Long Run Investment

Paolo Guasoni

Boston University and Dublin City University

January 21-23, 2019
18^{th} Winter School on Mathematical Finance
Congrescentrum De Werelt, Lunteren

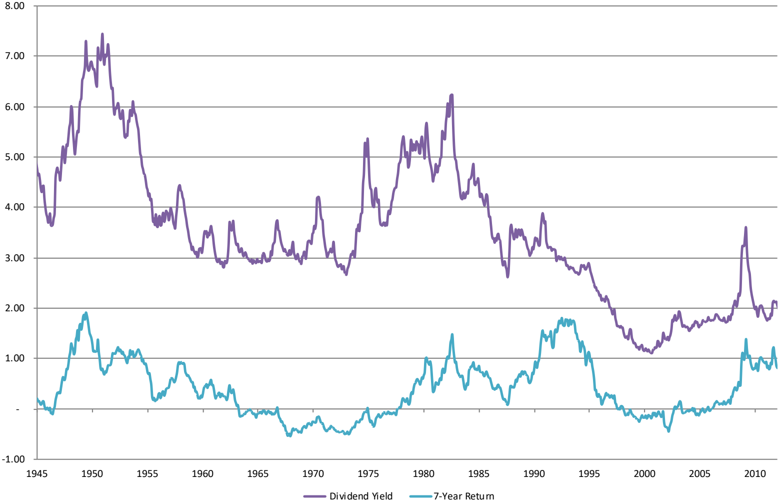
Outline

- Long Run and Stochastic Investment Opportunities
  (based on work with Scott Robertson).
- Shortfall Aversion
  (based on work with Gur Huberman and Dan Ren).
- Consumption, Investment, and Healthcare
  (based on work with Yu-Jui Huang).
- Commodities and Stationary Risks
  (based on work with Gu Wang and Antonella Tolomeo).
- Leveraged Funds
  (based on work with Eberhard Mayerhofer).

One

Long Run and Stochastic Investment Opportunities

Independent Returns?



- Higher yields tend to be followed by higher long-term returns.
- Should not happen if returns independent!

Utility Maximization

- Basic portfolio choice problem: maximize utility from terminal wealth:

$$\max_{\pi} E[U(X_T^\pi)]$$

- Easy for logarithmic utility $U(x) = \log x$.
  Myopic portfolio $\pi = \Sigma^{-1}\mu$ optimal. Numeraire argument.
- Portfolio does not depend on horizon (even random!), and on the dynamics of the the state variable, but only its current value.
- But logarithmic utility leads to counterfactual predictions.
  And implies that unhedgeable risk premia are all zero.
- Power utility $U(x) = x^{1-\gamma}/(1-\gamma)$ is more flexible.
  Portfolio no longer myopic. Risk premia nonzero, and depend on γ.
- Power utility far less tractable.
  Joint dependence on horizon and state variable dynamics.
- Explicit solutions few and cumbersome.
- Goal: keep dependence from state variable dynamics, lose from horizon.
- Tool: assume long horizon.

Asset Prices and State Variables

- Safe asset $S^0_t = \exp\left(\int_0^t r(Y_s)ds\right)$, d risky assets, k state variables.

$$
\begin{aligned}
\frac{dS^i_t}{S^i_t} &= r(Y_t)dt + dR^i_t & 1 \le i \le d \\
dR^i_t &= \mu_i(Y_t)dt + \sum_{j=1}^{n} \sigma_{ij}(Y_t)dZ^j_t & 1 \le i \le d \\
dY^i_t &= b_i(Y_t)dt + \sum_{j=1}^{k} a_{ij}(Y_t)dW^j_t & 1 \le i \le k \\
d\langle Z^i, W^j\rangle_t &= \rho_{ij}(Y_t)dt & 1 \le i \le d, 1 \le j \le k
\end{aligned}
$$

- Z, W Brownian Motions.
- $\Sigma(y) = (\sigma\sigma')(y)$, $\Upsilon(y) = (\sigma\rho a')(y)$, $A(y) = (aa')(y)$.

