UiO : Department of Mathematics
University of Oslo

Lecture II

Analysis of the forward price dynamics – Infinite dimensional approach

Fred Espen Benth

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Overview

In collaboration with Paul Krühner (Vienna)

1. **Forward curve modelling – preliminaries**
2. **Representation of functionals of stochastic integrals in Hilbert space**
3. **Exponential models of forward prices**

Overview

1. **Forward curve modelling – preliminaries**

2. Representation of functionals of stochastic integrals in Hilbert space

3. Exponential models of forward prices

HJMM forward curve dynamics

- $t \mapsto F(t, T)$, $t \leq T$ forward price
  - Contract delivering at time T, time of maturity
  - Delivery of some commodity like power, gas, coffee, soybeans, gold....
  - Gas & power: delivery *period* rather than time....we'll come back to that
- "Musiela parametrization": $x = T - t$, time to maturity
- Define $f(t, x)$ random field on $\mathbb{R}_+ \times \mathbb{R}_+$,

$$f(t, x) := F(t, t + x)$$

- $t \mapsto f(t, x)$ stochastic process with values in a space of real-valued function on $\mathbb{R}_+$

- Stochastic partial differential equation (SPDE) for f:

$$df(t) = \partial_x f(t)\,dt + \beta(t)\,dt + \Psi(t)\,dL(t)\,, \qquad f(0) = f_0 \in H$$

- Here, $\partial_x = \partial/\partial x$, L is an H-valued Lévy process
  - H assumed to be a separable Hilbert space of functions on $\mathbb{R}_+$
  - Reason for ∂_x-term: time-dependency in second argument of F
- Must make sense out of the SPDE:

1. Stochastic integral?
2. Existence and uniqueness of solution?

Stochastic integration in Hilbert space

- Assumption: L is square integrable zero mean Lévy process in H, with covariance operator $\mathcal{Q} := \mathrm{Cov}(L)$
- Question: For which Ψ can we define

$$\int_0^t \Psi(s)\, dL(s) \qquad ?$$

- Simple process Ψ: Let $0 = t_0 < t_1 < \cdots < t_m \leq t$, $A_j \in \mathcal{F}_{t_j}$ and $\Psi_j \in L(H)$, bounded operators on H

$$\Psi(s) = \sum_{j=0}^{m-1} 1_{A_j}(\omega) 1_{(t_j, t_{j+1}]}(s) \Psi_j$$

- Stochastic integral of simple process:

$$\int_0^t \Psi(s)\,dL(s) = \sum_{j=0}^{m-1} 1_{A_j}\Psi_j(L(t_{j+1}) - L(t_j)) \in H$$

- Isometry:

$$\mathbb{E}[|\int_0^t \Psi(s)\,dL(s)|^2] = \mathbb{E}[\int_0^t \|\Psi(s)\mathcal{Q}^{1/2}\|_{\text{HS}}^2\,ds] < \infty$$

- Hilbert-Schmidt operators $\mathcal{T} \in L(H)$:

$$\|\mathcal{T}\|_{\text{HS}}^2 := \sum_{n=1}^{\infty} |\mathcal{T}e_n|^2\,, \qquad \{e_n\}_{n\in\mathbb{N}} \text{ ONB in H}$$

- Complete the space of simple integrands under seminorm given by isometry
  - Integral becomes a square integrable zero mean martingale
- Characterization of the space of integrands, $\mathcal{L}_L^2(H)$:

Definition

$\Psi \in \mathcal{L}_L^2(H)$ if $s \mapsto \Psi(s)$ is a predictable stochastic process with values in $L(H)$ such that

$$\mathbb{E}[\int_0^t \|\Psi(s)\mathcal{Q}^{1/2}\|_{\mathrm{HS}}^2 \, ds] < \infty$$

- $t \mapsto X(t) \in H$ $t \leq T$ is predictable if it is measurable with respect to the σ-algebra on $[0, T] \times \Omega$ containing all sets $(s, t] \times A$, $A \in \mathcal{F}_s$

