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Implied correlation in the Lévy copula model[1]

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[1] Linders, D. & Schoutens, W. (2014) 'Basket option pricing and implied correlation in a Lévy copula model', Research report, AFI-1494, FEB KU Leuven.

Outline

Introduction

The financial market

- Today is time $t = 0$. We fix a maturity T.
- Price of stock j at time T is denoted by $S_j(T)$.
- The price of the basket at time T is denoted by $S(T)$:

$$S(T) = w_1 S_1(T) + \cdots + w_n S_n(T), \quad w_j \geq 0.$$

- Assumptions:
  - The market is arbitrage-free and there exists a pricing measure $\mathbb{Q}$.
  - The interest rate r is deterministic and constant.
- The price of a **basket option** with strike K and maturity T:

$$C[K,T] = \mathrm{e}^{-rT} \mathbb{E}_{\mathbb{Q}} \left[(S(T) - K)_+ \right].$$

Introduction

Basket options

- Basket option pricing in a Gaussian copula model:
  - Dhaene et al. (2002), Deelstra et al. (2004), Brigo et al. (2004), Carmona & Durrleman (2006), Korn & Zeytun (2013), Linders (2013).
  - Elliptical copula model: Valdez, Dhaene, Maj & Vanduffel (2009).
- Non-Gaussian setting:
  - Multivariate Variance Gamma model: Linders & Stassen (2014).
  - Jump diffusion model: Xu and Zheng (2014), Paletta et al. (2013).
  - Models with known joint characteristic function: Caldana et al. (2014).
  - Model-free upper bounds: Hobson et al. (2005), Chen et al. (2008).
- We consider basket option pricing in a multivariate Lévy model and show how to calibrate this model using traded single-name and multi-name derivatives.

- **Lévy copula model**
  - Extend the Gaussian copula model to a Lévy setting.
  - Based on the generic one-factor model introduced in Albrecher, Ladoucette & Schoutens (2007).
- **Approximate basket option pricing**
  - Three-moments-matching approximation.
  - Gaussian case: Brigo, Mercurio, Rapisarda & Scotti (2004).
- **Implied Lévy correlation**
  - Index options are traded and their prices can be observed.
  - Match the (approximate) model index option price with the market price
  - The *Implied Lévy correlation smile* solves some issues of implied Gaussian correlation: Linders & Schoutens (2014a).

The Lévy copula model

Lévy processes

- Assume that L is *infinitely divisible* and has characteristic function ϕ_L.
- **Examples:** Normal, Variance Gamma, NIG, Meixner, ...
- Define $n+1$ independent Lévy processes.
  - X is a process based on the distribution L.
  - $X_1, X_2, \ldots, X_n$ are independent processes based on L.
  - Characteristic function:

$$\mathbb{E}\left[\mathrm{e}^{iuX(t)}\right] = \mathbb{E}\left[\mathrm{e}^{iuX_j(t)}\right] = \phi_L(u)^t$$

- We assume that these are all standardized processes:
  - $\mathbb{E}[X(1)] = \mathbb{E}[X_j(1)] = 0$;
  - $\mathrm{Var}[X(1)] = \mathrm{Var}[X_j(1)] = 1$.

The Lévy copula model

Building correlated Lévy processes

- The log return of stock j is modeled using the r.v. A_j:

$$A_j = X(\rho) + X_j(1-\rho), \quad j = 1, 2, \ldots, n,$$

  - $\rho \in [0, 1]$,
  - log return = systematic component + idiosyncratic component
- Correlation:

$$\text{Corr}\left[A_i, A_j\right] = \rho, \qquad \text{for } i \neq j.$$

  - The dependence is modeled by a correlation parameter ρ:
    - ⋆ *single* correlation parameter;
    - ⋆ *positive* correlation parameter.

