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Rough volatility

Lecture 4: Affine models, the rough Heston model in particular

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Outline of lecture 4

- Forward variance models
- Affine forward variance models
- The characteristic function of an affine forward variance model
  - The rough Heston characteristic function
- Rational approximation of the rough Heston solution
- Numerical experiments

Forward variance models

Following [Bergomi and Guyon][2], forward variance models may be written in the form

$$dS_t = S_t \sqrt{v_t} \left(\rho dW_t + \sqrt{1-\rho^2}\, dW_t^\perp\right)$$
$$d\xi_t(u) = \eta_t(u;\omega)\, dW_t,$$

where $W, W^\perp$ are independent Brownian motions, the $\mathbb{R}_{\geqslant 0}$-valued stochastic process $\eta_t(u;\omega)$ is progressively measurable for all $u > 0$ and ξ is linked to the instantaneous variance v by

$$\xi_t(T) = \mathbb{E}\left[v_T \middle| \mathcal{F}_t\right].$$

- Models of this form were also studied by Hans Bühler as *variance curve models*.

- If v is continuous and uniformly integrable, we can recover v_t from $\xi_t(u)$ as $v_t = \lim_{u \downarrow t} \xi_t(u)$. For our purposes, $v_t = \xi_t(t)$.
- The initial conditions of a forward variance model are the initial stock price S_t and the initial forward variance curve $\xi_t(u)_{u \geq t}$.

Important remark

As noted by [Bergomi and Guyon][3], all conventional finite-dimensional Markovian stochastic volatility models may be cast as forward variance models.

Example: The classical Heston model

The classical Heston stochastic volatility model may be written as

$$\begin{aligned} \frac{dS_t}{S_t} &= \sqrt{v_t}\, dZ_t \\ dv_t &= -\lambda\,(v_t - \bar{v})\, dt + \eta\, \sqrt{v_t}\, dW_t \end{aligned}$$

with $\mathbb{E}\,[dZ_t\, dW_t] = \rho\, dt$ and where λ is the speed of reversion of v_t to its long term mean $\bar{v}$.

Forward variance in the Heston model

With $\xi_t(u) = \mathbb{E}\,[v_u \,|\, \mathcal{F}_t]$, take expectations of the SDE for v_t to get

$$d\xi_t(u) = -\lambda\,(\xi_t(u) - \bar{v})\, du.$$

This ODE has the solution

$$\xi_t(u) = (\xi_t(t) - \bar{v})\, e^{-\lambda\,(u-t)} + \bar{v} = (v_t - \bar{v})\, e^{-\lambda\,(u-t)} + \bar{v}.$$

The Heston model in forward variance form

For each u, $\xi_t(u)$ is a conditional expectation and so a martingale in t. It is then immediate from the last equation that

$$d\xi_t(u) = e^{-\lambda\,(u-t)}\, \widehat{dv_t} = \eta\, e^{-\lambda\,(u-t)}\, \sqrt{v_t}\, dW_t$$

where $\widehat{dv_t}$ denotes the martingale part of dv_t.

- It is easy to check explicitly that all drift (*i.e.* dt) terms cancel.

The rough Heston model

By considering the limit of a simple Hawkes process-based model of order flow, [El Euch and Rosenbaum][4] derive a rough Heston model. The equation for variance in this model takes the form

$$v_u = \theta(u) - \frac{1}{\Gamma(H+1/2)} \int_t^u (u-s)^{H-1/2} \lambda \, v_s \, ds + \frac{1}{\Gamma(H+1/2)} \int_t^u (u-s)^{H-1/2} \eta \sqrt{v_s} \, dW_s.$$

- $H \in (0, 1/2]$ is the Hurst exponent of the volatility, $\lambda > 0$ is the mean reversion parameter, $\eta > 0$ is the volatility of volatility parameter.
- The function θ is assumed to be continuous and represents a time-dependent mean reversion level.
- The rough Heston model is a natural fractional generalization of the classical Heston model which is recovered when $H = 1/2$.

Forward variance in the rough Heston model

- We will consider only the special case $\lambda = 0$. In this case, $\xi_t(u) = \mathbb{E}\left[v_u \middle| \mathcal{F}_t\right] = \theta(u)$.
- It follows that

$$v_u = \xi_t(u) + \frac{1}{\Gamma(H+1/2)} \int_t^u (u-s)^{H-1/2} \eta \sqrt{v_s} \, dW_s.$$

- Also

$$v_u = \xi_{t+h}(u) + \frac{1}{\Gamma(H+1/2)} \int_{t+h}^u (u-s)^{H-1/2} \eta \sqrt{v_s} \, dW_s.$$

The rough Heston model in forward variance form

Subtracting these two equations gives

$$\xi_{t+h}(u) - \xi_t(u) = \frac{1}{\Gamma(H+1/2)} \int_t^{t+h} (u-s)^{H-1/2} \eta \sqrt{v_s} \, dW_s.$$

Taking the limit $h \to 0$, we obtain

$$d\xi_t(u) = \frac{\eta}{\Gamma(H+1/2)} (u-t)^{H-1/2} \sqrt{v_t} \, dW_t,$$

the rough Heston model in forward variance form.

Non-Markovianity of the rough Heston model

- Note that the limit $u \to t$ of the rough Heston model makes no sense.
  - This reflects the fact that the rough Heston model is not Markovian.
  - There is no SDE for v_t and no corresponding PDE.
  - On the other hand, we can write an SDE for each $\xi_t(u)$, $u > t$.
    - We can even apply Itô's Lemma!
- The rough Heston model is Markovian in the infinite-dimensional forward variance curve $\xi_t(u),\ u > t$.

Affine processes

The following explanation is due to Martin Keller-Ressel:

> An *affine process* can be described as a Markov process whose log-characteristic function is an affine function of its initial state vector.

And here's a definition of the word *affine* from Wikipedia:

> In geometry, an affine transformation or affine map or an affinity (from the Latin, *affinis*, "connected with") between two vector spaces (strictly speaking, two affine spaces) consists of a linear transformation followed by a translation:
> $$x \mapsto A\,x + b$$

Affine CGF

Let $X_t = \log S_t$. According to Definition 2.2 of [Gatheral and Keller-Ressel][6], we say that a forward variance model has an *affine cumulant generating function* determined by $g(t;u)$, if its conditional cumulant generating function is of the form

(1)
$$\log \mathbb{E}\left[e^{u(X_T - X_t)} \middle| \mathcal{F}_t\right] = \int_t^T g(T-s;u)\,\xi_t(s)ds.$$

for all $u \in [0,1]$, $0 \le t \le T$ and $g(.\,;u)$ is $\mathbb{R}_{\leqslant 0}$-valued and continuous on $[0,T]$ for all $T > 0$ and $u \in [0,1]$.