- Going back to the HJMM dynamics

$$df(t) = \partial_x f(t)\,dt + \beta(t)\,dt + \Psi(t)\,dL(t)$$

- Suppose that $t \mapsto \beta(t)$ is a predictable H-valued process, integrable with respect to time a.s.
  - Integral $\int_0^t \beta(s)\,ds \in H$ defined in Bochner sense
- Problem: ∂_x is typically only densely defined on appropriate Hilbert spaces, e.g. unbounded operator
  - We loose smoothness by differentiating
  - E.g., C^n-functions become C^{n-1} after differentiation

C_0-semigroups and generators

Definition

We say that $\{\mathcal{S}(t)\}_{t\geq 0}$ is a C_0-semigroup on H if

1. $S(t) \in L(H)$ for every $t \geq 0$
2. $S(0) = \mathrm{Id}$
3. $\mathcal{S}(t)\mathcal{S}(s) = \mathcal{S}(t+s), t, s \geq 0$
4. $\lim_{t\downarrow 0} \mathcal{S}(t)f = f, f \in H$

- $\mathcal{A} : Dom(\mathcal{A}) \subset H \to H$ is called the *generator* of $\mathcal{S}$ if

$$\lim_{t\downarrow 0} t^{-1}(\mathcal{S}(t)f - f) = \mathcal{A}f$$

Filipovic space H_w

- Define H_w as the space of real-valued absolutely continuous functions on $\mathbb{R}_+$, with finite norm

$$|f|_w^2 := f^2(0) + \int_0^\infty w(x)(f'(x))^2\,dx$$

- f' is the weak derivative of f, w an increasing function with $w(0) = 1$ and

$$\int_0^\infty w^{-1}(x)\,dx < \infty$$

- Typically: $w(x) = \exp(\alpha x)$, $\alpha > 0$.

Theorem

H_w is a separable Hilbert space, where ∂_x is densely defined generator of the C_0-semigroup $\mathcal{S}(t)g = g(\cdot + t)$. Moreover, the the semigroup is quasi-contractive and uniformly bounded

$$\|\mathcal{S}(t)\|_{op} \leq e^{kt}\,, \qquad \|\mathcal{S}(t)\|_{op} \leq K$$

for positive constants k, K.

Proof.

See Filipovic (2001). ∎

Unique solution of the HJMM SPDE

- A mild solution $f \in H_w$ of the HJMM dynamics

$$f(t) = \mathcal{S}(t)f_0 + \int_0^t \mathcal{S}(t-s)\beta(s)\,ds + \int_0^t \mathcal{S}(t-s)\Psi(s)\,dL(s)$$

- Integrals are well-defined by bounds on operator norms of $\mathcal{S}(t)$
- $\mathcal{S}(t-s)\Psi(s) \in L(H_w)$, and

$$\|\mathcal{S}(t-s)\Psi(s)\mathcal{Q}^{1/2}\|_{\mathrm{HS}} \leq \|\mathcal{S}(t-s)\|_{\mathrm{op}}\|\Psi(s)\mathcal{Q}^{1/2}\|_{\mathrm{HS}}$$

- If $f \in Dom(\partial_x)$, then f is strong solution
  - In general f is only weakly differentiable, and $\partial_x f \in H_w$!
  - Mild solution is unique

Markovian HJMM-dynamics

- Consider

$$df(t) = \partial_x f(t)\,dt + b(t,f(t-))\,dt + \psi(t,f(t-))\,dL(t)$$

with global Lipschitz-continuity

$$|b(t,f)-b(t,g)|_w \leq C|f-g|_w\,, \quad \|\psi(t,f)-\psi(t,g)\|_{\text{op}} \leq C|f-g|_w$$

