Pricing options under processes with unknown characteristic functions

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General framework

- We consider an asset price which is modelled as,

$$S_t = \mathbb{1}_{\{t<\zeta\}} e^{X_t}.$$

- The asset price is an exponential function of some process X_t.
- We allow for default in the asset price, and assume the default occurs with some intensity $\gamma(s, X_s)$.
- We are interested in pricing derivatives with the log-underlying some stochastic process X_t.

European options

Consider a European option with maturity time T. The payoff at T is given by $\Phi(T, S_T)$. The option value $v(t,x)$ is defined by

$$v(t,x) = E\left[e^{-\int_t^T r ds}\Phi(T, S_T)|X_t = x\right], \ t \in [0, T].$$

This can be rewritten using $S_t = \mathbb{1}_{\{\zeta > t\}} e^{X_t}$ as

$$v(t,x) = \mathbb{1}_{\{\zeta > t\}} E\left[e^{-\int_t^T (r+\gamma(s,X_s))ds}\phi(T, X_T)|X_t = x\right], \ t \in [0, T],$$

where we have defined $\phi(x) := \Phi(e^x)$.

Bermudan put option

Consider M exercise moments $\{t_1, ..., t_M\}$ with payoff at exercise time t_m to be $\phi(t_m, x)$. The option value $v(t, x)$ is defined recursively as

$$v(t_M, x) = \mathbb{1}_{\{\zeta > t_M\}}\phi(t_M, x),$$

and

$$\begin{cases} c(t,x) = E\left[e^{\int_t^{t_m}(r+\gamma(s,X_s))ds} v(t_m, X_{t_m}) | X_t = x\right], \ t \in [t_{m-1}, t_m[\\ v(t_{m-1}, x) = \mathbb{1}_{\{\zeta > t_{m-1}\}} \max\{\phi(t_{m-1}, x), c(t_{m-1}, x)\}, \ m \in \{2, \dots, M\}, \end{cases}$$

followed by

$$v(0, x) = c(0, x).$$

Computing the expected value

In order to compute the option price, we must evaluate functions of the form

$$u(t,x) := E\left[e^{-\int_t^T \gamma(s,X_s)ds}\phi(T,X_T)|X_t = x\right].$$

The function u can be represented as an integral with respect to the transition distribution of the defaultable log-price process $\log S$:

$$u(t,x) = \int_{\mathbb{R}} \phi(y)\Gamma(t,x;T,dy).$$

The characteristic function of $\log S$ is given by

$$\hat{\Gamma}(t,x;T,\xi) := \mathcal{F}(\Gamma(t,x;T,\cdot))(\xi) = \int_{\mathbb{R}} e^{i\xi y}\Gamma(t,x;T,dy), \quad \xi \in \mathbb{R}.$$

Least Squares Monte Carlo

1. Generate paths using a Monte Carlo simulation
2. Calculate continuation value in a backwards recursive manner for every path at every time step using a least-squares regression:

$$\hat{c}(t_m, x(\omega)) = \sum_{k=0}^{K} \alpha_{t_m}(k)\psi_k(x(\omega)),$$

   with ψ_k, $k = 0, ..., K$ a set of basis functions and the coefficients α_{t_m} chosen by fitting a regression between the discounted future payoffs and the current underlying values.
3. Set up cash flow matrix by comparing exercise and continuation: $\max(v(t_m, x(\omega)), c(t_m, x(\omega))$
4. Finally, the option price is the sum of discounted cash flows averaged over the paths.

Approximation for expected values

With the COS method we calculate expected values (integrals):

$$u(t,x) = \int_{\mathbb{R}} \phi(T,y)\Gamma(t,x;T,dy),$$
$$\approx \sum_{k=0}^{N-1}{}' \operatorname{Re}\left(e^{-ik\pi\frac{a}{b-a}}\hat{\Gamma}\left(t,x;T,\frac{k\pi}{b-a}\right)\right)V_k(T),$$

by truncating the integration to $[a,b]$, replacing the distribution with its cosine expansion and truncating the summation to N terms. Here $V_k(T)$ is the Fourier-cosine coefficient of the payoff function

$$V_k(T) = \frac{2}{b-a}\int_a^b \cos\left(k\pi\frac{y-a}{b-a}\right)\phi(T,y)dy,$$

and $\hat{\Gamma}$ is the characteristic function.