Assumption

$r \in C^\gamma(E,\mathbb{R})$, $b \in C^{1,\gamma}(E,\mathbb{R}^k)$, $\mu \in C^{1,\gamma}(E,\mathbb{R}^n)$, $A \in C^{2,\gamma}(E,\mathbb{R}^{k\times k})$, $\Sigma \in C^{2,\gamma}(E,\mathbb{R}^{n\times n})$ and $\Upsilon \in C^{2,\gamma}(E,\mathbb{R}^{n\times k})$. The symmetric matrices A and Σ are strictly positive definite for all $y \in E$. Set $\bar{\Sigma} = \Sigma^{-1}$

State Variables

- Investment opportunities:
  safe rate r, excess returns μ, volatilities σ, and correlations ρ.
- State variables: anything on which investment opportunities depend.
- Example with predictable returns:

$$\begin{aligned} dR_t &= Y_t dt + \sigma dZ_t \\ dY_t &= -\lambda Y_t dt + dW_t \end{aligned}$$

- State variable is expected return. Oscillates around zero.
- Example with stochastic volatility:

$$\begin{aligned} dR_t &= \nu Y_t dt + \sqrt{Y_t} dZ_t \\ dY_t &= \kappa(\theta - Y_t) dt + a\sqrt{Y_t} dW_t \end{aligned}$$

- State variable is squared volatility. Oscillates around positive value.
- State variables are generally stationary processes.

(In)Completeness

- $\Upsilon'\Sigma^{-1}\Upsilon$: covariance of hedgeable state shocks:
  Measures degree of market completeness.
- $A = \Upsilon'\Sigma^{-1}\Upsilon$: complete market.
  State variables perfectly hedgeable, hence replicable.
- $\Upsilon = 0$: fully incomplete market.
  State shocks orthogonal to returns.
- Otherwise state variable partially hedgeable.
- One state: $\Upsilon'\Sigma^{-1}\Upsilon/a^2 = \rho'\rho$.
  Equivalent to R^2 of regression of state shocks on returns.

Well Posedness

Assumption

There exists unique solution $\left(P^{(r,y)}\right)_{r\in\mathbb{R}^n, y\in E}$ *to martingale problem:*

$$L = \frac{1}{2}\sum_{i,j=1}^{n+k} \tilde{A}^{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n+k} \tilde{b}^i(x)\frac{\partial}{\partial x_i} \quad \tilde{A} = \begin{pmatrix} \Sigma & \Upsilon \\ \Upsilon' & A \end{pmatrix} \quad \tilde{b} = \begin{pmatrix} \mu \\ b \end{pmatrix}$$

- $\Omega = C([0,\infty), \mathbb{R}^{n+k})$ with uniform convergence on compacts.
- $\mathcal{B}$ Borel σ-algebra, $(\mathcal{B}_t)_{t\geq 0}$ natural filtration.

Definition

$(P^x)_{x\in\mathbb{R}^n\times E}$ on $(\Omega,\mathcal{B})$ solves martingale problem if, for all $x \in \mathbb{R}^n \times E$:

- $P^x(X_0 = x) = 1$
- $P^x(X_t \in \mathbb{R}^n \times E, \forall t \geq 0) = 1$
- $f(X_t) - f(X_0) - \int_0^t (Lf)(X_u)du$ is P^x-martingale for all $f \in C_0^2(\mathbb{R}^n \times E)$

Trading and Payoffs

Definition

Trading strategy: process $(\pi_t^i)_{t \geq 0}^{1 \leq i \leq d}$, adapted to $\mathcal{F}_t = \mathcal{B}_{t+}$, the right-continuous envelope of the filtration generated by (R, Y), and R-integrable.

- Investor trades without frictions. Wealth dynamics:

$$\frac{dX_t^\pi}{X_t^\pi} = r(Y_t)dt + \pi_t' dR_t$$

- In particular, $X_t^\pi \geq 0$ a.s. for all t.
  Admissibility implied by R-integrability.

Equivalent Safe Rate

- Maximizing power utility $E\left[(X_T^\pi)^{1-\gamma}\right]/(1-\gamma)$ equivalent to maximizing the certainty equivalent $E\left[(X_T^\pi)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$.
- Observation: in most models of interest, wealth grows exponentially with the horizon. And so does the certainty equivalent.
- Example: with r, μ, Σ constant, the certainty equivalent is exactly $\exp\left((r + \frac{1}{2\gamma}\mu'\Sigma^{-1}\mu)T\right)$. Only total Sharpe ratio matters.
- Intuition: an investor with a long horizon should try to maximize the rate at which the certainty equivalent grows:

$$\beta = \max_\pi \liminf_{T\to\infty} \frac{1}{T} \log E\left[(X_T^\pi)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

- Imagine a “dream” market, without risky assets, but only a safe rate ρ.
- If $\beta > \rho$, an investor with long enough horizon prefers the dream.
- If $\beta < \rho$, he prefers to wake up.
- At $\beta = \rho$, his dream comes true.