The Lévy copula model

Properties

- A_j is standardized:

$$\mathbb{E}[A_j] = 0, \qquad \mathsf{Var}[A_j] = 1.$$

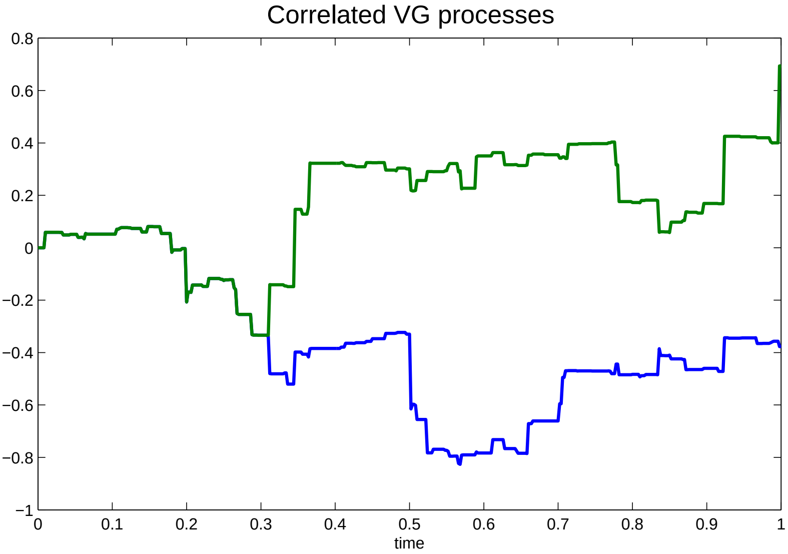
- Distribution:

$$A_j \stackrel{\mathsf{d}}{=} L.$$

- Conditioned on the systematic component, the stocks are independent.

The Lévy copula model

Correlated VG processes



- The Lévy copula model:

$$S_j(T) = S_j(0)\mathrm{e}^{\mu_j T + \sigma_j \sqrt{T} A_j}, \quad j = 1, 2, \ldots, n,$$

  where $\mu_j \in \mathbb{R}$ and $\sigma > 0$.
- If A_j is based on standard Brownian motions we find back the Gaussian copula case.
- σ_j can be interpreted as the **Lévy implied volatility**; Corcuera, Guillaume, Leoni & Schoutens (2009).
- One-factor models:
  - Vasicek (1987), Moosbrucker (2006 a,b), Baxter (2007), Itkin & Lipton (2014).

The Lévy copula model

Risk-neutral stock price processes

- Choose μ_j such that

$$\mathbb{E}[S_j(T)] = \mathrm{e}^{(r-q_j)T} S_j(0).$$

- Risk-neutral dynamics:

$$S_j(T) = S_j(0)\mathrm{e}^{(r-q_j-\omega_j)T+\sigma_j\sqrt{T}A_j}, \quad j = 1,2,\ldots,n, \tag{1}$$

  - ω_j is a mean-correction:

$$\omega_j = \frac{1}{T}\log\phi_L\left(-\mathrm{i}\sigma_j\sqrt{T}\right).$$

Approximate basket option pricing

- The price of the basket at time T is denoted by $S(T)$:

$$S(T) = w_1 S_1(T) + \cdots + w_n S_n(T), \quad w_j \geq 0.$$

- A **basket option** with strike K and maturity T pay at time T:

$$\left(S(T) - K\right)_+ .$$

- The price is denoted by $C[K,T]$ and given by:

$$C[K,T] = \mathrm{e}^{-rT} \mathbb{E}_{\mathbb{Q}} \left[\left(S(T) - K\right)_+ \right].$$

- In the Lévy copula model, the cdf of the r.v. $S(T)$ is unknown.

Approximate basket option pricing

The approximate basket

- Define the random variable A such that

$$A \stackrel{d}{=} L.$$

- We replace $S(T)$ by the more tractable r.v. $\tilde{S}(T)$.

$$\tilde{S}(T) = \bar{S}(T) + \lambda,$$

where $\lambda \in \mathbb{R}$ and

$$\bar{S}(T) = S(0)\mathrm{e}^{(\bar{\mu}-\bar{\omega})T+\bar{\sigma}\sqrt{T}A}.$$

- Black & Scholes case: Brigo, Mercurio, Rapisarda & Scotti (2004).

Approximate basket option pricing

Three-moments-matching approximation

- The first three moments of $S(T)$: m_1, m_2 and m_3.
  - Can be determined if ϕ_L is known[2].
- Determine $\bar{\mu}$, $\bar{\sigma}$ and λ such that

$$
\begin{aligned}
\mathbb{E}\left[\tilde{S}(T)\right] &= m_1, \\
\mathbb{E}\left[\tilde{S}(T)^2\right] &= m_2, \\
\mathbb{E}\left[\tilde{S}(T)^3\right] &= m_3.
\end{aligned}
$$

- Mean-correcting parameter: $\bar{\omega} = \frac{1}{T} \log \phi_L \left(-\mathrm{i}\bar{\sigma}\sqrt{T} \right)$.

[2] see Lemma 2 in Linders & Schoutens (2014b)

Approximate basket option pricing

The characteristic function

- Approximate $C[K,T]$
  - Replace the original basket $S(T)$ by the approximation $\tilde{S}(T)$:

$$C^{MM}[K,T] = \mathrm{e}^{-rT}\mathbb{E}\left[\left(\tilde{S}(T) - K\right)_+\right]$$

  - Use that $\tilde{S}(T) = \bar{S}(T) + \lambda$:

$$C^{MM}[K,T] = \mathrm{e}^{-rT}\mathbb{E}\left[\left(\bar{S}(T) - (K-\lambda)\right)_+\right].$$

- New problem: price an option on $\bar{S}(T)$ with shifted strike $K - \lambda$.
- Knowledge of the characteristic function $\phi_{\log \bar{S}(T)}$ is needed:

$$\phi_{\log \bar{S}(T)}(u) = \exp\left\{\mathrm{i}u\left(\log S(0) + (\bar{\mu} - \bar{\omega})T\right)\right\}\phi_L\left(u\bar{\sigma}\sqrt{T}\right).$$

The Carr-Madan formula

Option pricing via the characteristic function

- Consider the r.v. X and assume that $\mathbb{E}\left[X^{\alpha+1}\right]$ is finite for $\alpha > 0$.
- Then it was proven in Carr & Madan (1998):

$$e^{-rT}\mathbb{E}\left[(X-K)_+\right] = \frac{e^{-\alpha\log(K)}}{\pi}\int_0^{+\infty}\exp\left\{-iv\log(K)\right\}g(v)dv,$$

where

$$g(v) = \frac{e^{-rT}\phi_{\log X}\left(v-(\alpha+1)i\right)}{\alpha^2+\alpha-v^2+i(2\alpha+1)v}.$$

- The key ingredient to determine option prices is a closed form of the characteristic function.
- Fast implementation using the FFT.

Basket option pricing in the Lévy copula model

Numerical illustration: VG copula model

- The Variance Gamma (VG) copula model
  - L has a VG distribution:

$$L \stackrel{d}{=} VG(\sigma = 0.57, \nu = 0.75, \theta = -0.95, \mu = 0.95).$$

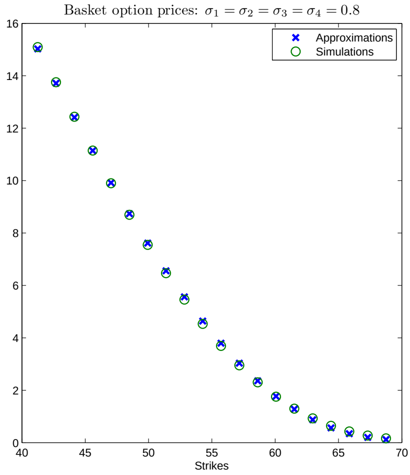
- Consider a basket option paying at maturity T:

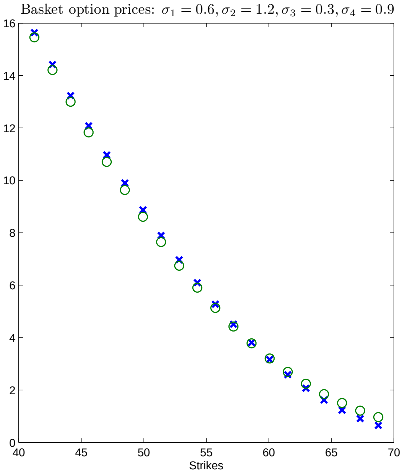
$$\left(\frac{1}{4} \left(S_1(T) + S_2(T) + S_3(T) + S_4(T) \right) - K \right)_+ .$$

  - $T = 0.5$, $\rho = 0$;
  - $S_1(0) = 40,\ S_2(0) = 50,\ S_3(0) = 60,\ S_4(0) = 70.$
- We compare simulated basket option prices with the corresponding approximations.

