UiO : Department of Mathematics
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Lecture III

Derivatives pricing in energy markets – an infinite dimensional approach

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Overview

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1. **Representing energy forwards in Hilbert space**

2. **Analysis of options on energy forwards**

Overview

1. **Representing energy forwards in Hilbert space**

2. Analysis of options on energy forwards

Representing energy forwards in Hilbert space

- So far dealt with forward contracts delivering at a fixed time
  - Forward price $t \mapsto f(t,x)$, x time to delivery
- Energy markets: forwards deliver over a period
  - Power, gas, temperature
  - Delivery of gas and power over an agreed period, a month say
  - Measurement of temperature index over an agreed period (CDD, HDD, CAT)
- Interpreted $t \mapsto f(t)$ as Hilbert-valued stochastic process
- Question: can energy forward prices be viewed as Hilbert-valued stochastic processes?
  - ...or rather HOW?

- Power forwards/futures: delivery over period $[T_1, T_2]$
- Assume constant risk-free interest rate $r > 0$
- Forward-style: settlement at T_2

$$\mathcal{F}(t, T_1, T_2) = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} F(t, T)\, dT\,, t \leq T_1$$

- Futures-style: balancing (margin) account during settlement

$$\mathcal{F}(t, T_1, T_2) = \int_{T_1}^{T_2} \frac{e^{-rT}}{\int_{T_1}^{T_2} e^{-rs}\, ds} F(t, T)\, dT\,, t \leq T_1$$

- NordPool: both forward- and futures-style contracts traded
  - Forwards when long delivery period, futures when short

- Temperature futures on CDD, HDD and CAT indices

$$\mathcal{F}(t, T_1, T_2) = \int_{T_1}^{T_2} F(t, T)\, dT\,, t \leq T_1$$

- CAT=cumulative average temperature
  - Daily average: average of minimum and maximum
- CDD=cooling degree day

$$\mathrm{CDD}(t) = \max(\mathcal{T}(t) - 18^\circ, 0)$$

- HDD=heating degree day
  - HDD "call option" on temperature with strike 18°
  - CDD "put option"

- General expression for energy forward/futures prices

$$\mathcal{F}^{\widetilde{\omega}}(t, T_1, T_2) = \int_{T_1}^{T_2} \widetilde{\omega}(T, T_1, T_2) F(t, T)\, dT \,, t \leq T_1$$

- $T \mapsto \widetilde{\omega}(T, T_1, T_2)$ weight function

$$\widetilde{\omega}(T, T_1, T_2) = 1 \,, \text{CAT, CCC, HDD, gas}$$

$$\widetilde{\omega}(T, T_1, T_2) = \frac{1}{T_2 - T_1} \,, \text{power forward}$$

$$\widetilde{\omega}(T, T_1, T_2) = \frac{e^{-rT}}{\int_{T_1}^{T_2} e^{-rs}\, ds} \,, \text{power futures}$$

- Let $\ell = T_2 - T_1$: *length of delivery*, and $x = T_1 - t \geq 0$, *time to start of delivery*
- With $f(t, y) := F(t, t + y)$, $y \geq 0$

$$F_\ell^\omega(t, x) := \mathcal{F}^{\widetilde{\omega}}(t, t + x, t + x + \ell) = \int_x^{x+\ell} \omega_\ell(t, x, y) f(t, y)\, dy$$

- Weight function

$$\omega_\ell(t, x, y) = \widetilde{\omega}_\ell(t + y, t + x, t + x + \ell)$$

- Example: power futures

$$\omega_\ell(t, x, y) = \frac{1}{1 - e^{-r\ell}} e^{-r(y-x)}$$

- Suppose $\omega_\ell(x,y) := \omega_\ell(y-x)$, and assume $z \mapsto \omega_\ell(z)$ is positive, bounded and measurable.
- Musiela representation of energy forward

$$F_\ell^\omega(t,x) = \int_x^{x+\ell} \omega_\ell(y-x) f(t,y)\, dy$$

- F_ℓ^ω representable as a linear operator on H_w, which is want we analyse next:
- Simple integration-by-parts

$$F_\ell^\omega(t,x) = \mathcal{W}_\ell(\ell) f(t,x) + \int_0^\infty q_\ell^\omega(x,y) \partial_y f(t,y)\, dy$$

