UiO : **Department of Mathematics**
University of Oslo

Lecture I

Modelling the forward price dynamics in energy markets

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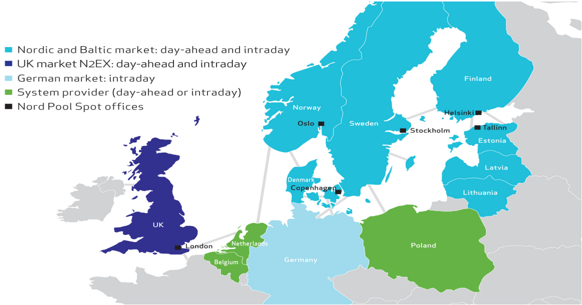
Overview

In collaboration with Paul Krühner (Vienna)

1. **Power markets: an brief introduction**
2. **Hilbert-valued Lévy processes by subordination**
3. **Examples**
4. **Some final notes on *H*-valued Lévy processes**

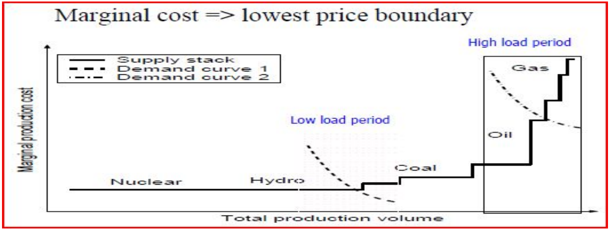
Overview

- Typically, power markets organize trade in
  - Hourly spot electricity, next-day physical delivery
  - Forward and futures contracts on spot
  - European options on forwards
- Examples: EEX, NordPool, APX, ICE...

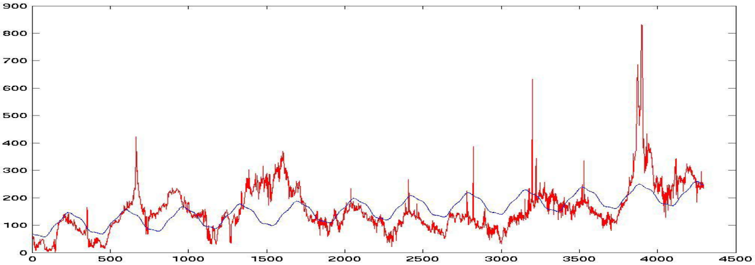


The spot market

- An hourly market with physical delivery of electricity
- Participants hand in bids the *day ahead*
  - Volume and price bids for each of the 24 hours next day
  - Maximum amount of bids within technical volume and price limits
- The exchange creates demand and production curves for each hour of the next day



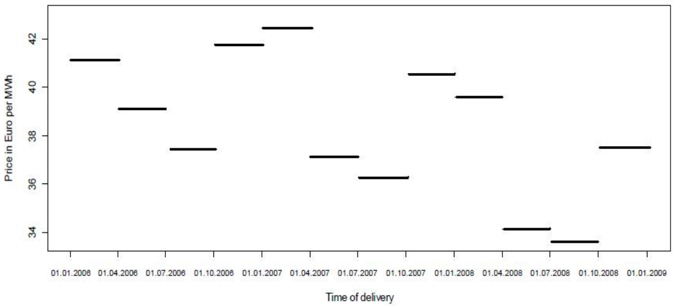
- The spot price is the equilibrium
  - Price for delivery of electricity at a specific hour next day
  - The *daily* spot price is the average of the 24 hourly prices
- Reference price for the forward market
- Historical spot price at NordPool from the beginning in 1992 (NOK/MWh)



The forward and futures market

- Contracts with “delivery” of electricity over a period
  - Financially settled: The money-equivalent of receiving electricity is paid to the buyer
  - The reference is the hourly spot price in the delivery period
- Delivery periods: next day, week, month, quarter, year
- Overlapping delivery periods (!)
- Base and peak load contracts
- European call and put options on these forwards

- The forward curve at NordPool, 1 January, 2006 (base load quarterly contracts)
- Constructed from observed prices of various delivery length