- The restriction $u \in [0,1]$ is for mathematical convenience. We will later allow complex u.

When is a forward variance model affine?

Theorem 2.4 of [Gatheral and Keller-Ressel][6] states that a forward variance model has an affine CGF if and only if it takes the form

$$\begin{aligned}\frac{dS_t}{S_t} &= \sqrt{v_t}\, dZ_t \\ d\xi_t(u) &= \sqrt{v_t}\, \kappa(u-t)\, dW_t\end{aligned}$$

for some deterministic, non-negative decreasing kernel κ, which satisfies $\int_0^T \kappa(r)dr < \infty$ for all $T > 0$.

- Essentially, the only affine stochstic volatility model is the Heston model, up to a choice of kernel.

Moreover, $g(.\,, u) : \mathbb{R}_{\geqslant 0} \to \mathbb{R}_{\leqslant 0}$ in the definition (1) of the CGF is the unique global continuous solution of the convolution Riccati equation

$$g(t,u) = R_V\left(u, \int_0^t \kappa(t-s)g(s,u)ds\right) = R_V\left(u, (\kappa \star g)(t,u)\right), \quad t \geq 0$$

where

$$R_V(u,w) = \frac{1}{2}(u^2 - u) + \rho\, u\, w + \frac{1}{2}\, w^2.$$

Alternatively, $g(t,u)$ can be written as

$$g(t,u) = R_V(u, f(t,u)),$$

where $f(t,u)$ is the unique global solution of the non-linear Volterra equation

$$f(t,u) = \int_0^t \kappa(t-s) R_V(u, f(t,s))ds.$$

Derivation of the Riccati equation

From the definition definition (1) of the CGF,

$$M_t = \mathbb{E}\left[e^{u\, X_T} \middle| \mathcal{F}_t\right] = \exp\left\{ u\, X_t + \int_t^T \xi_t(s)\, g(T-s; u)\, ds \right\} =: \exp\{u\, X_t + G_t\}$$

is a conditional expectation and thus a martingale in t.

Applying Itô's Lemma to M gives

$$\frac{dM_t}{M_t} = u\, dX_t + dG_t + \frac{u^2}{2}\, d\langle X\rangle_t + \frac{1}{2}\, d\langle G\rangle_t + u\, d\langle X, G\rangle_t.$$

Now

$$\begin{aligned} dX_t &= -\frac{1}{2}\, v_t\, dt + \sqrt{v_t}\, dZ_t \\ dG_t &= -\xi_t(t)\, g(T-t;u)\, dt + \int_t^T d\xi_t(s)\, g(T-s;u)\, ds \\ &= -v_t\, g(T-t;u)\, dt + \int_t^T \kappa(s-t)\, \sqrt{v_t}\, dW_t\, g(T-s;u)\, ds. \end{aligned}$$

We compute

$$\begin{aligned} d\langle X\rangle_t &= v_t\, dt \\ d\langle G\rangle_t &= v_t\, dt \left(\int_t^T \kappa(s-t)\, g(T-s;u)\, ds \right)^2 \\ d\langle X, G\rangle_t &= \rho\, v_t\, dt \int_t^T \kappa(s-t)\, g(T-s;u)\, ds. \end{aligned}$$

Imposing $\mathbb{E}\left[dM_t\right] = 0$ and letting $\tau = T - t$ gives

$$0 = v_t\, dt \left\{ -\frac{1}{2}\, u + \frac{1}{2}\, u^2 - g(\tau;u) + \rho\, u\, (\kappa \star g)(\tau,u) + \frac{1}{2}\, (\kappa \star g)(\tau,u)^2 \right\}$$

where the convolution integral is given by

$$(\kappa \star g)(\tau,u) = \int_0^\tau \kappa(\tau - s)\, g(s;u)\, ds.$$

- It is almost obvious why the CGF is affine if an only if the forward variance process is of the form $d\xi_t(u) = \sqrt{v_t}\, \kappa(u-t)\, dW_t$.

The convolution Riccati equation

Rearranging gives

$$g(\tau;u) = \frac{1}{2}\, u(u-1) + \rho\, u\, (\kappa \star g)(\tau;u) + \frac{1}{2}\, (\kappa \star g)(\tau;u)^2 = R_V(u, (\kappa \star g)),$$

as required.

Example: The classical Heston model

In this case, $\kappa(\tau) = \eta\, e^{-\lambda\,\tau}$.

Then

$$\eta\, h(\tau; u) := (\kappa \star g)(\tau; u) = \eta \int_0^\tau e^{-\lambda\,(\tau - s)}\, g(s; u)\, ds.$$

Also, $\partial_\tau h(\tau; u) = -\lambda\, h(\tau; u) + g(\tau; u)$. The convolution Riccati equation then becomes

(2)

$$\partial_\tau h(\tau; u) = \frac{1}{2}\, u\,(u-1) - (\lambda - \rho\,\eta\,u)\, h(\tau; u) + \frac{1}{2}\,\eta^2\, h(\tau; u)^2$$

consistent with the classical derivation in (for example) Chapter 2 of [The Volatility Surface][5].

The Heston characteristic function

- The classical Heston Riccati ODE (2) may be solved in closed form as in as in equation (2.12) of [The Volatility Surface][5].
- The characteristic function is then given by

$$\varphi_t^T(a) = \mathbb{E}\left[e^{\mathrm{i}aX_T}\,\middle|\,\mathcal{F}_t\right] = \exp\left\{\mathrm{i}aX_t + \int_t^T \xi_t(s)\, g(T-s; \mathrm{i}a)\, ds\right\}$$

  where

$$g(\tau; u) = \partial_\tau h(\tau; u) + \lambda\, h(\tau; u).$$

Example: The rough Heston model (with $\lambda = 0$)

In this case, with $\alpha = H + \frac{1}{2}$, $\kappa(\tau) = \frac{\eta}{\Gamma(\alpha)}\,\tau^{\alpha-1}$ and

$$\eta\, h(\tau; u) := (\kappa \star g)(\tau; u) = \frac{\eta}{\Gamma(\alpha)} \int_0^\tau (\tau - s)^{\alpha-1}\, g(s; u)\, ds = \eta\, I^\alpha g(\tau; u).$$

where D^α and I^α represent fractional differential and integral operators respectively.

Inverting this gives $g(\tau; u) = D^\alpha h(\tau; u)$.

The convolution integral Riccati equation then reads

(3)

$$D^\alpha h(\tau; u) = \frac{1}{2}\, u\,(u-1) + \rho\,\eta\,u\, h(\tau; u) + \frac{1}{2}\,\eta^2\, h(\tau; u)^2,$$

consistent with [El Euch and Rosenbaum][6].