- Define

$$\mathcal{W}_\ell(u) = \int_0^u \omega_\ell(v)\,dv\,,u \geq 0$$

$$q_\ell^\omega(x,y) = (\mathcal{W}_\ell(\ell) - \mathcal{W}_\ell(y-x))\,1_{[0,\ell]}(y-x)$$

- Consider the integral operator $\mathcal{I}_\ell^\omega$

$$\mathcal{I}_\ell^\omega(g) = \int_0^\infty q_\ell^\omega(\cdot,y)g'(y)\,dy$$

Proposition

$\mathcal{I}_\ell^\omega$ is a bounded linear operator on H_w

Proof.

- $\mathcal{I}_\ell^\omega$ well-defined on H_w: By Cauchy-Schwartz,

$$|\int_0^\infty q_\ell^\omega(x,y)g'(y)\,dy|^2 \leq \int_0^\infty w^{-1}(y)(q_\ell^\omega(x,y))^2\,dy \int_0^\infty w(y)(g'(y))^2\,dy$$

First term finite since ω_ℓ is bounded. Second term finite since $g \in H_w$.

- $\mathcal{I}_\ell^\omega \in H_w$ for $g \in H_w$: Let $\xi(x) := \mathcal{I}_\ell^\omega(g)(x)$,

$$\xi(x) = \int_x^{x+\ell} (\mathcal{W}_\ell(\ell) - \mathcal{W}_\ell(y-x))g'(y)\,dy$$

Proof.

Proof cont'd....

Direct calculation shows that ξ has weak derivative

$$\xi'(x) = \int_x^{x+\ell} \omega_\ell(y-x)g'(y)\,dy - \mathcal{W}_\ell(\ell)g'(x)$$

By boundedness of ω_ℓ, it follows from Cauchy-Schwartz,

$$|\mathcal{I}_\ell^\omega(g)|_w \leq C|g|_w < \infty$$

for some constant $C > 0$. ∎

Wrapping up: energy forwards

- Given a model for $t \mapsto f(t) \in H_w$:
  - Fixed-delivery forward price curve
  - Recall models in Lecture II
- Realize dynamics for energy forwards in H_w

$$F_\ell^\omega(t) = \mathcal{W}_\ell(\ell) f(t) + \mathcal{I}_\ell^\omega(f(t))$$

- More compact notation

$$F_\ell^\omega(t) = \mathcal{D}_\ell^\omega(f(t)), \qquad \mathcal{D}_\ell^\omega = \mathcal{W}_\ell(\ell)\mathrm{Id} + \mathcal{I}_\ell^\omega \in L(H_w)$$

Overview

1 Representing energy forwards in Hilbert space

2 **Analysis of options on energy forwards**

Analysis of options on energy forwards

- European options on energy forwards:
  - Energy forward price $t \mapsto \mathcal{F}^{\widetilde{\omega}}(t, T_1, T_2)$, $t \leq T_1$
  - Exercise time $0 < \tau \leq T_1$
  - Payoff at exercise: $p : \mathbb{R} \to \mathbb{R}$ measurable function of at most linear growth

$$p(\mathcal{F}^{\widetilde{\omega}}(\tau, T_1, T_2))$$

- Recall representation of $\mathcal{F}^{\widetilde{\omega}}(t, T_1, T_2)$, in compact form

$$\mathcal{F}^{\widetilde{\omega}}(t, T_1, T_2) := F^{\omega}_{T_2 - T_1}(t, T_1 - t)$$

where, for $f(t) \in H_w$,

$$F^{\omega}_{\ell}(t) = \mathcal{D}^{\omega}_{\ell}(f(t))$$

Lemma

Define $\mathcal{P}_\ell^\omega : \mathbb{R}_+ \times H_w \to \mathbb{R}$ *as*

$$\mathcal{P}_\ell^\omega(x, g) = p \circ \delta_x \circ \mathcal{D}_\ell^\omega(g)$$

Then

$$\sup_{x \geq 0} |\mathcal{P}_\ell^\omega(x, g)| \leq c(1 + |g|_w)$$

for a constant $c > 0$ *Moreover,*

$$p(\mathcal{F}^{\widetilde{\omega}}(\tau, T_1, T_2)) = \mathcal{P}^\omega_{T_2 - T_1}(T_1 - \tau, f(\tau))$$

- Note: $\mathcal{P}_\ell^\omega(x, \cdot)$ is a *nonlinear* functional on H_w.