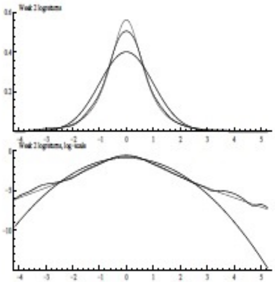


The option market

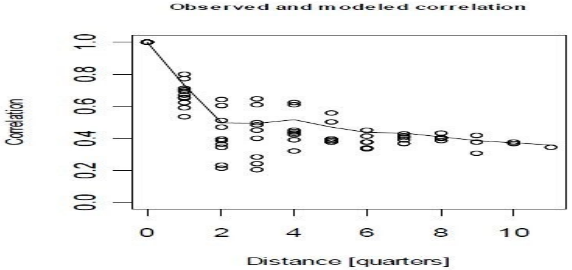
- European call and put options on electricity forwards
  - Quarterly and yearly delivery periods
- OTC market for electricity derivatives huge
  - Average-type (Asian) options, swing options, quanto options, spread options ....

Some brief empirical insight

- Probability density of returns is non-Gaussian
- Example: weekly and monthly contracts (Frestad 2008)
  - Fitted normal and NIG
  - "True" and logarithmic frequency axis
  - NIG=normal inverse Gaussian distribution

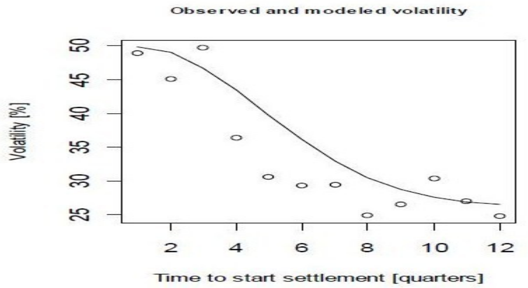


- Correlation structure of quarterly contracts at NordPool (Andersen et al (2010))
  - Correlation as a function of distance *between* start-of-delivery



- High degree of "idiosyncratic" risk
  - Quarterly contracts: 6 noise sources explain 96%, 7 explain 98%

- Observed Samuelson effect on (log-)returns
  - Volatility of forwards decrease with time to maturity
- Plot of Nordpool quarterly contracts, empirical volatility



Forward prices as an HJMM-dynamics

- These lectures: focus on the forward dynamics
- HJMM forward price $f(t,x)$, $t \geq, x \geq 0$,

$$df(t,x) = \partial_x f(t,x) + b(t)\,dt + dM(t,x) \qquad f(0,x) = f_0(x)$$

- $t \mapsto M(t)$ square-integrable martingale with values in a separable Hilbert space H
- Drift process $t \mapsto b(t)$
  - Equal to zero in risk-neutral setting
- H space of real-valued "smooth" functions on $\mathbb{R}_+$
  - E.g., some Sobolev-type space (Filipovic space)

- "Standard model" (fixed income, energy, commodities):

$$dM(t) = \sigma(t)\, dB(t)$$

- B $\mathbb{R}^d$-valued Brownian motion, σ "nice" operator-valued process from $\mathbb{R}^d$ to H
- Our aims:
  1. $M = L$, Hilbert-valued Lévy process (this lecture)
  2. Analyse the HJMM dynamics (Lectures II & III)
  3. Introduce a stochastic volatility process σ and let B be Wiener process in H (Lecture IV)
  4. Ambit fields and Volterra process in Hilbert space (Lecture V)

Overview

Aim

- Define a random field $L(t,x)$, $t,x \geq 0$, such that

1. $t \mapsto L(t,x)$ Lévy process
2. $x \mapsto L(t,x)$ random field with dependency (correlation) structure

- Typically, would like $L(\cdot,x)$ to be NIG Lévy process
- Method: subordinating Hilbert-valued Brownian motions

