Rectangular and Coherent Sets of Indistinguishable Models

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Outline

- Application I: Robust Investment
- Literature
- Model
  - Deterministic
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- Application II: Bang-Bang control
- Conclusion
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Objective

- Robustness
- Set of alternative models
- Optimisation
- Research on quantification of uncertainty
  - Characteristics of indistinguishable set
  - Link with Type I and II error, test horizon
  - All deviations $|\lambda(t, \omega)| \leq \frac{2.48}{\sqrt{T}}$

Robust Investment

- Merton (1969): Agent generates utility from terminal wealth

$$\max_{\pi} \min_{\mathbb{L} \in \mathcal{L}} \mathbb{E}^{\mathbb{L}}[u(X(T))]$$

- Allocate π to risky asset S with

$$dS_t = \mu S_t dt + \sigma S_t (dW_t + \lambda dt)$$

- Rest on bank account B with riskfree rate r
- Equity premium puzzle
- Robust optimal investment strategy subject to $\lambda^2 \leq k^2$

$$\pi^* = \max\left(\frac{\mu - r - \sigma k}{\gamma \sigma^2}, 0\right)$$

Strategies

Table: Robust Optimal Investments

$\gamma \setminus \mathcal{T}$	1	100	200	π^*_{M}
0.1	0%	12.5%	466.5%	1562.5%
0.2	0%	6.25%	233.2%	781.25%
0.5	0%	2.5%	93.3%	312.5%
1	**0%**	**1.25%**	**46.65%**	**156.25%**
3	0%	0.42%	15.55%	52.1%
5	0%	0.25%	9.33%	31.25%

The optimal robust investment strategy by the addition of a constraint is shown, π^*_{C} for several values of $\mathcal{T} = \{1, 100, 200\}$ with $k = \frac{2.48}{\sqrt{\mathcal{T}}}$. The last column shows the classical Merton solution without model uncertainty.

Literature Uncertainty

- Risk versus uncertainty
- The Ellsberg paradox (Ellsberg, 1961)
- Bayesian prior, posterior (Thomas Bayes, 1701 1761)
- Multiple prior model from Gilboa and Schmeidler (1989)
- Extension of the standard multiple prior approach Garlappi et al. (2007)

Literature Uncertainty

- Robustness: Hansen, Sargent and Tallarini (1999); Hansen, Sargent and Turmuhambetova (2006); Hansen, Sargent and Wang (2002); Hansen and Sargent (2008); Hansen and Sargent (2015)
- ϕ-Divergence: Ben-Tal, Den Hertog, De Waegenaere, Melenberg and Rennen (2013)
- Model Confidence Set: Hansen, Lunde and Nason (2011)
- Confidence Interval for Parameters

Literature Uncertainty

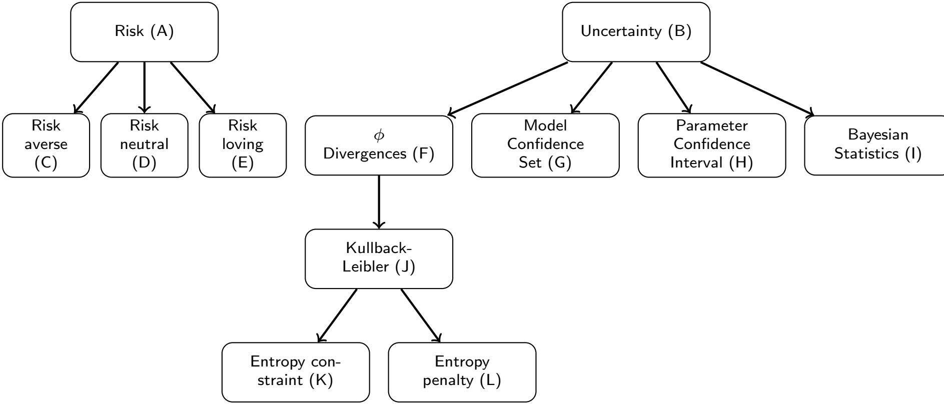
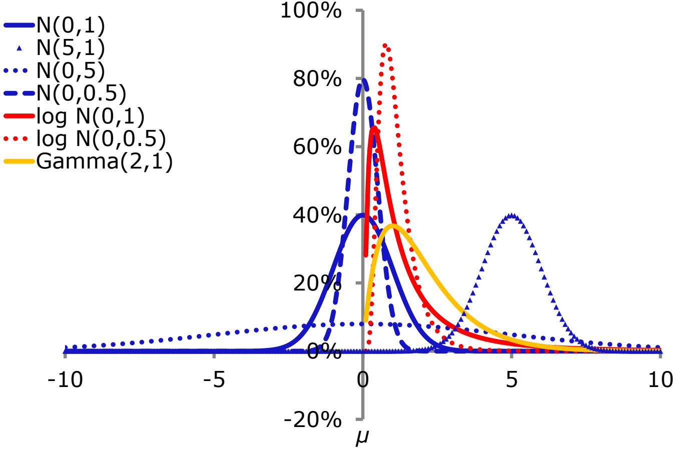