Equivalent Annuity

- Exponential utility $U(x) = -e^{-\alpha x}$ leads to a similar, but distinct idea.
- Suppose the safe rate is zero.
- Then optimal wealth typically grows linearly with the horizon, and so does the certainty equivalent.
- Then it makes sense to consider the *equivalent annuity*:

$$\beta = \max_{\pi} \liminf_{T\to\infty} -\frac{1}{\alpha T} \log E\left[e^{-\alpha X_T^\pi}\right]$$

- The dream market now does not offer a higher safe rate, but instead a stream of fixed payments, at rate ρ. The safe rate remains zero.
- The investor is indifferent between dream and reality for $\beta = \rho$.
- For positive safe rate, use definition with discounted quantities.
- Undiscounted equivalent annuity always infinite with positive safe rate.

Solution Strategy

- Duality Bound.
- Stationary HJB equation and finite-horizon bounds.
- Criteria for long-run optimality.

Stochastic Discount Factors

Definition

Stochastic discount factor: strictly positive adapted $M = (M_t)_{t\geq 0}$, such that:

$$E^y_P\left[M_t S^i_t \middle| \mathcal{F}_s\right] = M_s S^i_s \qquad \text{for all } 0 \leq s \leq t, 0 \leq i \leq d$$

Martingale measure: probability Q, such that $Q|_{\mathcal{F}_t}$ and $P^y|_{\mathcal{F}_t}$ equivalent for all $t \in [0, \infty)$, and discounted prices S^i/S^0 Q-martingales for $1 \leq i \leq d$.

- Martingale measures and stochastic discount factors related by:

$$\frac{dQ}{dP^y}\Big|_{\mathcal{F}_t} = \exp\left(\int_0^t r(Y_s)ds\right) M_t$$

- Local martingale property: all stochastic discount factors satisfy

$$M^\eta_t = \exp\left(-\int_0^t r dt\right) \mathcal{E}\left(-\int_0^\cdot (\mu'\Sigma^{-1} + \eta'\Upsilon'\Sigma^{-1})\sigma dZ + \int_0^\cdot \eta' a dW\right)_t$$

  for some adapted, $\mathbb{R}^k$-valued process η.
- η represents the vector of *unhedgeable risk premia*.
- Intuitively, the Sharpe ratios of shocks orthogonal to dR.

Duality Bound

- For any payoff $X = X_T^\pi$ and any discount factor $M = M_T^\eta$, $E[XM] \leq x$. Because XM is a local martingale.
- Duality bound for power utility:

$$E\left[X^{1-\gamma}\right]^{\frac{1}{1-\gamma}} \leq xE\left[M^{1-1/\gamma}\right]^{\frac{\gamma}{1-\gamma}}$$

- Proof: exercise with Hölder's inequality.
- Duality bound for exponential utility:

$$-\frac{1}{\alpha}\log E\left[e^{-\alpha X}\right] \leq \frac{x}{E[M]} + \frac{1}{\alpha}E\left[\frac{M}{E[M]}\log\frac{M}{E[M]}\right]$$

- Proof: Jensen inequality under risk-neutral densities.
- Both bounds true for any X and for any M. Pass to sup over X and inf over M.
- Note how α disappears from the right-hand side.
- Both bounds given in terms of certainty equivalents.
- As $T \to \infty$, bounds for equivalent safe rate and annuity follow.

Long Run Optimality

Definition (Power Utility)

An admissible portfolio π is *long run optimal* if it solves:

$$\max_{\pi} \liminf_{T\to\infty} \frac{1}{T} \log E\left[(X_T^\pi)^{1-\gamma}\right]^{\frac{1}{1-\gamma}}$$

The risk premia η are long run optimal if they solve:

$$\min_{\eta} \limsup_{T\to\infty} \frac{1}{T} \log E\left[(M_T^\eta)^{1-1/\gamma}\right]^{\frac{\gamma}{1-\gamma}}$$

Pair (π, η) long run optimal if both conditions hold, and limits coincide.

- Easier to show that (π, η) long run optimal together.
- Each η is an upper bound for all π and vice versa.