Hilbert-valued Lévy process

- First, some preparations.....:
- Given $(\Omega, \mathcal{F}, P)$ a probability space
- Let H be a separable Hilbert space
  - $\langle \cdot, \cdot \rangle$ inner product and $|\cdot|$ norm
  - $\{e_n\}_{n\in\mathbb{N}}$ orthonormal basis (ONB)
- A measurable map $X : \Omega \to H$ is called an H-valued random variable
  - The law of X is $P(X \in E), E \in \mathcal{B}(H)$
- If $|X|$ is P-integrable, define $\mathbb{E}[X] \in H$ as the Bochner integral

$$\langle \mathbb{E}[X], f \rangle = \langle \int_\Omega X(\omega)\, dP(\omega), f \rangle = \mathbb{E}[\langle X, f \rangle]$$

- $U(t), t \geq 0$ is called a H-valued Lévy process if:

1. $U(t) \in H$ with $U(0) = 0$,
2. Independent increments,
3. Law of $U(t) - U(s)$ depends on $t - s$ only, where $t \geq s$,
4. U is stochastically continuous

- Choose version of U with cadlag paths

- Lévy-Kintchine triplet (b, Q, ν) of U:

$$\varphi(x) := \log \mathbb{E}[\exp(\mathrm{i}\langle x, U(1)\rangle)]$$

$$\varphi(x) = \mathrm{i}\langle b, x\rangle - \frac{1}{2}\langle Qx, x\rangle + \int_{H\backslash\{0\}} \mathrm{e}^{\mathrm{i}\langle x,z\rangle} - 1 - \mathrm{i}1_{|z|<1}\langle x, z\rangle\, \nu(dz)$$

- $b \in H$, the *drift* of U, ν is the Lévy measure, i.e. measure on $H\backslash\{0\}$

$$\int_{H\backslash\{0\}} \min(1, |z|^2)\, \nu(dz) < \infty$$

- Q is a non-negative definite trace class operator on H

$$\mathrm{Tr}(Q) := \sum_{n=1}^{\infty} \langle Qe_n, e_n\rangle < \infty$$

- $t \mapsto \langle U(t), f\rangle$ is an $\mathbb{R}$-valued Lévy process with cadlag paths
- Characteristic function $\varphi_f(\theta) := \varphi(\theta f)$

$$\varphi_f(\theta) = \mathrm{i}\theta\langle b, f\rangle - \frac{1}{2}\theta^2\langle Qf, f\rangle + \int_{H\setminus\{0\}} \mathrm{e}^{\mathrm{i}\theta\langle f,z\rangle} - 1 - \mathrm{i}\theta 1_{|z|<1}\langle f, z\rangle \nu(dz)$$

- Lévy measure of $\langle U(t), f\rangle$ is image of ν under projection

$$g \mapsto \langle f, g\rangle$$

H-valued Wiener process

- A mean-zero H-valued Lévy process W with continuous paths is called a Wiener process
- W has Lévy-Kintchine triplet $(0, Q, 0)$, Q called the covariance operator
- $t \mapsto \langle W(t), f \rangle$ is an $\mathbb{R}$-valued Wiener process for every $f \in H$

$$\mathbb{E}[\langle W(t), f \rangle \langle W(s), g \rangle] = \min(s, t) \langle Qf, g \rangle$$

- $\langle W(t), f \rangle$ is a mean-zero Gaussian random variable on $\mathbb{R}$ with variance $t \langle Qf, f \rangle = t|Q^{1/2} f|^2$.

Covariance operator of U

- $U(t)$ is said to be square integrable if $|U(t)|$ is square integrable, which is equivalent to $\int_{H\setminus\{0\}} |z|^2 \nu(dz) < \infty$
- The covariance operator of a square integrable U is defined as

$$\langle \mathrm{Cov}(U)f, g\rangle = \mathbb{E}[\langle U(1) - \mathbb{E}[U(1)], f\rangle\langle U(1) - \mathbb{E}[U(1)], g\rangle]$$

- It holds

$$\mathrm{Cov}(U) = Q + \int_{H\setminus\{0\}} (z \otimes z)\, nu(dz)\,, (z \otimes z)(f) = \langle z, f\rangle z$$

