanilla option prices; see Algorithm 1. Assuming a normal distribution for L, this volatility parameter is denoted by σ_j^{BLS}, whereas the notation σ_j^{VG}, $j = 1, 2, \ldots, n$ is used for the VG model. For June 20, 2008, the parameter vector Θ for the VG copula model is given in Table 10 and the implied volatilities are listed in Table 9. Figure 3 shows the model (Gaussian and VG) and market prices for General Electric and IBM, both members of the Dow Jones, based on the implied volatility parameters listed in Table 9. We observe that the Variance Gamma copula model is more suitable in capturing the dynamics of the components of the Dow Jones than the Gaussian copula model.

Given the volatility parameters for the Variance Gamma case and the normal case, listed in Table 9, the implied correlation defined by equation (32), can be determined based on the available Dow Jones index options on June 20, 2008. For a given index strike K, the moneyness π is defined as $\pi = \frac{K}{S(0)}$. The implied Gaussian correlation (also called Black & Scholes correlation) is denoted by $\rho^{BLS}[\pi]$ and the corresponding implied Lévy correlation, based on a VG distribution, is denoted by $\rho^{VG}[\pi]$. In order to more closely match the vanilla option curves, we take into account the implied volatility smile and use a volatility parameter with moneyness π for each stock j, which we denote by $\sigma_j[\pi]$. For a detailed and step-by-step plan for the calculation of these volatility parameters, we refer to Linders and Schoutens (2014).

Figure 4 shows that both the implied Black & Scholes and implied Lévy correlation depend on the moneyness π. However, for low strikes, we observe that $\rho^{VG}[\pi] < \rho^{BLS}[\pi]$, whereas the opposite inequality holds for large strikes, making the implied Lévy correlation curve less steep than its Black & Scholes counterpart. In Linders and Schoutens (2014), the authors discuss the shortcomings of the implied Black & Scholes correlation and show that implied Black & Scholes correlations can become larger than one for low strike prices. Considering our more general approach and using the implied Lévy correlation solves, at least to some extent, this problem. Indeed, the region where the implied correlation stays below 1 is much larger for the flatter implied Lévy correlation curve than for its Black & Scholes counterpart. We also observe that near the at-the-money strikes, VG and Black & Scholes correlation estimates are comparable, which may be a sign that in this region, the use of implied Black & Scholes correlation (as defined in Linders and Schoutens (2014)) is justified. Figure 5 shows implied correlation curves for March, April, July and August, 2008. In all these situations, the time to maturity is close to 30 days. The calibrated parameters for each trading day are listed in 10.

Table 9: Implied Variance Gamma volatilities σ_j^{VG} and implied Black & Scholes volatilities σ_j^{BLS} for June 20, 2008.

Stock	σ_j^{VG}	σ_j^{BLS}
Alcoa Incorporated	0.6509	0.5743
American Express Company	0.4923	0.4477
American International Group	0.5488	0.4849
Bank of America	0.6003	0.5482
Boeing Corporation	0.3259	0.2927
Caterpillar	0.3009	0.2671
JP Morgan	0.5023	0.4448
Chevron	0.3252	0.3062
Citigroup	0.6429	0.5684
Coca Cola Company	0.2559	0.2343
Walt Disney Company	0.3157	0.2810
DuPont	0.2739	0.2438
Exxon Mobile	0.2938	0.2609
General Electric	0.3698	0.3300
General Motors	0.9148	0.8092
Hewlet-Packard	0.3035	0.2704
Home Depot	0.3604	0.3255
Intel	0.4281	0.3839
IBM	0.2874	0.2509
Johnson & Johnson	0.1741	0.1592
McDonald's	0.2508	0.2235
Merck & Company	0.3181	0.2896
Microsoft	0.3453	0.3068
3M	0.2435	0.2202
Pfizer	0.2779	0.2572
Procter & Gamble	0.1870	0.1671
AT&T	0.3013	0.2688
United Technologies	0.2721	0.2434
Verizon	0.3116	0.2847
Wal-Mart Stores	0.2701	0.2397

7.2 Double Exponential

In the previous subsection, we showed that the Lévy copula model allows for determining more robust implied correlation estimates. However, calibrating this model can be a computational

Table 10: Calibrated VG parameters for different trading days.