An aside: Fractional calculus

Define the fractional integral and differential operators:

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\, ds; \quad D^\alpha f(t) = \frac{d}{dt} I^{1-\alpha} f(t).$$

The fractional integral is a natural generalization of the ordinary integral using the Cauchy formula for repeated integration:

$$\begin{aligned} I^n f(t) &:= \int_0^t dt_1 \int_0^{t_1} \dots dt_{n-1} \int_0^{t_{n-1}} f(t_n)\, dt_n \\ &= \frac{1}{(n-1)!} \int_0^t (t-s)^{n-1} f(s)\, ds. \end{aligned}$$

The proof follows easily by induction when you notice that

$$D\, I^n f(t) = \frac{1}{(n-1)!} \int_0^t (n-1)\,(t-s)^{n-2} f(s)\, ds = I^{n-1} f(t)$$

and of course that $D\, I f(t) = f(t)$.

A microstructural foundation of the rough Heston model

- The rough Heston model emerges as the continuous time limit of a Hawkes process-based model of order flow.
  - Buys(sells) make the price go up(down)
  - Buys(sells) induce more buys(sells). That is, the processes are self-exciting.
  - Excitations decay as a power law.
- We explain the relationship between the class of Affine Forward Intensity (AFI) models of order flow and their limiting Affine Forward Variance (AFV) models in Lecture 5.
- This gives us a clue as to why rough volatility appears to be universal.

The rough Heston characteristic function

- There exist a number of standard numerical techniques, such as the Adams scheme, for solving fractional differential equations such as the rough Heston fractional Riccati equation.
  - These techniques are all slow!
- Recently, [Gatheral and Radoičić][7] showed how to approximate the solution of the rough Heston fractional Riccati equation by a rational function.
  - This approximation solution is just as fast as the classical Heston solution and appears to be more accurate than the Adams scheme for any reasonable number of time steps!

Rational approximation to the rough Heston solution

Wlog, set $\nu = 1$ and $x = t$. Then the rough Heston fractional Riccati ODE reads

$$\begin{aligned} D^\alpha h(a,x) &= -\frac{1}{2}\, a(a+\mathrm{i}) + \mathrm{i}\,\rho\, a\, h(a,x) + \frac{1}{2}\, h(a,x)^2 \\ &= \frac{1}{2}\left(h(a,x) - r_-\right)\,\left(h(a,x) - r_+\right) \end{aligned}$$

with

$$A = \sqrt{a\,(a+i) - \rho^2\, a^2}; \quad r_\pm = \{-\mathrm{i}\,\rho\, a \pm A\}\,.$$

The idea is to paste together short- and long-time expansions of the solution using a rational (Padé approximation).

Short-time expansion

From the exponentiation theorem of [Alòs, Gatheral and Radoičić][1]</sup>, $h(a,x)$ can be written as

$$h(a,x) = \sum_{j=0}^{\infty} \frac{\Gamma(1+j\,\alpha)}{\Gamma(1+(j+1)\,\alpha)} \beta_j(a)\, x^{(j+1)\,\alpha}$$

with

$$\begin{aligned} \beta_0(a) &= -\frac{1}{2}\, a(a+\mathrm{i}) \\ \beta_k(a) &= \frac{1}{2} \sum_{i,j=0}^{k-2} \mathbb{1}_{i+j=k-2}\, \beta_i(a)\, \beta_j(a) \frac{\Gamma(1+i\,\alpha)}{\Gamma(1+(i+1)\,\alpha)}\, \frac{\Gamma(1+j\,\alpha)}{\Gamma(1+(j+1)\,\alpha)} \\ &\quad + \mathrm{i}\,\rho\, a\, \frac{\Gamma(1+(k-1)\,\alpha)}{\Gamma(1+k\,\alpha)} \beta_{k-1}(a). \end{aligned}$$

- We will explain the exponentiation theorem in Lecture 5.

Solving the rough Heston Riccati equation for long times

- In analogy with the classical Heston solution, we expect that for a suitable range of a,

$$\lim_{x\to\infty} h(a,x) = r_-.$$

- In that case, for large x, we could linearize the fractional Riccati equation as follows.

$$\begin{aligned} D^\alpha h(a,x) &= \frac{1}{2}\left(h(a,x) - r_-\right)\,\left(h(a,x) - r_+\right) \\ &\approx -\frac{1}{2}\left(r_+ - r_-\right)\left(h(a,x) - r_-\right) \\ &= -A\,\left(h(a,x) - r_-\right). \end{aligned}$$

- The above linear fractional differential equation has the exact solution

$$h_\infty(a,x) = r_- \left[1 - E_\alpha(-A\,x^\alpha)\right],$$

where $E_\alpha(\cdot)$ is the Mittag-Leffler function.

- As $x \to \infty$,

$$E_\alpha(-A\,x^\alpha) = -\frac{1}{A}\,\frac{x^{-\alpha}}{\Gamma(1-\alpha)} + \mathcal{O}\left(|A\,x^\alpha|^{-2}\right).$$

- Thus, as $x \to \infty$,

$$h_\infty(a,x) - r_- = \frac{r_-}{A}\,\frac{x^{-\alpha}}{\Gamma(1-\alpha)} + \mathcal{O}\left(|A\,x^\alpha|^{-2}\right).$$

Large x expansion

- The form of the asymptotic solution motivates the following expansion of h for large x:

$$h(a,x) = r_- \sum_{k=0}^{\infty} \gamma_k\,\frac{x^{-k\alpha}}{A^k\,\Gamma(1-k\alpha)}.$$

- The coefficients γ_k satisfy the recursion

$$\gamma_1 = -\gamma_0 = -1$$
$$\gamma_k = -\gamma_{k-1} + \frac{r_-}{2A}\sum_{i,j=1}^{\infty} \mathbb{1}_{i+j=k}\,\gamma_i\,\gamma_j\,\frac{\Gamma(1-k\alpha)}{\Gamma(1-i\alpha)\,\Gamma(1-j\alpha)}.$$

Rational approximation

- Now we have small- and large-x expansions we can compute global rational approximations to $h(a,x)$ of the form

$$h^{(m,n)}(a,x) = \frac{\sum_{i=1}^{m} p_m y^m}{\sum_{j=0}^{n} q_n y^n}$$

with $y = x^\alpha$ that match these expansions up to order m and n respectively.

- Only the diagonal approximants $h^{(n,n)}$ are admissible approximations of h.