Definition (Exponential Utility)

Portfolio π and risk premia η *long run optimal* if:

$$\max_{\pi} \liminf_{T\to\infty} -\log E\left[e^{-\alpha X_T^\pi}\right] \qquad \min_{\eta} \limsup_{T\to\infty} E\left[\frac{M}{E[M]} \log \frac{M}{E[M]}\right]$$

HJB Equation

- $V(t, x, y)$ depends on time t, wealth x, and state variable y.
- Itô's formula:

$$dV(t, X_t, Y_t) = V_t dt + V_x dX_t + V_y dY_t + \frac{1}{2}\left(V_{xx} d\langle X\rangle_t + V_{xy} d\langle X, Y\rangle_t + V_{yy} d\langle Y\rangle_t\right)$$

- Vector notation. V_y, V_{xy} k-vectors. V_{yy} $k \times k$ matrix.
- Wealth dynamics:

$$dX_t = (r + \pi_t' \mu_t) X_t dt + X_t \pi_t' \sigma_t dZ_t$$

- Drift reduces to:

$$V_t + xV_x r + V_y b + \frac{1}{2}\operatorname{tr}(V_{yy} A) + x\pi'\left(\mu V_x + \Upsilon V_{xy}\right) + \frac{x^2}{2} V_{xx} \pi' \Sigma \pi$$

- Maximizing over π, the optimal value is:

$$\pi = -\frac{V_x}{xV_{xx}} \Sigma^{-1} \mu - \frac{V_{xy}}{xV_{xx}} \Sigma^{-1} \Upsilon$$

- Second term is new. Interpretation?

Intertemporal Hedging

- $\pi = -\frac{V_x}{xV_{xx}}\Sigma^{-1}\mu - \Sigma^{-1}\Upsilon\frac{V_{xy}}{xV_{xx}}$
- First term: optimal portfolio if state variable frozen at current value.
- Myopic solution, because state variable will change.
- Second term hedges shifts in state variables.
- If risk premia covary with Y, investors may want to use a portfolio which covaries with Y to control its changes.
- But to reduce or increase such changes? Depends on preferences.
- When does second term vanish?
- Certainly if $\Upsilon = 0$. Then no portfolio covaries with Y.
  Even if you want to hedge, you cannot do it.
- Also if $V_y = 0$, like for constant r, μ and σ.
  But then state variable is irrelevant.
- Any other cases?

HJB Equation

- Maximize over π, recalling $\max(\pi' b + \frac{1}{2}\pi' A \pi = -\frac{1}{2} b' A^{-1} b)$.
  HJB equation becomes:

$$V_t + xV_x r + V_y b + \frac{1}{2}\operatorname{tr}(V_{yy}A) - \frac{1}{2}(\mu V_x + \Upsilon V_{xy})' \frac{\Sigma^{-1}}{V_{xx}} (\mu V_x + \Upsilon V_{xy}) = 0$$

- Nonlinear PDE in $k+2$ dimensions. A nightmare even for $k=1$.
- Need to reduce dimension.
- Power utility eliminates wealth x by homogeneity.

Homogeneity

- For power utility, $V(t,x,y) = \frac{x^{1-\gamma}}{1-\gamma} v(t,y)$.

$$V_t = \frac{x^{1-\gamma}}{1-\gamma} v_t \qquad V_x = x^{-\gamma} v \qquad V_{xx} = -\gamma x^{-\gamma-1} v$$

$$V_{xy} = x^{-\gamma} v_y \qquad V_y = \frac{x^{1-\gamma}}{1-\gamma} v_y \qquad V_{yy} = \frac{x^{1-\gamma}}{1-\gamma} v_{yy}$$

- Optimal portfolio becomes:

$$\pi = \frac{1}{\gamma}\Sigma^{-1}\mu + \frac{1}{\gamma}\Sigma^{-1}\Upsilon\frac{v_y}{v}$$

- Plugging in, HJB equation becomes:

$$v_t + (1-\gamma)\left(r + \frac{1}{2\gamma}\mu'\Sigma^{-1}\mu\right) v + \left(b + \frac{1-\gamma}{\gamma}\Upsilon'\Sigma^{-1}\mu\right) v_y + \frac{1}{2}\operatorname{tr}(v_{yy}A) + \frac{1-\gamma}{\gamma}\frac{v_y'\Upsilon'\Sigma^{-1}\Upsilon v_y}{2v} = 0$$

- Nonlinear PDE in $k+1$ variables. Still hard to deal with.