Figure: Overview of Uncertainty Sets

Literature Uncertainty



Motivation

- Goal: we would like to identify and characterise the set of alternative models *ex ante*
- Independent from objective problem
- Consider a large class of alternatives
- Statistically indistinguishable models:
    - Based on Type I and II error and test horizon
- Outline
    - Model
    - Stochastic example
    - Deterministic

Model

- SDEs of form

$$dX = \mu(t,\omega)\,dt + \sigma(t,\omega)\,dW(t)$$

- Possible alternative models $dW_t + \lambda(t,\omega)\,dt$
- Ex ante $(t = 0)$
- Those $\lambda(t,\omega)$ indistinguishable from $\lambda = 0$
- $dW_t + \lambda(t,\omega)dt$ not only adjusting the mean of the probability distribution
- Stochastic example

Stochastic Example

- Mean repelling process: $\lambda(t,\omega) = a \tanh(aW(t))$
- Alternative fatter tails
- Let

$$: (\mathcal{T}) \quad = \quad \frac{1}{2}\left(e^{-\frac{1}{2}a^2\mathcal{T}+aW(\mathcal{T})} + e^{-\frac{1}{2}a^2\mathcal{T}-aW(\mathcal{T})}\right)$$

- Under $\mathbb{L}$[1] a mixture of $N(a\mathcal{T},\mathcal{T})$ and $N(-a\mathcal{T},\mathcal{T})$, together not normal, mean 0 and variance $\mathcal{T}+(a\mathcal{T})^2$
- Under $\mathbb{P}$ always $W(\mathcal{T}) \sim N(0,\mathcal{T})$

[1] $\mathbb{P}$ and $\mathbb{L}$ stand for baseline and alternative.

Model (2)

- Test $H_0 : \mathbb{P}$ versus $H_A : \mathbb{L}$
- Hence, $L(\mathcal{T})$ equals the likelihood ratio test statistic
- Radon-Nikodym derivative (Girsanov)

$$L(\mathcal{T}) = \exp\left\{-\frac{1}{2}\int_0^{\mathcal{T}} \lambda(t,\omega)^2 dt + \int_0^{\mathcal{T}} \lambda(t,\omega) dW^{\mathbb{P}}(t,\omega)\right\}$$

- Value $L(\mathcal{T},\omega)$ determined by realisation ω
- Test if model $\mathbb{P}$ should be rejected in favour of model $\mathbb{L}$
- Two simple hypotheses, Neyman-Pearson Lemma most powerful test is likelihood ratio test

Model (3)

- Type I error: incorrectly rejecting model $\mathbb{P}$

$$\mathbb{P}[L(\mathcal{T}) \geq \gamma] = \alpha$$

- Type II error: incorrectly rejecting model $\mathbb{L}$

$$\mathbb{L}[L(\mathcal{T}) < \gamma] = \beta$$

- Power: probability of accepting model $\mathbb{L}$ when model $\mathbb{L}$ is the true model

$$\begin{aligned} \mathbb{L}[L(\mathcal{T}) \geq \gamma] &= 1 - \beta \\ &= \mathbb{E}^{\mathbb{P}}\left[L(\mathcal{T})\mathbb{1}(L(\mathcal{T}) \geq \gamma)\right] \end{aligned}$$

Deterministic

- Radon-Nikodym, without ω, log normal
- E.g. $\alpha = 0.05$, then $\Phi^{-1}(\alpha) = -1.64$
- If we take $\beta = 0.20$ then power is 0.80 and we have $\Phi^{-1}(0.80) = 0.84$
- Hence, the class of all indistinguishable models is then given by all models that satisfy

$$\left(\int_0^{\mathcal{T}} \lambda(t)^2 \, dt \right)^{\frac{1}{2}} \leq 0.84 - (-1.64) = 2.48$$

$\mathcal{T}$

- Future moment in time at which test would *hypothetically* be performed
- Extra amount of data that one would take into consideration
- Time period during which model would remain the same

Rectangularity

- Rectangularity $\Leftrightarrow$ time-consistency $\Leftrightarrow$ m-stability $\Leftrightarrow$ BSDEs
- Power $\Leftrightarrow$ CVaR $\Leftrightarrow$ Coherent risk measure
- For time-consistent coherent risk measures (Barrieu and El Karoui (2007))
  - $|\lambda(t, \omega)| \leq k$
- Optimal solution at any time-point t does not depend on history between $[0, t]$
- Optimal policy devised at time 0 for $t > 0$ is still valid at time t given information $\mathcal{F}_t$
- Intersect classes

Optimisation

- Recall

$$L(\mathcal{T}) = \exp\left\{-\frac{1}{2}\int_0^{\mathcal{T}} \lambda(t,\omega)^2 dt + \int_0^{\mathcal{T}} \lambda(t,\omega) dW^{\mathbb{P}}(t,\omega)\right\}$$

- Optimisation problem

$$\max_{\gamma, |\lambda(t,\omega)|\leq k} \mathbb{E}\left[L(\mathcal{T})\mathbb{1}(L(\mathcal{T}) \geq \gamma)\right] \quad \text{(MP)}$$
$$\text{s.t. } \mathbb{E}\left[\mathbb{1}(L(\mathcal{T}) \geq \gamma)\right] = \alpha$$
$$dL = \lambda(t,\omega) L dW, \; L_0 = 1$$

- Maximum for $|\lambda(t,\omega)| \equiv k \Rightarrow$ log Normal

Stochastic

- Given that the optimal $L^*(\mathcal{T})$ is a lognormal martingale with volatility k
- The optimised power at time $t = 0$ and $L(0) = 1$ is therefore equal to

$$\mathbb{E}\left[L^*(\mathcal{T})\mathbb{1}\left(L^*(\mathcal{T}) > \gamma^*\right)\right] = \mathbb{L}\left[L^*(\mathcal{T}) > \gamma^*\right] = \Phi\left(\Phi^{-1}(\alpha) + k\sqrt{\mathcal{T}}\right)$$

- Set of indistinguishable models $|\lambda(t,\omega)^*| \leq k = 2.48/\sqrt{\mathcal{T}}$

Theorem (Rectangular and Coherent Sets of Indistinguishable Models)

Consider a baseline model $dX(t) = \mu(t,\omega)dt + \sigma(t,\omega)dW(t)$. *The set of all models with* $dW(t) + \lambda(t,\omega)dt$ *and* $|\lambda(t,\omega)| \leq k$ *is rectangular and coherent, where*

$$k = \frac{\Phi^{-1}(1-\beta) - \Phi^{-1}(\alpha)}{\sqrt{\mathcal{T}}} \tag{1}$$

forms an indistinguishable set for a Type I error of α, *a Type II error of* β *and a test horizon* $\mathcal{T}$.

Bounds on Divergences

- ϕ-Divergences/f-Divergences (non-symmetric distance measures)
- Robust results in optimisation problems
- Continuous ϕ-divergence

$$D_\phi(L(\mathcal{T},\omega)) \;=\; \mathbb{E}^{\mathbb{L}}\left[\phi\left(\frac{1}{L(\mathcal{T})}\right)\right] = \mathbb{E}^{\mathbb{P}}\left[L(\mathcal{T})\phi\left(\frac{1}{L(\mathcal{T})}\right)\right]$$

- For each measure $\phi(\cdot)$ given and convex
- Size of the uncertainty quantified by c

$$D_\phi(L(\mathcal{T},\omega)) \;\leq\; c$$

- The divergences are expressed in terms of the ratio $x = \frac{1}{L(\mathcal{T},\omega)}$ under the measure $\mathbb{L}$

Bounds on Divergences

Divergence	$\phi(x)$	$c \Leftrightarrow \lvert\lambda(t,\omega)\rvert \equiv k$	$k\sqrt{\mathcal{T}} = 2.48$
Kullback-Leibler	$x \ln x - x + 1$	$\frac{1}{2}k^2\mathcal{T}$	3.08
Burg entropy	$-\ln x + x - 1$	$\frac{1}{2}k^2\mathcal{T}$	3.08
J-divergence	$(x-1)\ln x$	$k^2\mathcal{T}$	6.15
χ^2-divergence	$\frac{1}{x}(x-1)^2$	$e^{k^2\mathcal{T}} - 1$	467.90
Modified χ^2-divergence	$(x-1)^2$	$e^{k^2\mathcal{T}} - 1$	467.90
Hellinger distance	$(\sqrt{x}-1)^2$	$2 - 2e^{-\frac{1}{8}k^2\mathcal{T}}$	1.07
Variation distance	$\lvert x-1\rvert$	$4N(\frac{1}{2}k\sqrt{\mathcal{T}}) - 2$	1.57
χ-divergence of order $\vartheta > 1$	$\lvert x-1\rvert^{\vartheta}$	*numerical*	*Table 2*
Cressie-Read $\vartheta \neq 0,1$	$\frac{1-\vartheta+\vartheta x - x^{\vartheta}}{\vartheta(1-\vartheta)}$	*expr*	*Table 2*

Bounds on Divergences

Divergence $\setminus \vartheta$	1.5	2.0	2.5	3.0
χ-divergence of order ϑ	10.40	467.90	1.02×10^6	1.03×10^8
Cressie-Read	12.05	233.95	2.72×10^4	1.72×10^7

Bang-Bang

- Baseline model: Brownian motion
- Stochastic deviation $\lambda(t,\omega) = a \cdot \operatorname{sgn}\left(W(t,\omega)\right)$
- $a > 0$
  - Mean repelling process
  - Increase the variance of $X(T)$ under model $\mathbb{L}$
- $a < 0$
  - mean reverting process
  - Decrease the variance of $X(T)$ under model $\mathbb{L}$

Bang-Bang

- Under $\mathbb{P}$
  - $dX(t) = dW(t)$
  - Model X normal distributed
- Under $\mathbb{L}$
  - $dX(t) = a \cdot \operatorname{sgn}(X(t))dt + dW(t)$
  - Model X unknown distribution and variance changed
- Distribution
  - $\ln L(\mathcal{T}) \sim^{\mathbb{P}} N\left(-\frac{1}{2}a^2\mathcal{T}, a^2\mathcal{T}\right)$
  - $\ln L(\mathcal{T}) \sim^{\mathbb{L}} N\left(\frac{1}{2}a^2\mathcal{T}, a^2\mathcal{T}\right)$
- Explicit bound $|a| \leq \frac{\Phi^{-1}(\beta) - \Phi^{-1}(1-\alpha)}{\sqrt{\mathcal{T}}}$
- Does satisfy rectangular and coherence axioms

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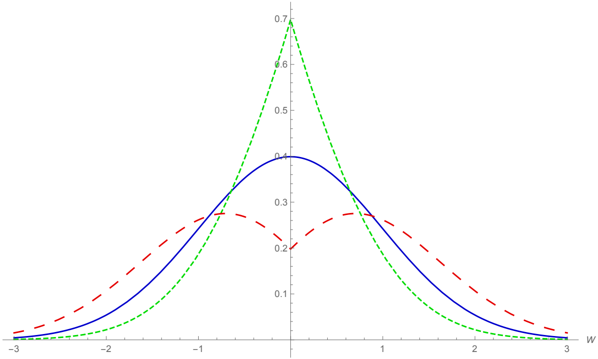

Conclusion

- Quantify uncertainty
- Most powerful test
- Ex ante
- For given size and power
- Stochastic deviation from the drift

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