$h^{(3,3)}$ is the best

- From various numerical experiments, the particular approximation $h^{(3,3)}$ seems to be amazingly close to the true solution for reasonable choices of model parameters.
- Though the excellent quality of the global approximation $h^{(3,3)}$ might at first seem very surprising, it is consistent with many Padé approximation stories from the literature.
- In our case, $h^{(3,3)}$ is clearly better than either $h^{(2,2)}$ or $h^{(4,4)}$.
  - $h^{(5,5)}$ is another very good approximation, but still not as good as $h^{(3,3)}$. $h^{(5,5)}$ is obviously also slower to compute.
  - Higher order approximations may turn out to beat $h^{(3,3)}$. However, $h^{(3,3)}$ may still be best in practice if speed of computation is taken into account.

Computing $h^{(3,3)}$

- We have the series expansion of h for small y:
$$h_s(y) = b_1\, y + b_2\, y^2 + b_3\, y^3 + \mathcal{O}(y^4).$$
- We have the series expansion of h for large y:
$$h_\ell(y) = g_0 + \frac{g_1}{y} + \frac{g_2}{y^2} + \mathcal{O}\left(\frac{1}{y^3}\right).$$
- Matching the coefficients of the rational approximation
$$h^{(3,3)}(y) = \frac{p_1\, y + p_2\, y^2 + p_3\, y^3}{1 + q_1\, y + q_2\, y^2 + q_3\, y^3}$$
to $h_s(y)$ and $h_\ell(y)$ respectively gives a linear system of six equations with six unknowns.

Some R-code

In [1]:
```
setwd("./LRV")
```

In [2]:
```
source("BlackScholes.R")
source("Lewis.R")
source("roughHestonPade.R")
```

In [3]:
```
library(repr)
options(repr.plot.height=5)
```

R implementation of the rational approximation

The code below shows how the solution is implemented.

In [4]:
```
d.h.Pade33
```

```
function (params)
function(a, x) {
    H <- params$H
    rho <- params$rho
    eta <- params$eta
    al <- H + 0.5
    aa <- sqrt(a * (a + (0 + (0+1i))) - rho^2 * a^2)
    rm <- -(0 + (0+1i)) * rho * a - aa
    rp <- -(0 + (0+1i)) * rho * a + aa
    b1 <- -a * (a + (0+1i))/(2 * gamma(1 + al))
    b2 <- (1 - a * (0+1i)) * a^2 * rho/(2 * gamma(1 + 2 * al))
    b3 <- gamma(1 + 2 * al)/gamma(1 + 3 * al) * (a^2 * (0+1i +
        a)^2/(8 * gamma(1 + al)^2) + (a + (0+1i)) * a^3 * rho^2/(2 *
        gamma(1 + 2 * al)))
    g0 <- rm
    g1 <- -rm/(aa * gamma(1 - al))
    g2 <- rm/aa^2/gamma(1 - 2 * al) * (1 + rm/(2 * aa) * gamma(1 -
        2 * al)/gamma(1 - al)^2)
    den <- g0^3 + 2 * b1 * g0 * g1 - b2 * g1^2 + b1^2 * g2 +
        b2 * g0 * g2
    p1 <- b1
    p2 <- (b1^2 * g0^2 + b2 * g0^3 + b1^3 * g1 + b1 * b2 * g0 *
        g1 - b2^2 * g1^2 + b1 * b3 * g1^2 + b2^2 * g0 * g2 -
        b1 * b3 * g0 * g2)/den
    q1 <- (b1 * g0^2 + b1^2 * g1 - b2 * g0 * g1 + b3 * g1^2 -
        b1 * b2 * g2 - b3 * g0 * g2)/den
    q2 <- (b1^2 * g0 + b2 * g0^2 - b1 * b2 * g1 - b3 * g0 * g1 +
        b2^2 * g2 - b1 * b3 * g2)/den
    q3 <- (b1^3 + 2 * b1 * b2 * g0 + b3 * g0^2 - b2^2 * g1 +
        b1 * b3 * g1)/den
    p3 <- g0 * q3
    y <- x^al
    h.pade <- (p1 * y + p2 * y^2 + p3 * y^3)/(1 + q1 * y + q2 *
        y^2 + q3 * y^3)
    res <- 1/2 * (h.pade - rm) * (h.pade - rp)
    return(res)
}
```

Computing option prices from the characteristic function

It turns out (see [Carr and Madan][3] and [Lewis][8]) that it is quite straightforward to get option prices by inverting the characteristic function of a given stochastic process (if it is known in closed-form).

Formula (5.6) of [The Volatility Surface][5] is a special case of formula (2.10) of Lewis (as usual we assume zero interest rates and dividends):

Formula (2.10) of Lewis

(7)

$$C(S,K,t,T) = S - \sqrt{SK}\frac{1}{\pi}\int_0^\infty \frac{du}{u^2+\frac{1}{4}} \operatorname{Re}\left[e^{-iuk}\varphi_t^T\,(u-i/2)\right]$$

with $k = \log\left(\frac{K}{S}\right)$.

- An anologous formula holds for puts.

R implementation of the Lewis formula

In [5]:
```
option.OTM.raw
```

```
function (phi, k, t)
{
    integrand <- function(u) {
        Re(exp(-(0+1i) * u * k) * phi(u - (0+1i)/2, t)/(u^2 +
            1/4))
    }
    k.minus <- (k < 0) * k
    res <- exp(k.minus) - exp(k/2)/pi * integrate(integrand,
        lower = 0, upper = Inf, rel.tol = 1e-08)$value
    return(res)
}
```

Some notable features of R

- Complex arithmetic with $1\mathrm{i}$.
- Functional programming:
  - This is what allows us to code a function which is called as:

```
impvol.phi(phiHeston(paramsBCC))(0,1)
```

  - We can define a function that returns a function (and so on indefinitely).
  - We could even conveniently define a new function:

```
impvolBCC <-
    impvol.phi(phiHeston(paramsBCC))
```

- We can conceptually separate parameters and variables rather than having to carry all the parameters around with each function call.