- Under linear growth of b and ψ (Filipovic et al (2010)): there exists a unique H_w-valued adapted cadlag mild solution,

$$f(t) = \mathcal{S}(t)f_0 + \int_0^t \mathcal{S}(t-s)b(s,f(s))\,ds + \int_0^t \mathcal{S}(t-s)\psi(s,f(s-))\,dL(s)$$

Overview

1 Forward curve modelling – preliminaries

2 **Representation of functionals of stochastic integrals in Hilbert space**

3 Exponential models of forward prices

- We are interested in $t \mapsto F(t,T)$, i.e.,

$$F(t,T) = f(t, T-t) = \delta_{T-t} f(t), f(t) \in H_w$$

- $\delta_x, x \geq 0$ evaluation map on H_w

$$\delta_x : H_w \to \mathbb{R}, \qquad \delta_x(g) = g(x)$$

- From mild solution:

$$\delta_{T-t} \int_0^t \mathcal{S}(t-s)\Psi(s)\, dL(s) = ???$$

Lemma

$\delta_x \in H_w^*$, and for any $g \in H_w$, $\delta_x(g) = \langle g, h_x \rangle_w$, where $h_x \in H_w$,

$$h_x(y) = 1 + \int_0^{x \wedge y} w^{-1}(z)\,dz$$

Moreover, the dual $\delta_x^* : \mathbb{R} \to H_w$, $c \mapsto c h_x$ and $\|\delta_x\|_{op}^2 = h_x(x)$.

Proof.

$$\begin{aligned} \langle g, h_x \rangle_w &= g(0)h_x(0) + \int_0^\infty w(z)g'(z)h_x'(z)\,dz \\ &= g(0) + \int_0^x w(z)g'(z)w^{-1}(z)\,dz = g(x) \end{aligned}$$

- Focus on $L = W$, H-valued Wiener process

Proposition

Assume $\mathcal{L} \in L(H, \mathbb{R}^n)$ *and* $\Phi \in \mathcal{L}^2_W(H)$. *Then there exists a standard n-dimensional Brownian motion B such that*

$$\mathcal{L} \int_0^t \Phi(s)\, dW(s) = \int_0^t \sigma(s)\, dB(s)$$

where $\sigma(s) = (\mathcal{L}\Phi(s)\mathcal{Q}\Phi^*(s)\mathcal{L}^*)^{1/2}$.

Proof.

By Peszat and Zabczyk (2007), $X(t) := \mathcal{L}\int_0^t \Psi(s)\,dW(s)$ is a continuous $\mathbb{R}^n$-valued process, with operator angle bracket (for $X(t) \otimes X(t) = X(t)X(t)'$)

$$\langle\langle X\rangle\rangle(t) = \int_0^t \mathcal{L}\Psi(s)\mathcal{Q}\Psi^*(s)\mathcal{L}^*\,ds$$

On given filtered probability space there exists an n-dimensional standard Brownian motion, result follows by Jacod (1979). ∎

- Analogous to Lévy's characterisation of Brownian motion

- Example related to forward price dynamics (with $\beta = 0, n = 1$):

$$F(t,T) = \delta_{T-t} f(t) = \delta_{T-t}\mathcal{S}(t) f_0 + \delta_{T-t}\int_0^t \mathcal{S}(t-s)\Psi(s)\,dW(s)$$

- It holds

$$\delta_{T-t}\mathcal{S}(t-s) = \delta_{T-s} = \delta_0\mathcal{S}(T-s)$$

- In representation: choose $\mathcal{L} = \delta_0$ and $\Phi(s) = \mathcal{S}(T-s)\Psi(s)$

$$F(t,T) = f_0(T) + \int_0^t ((\Psi(s)\mathcal{Q}\Psi^*(s)h_{T-s})(T-s))^{1/2}\,dB(s)$$

- Example of spot price dynamics (for $\beta = 0, n = 1$)

$$S(t) = \delta_0 f(t) = \delta_0 \mathcal{S}(t) f_0 + \delta_0 \int_0^t \mathcal{S}(t-s)\Psi(s)\,dW(s)$$