			VG Parameters		
	$S(0)$	T	σ	ν	θ
March 25, 2008	125.33	25 days	0.2981	0.5741	-0.1827
April 18, 2008	128.49	29 days	0.3606	0.5247	-0.2102
June 20, 2008	118.43	29 days	0.3587	0.4683	-0.1879
July 18, 2008	114.97	29 days	0.2639	0.5222	-0.1641
August 20, 2008	114.17	31 days	0.2467	0.3770	-0.1887

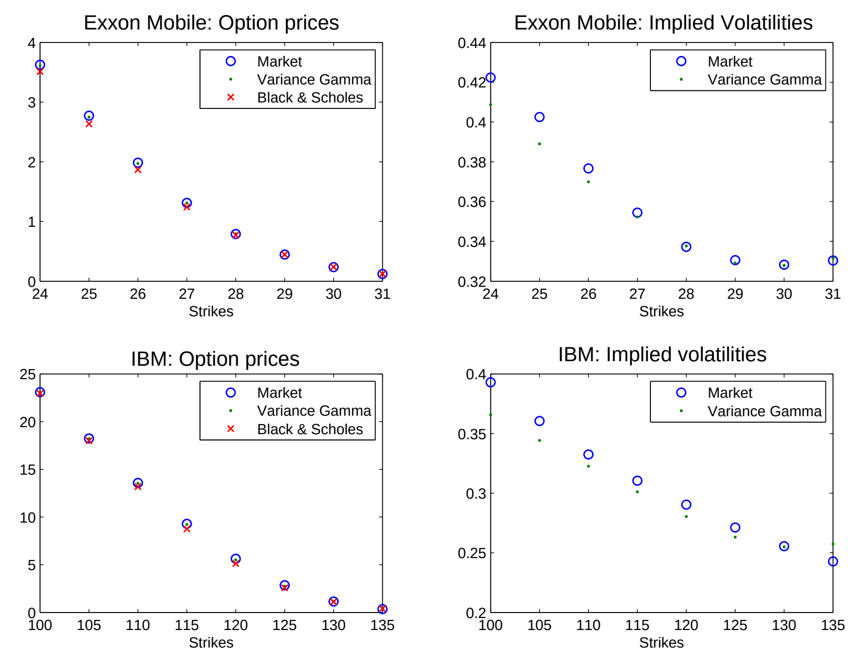

Figure 3: Option prices and implied volatilities (model and market) for Exxon Mobile and IBM on June 20, 2008 based on the parameters listed in Table 9. The time to maturity is 30 days.

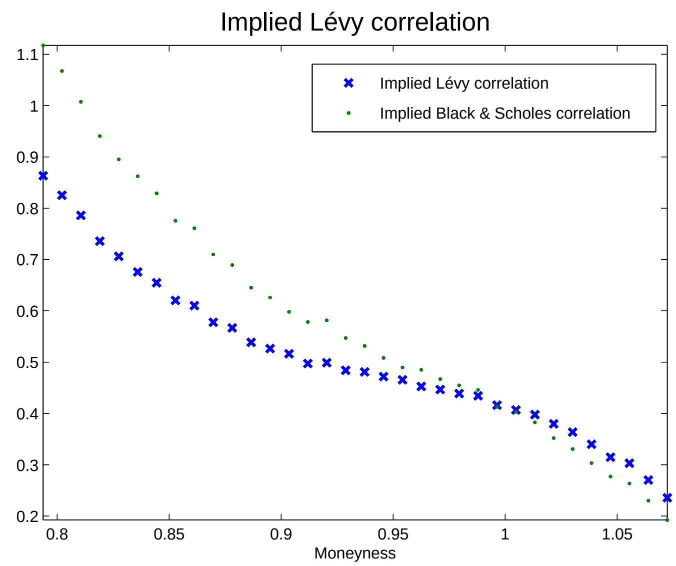

Figure 4: Implied correlation smile for the Dow Jones, based on a Gaussian (dots) and a Variance Gamma copula model (crosses) for June 20, 2008.