Lévy processes

- With exponential Lévy processes the asset price is modelled as an exponential function of a Lévy process L_t,

$$S_t = e^{L_t}.$$

- Each Lévy process can be characterised by a triplet (μ, σ, ν) with $\mu \in \mathbb{R}$, $\sigma \leq 0$ and ν a measure with $\nu(0) = 0$ and

$$\int_{\mathbb{R}} \min(1, |x|^2)\nu(dx) < \infty.$$

- For the Lévy process we have an explicit form of the characteristic function (Lévy-Khinchine formula)

$$\hat{\Gamma}(t, x, T, \zeta) = e^{(T-t)\left(i\mu\zeta - \frac{1}{2}\sigma^2\zeta^2 + \int_{\mathbb{R}}(e^{i\zeta x} - 1 - i\zeta x \mathbb{1}_{\{|x|<1\}}\nu(dx)\right)}.$$

Motivation Fourier methods

- Fourier methods are methods that are
  - computationally fast,
  - not restricted to Gaussian-based models,
  - work as long as we have the characteristic function (available for Lévy processes and Heston model).
- Problem: there are interesting dynamics, with lots of flexibility, for which we do not have explicit characteristic functions.
- We have to resort to techniques to approximate them.

Local Lévy process

We consider a defaultable asset S whose risk-neutral dynamics are given by:

$$S_t = \mathbb{1}_{\{t<\zeta\}} e^{X_t},$$
$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t + \int_{\mathbb{R}} d\tilde{N}_t(t, X_{t-}, dz)z,$$
$$d\tilde{N}_t(t, X_{t-}, dz) = dN_t(t, X_{t-}, dz) - \nu(t, X_{t-}, dz)dt,$$
$$\zeta = \inf\{t \geq 0 : \int_0^t \gamma(s, X_s)ds \geq \varepsilon\}, \quad (1)$$

where $\tilde{N}_t(t, x, dz)$ is a compensated random measure with state-dependent Lévy measure $\nu(t, x, dz)$ and $\varepsilon \sim \mathrm{Exp}(1)$ and is independent of X.

Approximating the characteristic function (Ruijter, Oosterlee, 2015)

- Suppose we have no jumps, i.e. $dX_t = \mu(X_t)dt + \sigma(X_t)dW_t$.
- Discretize the process, by e.g. Euler scheme, Milstein scheme or order 2.0 weak Taylor scheme.
- In a general form,

$$X_{m+1}^{\Delta} = x + m(x)\Delta t + s(x)\Delta W_{m+1} + k(x)(\Delta W_{m+1})^2, \quad X_m^{\Delta} = x.$$

- The characteristic function of X_{m+1}^{Δ} given $X_m^{\Delta} = x$ is given by

$$\hat{\Gamma}(t_m, x; t_{m+1}, \zeta) = e^{i\zeta x + i\zeta m(x)\Delta t - \frac{\frac{1}{2}\zeta^2 s^2(x)\Delta t}{1 - 2i\zeta k(x)\Delta t}} (1 - 2i\zeta k(x)\Delta t)^{-\frac{1}{2}}.$$

Adjoint expansion of the characteristic function

(Pagliarani, Pascucci, Riga, 2013)

The density $\Gamma(t, x; T, y)$ of a process solves the Cauchy problem

$$\begin{cases} L(t,x)\Gamma(t,x;T,y) = 0, & t \in [0,T[, \ x \in \mathbb{R}, \\ \Gamma(T,\cdot;T,y) = \delta_y, & x \in \mathbb{R}, \end{cases} \quad (2)$$

where $L(t, x)$ is the integro-differential operator (of the process)

$$\begin{aligned} L(t,x) = \partial_t + r\partial_x + \gamma(t,x)(\partial_x - 1) \\ + \frac{\sigma^2(t,x)}{2}(\partial_{xx} - \partial_x) - \int_{\mathbb{R}} \nu(t,x,dz)(e^z - 1 - z)\partial_x \\ + \int_{\mathbb{R}} \nu(t,x,dz)(e^{z\partial_x} - 1 - z\partial_x). \end{aligned}$$

A Taylor expansion of the coefficients

Use an expansion of the space-dependent coefficients in the operator L around some point $\bar{x}$.
Consider for simplicity only a local-volatility. Define

$$a(t,x) := \frac{\sigma^2(t,x)}{2}, \quad a_k = \frac{\partial_x^k a(\bar{x})}{k!}$$

The nth-order approximation of L is

$$L_n = L_0 + \sum_{k=1}^{n} \Big((x-\bar{x})^k a_k (\partial_{xx} - \partial_x) \Big),$$
$$L_0 = \partial_t + r\partial_x + a_0(\partial_{xx} - \partial_x).$$