The rough Heston smile

In [6]:
```
params.rHeston <- list(H=0.05,nu=0.4,rho=-.65,eta=0.4)
xiCurve <- function(t){0.025+0*t}
```

In [7]:
```
phi <- phiRoughHestonDhApprox(params.rHeston, xiCurve, dh.approx= d.h.Pade33, n=20)
```

In [8]:
```
vol <- function(k){
    sapply(k,function(x){impvol.phi(phi)(x,1)})}
system.time(curve(vol(x),from=-.4,to=.4,col="red"))
```

```
   user  system elapsed
  1.222   0.044   1.270
```

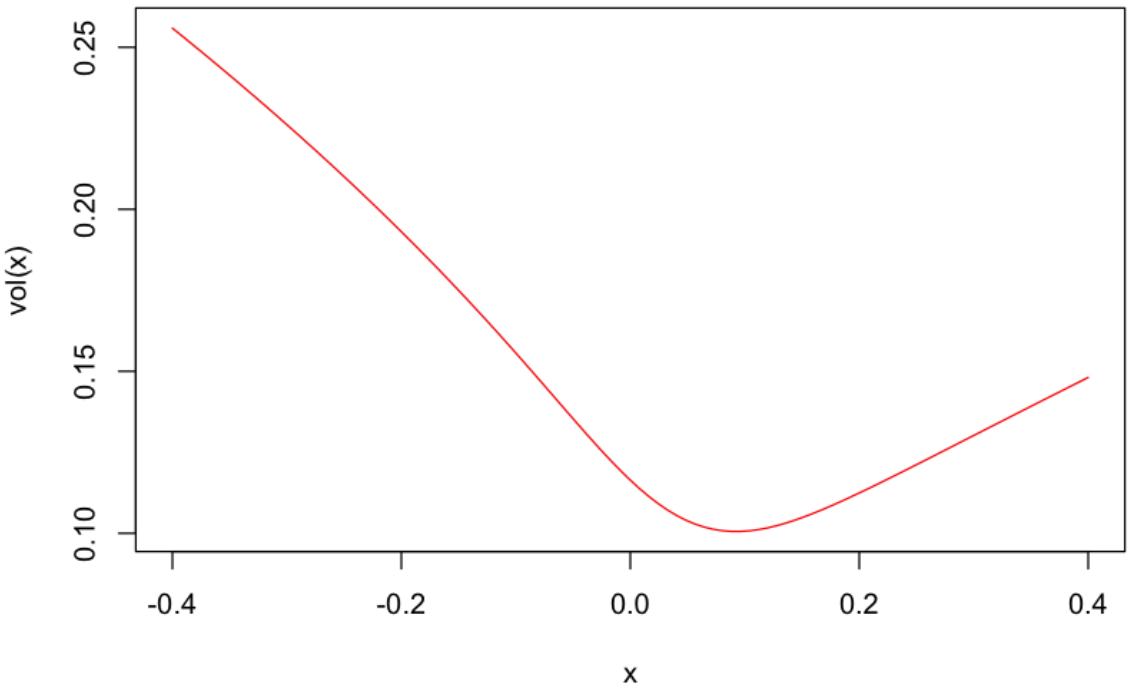

Figure 1: The 1-year rough Heston smile using the approximation $h^{(3,3)}$.

How does $h^{(3,3)}$ compare with $h^{(2,2)}$ and $h^{(4,4)}$?

In [9]:
```
phi2 <- phiRoughHestonDhApprox(params.rHeston, xiCurve, dh.approx= d.h.P
ade22, n=20)
phi4 <- phiRoughHestonDhApprox(params.rHeston, xiCurve, dh.approx= d.h.P
ade44, n=20)
```

In [10]:
```
vol2 <- function(k){sapply(k,function(x){impvol.phi(phi2)(x,1)})}
vol4 <- function(k){sapply(k,function(x){impvol.phi(phi4)(x,1)})}
```

In [11]:
```
curve(vol(x),from=-.4,to=.4,col="red")
curve(vol2(x),from=-.4,to=.4,col="blue",add=T,lty=2)
curve(vol4(x),from=-.4,to=.4,col="green4",add=T,lty=3)
```

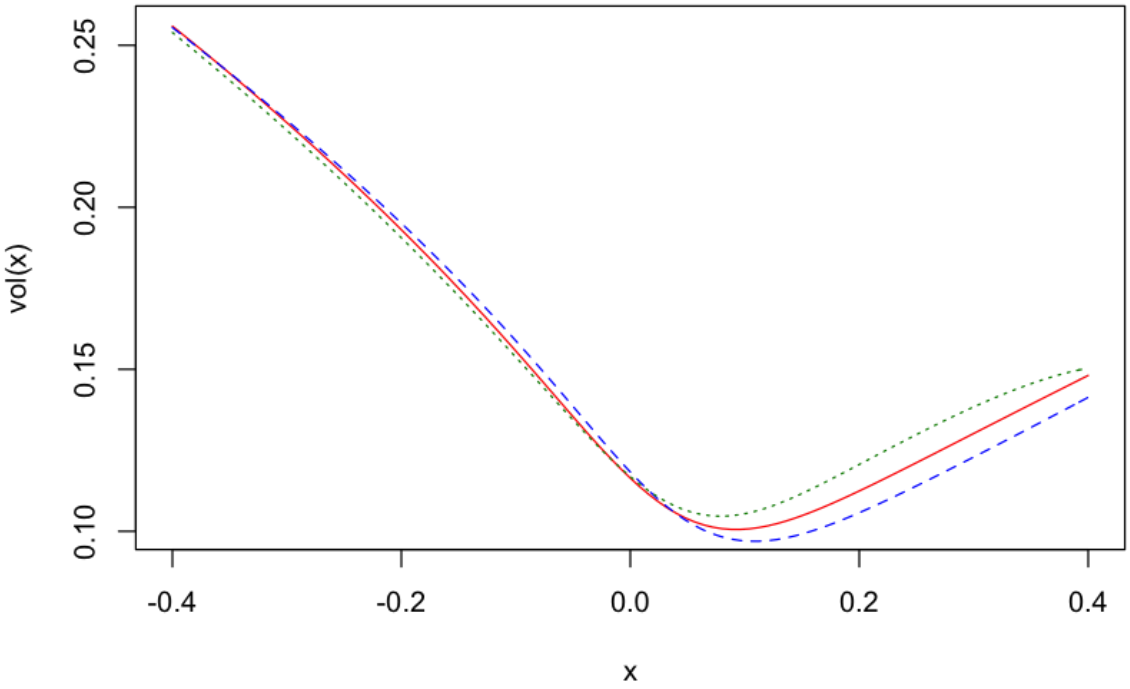

Figure 2: The 1-year rough Heston smile in red with approximation $h^{(3,3)}$. The blue dashed line is $h^{(2,2)}$, and the green dotted line $h^{(4,4)}$.