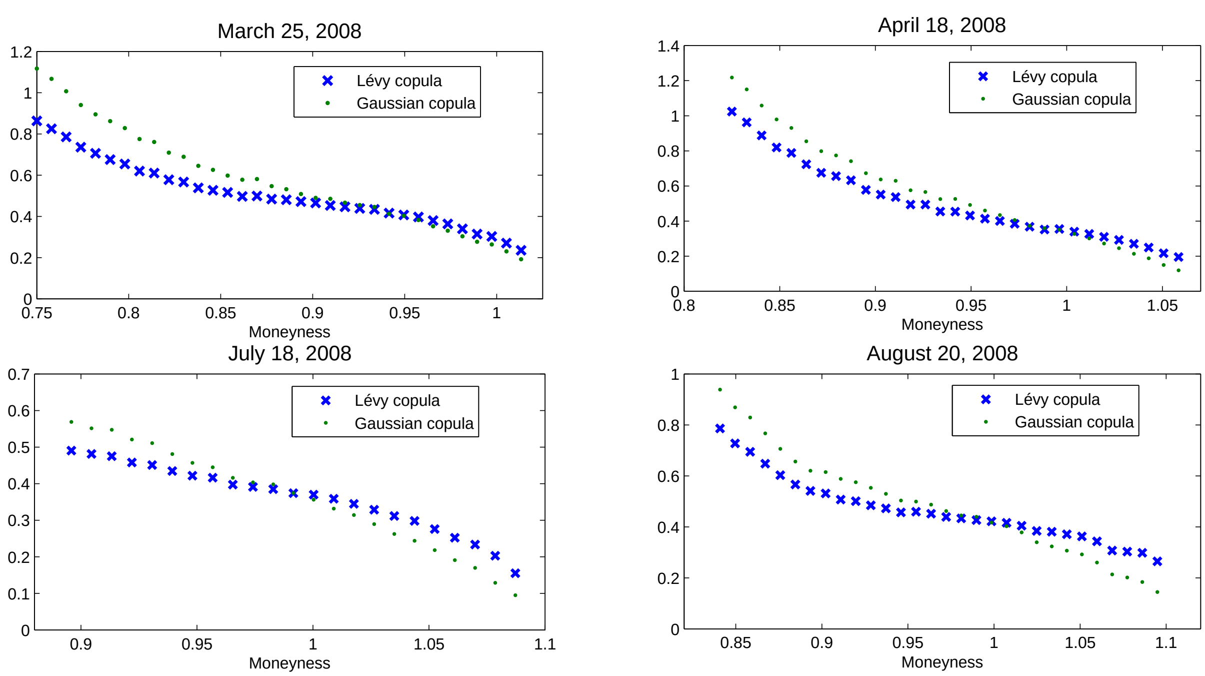

Figure 5: Implied correlation smile for the Dow Jones, based on a Gaussian (dots) and a Variance Gamma copula model (crosses) for different trading days.

challenging task. Indeed, in case we deal with the Dow Jones Industrial Average, there are 30 underlying stocks and each stock has approximately 5 traded option prices. Calibrating the parameter vector Θ and the volatility parameters σ_j has to be done simultaneously. This contrasts sharply with the Gaussian copula model, where the calibration can be done stock per stock.

In this subsection we consider a model with the computational attractive calibration property of the Gaussian copula model, but without imposing any normality assumption on the marginal log returns. To be more precise, given the convincing arguments exposed in Figure 5 we would like to keep L a $VG(\sigma, \nu, \theta, \mu)$ distribution. However, we do not calibrate the parameter vector $\Theta = (\sigma, \nu, \theta, \mu)$ to the vanilla option curves, but we fix these parameters upfront as follows

$$\begin{aligned} \mu &= 0, \\ \theta &= 0, \\ \nu &= 1, \\ \sigma &= 1. \end{aligned}$$