Notice that

$$L_h - L_{h-1} = (x-\bar{x})^h a_h (\partial_{xx} - \partial_x).$$

Cauchy problems of the expansion

The nth-order approximation of Γ is defined as

$$\Gamma^{(n)}(t,x;T,y)=\sum_{k=0}^{n}G^k(t,x;T,y),$$

with G^0 solving

$$\begin{cases} L_0 G^0(t,x;T,y)=0, \\ G^0(T,\cdot;T,y)=\delta_y. \end{cases}$$

and G^k for $k\geq 1$ defined through

$$\begin{cases} L_0 G^k(t,x;T,y)=-\displaystyle\sum_{h=1}^{k}(L_h-L_{h-1})G^{k-h}(t,x;T,y), \\ G^k(T,x;T,y)=0. \end{cases}$$

for $t\in[0,T[,\ x\in\mathbb{R}$

Solving the Adjoint Cauchy problems in Fourier space

The nth-order approximation of the characteristic function $\hat{\Gamma}$ is defined to be

$$\hat{\Gamma}^{(n)}(t,x;T,\xi) = \sum_{k=0}^{n} \mathcal{F}\left(G^k(t,x;T,\cdot)\right)(\xi) := \sum_{k=0}^{n} \hat{G}^k(t,x;T,\xi), \ \xi \in \mathbb{R}.$$

Note that Fourier transform is taken with respect to (T,y), but L acts on (t,x). We will:

- Define the functions $G^0(t,x;\cdot,\cdot)$ and $G^k(t,x;\cdot,\cdot)$, $k \geq 1$ through the Cauchy problems with the adjoint operator $\tilde{L}_0^{(T,y)}$ and $\tilde{L}_h^{(T,y)} - \tilde{L}_{h-1}^{(T,y)}$.
- Solve the adjoint Cauchy problems in the Fourier space. This immediately gives $\hat{\Gamma}$.

Theorem (Dual formulation)

The function $G^0(t, x; \cdot, \cdot)$ is defined through the following dual Cauchy problem

$$\begin{cases} \tilde{L}_0^{(T,y)} G^0(t, x; T, y) = 0 & T > t, \ y \in \mathbb{R}, \\ G^0(T, x; T, \cdot) = \delta_x. \end{cases}$$

For any $k \geq 1$the function $G^k(t, x; \cdot, \cdot)$ is defined through

$$\begin{cases} \tilde{L}_0^{(T,y)} G^k(t, x; T, y) = -\sum_{h=1}^{k} \left(\tilde{L}_h^{(T,y)} - \tilde{L}_{h-1}^{(T,y)} \right) G^{k-h}(t, x; T, y) \\ G^k(T, x; T, y) = 0 \end{cases}$$

with $\tilde{L}_0^{(T,y)}$ and $\tilde{L}_h^{(T,y)} - \tilde{L}_{h-1}^{(T,y)}$ being the adjoint operators.

Solution in Fourier space

We have

$$\tilde{L}_0^{(T,y)} = -\partial_T - r\partial_y + a_0(\partial_{yy} + \partial_y).$$

Then

$$\mathcal{F}\left(\tilde{L}_0^{(T,\cdot)} G^k(t,x;T,\cdot)\right)(\xi) = \psi(\xi)\hat{G}^k(t,x;T,\xi) - \partial_T \hat{G}^k(t,x;T,\xi),$$

where

$$\psi(\xi) = i\xi r + a_0(-\xi^2 - i\xi).$$

Then the solution to the adjoint Cauchy problems is given by

$$\hat{G}^0(t,x;T,\xi) = e^{i\xi x + (T-t)\psi(\xi)},$$

$$\hat{G}^k(t,x;T,\xi) = -\int_t^T e^{\psi(\xi)(T-s)} \mathcal{F}\left(\sum_{h=1}^{k}\left(\tilde{L}_h^{(s,\cdot)} - \tilde{L}_{h-1}^{(s,\cdot)}\right) G^{k-h}(t,x;s,\cdot)\right)(\xi) ds.$$

The characteristic function

The approximation of order n of the characteristic function is of the form

$$\hat{\Gamma}^{(n)}(t, x; T, \xi) := e^{i\xi x} \sum_{h=0}^{n} (x - \bar{x})^h g_{n,h}(t, T, \xi),$$

where the coefficients $g_{n,h}$, with $0 \leq h \leq n$, depend only on t, T and ξ, but not on x.

Back to the Bermudan option valuation [1/2]

Remember we had to value the continuation value of the form:

$$\hat{c}(t,x) = e^{-r(t_{m+1}-t)} \sum_{k=0}^{N-1}{}' \operatorname{Re}\left(e^{-ik\pi\frac{a}{b-a}}\hat{\Gamma}\left(t,x;t_{m+1},\frac{k\pi}{b-a}\right)\right) V_k(t_{m+1}),$$

$$V_k(t_m) = \frac{2}{b-a}\int_a^b \cos\left(k\pi\frac{y-a}{b-a}\right)\max\{\phi(t_m,y),c(t_m,y)\}dy.$$

We can rewrite

$$V_k(t_m) = \frac{2}{b-a}\int_{x_m^*}^b \cos\left(k\pi\frac{y-a}{b-a}\right)c(t_m,y)dy + C_k,$$

with x_m^* being the early-exercise point such that
$c(t_m,x_m^*) = \phi(t_m,x_m^*)$.