- Problem: cannot simply take $T = t$ in forward price, as B is (implicitly) T-dependent.
- Use a "trick" of Filipovic et al (2010): There exists an extension $\overline{\mathcal{S}}$ of $\mathcal{S}$ on a Hilbert space $\overline{H}_w$ such that
    1. $H_w \subset \overline{H}_w$,
    2. $\overline{\mathcal{S}}|_{H_w} = \mathcal{S}$,
    3. $\overline{\mathcal{S}}$ is a C_0-group
- Crucial property: $\mathcal{S}$ is quasi-contractive

- For a fixed $T > t$,

$$\delta_0 \mathcal{S}(t-s) = \delta_0 \overline{\mathcal{S}}(t-T)\mathcal{S}(T-s)$$

- Using representation and properties of $\overline{\mathcal{S}}$:

$$S(t) = f_0(t) + \int_0^t ((\Psi(s)\mathcal{Q}\Psi^*(s)h_{t-s})(t-s))^{1/2}\, dB(s)\,, \quad t \leq T$$

- Volterra process with volatility modulation
  - Barndorff-Nielsen et al. (2013): modelling power spot EEX
  - Time independent Ψ: Lévy semistationary spot dynamics

Representation and subordinated Wiener processes

- Suppose W is H-Wiener process and Θ an independent real-valued subordinator with finite moment,

$$L(t) := W(\Theta(t))$$

- L a mean-zero square-integrable Lévy process (Lecture I)
- Suppose W Wiener relative to (right-continuous) filtration $\{\mathcal{F}_t\}_{t\geq 0}$

$$\mathcal{G}_t := \cap_{s>t}\mathcal{F}_{\Theta(s)}$$

- $\{\mathcal{G}\}_{t\geq 0}$ time-changed filtration, L Lévy relative to $\mathcal{G}_t$

Proposition

Let $\Phi \in \mathcal{L}^2_L(H;\mathcal{G})$*. There exists an isometric embedding*

$$\Gamma_\Theta : \mathcal{L}^2_L(H;\mathcal{G}) \to \mathcal{L}^2_W(H;\mathcal{F})$$

such that $\Gamma_\Theta(\Phi) \in \mathcal{L}^2_W(H;\mathcal{F})$ *and*

$$\int_0^t \Phi(s)\,dL(s) = \int_0^{\Theta(t)} \Gamma_\Theta(\Phi)(s)\,dW(s)$$

- RHS is a time-changed dW-integral, dW-integral with respect to $\mathcal{F}$
- Proof goes by density argument, after showing the result on elementary integrands.......

Proof.

For Φ elementary, e.g, $0 = s_0 < s_1 < ..$, Y_j square-integrable $\mathcal{G}_{s_j}$-measurable, $\varphi_j \in L(H)$ and $\Phi = \sum_{j=0}^{n-1} Y_j 1_{(s_j, s_{j+1}]}\varphi_j$, let

$$\Gamma_\Theta(\Phi) = \sum_{j=0}^{n-1} Y_j 1_{(\Theta(s_j),\Theta(s_{j+1})]}\varphi_j \in \mathcal{L}^2_W(H;\mathcal{F})$$

By definition

$$\begin{aligned}\int_0^t \Phi(s)\,dL(s) &= \sum_{j=0}^{n-1} Y_j\varphi_j\left(W(\Theta(t)\wedge\Theta(s_{j+1})) - W(\Theta(t)\wedge\Theta(s_j))\right)\\ &= \int_0^{\Theta(t)} \Gamma_\Theta(\Phi)(s)\,dW(s)\end{aligned}$$

Proposition

Assume $\mathcal{L} \in L(H, \mathbb{R}^n)$ and $\Phi \in \mathcal{L}_L^2(H)$, with $L = W(\Theta(t))$. Then there exists a n-dimensional mean-zero square integrable Lévy process N such that

$$\mathcal{L} \int_0^t \Phi(s)\, dL(s) = \int_0^t \sigma(s)\, dN(s)$$

where $\sigma(s) = (\mathcal{L}\Phi(s)\mathcal{Q}\Phi^*(s)\mathcal{L}^*)^{1/2}$.