In this setting, L is a standardized distribution and its characteristic function ϕ_L is given by

$$\phi_L(u) = \frac{1}{1 + \frac{u^2}{2}}, \quad u \in \mathbb{R}.$$

From its characteristic function, we see that L has a *Standard Double Exponential distribution*, also called Laplace distribution, and its pdf f_L is given by

$$f_L(u) = \frac{1}{2\sqrt{2}} \mathrm{e}^{-\frac{|u|}{\sqrt{2}}}, \quad u \in \mathbb{R}.$$

The Standard Double Exponential distribution is symmetric and centered around zero, while it has variance 1. Note, however, that it is straightforward to generalize this distribution such that it has center μ and variance σ^2. Moreover, the kurtosis of this Double Exponential distribution is 6.

By using the Double Exponential distribution instead of the, more general, Variance Gamma distribution, a little bit flexibility is lost for modeling the marginals. However, the Double Exponential distribution is still a much better distribution for modeling the stock returns than the normal distribution. Moreover, in this simplified setting, the only parameters to be calibrated are the marginal volatility parameters, which we denote by σ_j^{DE}, and the correlation parameter ρ^{DE}. Similar to the Gaussian copula model, calibrating the volatility parameter σ_j^{DE} only requires the option curve of stock j. As a result, the time to calibrate the Double Exponential copula model is comparable to its Gaussian counterpart and much shorter than the general Variance Gamma copula model.

Consider the DJ on March 25, 2008. The time to maturity is 25 days. We determine the implied marginal volatility parameter for each stock in a Variance Gamma copula model and a Double Exponential framework. The results are listed in Table 11. One can observe that the implied volatility in each of these models is more or less the same. Given this information, we can determine the prices $C^{VG}[K, T]$ and $C^{DE}[K, T]$ for a basket option in a Variance-Gamma and a Double Exponential copula model, respectively. For this illustration, we have put $\rho = 0$. The option prices $C^{VG}[K,T]$ and $C^{DE}[K,T]$ are shown in Figure 6 for different choices of K and a time to maturity of 25 days. For out-of-the-money strikes K, we find that $C^{DE}[K,T]$ is significantly bigger than $C^{VG}[K,T]$, whereas the difference between the prices is smaller for in-the-money strikes. This difference between the Variance Gamma and the Double Exponential copula model for out-of-the-money strikes has its impact when determining implied correlation estimates. Indeed, it is shown in Figure 7 that the implied Variance Gamma correlation is larger than its Double Exponential counterpart for a moneyness bigger than one, whereas both implied correlation estimates are relatively close to each other in the other situation.

Table 11: Volatility parameters of the Variance Gamma and the Double Exponential copula model on March 25, 2008.

Stocks	σ_j^{VG}	σ_j^{DE}
Alcoa Incorporated	0.5255	0.5187
American Express Company	0.4636	0.4498
Kraft Foods	0.5027	0.4957
Bank of America	0.4270	0.4232
Boeing Corporation	0.3053	0.3013
Caterpillar	0.3652	0.3648
JP Morgan	0.4968	0.4765
Chevron	0.3145	0.3139
Citigroup	0.6862	0.6440
Coca Cola Company	0.2146	0.2158
Walt Disney Company	0.2755	0.2842
DuPont	0.3152	0.3190
Exxon Mobile	0.3087	0.3054
General Electric	0.2849	0.2865
General Motors	0.8372	0.7914
Hewlet-Packard	0.3110	0.3077
Home Depot	0.4093	0.4001
Intel	0.4325	0.4259
IBM	0.2985	0.2962
Johnson & Johnson	0.1937	0.1958
McDonald's	0.2635	0.2642
Merck & Company	0.3524	0.3472
Microsoft	0.3093	0.3049
3M	0.2489	0.2490
Pfizer	0.2748	0.2763
Practer & Gamble	0.1780	0.1780
AT&T	0.3155	0.3105
United Technologies	0.2821	0.2830
Verizon	0.3208	0.3164
Wal-Mart Stores	0.2609	0.2674