Subordinator

- Let Θ be an $\mathbb{R}$-valued Lévy process with increasing paths
- Lévy-Kintchine triplet $(a, 0, F)$, $a \geq 0$ and F Lévy measure concentrated on $\mathbb{R}_+$

$$\psi(x) := \log \mathbb{E}[\exp(\mathrm{i}x\Theta(1))] = \mathrm{i}ax + \int_0^\infty (\mathrm{e}^{\mathrm{i}xz} - 1)\, F(dz)\,, x \in \mathbb{R}$$

- Paths of Θ is cadlag, bounded variation and supported on $\mathbb{R}_+$
- Assume Θ independent of U

Proposition

$L(t) := U(\Theta(t))$ is an H-valued Lévy process, where the characteristic functional is $\log\mathbb{E}[\exp(\mathrm{i}\langle x, L(1)\rangle)] = \psi(\varphi(x))$

Proof.

Independence of Θ and U and independent increments,

$$\mathbb{E}[\exp(\mathrm{i}\langle\sum_k(U(\Theta(t_{k+1}) - U(\Theta(t_k))), x_k\rangle))]$$
$$= \prod_k \mathbb{E}[\exp(i(\Theta(t_{k+1}) - \Theta(t_k))\varphi(x_k))]$$
$$= \prod_k \mathbb{E}[\exp((t_{k+1} - t_k)\psi(\varphi(x_k)))]$$

∎

Proposition

Characteristic triplet of $L(t) = U(\Theta(t))$ is (β, Γ, μ), where

$$\beta = ab + \int_{\mathbb{R}_+} \mathbb{E}[U(z)1_{|U(z)|<1}]\, F(dz)$$

$$\Gamma = aQ$$

$$\mu(A) = a\nu(A) + \int_{\mathbb{R}_+} P(U(z) \in A)\, F(dz)$$

where $A \subset H$ Borel.

Proof.

It holds that $|\mathbb{E}[U(z)1_{|U(z)|<1}]| \leq C\min(1, z)$. Direct calculation of $\psi(\varphi(x))$ shows the result. ∎

Example

- Let $U = W$, an H-valued Wiener process
- Recall Lévy-Kintchine triplet for W: $(b, Q, \nu) = (0, Q, 0)$
- Assume Θ driftless subordinator, $(a, 0, F) = (0, 0, F)$
- L has triplet $(\beta, 0, \mu)$ with

$$\beta = \int_{\mathbb{R}_+} \mathbb{E}[W(z)1_{|W(z)|<1}]\, F(dz)$$

$$\mu(A) = \int_{\mathbb{R}_+} P(W(z) \in A)\, F(dz)$$

- L is a pure-jump Lévy process

Covariance operator

- Covariance operator of the subordinated Lévy process
- Describes the "spatial covariance"
- Requires square integrability of L:
  - *Either* U is mean zero and square integrable, and Θ is integrable,
  - *or* U is square integrable and Θ is square integrable
- "If and only if" result

Proposition

Assume U has zero mean and is square integrable. If Θ is integrable, then L has mean zero and is square integrable, with

$$Cov(L) = \mathbb{E}[\Theta(1)]Cov(U)$$

Proof.

Double conditioning implies: zero mean $\mathbb{E}[L(t)] = 0$,

$$\mathbb{E}[|U(\Theta(t))|^2] = \mathbb{E}[\Theta(t)]\mathbb{E}[|U(1)|^2] < \infty$$

$$\begin{aligned} \langle \mathrm{Cov}(L)f, g\rangle &= \mathbb{E}\left[\langle L(\Theta(1)), f\rangle\langle L(\Theta(1)), g\rangle\right] \\ &= \mathbb{E}[\Theta(1)]\mathbb{E}[\langle U(1), f\rangle\langle U(1), g\rangle] \end{aligned}$$

Proposition

Assume U is square integrable. If Θ is square integrable, then L is square integrable and

$$\mathbb{E}[L(1)] = \mathbb{E}[\Theta(1)]\mathbb{E}[U(1)]$$
$$Cov(L) = \mathbb{E}[\Theta(1)]Cov(U) + Var(\Theta(1))\left\{\mathbb{E}[U(1)] \otimes \mathbb{E}[U(1)]\right\}$$

Proof.