Proof.

By linear growth of p:

$$|\mathcal{P}_\ell^\omega(x,g)| \leq c(1+|\mathcal{D}_\ell^\omega(g)(x)|)$$

Recall from Lecture II, proof of H_w being Banach algebra, the sup norm is bounded by H_w-norm. Since $\mathcal{D}_\ell^\omega \in L(H_w)$, the result follows. ∎

- Assume $\mathbb{E}[|f(t)|_w] < \infty$ for all $t \geq 0$
- Arbitrage-free option price dynamics for $t \leq \tau$

$$
\begin{aligned}
V(t) &= e^{-r(\tau-t)}\mathbb{E}[p(\mathcal{F}^{\widetilde{\omega}}(\tau, T_1, T_2)) \,|\, \mathcal{F}_t] \\
&= e^{-r(\tau-t)}\mathbb{E}[\mathcal{P}^{\omega}_{T_2-T_1}(T_1-\tau, f(\tau)) \,|\, \mathcal{F}_t]
\end{aligned}
$$

- The linear growth of the payoff p ensures that V is finite
- Assume Markovian HJMM dynamics with Lipschitz parameters

$$
df(t) = \partial_x f(t)\,dt + \psi(t, f(t-))\,dL(t)
$$

- Recall Lecture II for all assumptions...!

- Mild solution for $t \leq s$

$$f(s) = \mathcal{S}(s-t)f(t) + \int_t^s \mathcal{S}(s-u)\psi(u, f(u-))\,dL(u)$$

- Option price $V(t) := V(t, f(t))$, with

$$V(t,g) = e^{-r(\tau-t)}\mathbb{E}[\mathcal{P}^{\omega}_{T_2-T_1}(T_1-\tau, f(\tau))\,|\,f(t)=g]$$

Stability of option prices wrt current forward curve

Proposition

Suppose that the payoff function p is Lipschitz continuous. Then, for any $g, \widetilde{g} \in H_w$,

$$\sup_{0 \leq t \leq \tau} |V(t, g) - V(t, \widetilde{g})| \leq C|g - \widetilde{g}|_w$$

for a positive constant C depending on τ.

- Option price is not sensitive to small errors in the current forward curve
- Note: we have only discrete forward price observations available, and must construct/recover the curve from these

Proof.

By Lipschitz continuity of p and linearity of $\delta_x, \mathcal{D}_\ell^\omega$,

$$|\mathcal{P}_\ell^\omega(x,g) - \mathcal{P}_\ell^\omega(x,\widetilde{g})| \leq c\|\delta_x\|_{\mathrm{op}}|g-\widetilde{g}|_w$$

From lecture II, $\|\delta_x\|_{\mathrm{op}}^2 = h_x(x) \leq c$,

$$|\mathcal{P}_\ell^\omega(x,g) - \mathcal{P}_\ell^\omega(x,\widetilde{g})| \leq c|g-\widetilde{g}|_w$$

for some positive (generic) constant $c > 0$ independent of x. Thus,

$$|V(t,g) - V(t,\widetilde{g})| \leq c\mathbb{E}[|f^{t,g}(\tau) - f^{t,\widetilde{g}}(\tau)|_w]$$

where $f^{t,g}(t) = g$. ■

Proof.