Back to the Bermudan option valuation [2/2]

Inserting $\hat{c}(t,x)$ into the formula for $V_k(t_m)$ we find in vectorized form:

$$\hat{\mathbf{V}}(t_m) = \sum_{h=0}^{n} e^{-r(t_{m+1}-t_m)} \mathrm{Re}\left(\mathcal{M}^h(x_m^*, b)\boldsymbol{u}^h\right) + \mathbf{C}, \qquad (3)$$

with

$$M_{k,j}^h(x_m^*, b) = \frac{2}{b-a}\int_{x_m^*}^{b} e^{ij\pi\frac{x-a}{b-a}}(x-\bar{x})^h \cos\left(k\pi\frac{x-a}{b-a}\right) dx \qquad (4)$$

The matrix-vector multiplication $\mathcal{M}(x_m^*, b)\boldsymbol{u}$ can be calculated using a fast Fourier transform.

A quick example

Consider a process under the CEV-Merton dynamics with local vol. and Gaussian jumps.

Table: Prices for a European and a Bermudan Put option ($T = 1$ and 10 exercise dates) in the CEV-Merton model for the 2nd-order approximation of the characteristic function, and a Monte Carlo method.

	European		Bermudan	
K	MC 95% c.i.	Value	MC 95% c.i.	Value
0.8	0.02526-0.02622	0.02581	0.02617-0.02711	0.02520
1	0.08225-0.08395	0.08250	0.08480-0.08640	0.08593
1.2	0.1965-0.1989	0.1977	0.2097-0.2115	0.2132
1.4	0.3560-0.3589	0.3574	0.3946-0.3957	0.3954
1.6	0.5341-0.5385	0.5364	0.5930-0.5941	0.5932

Other applications: Gas storage pricing

- Optimal operation of a storage facility amounts to finding the optimal times to inject and withdraw gas, depending on the current and expected spot/futures prices.
- The contract then allows the holder to take an action u_n at any time t_n, $n = 1, \dots N - 1$.
- Injection at time t_n as a positive volume change Δv_n and a withdrawal as a negative volume change Δv_n.
- The volume in the storage tank satisfies a constraint, $v_n^{\min} \leq v_n \leq v_n^{\max}$.
- The withdrawal rate is assumed to satisfy, $\alpha^w(n, v_n) \leq \Delta v_n \leq \alpha^i(n, v_n)$, with α^w the (negative) maximum withdrawal rate, and α^i the (positive) maximum withdrawal rate.

Pricing gas contracts

- Define the set of allowed actions at time t_n given the volume v_n to be,

$$\mathcal{D}(n, v_n) := \left\{ \Delta v | v_{n+1}^{\min} \leq v_n + \Delta v \leq v_{n+1}^{\max}, \text{ and } \alpha^{w}(n, v_n) \leq \Delta v \leq \alpha^{i}(n, v_n) \right\}.$$

- Denote the value of a storage contract starting at time t_n with volume v_n by $u(n, S_n, v_n)$, the payoff after taking some action as $h(S_n, \Delta v)$ and define the continuation value $c(n, S_n, v_{n+1})$ as the value we attach to the contract after taking an allowed action $\Delta v \in \mathcal{D}(n, v_n)$,

$$c(n, S_n, v_{n+1}) := \mathbb{E}_n \left[e^{-r\Delta t} u(n+1, S_{n+1}, v_n + \Delta v) \right].$$

- Then we find the dynamic programming backwards recursion,

$$u(N, S_N, v_N) = q(S_N, v_N),$$
$$u(n, S_n, v_n) = \max_{\Delta v \in \mathcal{D}(n, v_n)} \left(h(S_n, \Delta v) + c(n, S_n, v_{n+1}) \right), \quad n = N-1, ..., 0.$$

- We can solve this using a Least Squares Monte Carlo or as usual the COS method approach.

The challenges

- The spot price is typically a complex process; seasonality, mean-reversion, price spikes should be included.
- Previous work included modelling the asset price as a time in-homogeneous exponential Lévy process (Safarov and Atkinson (2017)).
- Alternatively, the local Lévy model might be of interest to use.
- The gas storage value is very sensitive to the modeling assumptions. Therefore, a good asset model is of the essence.
- Future work: combining the COS method with an asset model for which we can approximate the characteristic function for fast and efficient gas storage valuation.

For Further Reading

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