Sensitivity of the rough Heston smile to η

First, a function to compute the 1-year smile:

In [12]:
```
vol <- function(params)function(k){ # A function to compute the 1-year s
mile
    phi <- phiRoughHestonDhApprox(params, xiCurve, dh.approx= d.h.Pade33
, n=20)
    sapply(k,function(x){impvol.phi(phi)(x,1)})}

myCol <- rainbow(6)
```

In [13]:
```
sub.eta <- function(eta.in){
    tmp <- params.rHeston
    tmp$eta <- eta.in
    return(tmp)
}
```

In [14]:
```
yrange <- c(0.05,.35)
curve(vol(params.rHeston)(x),from=-.5,to=.5,col=myCol[1],ylim=yrange,lwd
=2,ylab="Implied vol.",xlab="Log-strike k")
eta.vec <- params.rHeston$eta + c(0.1,0.2,0.3,0.4,0.5)
for (j in 1:5)
    {

    curve(vol(sub.eta(eta.vec[j]))(x),from=-.5,to=.5,col=myCol[j+1],lty=
2,add=T)
    }
```

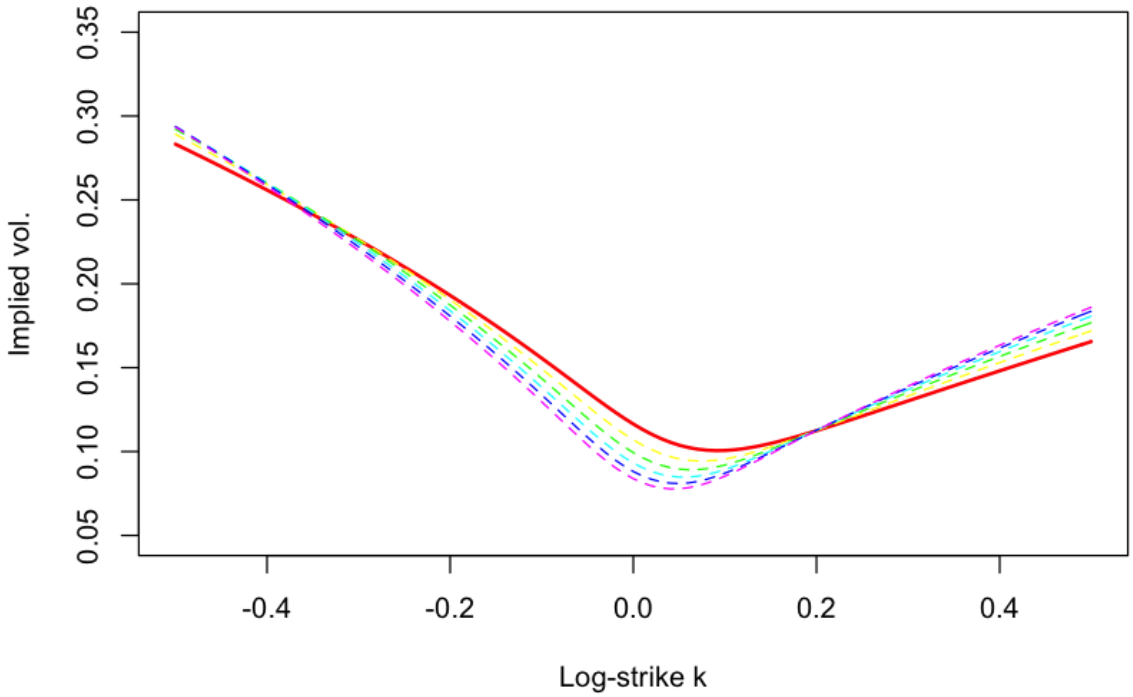

Figure 3: The dotted lines are smiles with $\eta \mapsto= \eta + \{0.1, 0.2, 0.3, 0.4, 0.5\}$.

Sensitivity of the rough Heston smile to ρ

In [15]:
```
sub.rho <- function(rho.in){
    tmp <- params.rHeston
    tmp$rho <- rho.in
    return(tmp)
}
```

In [16]:
```
yrange <- c(0.05,.35)
curve(vol(params.rHeston)(x),from=-.5,to=.5,col=myCol[1],ylim=yrange,lwd
=2,ylab="Implied vol.",xlab="Log-strike k")
rho.vec <- params.rHeston$rho - c(0.05,0.10,0.15,0.20,0.25)
for (j in 1:5)
    {

    curve(vol(sub.rho(rho.vec[j]))(x),from=-.5,to=.5,col=myCol[j+1],lty=
2,add=T)
    }
```

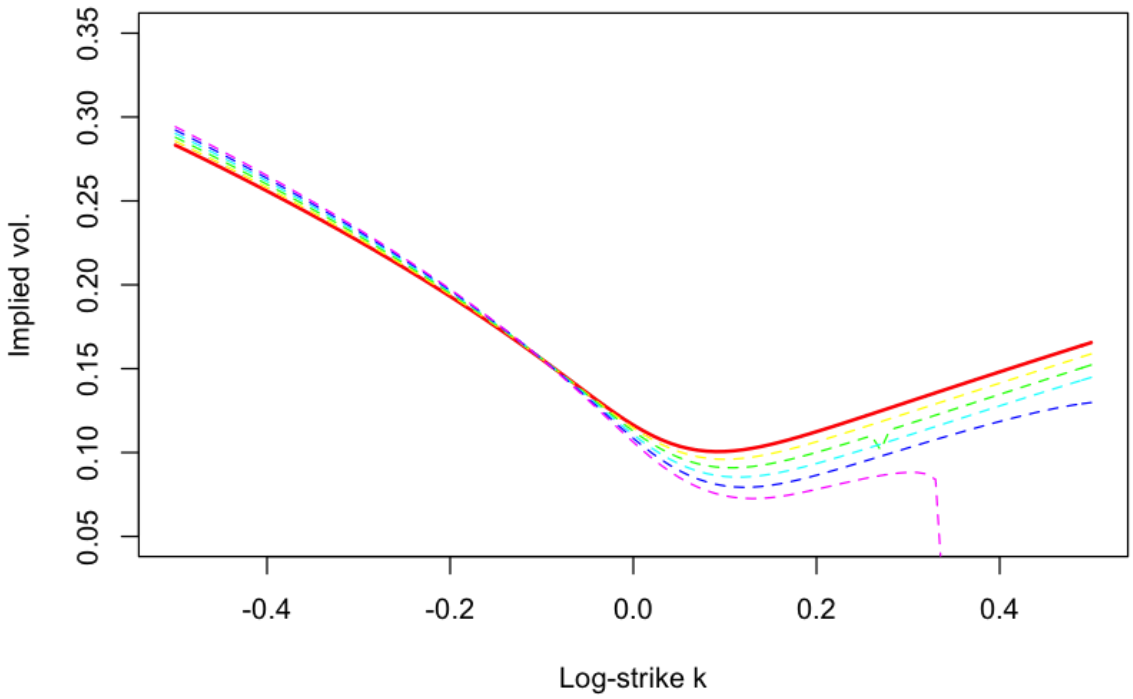

Figure 4: The dotted lines are smiles with $\rho \mapsto \rho - \{0.05, 0.10, 0.15, 0.2, 0.25\}$.