On H_w, the operator norm of $\mathcal{S}(t)$ is uniformly bounded in t:

$$|f^{t,g}(\tau)-f^{t,\widetilde{g}}(\tau)|_w^2 \leq c|g-\widetilde{g}|_w^2 + 2|\int_t^\tau \mathcal{S}(\tau-s)(\psi(s,f^{t,g}(s-))-\psi(s,f^{t,\widetilde{g}}(s-)))\,dL(s)|_w^2$$

Using Itô's isometry and Lipschitz of ψ

$$\mathbb{E}\left[|\int_t^\tau \mathcal{S}(\tau-s)(\psi(s,f^{t,g}(s-))-\psi(s,f^{t,\widetilde{g}}(s-)))\,dL(s)|_w^2\right]$$
$$\leq \int_t^\tau \mathbb{E}\left[\|\mathcal{S}(\tau-s)(\psi(s,f^{t,g}(s-))-\psi(s,f^{t,\widetilde{g}}(s-)))\mathcal{Q}^{1/2}\|_{L_{\mathrm{HS}}(H_w)}^2\right]ds$$
$$\leq c\int_t^\tau \mathbb{E}\left[|f^{t,g}(s)-f^{t,\widetilde{g}}(s)|_w^2\right]ds$$

Proof.

Hence,

$$\mathbb{E}[|f^{t,g}(\tau) - f^{t,\widetilde{g}}(\tau)|_w^2] \leq c|g - \widetilde{g}|_w^2 + c\int_t^\tau \mathbb{E}[|f^{t,g}(s) - f^{t,\widetilde{g}}(s)|_w^2]\, ds$$

We conclude by Gronwall's inequality,

$$\mathbb{E}[|f^{t,g}(\tau) - f^{t,\widetilde{g}}(\tau)|_w^2] \leq ce^{c(\tau - t)}|g - \widetilde{g}|_w^2$$

Pricing of options in Gaussian case

- Focus on simple Gaussian dynamics; $L = W$

$$f(\tau) = \mathcal{S}(\tau - t)f(t) + \int_t^\tau \mathcal{S}(\tau - s)\Psi(s)\,dW(s)$$

- Recalling representation analysis in Lecture II

$$\mathcal{F}^{\widetilde{\omega}}(\tau, T_1, T_2) = \delta_{T_1-t}\mathcal{D}^{\omega}_{T_2-T_1}f(t) + \int_t^\tau \sigma_{T_1,T_2}(s)\,dB(s)\,, t \leq \tau \leq T_1$$

with B being a real-valued Brownian motion and

$$\sigma^2_{T_1,T_2}(s) = (\delta_{T_1-s}\mathcal{D}^{\omega}_{T_2-T_1}\Psi(s)\mathcal{Q}\Psi^*(s)\mathcal{D}^{\omega,*}_{T_2-T_1}\delta^*_{T_1-s})(1)$$

Proposition

Suppose $\Psi : \mathbb{R}_+ \to L(H_w)$ *is deterministic. Then*

$$V(t,g) = e^{-r(\tau-t)}\mathbb{E}[p(m(g)+\xi X)]$$

where X *is a standard normal distributed random variable,*

$$\xi^2 := \int_t^\tau \sigma^2_{T_1,T_2}(s)\,ds\,, \qquad m(g) = \delta_{T_1-t}\mathcal{D}^{\omega}_{T_2-T_1}(g)$$

Proof.

Immediate, since Itô integral of the determinstic function $\sigma_{T_1,T_2}(s)$ is centered normally distributed. ■

Study of the volatility $\sigma_{T_1,T_2}(s)$

- From Lecture II:

$$\delta^*_{T_1-s}(1) = h_{T_1-s}(\cdot) = 1 + \int_0^{(T_1-s)\wedge\cdot} w^{-1}(z)\,dz$$

- Therefore, for $x \geq 0$ and $\ell = T_2 - T_1$,

$$\begin{aligned}\delta_x \mathcal{D}_\ell^{\omega,*}\delta^*_{T_1-s}(1) &= \mathcal{D}_\ell^{\omega,*}(h_{T_1-s})(x) = \langle \mathcal{D}_\ell^{\omega,*}(h_{T_1-s}), h_x\rangle \\ &= \langle h_{T_1-s}, \mathcal{D}_\ell^{\omega}(h_x)\rangle = \mathcal{D}_\ell^{\omega}(h_x)(T_1-s) \\ &= \mathcal{W}_\ell(\ell)h_{T_1-s}(x) + \int_0^x w^{-1}(z)q_\ell^\omega(T_1-s,z)\,dz\end{aligned}$$