- N being a subordinated n-dimensional Brownian motion

$$N(t) = B(\Theta(t))$$

- L and W have same covariance operator $\mathcal{Q}$ (modulo scaling by the expected value of $\Theta(1)$)

Proof.

Let Φ be elementary, and note

$$\mathcal{L}\Gamma_{\Theta}(\Phi) = \sum_{j=0}^{n-1} Y_j 1_{(\Theta(s_j),\Theta(s_{j+1})]}(\mathcal{L} \circ \varphi_j) = \Gamma_{\Theta}(\mathcal{L}\Phi)$$

Using previous proposition and representation for H-valued Wiener processes,

$$\mathcal{L} \int_0^t \Phi(s)\, dL(s) = \int_0^{\Theta(t)} \mathcal{L}\Gamma_{\Theta}(\Phi)(s)\, dW(s) = \int_0^{\Theta(t)} \Gamma_{\Theta}\sigma(s)\, dB(s)$$

■

Proof.

We have

$$\left(\mathcal{L}\Gamma_{\Theta}(\Phi(s))\mathcal{Q}\Gamma_{\Theta}(\Phi(s))^{*}\mathcal{L}^{*}\right)^{1/2}(s) = \Gamma_{\Theta}\sigma(s)$$

which shows last equality for σ. Again previous proposition

$$\int_0^{\Theta(t)} \Gamma_{\Theta}\sigma(s)\, dB(s) = \int_0^t \sigma(s)\, dN(s)$$

∎

- For $n = 1$, Θ inverse Gaussian subordinator, N is a real-valued NIG Lévy process (lecture I)

Overview

- Exponential models natural in commodity markets
  - Ensure positivity of prices
  - Returns (logreturns) conveniently modelled
- Power markets may have negative spot prices
  - Negative forward prices????
- Define forward price as

$$f(t) = \exp(g(t)) \, , t \geq 0$$

- g solution of HJMM dynamics

Proposition

H_w is a Banach algebra after an appropriate re-scaling of the $|\cdot|_w$-norm

Proof.

Let $k^2 := \int_0^\infty w^{-1}(x)\,dx < \infty$. First show that $|g|_\infty \leq c|g|_w$, for $|g|_\infty = \sup_{x\geq 0}|g(x)|$: recall for

$$h_x(y) = 1 + \int_0^{x\wedge y} w^{-1}(z)\,dz$$

we have $g(x) = \delta_x g = \langle h_x, g\rangle_w$. We find

$$|g(x)|^2 \leq |h_x|_w^2 |g|_w^2 = h_x(x)|g|_w^2 \leq (1+k^2)|g|_w^2$$

Proof.

Proof cont'd: Using the product rule for derivatives, Cauchy-Schwartz and estimate for uniform norm,

$$|fg|_w^2 \leq (5+4k^2)|f|_w^2|g|_w^2$$

Hence, H_w is closed under multiplication. Define the norm $\|f\|_w := \sqrt{5+4k^2}|f|_w$. Then,

$$\|fg\|_w \leq \|f\|_w\|g\|_w$$

- $\exp g \in H_w$ for any $g \in H_w$,

$$|\exp g|_w \leq C^{-1}\exp(C|g|_w) < \infty\,, C = \sqrt{5+4k^2}$$

- For $t \mapsto g(t) \in H_w$, solution of HJMM-dynamics,

$$F(t,T) := \delta_{T-t} f(t) = \delta_{T-t} \exp(g(t)) = \exp(\delta_{T-t} g(t)) \,, t \leq T$$