Sensitivity of the rough Heston 1 year smile to H

In [17]:
```
sub.H <- function(H.in){
    tmp <- params.rHeston
    tmp$H <- H.in
    return(tmp)
}
```

In [18]:
```
yrange <- c(0.05,.35)
curve(vol(params.rHeston)(x),from=-.5,to=.5,col=myCol[1],ylim=yrange,lwd
=2,ylab="Implied vol.",xlab="Log-strike k")
H.vec <- params.rHeston$H + seq(0.1,0.4,0.1)
for (j in 1:4)
    {
    curve(vol(sub.H(H.vec[j]))(x),from=-.5,to=.5,col=myCol[j+1],lty=2,ad
d=T)
    }
```

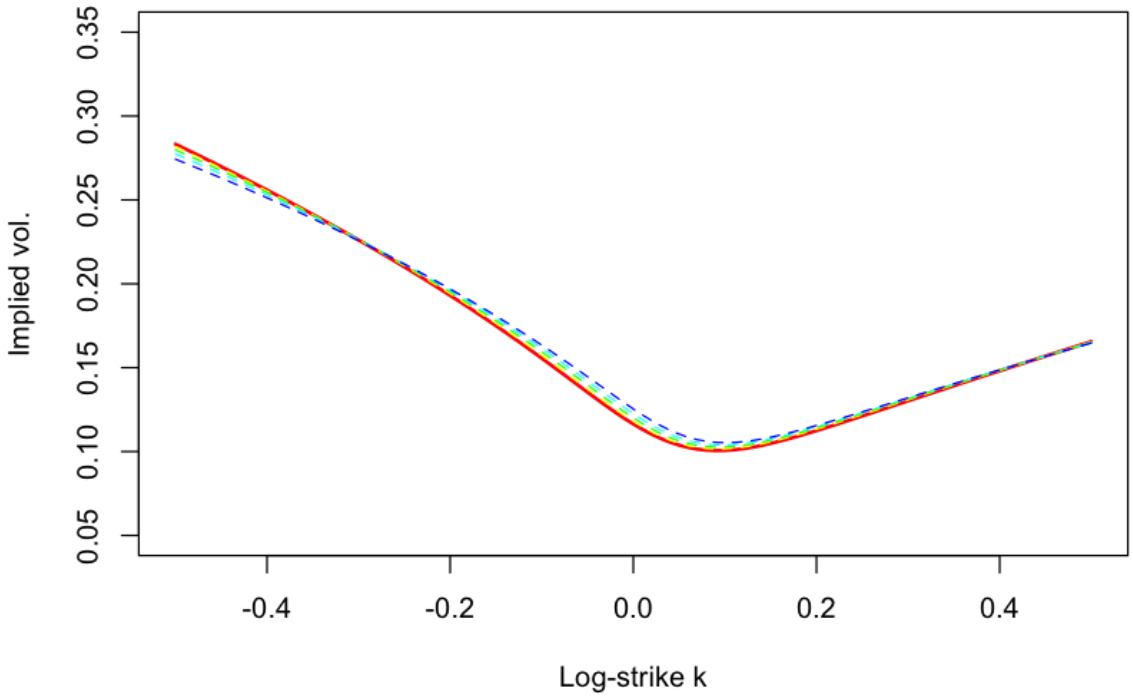

Figure 5: The dotted lines are 1 year smiles with $H \mapsto H + \{0.1, 0.2, 0.3, 0.4\}$.

Sensitivity of the rough Heston 1 week smile to H

A function to draw the 1-week smile:

In [19]:
```
vol <- function(params)function(k){
    phi <- phiRoughHestonDhApprox(params, xiCurve, dh.approx= d.h.Pade33
, n=20)
    sapply(k,function(x){impvol.phi(phi)(x,1/52)})}
```

In [20]:
```
yrange <- c(0.05,.4)
curve(vol(params.rHeston)(x),from=-.15,to=.15,col=myCol[1],ylim=yrange,l
wd=2,ylab="Implied vol.",xlab="Log-strike k")
H.vec <- params.rHeston$H + seq(0.1,0.4,0.1)
for (j in 1:4)
    {
    curve(vol(sub.H(H.vec[j]))(x),from=-.15,to=.15,col=myCol[j+1],lty=2,
lwd=2,add=T)
    }
```

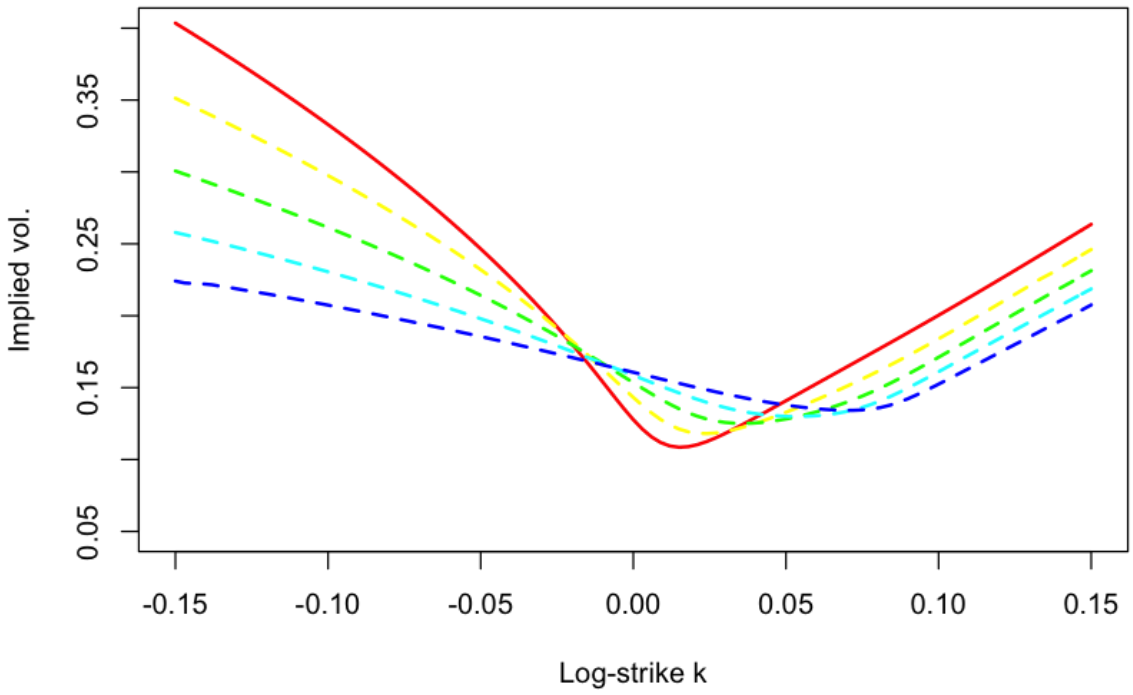

Figure 6: The dotted lines are 1 week smiles with $H \mapsto H + \{0.1, 0.2, 0.3, 0.4\}$. The smile flattens as we increase H.