- Hence,

$$\mathcal{D}_\ell^{\omega,*}\delta_{T_1-s}^*(1)=\mathcal{W}_\ell(\ell)h_{T_1-s}(\cdot)+\int_0^\cdot w^{-1}(z)q_\ell^\omega(T_1-s,z)\,dz\in H_w$$

- $\Sigma(s):=\Psi(s)\mathcal{Q}\Psi^*(s)\in L(H_w)$ is the modeller's choice
  - $\mathcal{Q}$ variance-covariance structure in "spatial" coordinate x
  - Ψ space-time volatility scaling
- Useful characterization: if $\mathcal{L}\in L(H_w)$,

$$\begin{aligned}\delta_x\mathcal{L}^*g &= \langle\mathcal{L}^*g,h_x\rangle=\langle g,\mathcal{L}h_x\rangle\\ &= g(0)\mathcal{L}(h_x)(0)+\int_0^\infty(\mathcal{L}h_x)'(y)w(y)g'(y)\,dy\end{aligned}$$

- Thus: $\mathcal{L}^*$ is essentially an integral operator on H_w ... and the same for $\mathcal{L}=(\mathcal{L}^*)^*$

The delta

- ....or, the sensitivity to the current forward curve
- Perturbing the current forward curve in a direction $h \in H_w$.
- Gateaux derivative, $D_h V(t, g)$, g current forward curve

$$D_h V(t, g) := \frac{d}{d\epsilon} V(t, g + \epsilon h)|_{\epsilon=0}$$

Proposition

Suppose $\Psi : \mathbb{R}_+ \to L(H_w)$ is deterministic. For any $h \in H_w$ it holds

$$D_h V(t, g) = \frac{1}{\xi} m(h) \mathbb{E}[p(m(g) + \xi X) X]$$

with m and ξ as defined earlier.

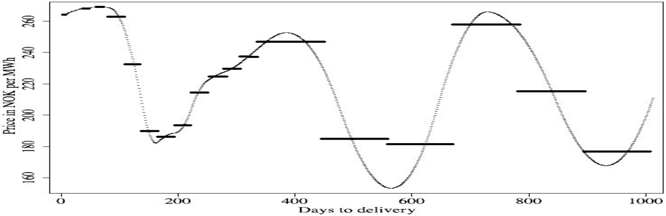
Proof.

Let φ denote the standard normal density function. Change of variables, Fubini and chain rule yield,

$$
\begin{aligned}
D_h V(t,g) &= D_h \int_{\mathbb{R}} p(m(g)+\xi x)\varphi(x)\,dx \\
&= \frac{1}{\xi} D_h \int_{\mathbb{R}} p(y)\varphi((y-m(g))/\xi)\,dy \\
&= \frac{1}{\xi} \int_{\mathbb{R}} p(y)\varphi'((y-m(g))/\xi)(-1/\xi)D_h m(g)\,dy
\end{aligned}
$$

$$
D_h m(g) = \frac{d}{d\epsilon}(m(g)+\epsilon m(h))|_{\epsilon=0} = m(h)
$$

∎



- Extract a smooth curve *g* from energy forward prices
- Functionals of the smooth curve, over discrete delivery periods
- No unique way to smoothen the forward curve
  - Delta provides a sensitivity measure

- At EEX and NordPool: European call and put options on monthly forward contracts
- Payoff of a call: $p(x) = \max(x - K, 0)$

Proposition

The price of a call option with strike K and exercise time $\tau \leq T_1$ is

$$V(t, g(t)) = \xi\varphi((m(g(t)) - K)/\xi) + (m(g(t)) - K)\Phi((m(g(t)) - K)/\xi)$$

with Φ being the cumulative normal distribution function. Moreover,

$$D_h V(t, g(t)) = m(h)\Phi((m(g(t)) - K)/\xi)$$

for any $h \in H_w$.

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