- Recall HJMM dynamics (mild solution),

$$g(t) = \mathcal{S}(t) g_0(t) + \int_0^t \mathcal{S}(t-s)\beta(s)\,ds + \int_0^t \mathcal{S}(t-s)\Psi(s)\,dL(s)$$

- Representations above, for L being subordinated Wiener process

$$g(t, T-t) = g_0(T) + \int_0^t \beta(s, T-s)\,ds + \int_0^t ((\Psi(s)\mathcal{Q}\Psi^*(s)h_{T-s})(T-s))^{1/2}\,dN(s)$$

- β models return/risk premium. If futures price is modelled under risk neutrality, then

$$t \mapsto F(t,T)\,,t \leq T \text{ martingale}$$

- Must impose condition on β:

$$\beta(t,T-t) = -\mathcal{K}(\sigma(t,T-t))\,,$$

where,

$$\sigma(t,T-t) = ((\Psi(t)\mathcal{Q}\Psi^*(t)h_{T-t})(T-t))^{1/2}$$

and, for $\ell(dz)$ being the Lévy measure,

$$\mathcal{K}(y) = \int_{\mathbb{R}} e^{yz} - 1 - yz1_{|z|<1}\,\ell(dz)$$

- Use Itô's Formula for jump processes

A kind of example: the Schwartz model

- Assume spot price dynamics of a commodity
  - Seasonal level set to 1 for simplicity

$$S(t) = \exp(X(t)), dX(t) = \rho(\theta - X(t))\,dt + dL(t)$$

- $\rho > 0$ speed of mean reversion, θ log-price level, L real-valued (driftless) Lévy process
  - Assume $L(1)$ has finite exponential moment
  - Denote by φ its log-MGF
- $t \mapsto F(t, T)$ with $t \leq T$ forward price
  - ...assuming X is modelled directly under the pricing measure
  - Otherwise, do a measure change (Esscher, say)

$$F(t, T) := \mathbb{E}[S(T) \mid \mathcal{F}_t]$$

- Calculating,

$$F(t,T) = \exp(e^{-\rho(T-t)}X(t) + \Theta(T-t))$$

where

$$\Theta(x) = \theta(1 - e^{-\rho x})) + \int_0^x \varphi(e^{-\rho s})\,ds\,, x \geq 0$$

- In Musiela parametrization, $x = T - t$,

$$f(t,x) = \exp(e^{-\rho x}X(t) + \Theta(x))$$

Lemma

If $w(x)\exp(-2\rho x) \in L^1(\mathbb{R}_+, \mathbb{R})$, then $f(t) \in H_w$

References

- Barndorff-Nielsen, Benth and Veraart (2013). Modelling energy spot prices by volatility modulated Lévy-driven Volterra processes. Bernoulli, 19, 803–845
- Benth, Kallsen and Meyer-Brandis (2007). A non-Gaussian Ornstein-Uhlenbeck process for electricity spot price modelling and derivatives pricing. Appl. Math. Finance 14, 153-169.
- Benth and Krühner (2015). Subordination of Hilbert space valued Lévy processes. Stochastics, 87, 458–476.
- Benth, Saltyte Benth and Koekebakker (2008). *Stochastic Modelling of Electricity and Related Markets*, World Scientific
- Filipovic (2001). Consistency Problems for Heath-Jarrow-Morton Interest Rate Models, Lecture Notes in Mathematics, Vol. 1760, Springer, Berlin.
- Filipovic, Tappe and Teichmann (2010). Jump-diffusions in Hilbert spaces: existence, stability and numerics. Stochastics, 82, 475–520.
- Jacod (1979). Calcul Stochastique et Problèmes de Martingales. Lecture Notes in Mathematics, Vol 714, Springer, Berlin
- Peszat and Zabczyk (2007). Stochastic Partial Differential Equations with Lévy Noise, Cambridge University Press, Cambridge.

UiO : Department of Mathematics
University of Oslo

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