Ease of calibration of rough volatility models

- Rough volatility models are typically very parsimonious.
- Moreover, from the above sensitivity analyses, the effect of changing each parameter is clear.
  - Contrast this with the classical Heston model where volatility of volatility and mean reversion are competing effects.

Dynamics of the rough Heston volatility surface

- All rough stochastic volatility models have essentially the same implications for the shape of the volatility surface.
- Recall from Lecture 2 that we can differentiate between models by examining how ATM skew depends on ATM volatility keeping model parameters fixed.
- In Figure 7, we that rough Heston dynamics are not consistent with empirical dynamics, in contract to rough Bergomi.

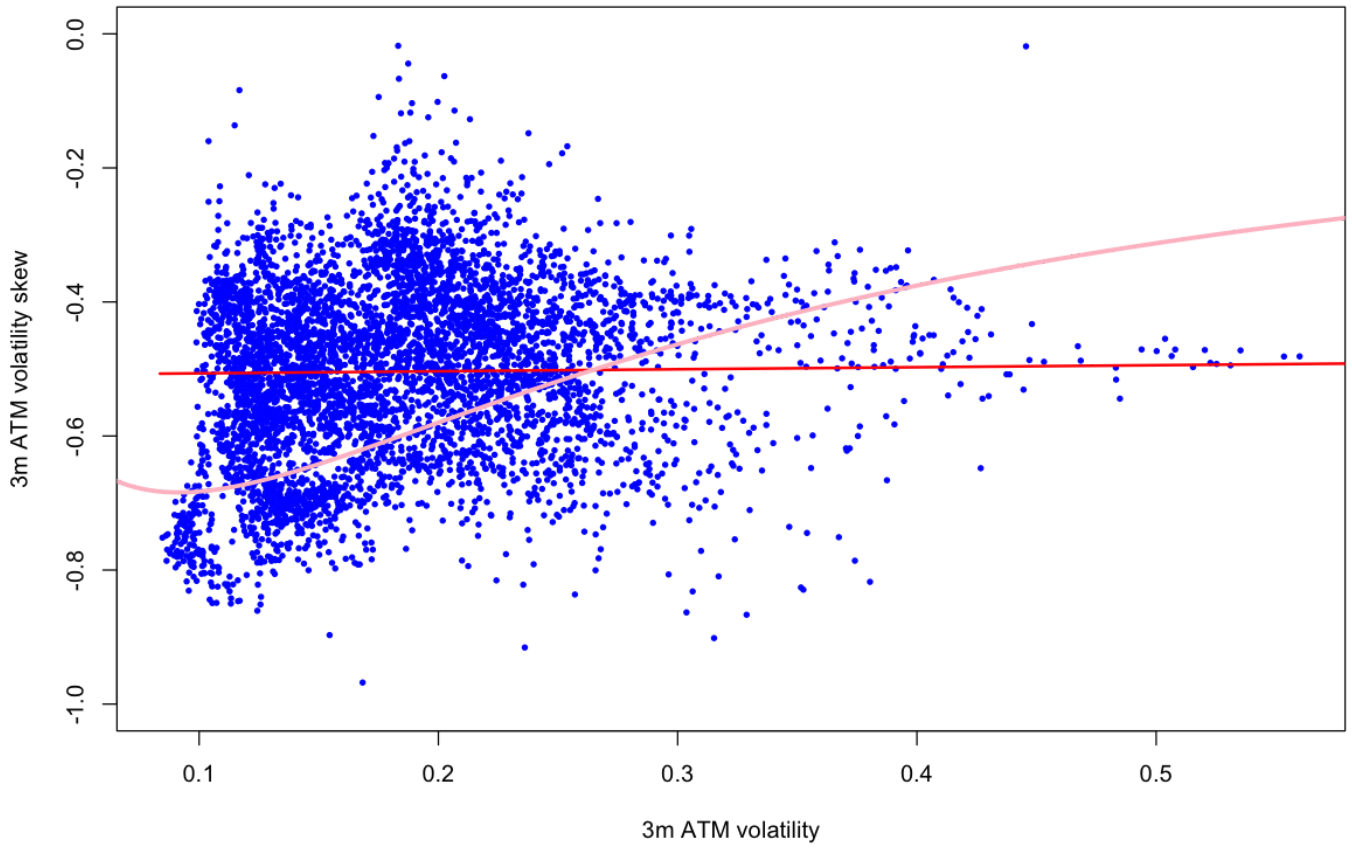

Figure 7: Blue points are empirical 3-month ATM volatilities and skews (from Jan-1996 to today); the red line is the rough Bergomi computation with the above parameters; the pink curve is the rough Heston computation.

Summary of Lecture 4

- We introduce forward variance and affine models.
- We showed how to compute the characteristic function for any affine forward variance model.
- We applied these general results to the classical Heston and rough Heston models.
- We explained the idea behind the Padé approximation to the rough Heston solution.
- We explored the effects of the rough Heston parameters on volatility smiles.

- However, though rough Heaton is highly tractable, its dynamics are unreasonable.

References

1. ^ Elisa Alòs, Jim Gatheral, and Radoš Radoičić, Exponentiation of conditional expectations under stochastic volatility, available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2983180, (2017).
2. ^ Lorenzo Bergomi and Julien Guyon, The smile in stochastic volatility models. *SSRN* (2011).
3. ^ Peter Carr and Dilip Madan, Option valuation using the Fast Fourier Transform, *Journal of Computational Finance* **2**(4), 61–73 (1999).
4. ^ Omar El Euch and Mathieu Rosenbaum, The characteristic function of rough Heston models, *Mathematical Finance*, forthcoming (2018).
5. ^Jim Gatheral, *The Volatility Surface: A Practitioner's Guide*, John Wiley and Sons, Hoboken, NJ (2006).
6. ^ Jim Gatheral and Martin Keller-Ressel, Affine forward variance models, available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3105387 (2017).
7. ^ Jim Gatheral and Radoš Radoičić, Rational approximation of the rough Heston solution, available at https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3191578 (2018).
8. ^ Alan L. Lewis, *Option Valuation under Stochastic Volatility with Mathematica Code*, Finance Press: Newport Beach, CA (2000).

In [ ]: