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Rough volatility

Lecture 5: A microstructural foundation for rough volatility, the exponentiation theorem

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Outline of Lecture 5

- A microstructural foundation for affine stochastic volatility models
- Exponentiation of conditional expectations
- Leverage swap computation
- Calibration of rough Heston parameters

A microstructural foundation for affine stochastic volatility models

- [Jaisson and Rosenbaum][7]</sup> first showed that affine stochastic volatility models could arise as limits of Hawkes process-based models of order flow.
- In the following, we both generalize and hopefully shed light on their argument.

Hawkes processes

- Dating from the 1970's, Hawkes processes are jump processes where the jump arrival rate is self-exciting.
- One of the first applications was to the modeling of earthquakes.

The Hawkes process-based microstructure model of Jaisson and Rosenbaum

[Jaisson and Rosenbaum][6]</sup> consider the following simple model of price formation:

- Order arrivals are modeled as a counting process
  - Buy order arrivals cause the price to increase
  - Sell order arrivals cause the price to decrease
  - All orders are unit size

- The order arrival process is self-exciting
  - The price process is a bivariate Hawkes process.

The stock price process

Specifically, with $X_t = \log S_t$,

$$dX_t = m_X dt + dN_t^+ - dN_t^-$$

where $N^\pm$ are counting processes with arrival rates $\lambda_t^\pm$, and m_X is determined by the martingale condition on $S = e^X$.

The order arrival rate process

$$\lambda_t = \mu + \int_0^t \varphi(t-s)\, d\mathbf{N}_s.$$

where $\lambda = \{\lambda^+, \lambda^-\}$ and $\mathbf{N} = \{N^+, N^-\}$. The kernel φ is a 2×2 matrix.

- The order arrival process is self-exciting.
  - As orders arrive, the order arrival rate increases.
  - In the absence of new orders, the order arrival rate decays according to some Hawkes kernel φ.

- [Jaisson and Rosenbaum][6]</sup> show that that in a suitable scaling limit, and with a suitable choice of the kernel φ, this model tends to the rough Heston model.

Affine forward intensity (AFI) models

- In analogy to stochastic volatility models in forward variance form, [Gatheral and Keller-Ressel][6] define the forward intensity model

$$\begin{aligned} dX_t &= -\lambda_t m_X dt + dJ_t^+ - dJ_t^-, \\ d\xi_t(T) &= \kappa(T-t)\left(\gamma^+ \, d\widetilde{J}_t^+ + \gamma^- \, d\widetilde{J}_t^-\right). \end{aligned}$$

  where κ is an integrable, decreasing non-zero kernel.
  - $\gamma^\pm$ are positive constants
  - jumps can have various sizes; the jump size measures are $\zeta_\pm$
  - m_X is determined by the martingale condition on $S = e^X$

- The $\widetilde{J}_t^\pm$ denote the *compensated* order flow processes, i.e.

$$\widetilde{J}_t^\pm := J_t^\pm - m_\pm \int_0^t \lambda_s ds,$$

  where

$$m_\pm = \int_{\mathbb{R}_{\geqslant 0}} x \, \zeta_\pm(dx).$$

Variance and jump intensity

Denote the variance per unit time of the process X_t by v_t. Then

$$v_t \, dt = \mathrm{var}[dJ_t^+ - dJ_t^-] = \lambda_t \, \left\{v^+ + v^-\right\} \, dt =: \lambda_t \, v_J \, dt,$$

where

$$v^\pm = \int_{\mathbb{R}_{\geqslant 0}} x^2 \, \zeta_\pm(dx) - m_\pm^2$$

are the variance of positive and negative jump sizes respectively.

Continuing the analogy with stochastic volatility, $\xi_t(u)$ is linked to v_t by

$$\xi_t(u) = \mathbb{E}\left[v_u \,|\, \mathcal{F}_t\right].$$

Setting

$$J_t^X = J_t^+ - J_t^-, \qquad \widetilde{J}_t^v = \gamma^+ \widetilde{J}_t^+ + \gamma^- \widetilde{J}_t^-,$$

the affine forward intensity (AFI) model may be rewritten as

$$\begin{aligned} dX_t &= -\lambda_t m_X dt + dJ_t^X, \\ d\xi_t(T) &= \kappa(T-t) d\widetilde{J}_t^v. \end{aligned}$$

High-frequency limit of the AFI model

Consider new processes J^ϵ such that

$$\lambda^\epsilon = \frac{1}{\epsilon}\,\lambda; \quad \zeta^\epsilon(dx) = \zeta\left(\frac{dx}{\sqrt{\epsilon}}\right).$$

Thus in the limit $\epsilon \to 0$,

- jump sizes are very small and jumps are very frequent.
- the martingale component of dX_t may be approximated by $\sqrt{v_t}\,dZ_t$
- $d\tilde{J}^v_t$ may be approximated by dY_t for some diffusion process Y.

High frequency limit of the AFI model

In the limit, we obtain

$$\begin{aligned} dX_t &= -\frac{1}{2}\,v_t\,dt + \sqrt{v_t}\,dZ_t, \\ d\xi_t(T) &= \kappa(T-t)\,dY_t, \end{aligned}$$

where

$$\begin{aligned} \mathrm{var}[dY_t] = \mathrm{var}[d\tilde{J}^v_t] &= \lambda_t\,\left[\gamma^{+2}\,v^+ + \gamma^{-2}\,v^-\right]\,dt \\ &= v_t\,\left[\frac{\gamma^{+2}\,v^+ + \gamma^{-2}\,v^-}{v^+ + v^-}\right]\,dt. \end{aligned}$$

Then

$$d\xi_t(T) = \eta\,\kappa(T-t)\,\sqrt{v_t}\,dW_t$$

where

$$\eta^2 = \frac{\gamma^{+2}\,v^+ + \gamma^{-2}\,v^-}{v^+ + v^-}.$$

As for the correlation between dZ_t and dW_t, we first compute

$$\mathbb{E}\left[dJ^+_t\,d\tilde{J}^+_t\right] = \lambda_t\,v^+\,dt; \quad \mathbb{E}\left[dJ^-_t\,d\tilde{J}^-_t\right] = \lambda_t\,v^-\,dt$$

so

$$\begin{aligned} \mathbb{E}\left[dX_t\,d\tilde{J}^v_t\right] &= \lambda_t\,(\gamma^+\,v^+ - \gamma^-\,v^-)\,dt \\ &= \mathbb{E}\left[\sqrt{v_t}\,dZ_t\,\eta\,\sqrt{v_t}\,dW_t\right] =: \rho\,\eta\,v_t\,dt, \end{aligned}$$

where

$$\rho = \frac{1}{\sqrt{v^+ + v^-}}\,\frac{\gamma^+\,v^+ - \gamma^-\,v^-}{\sqrt{\gamma^{+2}\,v^+ + \gamma^{-2}\,v^-}}.$$

Example: The bivariate Hawkes process of of Jaisson and Rosenbaum

Consider the case of a bivariate Hawkes process (J^+, J^-) with unit jump size (i.e., $\zeta_\pm(dx) = \delta_1(dx)$). Then in the above limit, as $\epsilon \to 0$, the process converges to

$$dX_t = -\frac{1}{2}\, v_t\, dt + \sqrt{v_t}\, dZ_t,$$
$$d\xi_t(T) = \eta \sqrt{v_t}\, \kappa(T-t)\, dW_t,$$

where $\mathbb{E}\left[dZ_t\, dW_t\right] = \rho\, dt$ and

$$\eta^2 = \frac{1}{2}\left[\gamma^{+2} + \gamma^{-2}\right]; \quad \rho = \frac{\gamma^+ - \gamma^-}{\sqrt{2\,\left(\gamma^{+2} + \gamma^{-2}\right)}}.$$

Near instability of Hawkes kernel in the limit

- So far, we have shown how AFV models arise naturally as limits of AFI models.
- Now we show that in order to get stochastic (as opposed to constant) volatility, the AFI model Hawkes process needs to be nearly unstable.

Consider the (generalized) Hawkes process

$$\lambda_t = \mu + \int_0^t \varphi(t-s)\, dJ_s^v$$
$$= \mu + \hat{\gamma} \int_0^t \varphi(t-s)\, \lambda_s\, ds + \int_0^t \varphi(t-s)\, d\tilde{J}_s^v$$

where $\hat{\gamma} = \gamma^+ m_+ + \gamma^- m_-$.

Following [Bacry et al.][3]</sup>, we rewrite this last equation symbolically as

$$\lambda = \mu + \hat{\gamma}\,(\varphi \star \lambda) + \varphi \star d\tilde{J}^v.$$

Rearranging gives

$$(1 - \hat{\gamma}\,\varphi\star)\lambda = \mu + \varphi \star d\tilde{J}^v$$

and applying the Laplace transform gives

$$(1 - \hat{\gamma}\,\hat{\varphi})\hat{\lambda} = \hat{\mu} + \hat{\varphi}\,\widehat{d\tilde{J}^v}.$$

which may be rearranged as

$$\hat{\lambda} = \hat{\mu} + \hat{\psi}\,\hat{\mu} + \frac{1}{\hat{\gamma}}\,\hat{\psi}\,\widehat{\tilde{J}^v}$$

where

$$\hat{\psi} = \frac{\hat{\gamma}\hat{\varphi}}{1 - \hat{\gamma}\,\hat{\varphi}}.$$

Then

$$v_J\,\hat{\lambda} = v_J\,\hat{\mu} + \hat{\gamma}\,\hat{\kappa}\,\hat{\mu} + \hat{\kappa}\,\widehat{\tilde{J}^v}$$

where

$$\hat{\kappa} = \frac{v_J}{\hat{\gamma}}\,\hat{\psi} = \frac{v_J\,\hat{\varphi}}{1 - \hat{\gamma}\,\hat{\varphi}}.$$

Inverting the Laplace transform, and recalling that $v_t = v_J\,\lambda_t$, we obtain

$$v_u = v_J\,\mu + \hat{\gamma}\,\mu\int_0^u \kappa(u-s)\,ds + \int_0^u \kappa(u-s)\,d\tilde{J}^v_s.$$

Taking a conditional expectation wrt $\mathcal{F}_t$,

$$\xi_t(u) = \mathbb{E}\left[v_u \,|\, \mathcal{F}_t\right]$$
$$= v_J\,\mu + \hat{\gamma}\,\mu\int_0^u \kappa(u-s)\,ds + \eta\int_0^t \kappa(u-s)\,\sqrt{v_s}\,dW_s$$

and so $d\xi_t(u) = \kappa(u-t)\,d\tilde{J}^v_t$, the dynamics of an AFI model.

Now

$$\hat\kappa = \frac{v_J\,\hat\varphi}{1-\hat\gamma\,\hat\varphi} \implies \hat\varphi = \frac{\hat\kappa}{v_J + \hat\gamma\,\hat\kappa}.$$

Recall that the kernel of our generalized Hawkes process is $\hat\gamma\,\hat\varphi$. The stability condition is then

$$\hat\gamma \int_{\mathbb{R}_{\geqslant 0}} \varphi(\tau)\,d\tau = \hat\gamma\,\hat\varphi(0) = \frac{\hat\gamma\,\hat\kappa}{v_J + \hat\gamma\,\hat\kappa} \to 1 \text{ as } \epsilon \to 0$$

since in that limit, $v_J \sim \epsilon$ and $\hat\gamma \sim \sqrt{\epsilon}$.

Conversely, $\hat\gamma\,\hat\varphi(0) \to a < 1$ as $\epsilon \to 0$ only if $\kappa \sim \sqrt{\epsilon}$. Then in the limit, $\kappa \to 0$ and volatility is deterministic.

> **Near instability**
>
> The high frequency limit of the AFI model is a non-trivial AFV model if and only if the Hawkes process is nearly unstable.

Diamonds and the exponentiation theorem

- We now turn our attention to diamonds and the exponentiation theorem.
- The exponentiation theorem is effectively a generalization of both the Alòs decomposition formula and the Bergomi-Guyon expansion.
- Diamond functionals are generalizations of the Bergomi-Guyon autocovariance functionals.

The Alòs decomposition formula

Following [Alòs][1]</sup>, let $X_t = \log S_t/K$ and consider the price process

$$dX_t = \sigma_t\,dZ_t - \frac{1}{2}\,\sigma_t^2\,dt.$$

Now let $H(X_t, w_t(T))$ (H_t for short) be some function that solves the Black-Scholes equation.

- Specifically,

  $$-\partial_w H_t + \frac{1}{2}\,(\partial_{xx} - \partial_x)\,H_t = 0$$

  which is of course the gamma-vega relationship.
- Note in particular that ∂_x and ∂_w commute when applied to a solution of the Black-Scholes equation.

Variance swaps

We now specify the variance swap $w_t(T)$ as the integral of the expected future variance:

$$w_t(T) = \int_t^T \mathbb{E}\left[\sigma_u^2 \middle| \mathcal{F}_t\right] du = \int_t^T \xi_t(u)\, du,$$

where the $\xi_t(u)$ are forward variances.

Notice that

$$w_t(T) = M_t - \int_0^t \sigma_s^2 ds,$$

where the martingale $M_t := \mathbb{E}\left[\int_0^T \sigma_s^2 ds \middle| \mathcal{F}_t\right]$. Then it follows that

$$dw_t(T) = -\sigma_t^2\, dt + dM_t$$

The Itô Decomposition Formula

Applying Itô's Formula to H, taking conditional expectations, simplifying using the Black-Scholes equation and integrating, we obtain

The Itô Decomposition Formula of Alòs

(1)

$$\begin{aligned}\mathbb{E}\left[H_T \mid \mathcal{F}_t\right] = H_t &+ \mathbb{E}\left[\int_t^T \partial_{xw} H_s\, d\langle X, M\rangle_s \middle| \mathcal{F}_t\right] \\ &+ \frac{1}{2}\mathbb{E}\left[\int_t^T \partial_{ww} H_s\, d\langle M, M\rangle_s \middle| \mathcal{F}_t\right].\end{aligned}$$

- Note in particular that this decomposition is *exact*.

Diamond notation

Let A_t and B_t be semimartingales (here some combinations of X and M). Then

$$(A \diamond B)_t(T) = \mathbb{E}\left[\int_t^T d\langle A, B\rangle_s \middle| \mathcal{F}_t\right].$$

When $(A \diamond B)_t(T)$ appears before some solution H_t of the Black-Scholes equation, the dot $\cdot$ means act on H_t with the appropriate combintation of ∂_x and ∂_w.

So for example

$$(X \diamond M)_t(T) \cdot H_t = \mathbb{E}\left[\int_t^T d\langle X, M\rangle_s \middle| \mathcal{F}_t\right] \partial_{xw} H_t$$

and so on.

Diamond functionals as covariances

- Diamond (or autocovariance) functionals are intimately related to conventional covariances.
- Covariances are typically easy to compute using simulation.
- Diamond functionals are expressible directly in terms of the formulation of a model in forward variance form.

\end{frame}

Bergomi-Guyon in diamond notation

According to equation (13) of [Bergomi and Guyon][1]</sup>, in diamond notation, the conditional expectation of a solution of the Black-Scholes equation satisfies

$$\begin{aligned}&\mathbb{E}\left[H_T \,|\, \mathcal{F}_t\right]\\ &= \left\{1 + \epsilon\,(X \diamond M)_t + \frac{\epsilon^2}{2}\,(M \diamond M)_t \right.\\ &\qquad \left. + \frac{\epsilon^2}{2}\,\left[(X \diamond M)_t\right]^2 + \epsilon^2\,(X \diamond (X \diamond M))_t + \mathcal{O}(\epsilon^3)\right\} \cdot H_t\end{aligned}$$

- We notice that
$$\begin{aligned}\mathbb{E}\left[H_T \,|\, \mathcal{F}_t\right] = \exp &\left\{\epsilon\,(X \diamond M)_t + \frac{\epsilon^2}{2}\,(M \diamond M)_t \right.\\ &\qquad \left. + \epsilon^2\,(X \diamond (X \diamond M))_t + \mathcal{O}(\epsilon^3)\right\} \cdot H_t,\end{aligned}$$
the exponential of a sum of `connected diagrams'.
- Motivated by exponentiation results in physics, we are tempted to see if something like this holds to all orders.

Freezing derivatives

Freezing the derivatives in (1) gives us the approximation

$$\begin{aligned}\mathbb{E}\left[H_T \,|\, \mathcal{F}_t\right] &\approx H_t + \mathbb{E}\left[\int_t^T d\langle X, M\rangle_s \middle| \mathcal{F}_t\right] \partial_{xw} H_t \\ &\quad + \frac{1}{2}\,\mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t\right] \partial_{ww} H_t \\ &= H_t + (X\diamond M)_t(T)\, H_t + \frac{1}{2}\,(M\diamond M)_t(T)\, H_t.\end{aligned}$$

- In Theorem 3.3 of [Alòs] for example the error in this approximation is bounded in the context of European option pricing.

The idea of the exponentiation theorem

- The essence of the exponentiation theorem we prove in [Alòs, Gatheral and Radoičić][2]</sup> is that we may express $\mathbb{E}\left[H_T \,|\, \mathcal{F}_t\right]$ as an exact expansion consisting of infinitely many terms, with derivatives in each such term frozen.

Trees

- Terms such as $(X\diamond M)$, $(M\diamond M)$ and $X\diamond(X\diamond M)$ are naturally indexed by trees, each of whose leaves corresponds to either X or M.

- We end up with diamond trees reminiscent of Feynman diagrams, with analogous rules.

Forests

The forest recursion

Let $\mathbb{F}_0 = M$. Then the higher order forests $\mathbb{F}_k$ are defined recursively as follows:

$$\mathbb{F}_k = \frac{1}{2}\sum_{i,j=0}^{k-2} \mathbb{1}_{i+j=k-2}\, \mathbb{F}_i \qquad \diamond$$

The first few terms

Applying the recursion, we have

$$
\begin{aligned}
\mathbb{F}_0 &= M \\
\mathbb{F}_1 &= \quad X \diamond \mathbb{F}_0 \\
&= (X \diamond M) \\
\mathbb{F}_2 &= \frac{1}{2}(\mathbb{F}_0 \diamond \mathbb{F}_0) + X \\
\diamond \mathbb{F}_1 &= \frac{1}{2}(M \\
&\diamond M) + X \\
&\diamond (X \diamond M) \\
\mathbb{F}_3 &= (\mathbb{F}_0 \diamond \mathbb{F}_1) + X \\
&\diamond \mathbb{F}_2 \\
&= M \diamond (X \diamond M) \\
&+ X \diamond \frac{1}{2}(M \\
&\diamond M) + X \\
&\diamond (X \diamond (X \diamond M))
\end{aligned}
$$

The exponentiation theorem

The exponentiation theorem

Let H_t be any solution of the Black-Scholes equation such that $\mathbb{E}\left[H_T \,|\, \mathcal{F}_t\right]$ is finite and the integrals contributing to each forest $\mathbb{F}_k, k \geq 0$ exist.

Then

$$\mathbb{E}\left[H_T \,|\, \mathcal{F}_t\right] = e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \cdot H_t.$$

If H_t is a characteristic function

Consider the Black-Scholes characteristic function

$$\Phi_t^T(a) = e^{\mathrm{i}\, a\, X_t - \frac{1}{2} a\, (a+\mathrm{i})\, w_t(T)}$$

- Applying $\mathbb{F}_k$ to Φ just multiplies Φ by some deterministic factor.
- Then

$$e^{\sum_{k=1}^{\infty} \mathbb{F}_k} \Phi_t^T(a) = e^{\sum_{k=1}^{\infty} \tilde{\mathbb{F}}_k(a)} \Phi_t^T(a)$$

where $\tilde{\mathbb{F}}_k(a)$ is $\mathbb{F}_k$ with each occurrence of ∂_x replaced with $\mathrm{i}\, a$ and each occurrence of ∂_w replaced with $-\frac{1}{2}\, a\, (a + \mathrm{i})$.

The characteristic function under stochastic volatility

Applying the Exponentiation Theorem, we have the following lemma.

Let

$$\varphi_t^T(a) = \mathbb{E}\left[e^{\mathrm{i}\, a\, X_T} \,\middle|\, \mathcal{F}_t\right]$$

be the characteristic function of the log stock price. Then

$$\varphi_t^T(a) = e^{\sum_{k=1}^{\infty}\, \tilde{\mathbb{F}}_k(a)}\, \Phi_t^T(a).$$

The cumulant generating function under stochastic volatility

As a corollary, the cumulant generating function (CGF) is given by

$$\psi_t^T(a) = \log \varphi_t^T(a) = \mathrm{i}\, a\, X_t - \frac{1}{2} a\,(a+\mathrm{i})\, w_t(T) + \sum_{k=1}^{\infty} \tilde{\mathbb{F}}_k(a).$$

- An explicit expression for the CGF for *any* stochastic volatility model!

Variance and gamma swaps

The variance swap is given by the fair value of the log-strip:

$$\mathbb{E}\left[X_T \,|\, \mathcal{F}_t\right] = (-\mathrm{i})$$

$$\psi_t^{T\prime}(0) = X_t - \frac{1}{2}$$

$$w_t(T)$$

and the gamma swap (wlog set $X_t = 0$) by

$$\mathbb{E}\left[X_T\, e^{X_T} \,\middle|\, \mathcal{F}_t\right]$$

$$= (-\mathrm{i})\, \frac{d}{da}\, \psi_t^T(a)\Big|_{a=-\mathrm{i}}$$

$$.$$

- The point is that we can in principle compute such moments for any stochastic volatility model written in forward variance form, whether or not there exists a closed-form expression for the characteristic function.

The gamma swap

We can compute the gamma swap as

$$\mathbb{E}\left[X_T\, e^{X_T} \,\middle|\, \mathcal{F}_t\right]$$

$$= (-\mathrm{i})\, \frac{d}{da}$$

$$\psi_t^T(a)\Big|_{a=-\mathrm{i}}.$$

It is easy to see that only trees containing a single M leaf will survive in the sum after differentiation when $a = -\mathrm{i}$ so that

$$\sum_{k=1}^{\infty} \tilde{\mathbb{F}}_k{}'(-\mathrm{i})$$

$$= \frac{1}{2}\, \sum_{k=1}^{\infty}$$

$$(X\diamond)^k M$$

where $(X\diamond)^k M$ is defined recursively for $k > 0$ as $(X\diamond)^k M = X.$

$$\diamond(X\diamond)^{k-1} M$$

- For example, $(X\diamond)^3 M.$

  $$= (X$$

  $$\diamond(X\diamond(X$$

  $$\diamond M)))$$

Then the fair value of a gamma swap is given by

$$\mathcal{G}_t(T) = 2\, \mathbb{E}\left[X_T\, e^{X_T} \middle| \mathcal{F}_t\right] = w_t(T) + \sum_{k=1}^{\infty} (X\diamond)^k M.$$

- This expression allows for explicit computation of the gamma swap for any model written in forward variance form.

The leverage swap

We deduce that the fair value of a leverage swap is given by

(2)

$$\mathcal{L}_t(T) = \mathcal{G}_t(T) - w_t(T) = \sum_{k=1}^{\infty} (X\diamond)^k M.$$

- The leverage swap is expressed explicitly in terms of covariance functionals of the spot and vol. processes.
  - If spot and vol. processes are uncorrelated, the fair value of the leverage swap is zero.
- The leverage swap may be easily estimated from the volatility smile along the lines of [Fukasawa][5] or alternatively by integration if we have fitted some curve to the smile.
  - We use two different Vola Dynamics (http://www.voladynamics.com (http://www.voladynamics.com)) curves below.
- We will now use (2) to compute an explicit expression for the value of a leverage swap in the rough Heston model.

The rough Heston model in forward variance form

Recall that in forward variance form, the rough Heaston model reads

$$\frac{dS_t}{S_t} = \sqrt{v_t}\, dZ_t$$

$$d\xi_t(u) = \frac{\nu}{\Gamma(\alpha)}\, \frac{\sqrt{v_t}}{(u-t)^\gamma}\, dW_t.$$

- The rough Heston model (with $\lambda = 0$ turns out to be even more tractable than the classical Heston model!

Computation of autocovariance functionals

Apart from $\mathcal{F}_t$ measurable terms (abbreviated as `drift'), we have

$$\begin{aligned} dX_t &= \sqrt{v_t}\, dZ_t + \text{drift} \\ dM_t &= \int_t^T d\xi_t(u)\, du \\ &= \frac{\nu}{\Gamma(\alpha)} \sqrt{v_t} \left(\int_t^T \frac{du}{(u-t)^\gamma}\right) dW_t \\ &= \frac{\nu\,(T-t)^\alpha}{\Gamma(1+\alpha)} \sqrt{v_t}\, dW_t. \end{aligned}$$

The first order forest

There is only one tree in the forest $\mathbb{F}_1$.

$$\begin{aligned} \mathbb{F}_1 = (X \diamond M)_t(T) &= \mathbb{E}\left[\int_t^T d\langle X, M\rangle_s \middle| \mathcal{F}_t\right] \\ &= \frac{\rho\,\nu}{\Gamma(1+\alpha)} \mathbb{E}\left[\int_t^T \frac{v_s}{(T-s)^\alpha}\, ds \middle| \mathcal{F}_t\right] \\ &= \frac{\rho\,\nu}{\Gamma(1+\alpha)} \int_t^T \frac{\xi_t(s)}{(T-s)^\alpha}\, ds. \end{aligned}$$

The second order forest

There are two trees in $\mathbb{F}_2$. The first tree is

$$(M \diamond M)_t(T) = \mathbb{E}\left[\int_t^T d\langle M, M\rangle_s \middle| \mathcal{F}_t\right] = \frac{\nu^2}{\Gamma(1+\alpha)^2} \int_t^T \xi_t(s)\,(T-s)^{2\,\alpha}\, ds.$$

The second tree $(X \diamond (X \diamond M))_t(T)$ is more complicated.

Define for $j \geq 0$

$$I_t^{(j)}(T) := \int_t^T ds\, \xi_t(s)\,(T-s)^{j\,\alpha}.$$

In terms of $I_t^{(j)}(T)$

We may then rewrite the above expressions as

$$(X \diamond M)_t(T) = \frac{\rho\,\nu}{\Gamma(1+\alpha)}\, I_t^{(1)}(T)$$

$$(M \diamond M)_t(T) = \frac{\nu^2}{\Gamma(1+\alpha)^2}\, I_t^{(2)}(T).$$

A little more computation gives

$$(X\diamond(X\diamond M))_t(T) = \mathbb{E}\left[\int_t^T d\langle X, I^{(1)}\rangle_s \middle| \mathcal{F}_t\right]$$

$$= \frac{\rho^2\,\nu^2}{\Gamma(1+\alpha)}\,\frac{\Gamma(1+\alpha)}{\Gamma(1+2\,\alpha)}\int_t^T ds\,\mathbb{E}\left[\int_s^T \frac{v_s}{(T-s)^{2\,\alpha}}\,ds \middle| \mathcal{F}_t\right]$$

$$= \frac{\rho^2\,\nu^2}{\Gamma(1+\alpha)\,\Gamma(\alpha)}\int_t^T ds\,\mathbb{E}\left[\int_s^T v_s\,\frac{(T-u)^{\alpha}}{(u-s)^{\gamma}}\,du \middle|\right.$$

$$\left| \mathcal{F}_t \right]$$

$$= \frac{\rho^2 \nu^2}{\Gamma(1+2\alpha)} \int_t^T ds\, \xi_t(s) (T-s)^{2\alpha} = \frac{\rho^2 \nu^2}{\Gamma(1+2\alpha)} I_t^{(2)}(T).$$

One can be easily convinced that each tree in the level-k forest $\mathbb{F}_k$ is $I^{(k)}$ multiplied by a simple prefactor.

The third order forest

For example, continuing to the forest $\mathbb{F}_3$, we have the following.

$$(M\diamond(X\diamond M))_t(T) = \frac{\rho\,\nu^3\,\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\,\Gamma(1+3\alpha)} I_t^{(3)}(T)$$

$$(X\diamond(X\diamond(X\diamond M)))_t(T) = \frac{\rho^3\,\nu^3}{\Gamma(1+3\alpha)} I_t^{(3)}(T)$$

$$(X\diamond(M\diamond M))_t(T) = \frac{\rho\,\nu^3\,\Gamma(1+2\alpha)}{\Gamma(1+\alpha)^2\,\Gamma(1+3\alpha)} I_t^{(3)}(T).$$

In particular, we easily identify the pattern

$$(X\diamond)^k M_t = \frac{(\rho\,\nu)^k}{\Gamma(1+k\,\alpha)} I_t^{(k)}(T).$$

The leverage swap under rough Heston

Using (2), we have

$$\mathcal{L}_t(T) = \sum_{k=1}^{\infty} (X\diamond)^k M$$
$$= \sum_{k=1}^{\infty} \frac{(\rho\,\nu)^k}{\Gamma(1+k\,\alpha)} \int_t^T du\,\xi_t(u)\,(T-u)^{k\,\alpha}$$
$$= \int_t^T du\,\xi_t(u)\,\{E_\alpha(\rho\,\nu\,(T-u)^\alpha) - 1\}$$

where $E_\alpha(\cdot)$ denotes the Mittag-Leffler function.

- A closed-form formula for the leverage swap!

The normalized leverage swap

Given the form of the expression for the leverage swap, it is natural to normalize by the variance swap. We therefore define

$$L_t(T) = \frac{\mathcal{L}_t(T)}{w_t(T)}.$$

In the special case of the rough Heston model with a flat forward variance curve,

$$L_t(T) = E_{\alpha,2}(\rho\,\nu\,\tau^\alpha) - 1,$$

where $E_{\alpha,2}(\cdot)$ is a generalized Mittag-Leffler function, independent of the reversion level θ. We further define an nth order approximation to $L_t(T)$ as

$$L_t^{(n)}(T) = \sum_{k=1}^{n} \frac{(\rho\,\nu\,\tau^\alpha)^k}{\Gamma(2+k\,\alpha)}.$$

Implement the approximate formulae

In [1]:

A numerical example

We now perform a numerical computation of the value of the leverage swap using the forest expansion in the rough Heston model with the following parameters, calibrated in [Roughening Heston][4] to the SPX options market as of May 19, 2017:

$$H = 0.0474;$$
$$\nu = 0.2910;$$
$$\rho = -0.6710.$$

In [2]:

Plot of successive approximations

In [3]:

In [4]:

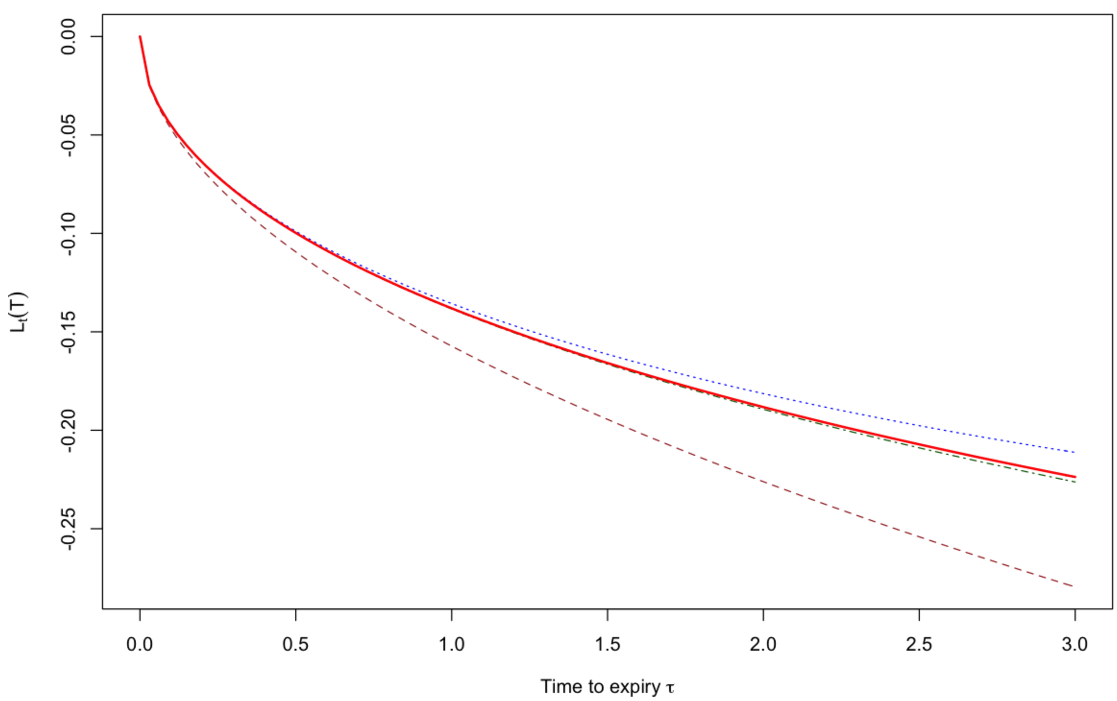

Figure 1: Successive approximations to the (absolute value of) the normalized rough Heston leverage swap. The solid red line is the exact expression $L_t(T)$; $L_t^{(1)}(T)$, $L_t^{(2)}(T)$, and $L_t^{(3)}(T)$ are brown dashed, blue dotted and dark green dash-dotted lines respectively.

Calibration of rough Heston using the leverage swap

We get leverage swap estimates from Vola Dynamics.

In [5]:

In [6]:

asOfTime	expiryTime	timeV	var	gam	lev	chi
20170519-160000.000-EDT	20170522-160000.000-EDT	0.008219178	4.611848e-05	4.529596e-05	-8.225198e-07	0.082512[illegible]
20170519-160000.000-EDT	20170524-160000.000-EDT	0.013698630	1.486447e-04	1.449538e-04	-3.690866e-06	0.042845[illegible]
20170519-160000.000-EDT	20170526-160000.000-EDT	0.019178082	2.525408e-04	2.442417e-04	-8.299132e-06	0.096151[illegible]
20170519-160000.000-EDT	20170530-160000.000-EDT	0.030136986	3.303599e-04	3.184876e-04	-1.187229e-05	0.031753[illegible]
20170519-160000.000-EDT	20170531-160000.000-EDT	0.032876712	3.890970e-04	3.741274e-04	-1.496953e-05	0.060300[illegible]
20170519-160000.000-EDT	20170602-160000.000-EDT	0.038356164	5.156240e-04	4.928132e-04	-2.281081e-05	0.048097[illegible]

In [7]:

Rough Heston parameter optimization

In [8]:

In [9]:

4669.56287459822

In [10]:

```
   user  system elapsed 
  0.024   0.000   0.024 
```

In [11]:

```
   user  system elapsed 
  0.032   0.000   0.032 
```

Notice how fast the calibration is!

The optimized parameters are:

In [12]:

$H
0.000835784396700211
$nu
0.373719584768268
$rho
-0.652211596123752

In [13]:

$H
0.0033274407426272
$nu
0.38222733264668
$rho
-0.652436353082184

Plot the Roughening Heston and optimized leverage swap fits

In [14]:

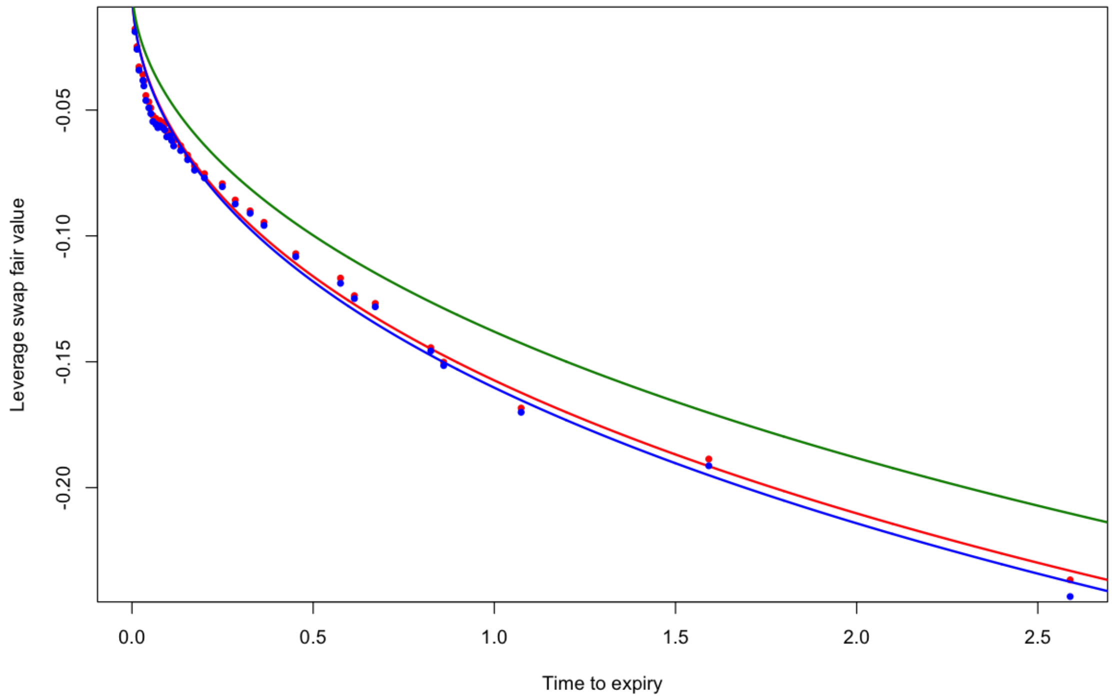

Figure 2: Vola Dynamics (http://www.voladynamics.com (http://www.voladynamics.com)) red and blue points are C14PM and C15PM estimates respectively; normalized l|everage swap fits with optimized parameters in red and blue respectively; with smile-calibrated parameters in green.

Summary of lecture 5

- There is a one-to-one correspondence between AFI models and AFV models.
  - Jaisson and Rosenbaum's rough Heston model is one example.
- To get a non-trivial stochastic volatility model as a limit, we need near-instability of the Hawkes kernel.
- Diamonds and the exponentiation theorem allow easy computation of model quantities that can be compared with market values
  - Easy calibration.
  - As many matching conditions as market option expirations.

Rough volatility summary

- Roughness of volatility appears to be universal.
  - The microstructural explanation is cool.
- Rough volatility models tend to be parsimonious yet consistent with both time series and implied volatility data.
- There should be many applications to trading.
- And of course the are many interesting mathematical problems.
- Rough volatility continues to be an active and fashionable research topic.

More resources: The Rough Volatility Network

- For an exhaustive list of papers and presentations on rough volatility and to keep up with the latest developments, see https://sites.google.com/site/roughvol/ (https://sites.google.com/site/roughvol/)

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In [ ]:
```