Square integrability: Note that $U(\theta) - \mathbb{E}[U(\theta)]$ is zero mean Lévy process. From Lévy-Kintchine, $\mathbb{E}[U(\theta)] = \theta\mathbb{E}[U(1)]$. ∎

Proof.

...cont'd

Algebra yields,

$$\mathbb{E}[|U(\theta)|^2] = \mathbb{E}[|U(\theta) - \mathbb{E}[U(\theta)]|^2] + |\mathbb{E}[U(\theta)]|^2$$
$$= \theta\mathbb{E}[|U(1) - \mathbb{E}[U(1)]|^2] - \theta^2|\mathbb{E}[U(1)]|^2$$

Hence, double conditioning

$$\mathbb{E}[|L(1)|^2] = \mathbb{E}[\Theta(1)]\mathbb{E}[|U(1) - E[U(1)]|^2] + \mathbb{E}[\Theta^2(1)]|\mathbb{E}[U(1)]|^2 < \infty$$

Integrability follows, and by double conditioning,

$$\mathbb{E}[L(1)] = \mathbb{E}[\Theta(1)]\mathbb{E}[U(1)]$$

Proof.

...cont'd

Covariance operator:

$$\begin{aligned}\langle \mathrm{Cov}(L)f, g\rangle &= \mathbb{E}[\langle L(1) - \mathbb{E}[L(1)], f\rangle\langle L(1) - \mathbb{E}[L(1), g\rangle] \\ &= \mathbb{E}[\langle L(1), f\rangle\langle L(1), g\rangle] - \langle \mathbb{E}[L(1)], f\rangle\langle \mathbb{E}[L(1)], g\rangle\end{aligned}$$

First expectation: double conditioning, using (with $m = \mathbb{E}[U(1)]$),

$$\begin{aligned}\langle U(\theta), f\rangle\langle U(\theta), g\rangle = \langle U(\theta) - m\theta, f\rangle\langle U(\theta) - m\theta, g\rangle \\ + \theta\langle m, g\rangle\langle U\theta) - m\theta, f\rangle \\ + \theta\langle m, f\rangle\langle U(\theta) - m\theta, g\rangle \\ + \theta^2\langle m, f\rangle\langle m, g\rangle\end{aligned}$$

Result follows ∎

Overview

The normal inverse Gaussian (NIG) distribution

- Aim: define H-valued NIG Lévy process. First, NIG on $\mathbb{R}$....
- A normal mean-variance mixture model:
  - Let Z be inverse Gaussian distributed

$$f_{IG}(z) = \frac{\delta}{\sqrt{2\pi}} z^{-3/2} \exp\left(\delta\gamma - \frac{1}{2}\left(\delta^2 z^{-1} + \gamma^2 z\right)\right), z > 0$$

  - Conditional distribution of X is normal:

$$X|Z \sim \mathcal{N}(\mu + \beta Z, Z)$$

- X is NIG with parameters α, β, μ and δ, where

$$\alpha = \sqrt{\gamma^2 + \beta^2}$$

- Density function

$$f_{\mathrm{NIG}}(x) = k \exp\left(\beta(x-\mu)\right) \frac{K_1\left(\alpha\sqrt{\delta^2+(x-\mu)^2}\right)}{\sqrt{\delta^2+(x-\mu)^2}}$$

- $K_1(x)$ modified Bessel function of the third kind with index one. Normalizing constant k is known
- μ location, β asymmetry, δ scale ("volatility"), α steepness
  - smaller α yields steeper distribution, and thus fatter tails

$$\delta > 0\,,\, 0 \leq |\beta| < \alpha$$

- Moment generating function (MGF): for $-\alpha-\beta \leq \theta \leq \alpha-\beta$

$$M_{\mathrm{NIG}}(\theta) = \exp\left(\theta\mu + \delta\sqrt{\alpha^2-\beta^2} - \delta\sqrt{\alpha^2-(\beta+\theta)^2}\right)$$