Basket option pricing in the Lévy copula model

Numerical illustration: VG copula model





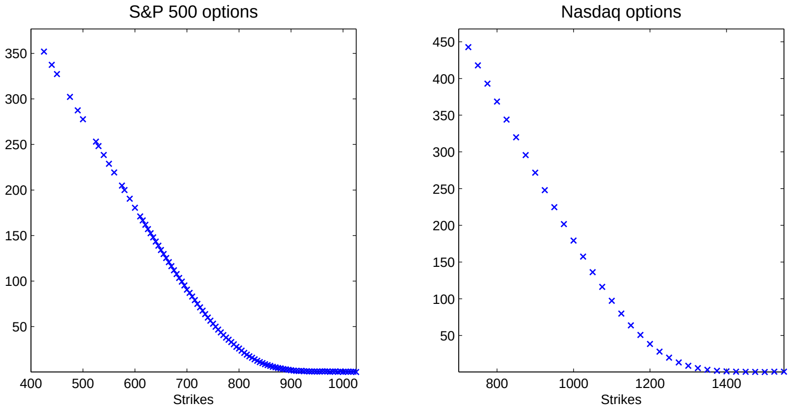
- Pricing date = February 19, 2009.
- We consider a basket option paying at maturity T

$$(w_1 S_1(T) + w_2 S_2(T) - K)_+ .$$

  - $T = 30$ days.
  - $S_1(T)$ denotes the price of the S&P 500 and $S_2(T)$ the price of the Nasdaq.
  - The weights are chosen such that $w_1 S_1(0) + w_2 S_2(0) = 100$.
- **Question:** how can we determine basket option prices consistent with the observed option curves of the S&P 500 and the Nasdaq?

Pricing basket options in the Lévy copula model

Market prices for S&P 500 and Nasdaq options with $T = 30$ days



Pricing basket options in the Lévy copula model

Calibrating the marginal parameters

- Remember that in the Lévy copula model:

$$S_j(T) = S_j(0)\mathrm{e}^{(r-q_j-\omega_j)T+\sigma_j\sqrt{T}A_j}, \qquad j = 1, 2.$$

  - The distributions A_1 and A_2 are described by the parameter vector Θ.
  - The characteristic function $\phi_{\log S_j(T)}$ is given in closed form.

- Model prices of vanilla options:

$$C_j^{model}[K, T; \Theta, \sigma_j] = \mathrm{e}^{-rT}\mathbb{E}\left[\left(S_j(T) - K\right)_+\right].$$

  - $C_j^{model}[K, T; \Theta, \sigma_j]$ can be determined using the Carr-Madan formula.

Pricing basket options in the Lévy copula model

Calibrating the marginal parameters

- Determine implied estimates for Θ and σ_1, σ_2 by minimizing the distance between the market and the model prices of available vanilla options.
- Calibration has to be done simultaneously:
  - Common parameter vector Θ.
  - Gaussian copula case: A_1 and A_2 are standard normal distributed and calibration can be done stock per stock.
- Alternative approach:
  - Fix the distribution of A and the parameter vector Θ upfront.

Pricing basket options in the Lévy copula model

Calibrating the marginal parameters

Table: Lévy copula models: Calibrated model parameters

Model	Error	Model Parameters			Volatilities	
		μ_{normal}	σ_{normal}		σ_1	σ_2
Normal	15.91%	0	1		0.2863	0.2762
		σ_{VG}	ν_{VG}	θ_{VG}		
VG	9.39 %	0.3640	0.7492	-0.3123	0.3876	0.3729
		$\alpha_{Meixner}$	$\beta_{Meixner}$			
Meixner	9.33%	1.5794	-1.6235		0.4015	0.3833
		α_{NIG}	β_{NIG}			
NIG	9.51 %	1.5651	-1.0063		0.4130	0.3941

Implied Lévy correlation

Calibration using multi-name derivatives

- Dow Jones Industrial Average (DJ)
  - 30 underlying stocks.
  - Options are traded for each component.
  - The parameter vector Θ and the volatility parameters $\sigma_1, \sigma_2, \ldots, \sigma_{30}$ can be calibrated.
- The only free parameter is the correlation ρ.
- Market prices for index options
  - Options on the Dow Jones are traded.
  - ρ can be calibrated to traded Dow Jones index options.
- Only one traded index option is required for fixing ρ.

Implied Lévy correlation

Definition[3]

- We observe the market price $C[K,T]$ for the index option with strike K and maturity T.
- Given the parameter ρ, we can determine the (approximate) model price: $C^{model}\left[K,T;\underline{\sigma},\Theta,\rho\right]$.

Definition (Implied correlation)

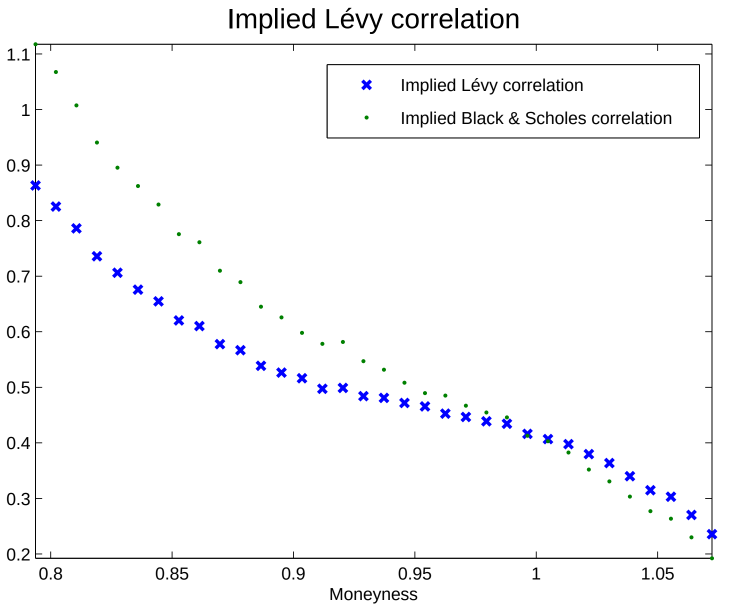
The implied correlation $\rho[\pi]$, with moneyness $\pi = \frac{K}{S(0)}$ is defined such that:

$$C^{model}\left[K,T;\underline{\sigma},\Theta,\rho\left[\pi\right]\right] = C[K,T].$$

[3] see Linders & Schoutens (2014a)

Implied Lévy correlation

The DJ implied correlation smile: June 20, 2008 and $T = 30$ days



Implied Lévy correlation

The implied correlation smile

- Implied Gaussian correlation
  - For low moneyness, Gaussian implied correlation goes beyond 1.
  - The lognormal distribution is not capable of describing the marginals adequately.
- Implied Lévy correlation
  - Better fit of the marginals and correlations which are below 1.
  - The Lévy implied correlation smile is flatter than its Gaussian counterpart.
- At-the-money implied Gaussian and Lévy correlations are more or less equal.

Conclusions

- We extended the classical Gaussian copula model to a Lévy copula model.
- We considered the problem of pricing basket options.
- The model can be calibrated using single-name and multi-name derivatives.
- Implied Lévy correlation solves some issues of the classical Gaussian correlation.
- Literature:
  - Gaussian copula model: Skintzi & Refenes (2005), CBOE (2009), Linders & Schoutens (2014a).
  - Implied correlation: Da Fonseca, Graselli & Tebaldi (2007), Tavin (2013), Balotta, Deelstra & Rayée (2014), Austing (2014).
  - Lévy base correlation: Garcia, Goossens, Masol & Schoutens (2009).

Thank you for your attention!

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