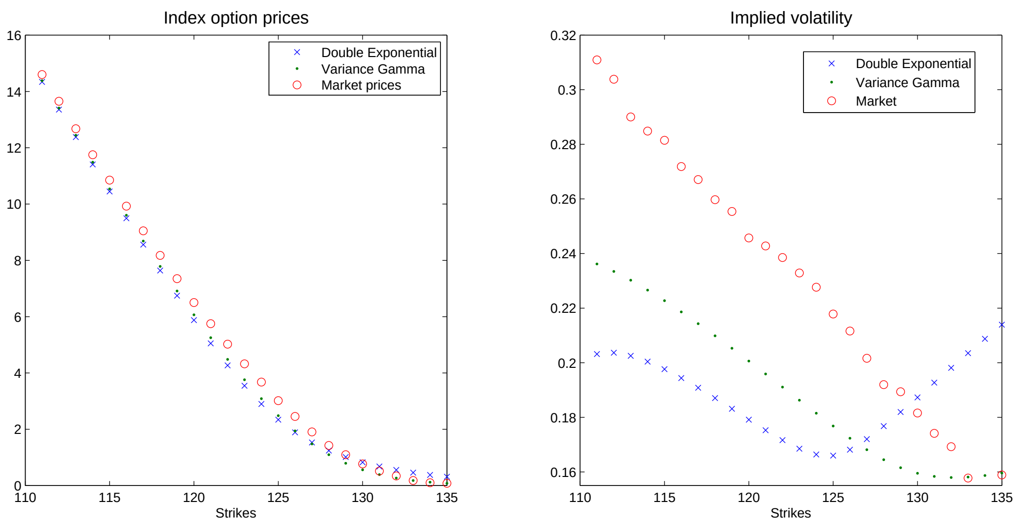

Figure 6: Dow Jones option prices (left) and implied volatilities (right) in the Variance Gamma and the Double Exponential copula model with $\rho = 0$ and volatility parameters given in Table 11, for March 25, 2008 and a time to maturity of 25 days.

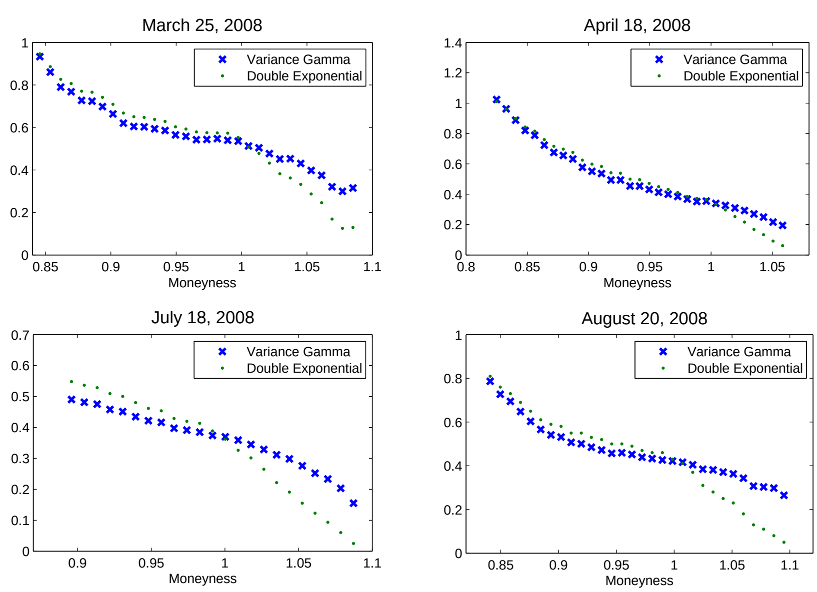

Figure 7: Implied correlation smiles in the Variance Gamma and the Double Exponential copula model.

8 Conclusion

In this paper we extended the classical Gaussian copula model to a Lévy copula model. We proposed two methods for approximating the price of a basket option. Both approximations consist of replacing the original r.v. describing the basket at maturity by a r.v. which has a more simple structure. Furthermore, we showed that the Carr-Madan formula can be used to determine the approximate basket option prices. Well-known distributions like the Normal, Variance Gamma, NIG, Meixner, ... can be used in the Lévy copula model. We calibrate these different models to market data and determine basket option prices for the different model settings. Our newly designed (approximate) basket option pricing formula can be used to define implied Lévy correlation. The Lévy copula model provides a flexible framework for deriving implied correlation estimates in different model settings. Indeed, by employing a Brownian motion and a Variance Gamma process in our Lévy copula model, we can determine Gaussian and VG implied correlation estimates, respectively. We observe that the VG implied correlation is an improvement of the Gaussian implied correlation.

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A Proof of Lemma 2

The proof for expression (27) is straightforward.
Starting from the multinomial theorem, we can write the second moment m_2 as follows

$$\begin{aligned} m_2 &= \mathbb{E}\left[(w_1S_1(T)+w_2S_2(T)+\ldots w_nS_n(T))^2\right] \\ &= \mathbb{E}\left[\sum_{i_1+i_2+\ldots+i_n=2}\frac{2}{i_1!i_2!\ldots i_n!}\prod_{j=1}^{n}(w_jS_j(T))^{i_j}\right]. \end{aligned}$$

Considering the cases $(i_n=0)$, $(i_n=1)$ and $(i_n=2)$ separately, we find

$$m_2 = \mathbb{E}\left[\left(\sum_{j=1}^{n-1}w_jS_j(T)\right)^2 + 2w_nS_n(T)\sum_{j=1}^{n-1}w_jS_j(T)+w_j^2S_n^2(T)\right].$$

Continuing recursively gives

$$m_2 = \sum_{j=1}^{n}\sum_{k=1}^{n}w_jw_k\mathbb{E}\left[S_j(T)S_k(T)\right]. \tag{33}$$

We then find that

$$\begin{aligned} m_2 &= \sum_{j=1}^{n}\sum_{k=1}^{n}w_jw_kS_j(0)S_k(0) \\ &\quad\times\mathbb{E}\left[\exp\left\{(2r-q_j-q_k-\omega_j-\omega_k)T+(\sigma_jA_j+\sigma_kA_k)\sqrt{T}\right\}\right] \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}w_jw_k\frac{\mathbb{E}\left[S_j(T)\right]\mathbb{E}\left[S_k(T)\right]}{\phi_L\left(-\mathrm{i}\sigma_j\sqrt{T}\right)\phi_L\left(-\mathrm{i}\sigma_k\sqrt{T}\right)}\mathbb{E}\left[\exp\left\{(\sigma_jA_j+\sigma_kA_k)\sqrt{T}\right\}\right]. \end{aligned}$$

In the last step, we used the expression $\omega_j = \frac{1}{T}\log\phi_L\left(\mathrm{i}\sigma_j\sqrt{T}\right)$. If we use expression (3) to decompose A_j and A_k in the common component $X(\rho)$ and the independent components $X_j(1-\rho)$ and $X_k(1-\rho)$, we find the following expression for m_2

$$m_2 = \sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k \frac{\mathbb{E}\left[S_j(T)\right]\mathbb{E}\left[S_k(T)\right]}{\phi_L\left(-\mathrm{i}\sigma_j\sqrt{T}\right)\phi_L\left(-\mathrm{i}\sigma_k\sqrt{T}\right)}\mathbb{E}\left[\mathrm{e}^{(\sigma_j+\sigma_k)X(\rho)}\mathrm{e}^{\sigma_j\sqrt{T}X_j(1-\rho)}\mathrm{e}^{\sigma_k\sqrt{T}X_k(1-\rho)}\right].$$

The r.v. $X(\rho)$ is independent from $X_j(1-\rho)$ and $X_k(1-\rho)$. Furthermore, the characteristic function of $X(\rho)$ is ϕ_L^{ρ}, which results in

$$\begin{aligned} m_2 &= \sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k \frac{\mathbb{E}\left[S_j(T)\right]\mathbb{E}\left[S_k(T)\right]}{\phi_L\left(-\mathrm{i}\sigma_j\sqrt{T}\right)\phi_L\left(-\mathrm{i}\sigma_k\sqrt{T}\right)}\phi_L\left(-\mathrm{i}(\sigma_j+\sigma_k)\sqrt{T}\right)^{\rho} \\ &\quad \times\mathbb{E}\left[\mathrm{e}^{\sigma_j\sqrt{T}X_j(1-\rho)}\mathrm{e}^{\sigma_k\sqrt{T}X_k(1-\rho)}\right] \end{aligned}$$

If $j \neq k$, $X_j(1-\rho)$ and $X_k(1-\rho)$ are i.i.d. with characteristic function $\phi_L^{1-\rho}$, which gives the following expression for m_2:

$$m_2 = \sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k \mathbb{E}\left[S_j(T)\right]\mathbb{E}\left[S_k(T)\right]\left(\frac{\phi_L\left(-\mathrm{i}(\sigma_j+\sigma_k)\sqrt{T}\right)}{\phi_L\left(-\mathrm{i}\sigma_j\sqrt{T}\right)\phi_L\left(-\mathrm{i}\sigma_k\sqrt{T}\right)}\right)^{\rho}.$$

If $j = k$, we find that

$$\mathbb{E}\left[\mathrm{e}^{\sigma_j\sqrt{T}X_j(1-\rho)}\mathrm{e}^{\sigma_k\sqrt{T}X_k(1-\rho)}\right] = \phi_L\left(-\mathrm{i}\left(\sigma_j+\sigma_k\right)\sqrt{T}\right),$$

which gives

$$m_2 = \sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k \mathbb{E}\left[S_j(T)\right]\mathbb{E}\left[S_k(T)\right]\frac{\phi_L\left(-\mathrm{i}(\sigma_j+\sigma_k)\sqrt{T}\right)}{\phi_L\left(-\mathrm{i}\sigma_j\sqrt{T}\right)\phi_L\left(-\mathrm{i}\sigma_k\sqrt{T}\right)}.$$

This proves expression (28) for m_2.

We can write m_3 as follows

$$\begin{aligned} m_3 &= \mathbb{E}\left[\left(\sum_{j=1}^{n} w_j S_j(T)\right)^3\right] \\ &= \mathbb{E}\left[\left(\sum_{j=1}^{n} w_j S_j(T)\right)^2 \sum_{l=1}^{n} w_l S_l(T)\right] \end{aligned}$$

Using expression (33), we find the following expression for m_3:

$$\begin{aligned} m_3 &= \mathbb{E}\left[\left(\sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k S_j(T) S_k(t)\right)\sum_{l=1}^{n} w_l S_l(T)\right] \\ &= \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} w_j w_k w_l \mathbb{E}\left[S_j(T) S_k(T) S_l(T)\right]. \end{aligned}$$

Similar calculations as for m_2 result in

$$\begin{aligned} m_3 &= \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{l=1}^{n} w_j w_k w_l \mathbb{E}\left[S_j(T)\right]\mathbb{E}\left[S_k(T)\right]\mathbb{E}\left[S_l(T)\right] \\ &\quad \times \frac{\phi_L\left(-\mathsf{i}(\sigma_j+\sigma_k+\sigma_l)\sqrt{T}\right)^{\rho}}{\phi_L\left(-\mathsf{i}\sigma_j\sqrt{T}\right)\phi_L\left(-\mathsf{i}\sigma_k\sqrt{T}\right)\phi_L\left(-\mathsf{i}\sigma_l\sqrt{T}\right)} A_{j,k,l}, \end{aligned}$$

where

$$A_{j,k,l} = \mathbb{E}\left[\mathsf{e}^{\sigma_j\sqrt{T}X_j(1-\rho)}\mathsf{e}^{\sigma_k\sqrt{T}X_k(1-\rho)}\mathsf{e}^{\sigma_l\sqrt{T}X_l(1-\rho)}\right].$$

Differentiating between the situations $(j=k=l)$, $(j=k, k\neq l)$, $(j\neq k, k=l)$, $(j\neq k, k\neq l, j=l)$ and $(j\neq k\neq l, j\neq l)$, we find expression (29).