H-valued NIG Lévy process

- Let Θ be a inverse Gaussian subordinator, having Lévy measure

$$F(dz) = \frac{s}{\sqrt{2\pi z^3}} e^{-c^2 z/2}\, dz\,, \qquad z > 0$$

- Drift is zero, s, c positive parameters.
- Define U to be a drifted Wiener process,

$$U(t) = bt + W(t)$$

- $b \in H$ and W H-valued Wiener process with covariance operator Q

- $L(t) = U(\Theta(t))$ has Lévy-Kintchine triplet $(\beta, 0, \mu)$,

$$\beta = \frac{sb}{c} - \int_{|z|>1} z\mu(dz)$$

$$\mu(A) = \int_0^\infty \Phi_z(A) F(dz), \qquad A \subset H$$

- Φ_z Gaussian measure on H with mean zb and covariance operator zQ
- Characteristic functional of L

$$\psi(\varphi(x)) = s\left(c - \sqrt{c^2 + \langle Qx, x\rangle - 2\mathrm{i}\langle x, b\rangle}\right)$$

Some properties of L

- Expectation and covariance operator

$$\mathbb{E}[L(1)] = \frac{s}{c} b\,, \qquad \mathrm{Cov}(L) = \frac{s}{c^3}(b \otimes b) + \frac{s}{c} Q$$

- $t \mapsto \langle L(t), f \rangle$ is $\mathbb{R}$-valued NIG Lévy process
  - Log-MGF (using $x = -\mathrm{i} f\theta$ in characteristic functional of L)

$$M_{\mathrm{NIG}}(\theta) = s\left(c - \sqrt{c^2 - \theta^2 \langle Qf, f \rangle - 2\theta \langle f, b \rangle}\right)$$

or

$$\mu = 0\,, \quad \delta = s\langle Qf, f \rangle^{1/2}\,, \quad \beta = \frac{\langle f, b \rangle}{\langle Qf, f \rangle}\,, \quad \alpha^2 = \frac{c^2}{\langle Qf, f \rangle} + \beta^2$$

- Multivariate NIG Lévy process:

$$t \mapsto (\langle L(t), f_1 \rangle, \ldots, \langle L(t), f_n \rangle)$$

- Log-MGF with $x = (-\mathrm{i})(f_1\theta_1 + \cdots f_n\theta_n)$

$$M_{\mathrm{NIG}}(\theta) = s\left(c - \sqrt{c^2 - \theta'\Sigma\theta - \beta'\theta}\right)$$

with

$$\beta' = (\langle f_1, b \rangle, \ldots, \langle f_n, b \rangle) \in \mathbb{R}^n, \quad \Sigma = \{\langle Qf_i, f_j \rangle\}_{i,j=1}^n \in \mathbb{R}^{n \times n}$$

Overview

- Suppose U is mean zero and square integrable
- Expansion of Lévy process U along basis $\{e_n\}_{n\in\mathbb{N}}$

$$U(t) = \sum_{n=1}^{\infty} \langle U(t), e_n \rangle e_n$$

- $U_n(t) = \langle U(t), e_n \rangle$ $\mathbb{R}$-valued Lévy process
  - Correlated Lévy processes
- Suppose $\{e_n\}_{n\in\mathbb{N}}$ is such that

$$\mathrm{Cov}(U)e_n = \lambda_n e_n \qquad \lambda_n \in \mathbb{R}_+$$

- $\{U_n(t)\}_{n\in\mathbb{N}}$ uncorrelated Lévy processes (but possibly dependent!)

- It holds

$$\lambda_n = \mathbb{E}[U_n^2(1)], \qquad \mathbb{E}[U_n(t)U_m(s)] = \lambda_n \delta_{nm} \min(t,s)$$

- If $U = W$, H-valued Wiener process, $\{W_n(t)\}_{n\in\mathbb{N}}$ independent $\mathbb{R}$-valued Wiener processes
- Define $B_n(t) := W_n(t)/\sqrt{\lambda_n}$. Then B_n is standard Brownian motion on $\mathbb{R}$ and

$$W(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} B_n(t) e_n$$

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