Long Run Asymptotics

- For a long horizon, use the guess $v(t, y) = e^{(1-\gamma)(\beta(T-t)+w(y))}$.
- It will never satisfy the boundary condition. But will be close enough.
- Here β is the equivalent safe, to be found.
- We traded a function $v(t, y)$ for a function $w(y)$, plus a scalar β.
- The HJB equation becomes:

$$\left(-\beta + r + \frac{1}{2\gamma}\mu'\Sigma^{-1}\mu\right) + \left(b + \frac{1-\gamma}{\gamma}\Upsilon'\Sigma^{-1}\mu\right) w_y + \frac{1}{2}\operatorname{tr}(w_{yy}A) + \frac{1-\gamma}{2}w_y'\left(A - \frac{1-\gamma}{\gamma}\Upsilon'\Sigma^{-1}\Upsilon\right) w_y = 0$$

- And the optimal portfolio:

$$\pi = \frac{1}{\gamma}\Sigma^{-1}\mu + \left(1 - \tfrac{1}{\gamma}\right)\Sigma^{-1}\Upsilon w_y$$

- Stationary portfolio. Depends on state variable, not horizon.
- HJB equation involved gradient w_y and Hessian w_{yy}, but not w.
- With one state, first-order ODE.
- Optimality? Accuracy? Boundary conditions?

Example

- Stochastic volatility model:

$$dR_t = \nu Y_t dt + \sqrt{Y_t} dZ_t$$
$$dY_t = \kappa(\theta - Y_t) dt + \varepsilon \sqrt{Y_t} dW_t$$

- Substitute values in stationary HJB equation:

$$\left(-\beta + r + \frac{\nu^2}{2\gamma} y\right) + \left(\kappa(\theta - y) + \frac{1-\gamma}{\gamma} \rho \varepsilon \nu y\right) w_y + \frac{\varepsilon^2}{2} w_{yy} + (1-\gamma) \frac{\varepsilon^2 y}{2} w_y^2 \left(1 - \frac{1-\gamma}{\gamma} \rho^2\right) = 0$$

- Try a linear guess $w = \lambda y$. Set constant and linear terms to zero.
- System of equations in β and λ:

$$-\beta + r + \kappa\theta\lambda = 0$$
$$\lambda^2 (1-\gamma) \frac{\epsilon^2}{2} \left(1 - \frac{1-\gamma}{\gamma} \rho^2\right) + \lambda \left(\frac{1-\gamma}{\gamma} \epsilon \nu \rho - \kappa\right) + \frac{\nu^2}{2\gamma} = 0$$

- Second equation quadratic, but only larger solution acceptable. Need to pick largest possible β.

Example (continued)

- Optimal portfolio is constant, but not the usual constant.

$$\pi = \frac{1}{\gamma}(\nu + \beta\rho\varepsilon)$$

- Hedging component depends on various model parameters.
- Hedging is zero if $\rho = 0$ or $\varepsilon = 0$.
- $\rho = 0$: hedging impossible. Returns do covary with state variable.
- $\varepsilon = 0$: hedging unnecessary. State variable deterministic.
- Hedging zero also if $\beta = 0$, which implies logarithmic utility.
- Logarithmic investor does not hedge, even if possible.
- Lives every day as if it were the last one.
- Equivalent safe rate:

$$\beta = \frac{\theta\nu^2}{2\gamma}\left(1 - \left(1 - \frac{1}{\gamma}\right)\frac{\nu\rho}{\kappa}\varepsilon\right) + O(\varepsilon^2)$$

- Correction term changes sign as γ crosses 1.

Martingale Measure

- Many martingale measures. With incomplete market, local martingale condition does not identify a single measure.
- For any arbitrary k-valued η_t, the process:

$$M_t = \mathcal{E}\left(-\int_0^\cdot (\mu'\Sigma^{-1} + \eta'\Upsilon'\Sigma^{-1})\sigma dZ + \int_0^\cdot \eta' a dW\right)_t$$

  is a local martingale such that MR is also a local martingale.
- Recall that $M_T = yU'(X_T^\pi)$.
- If local martingale M is a martingale, it defines a stochastic discount factor.
- $\pi = \frac{1}{\gamma}\Sigma^{-1}(\mu + \Upsilon(1-\gamma)w_y)$ yields:

$$\begin{aligned} U'(X_T^\pi) = (X_T^\pi)^{-\gamma} &= x^{-\gamma} e^{-\gamma\int_0^T(\pi'\mu - \frac{1}{2}\pi'\Sigma\pi)dt - \gamma\int_0^T \pi'\sigma dZ_t} \\ &= e^{-\int_0^T(\mu' + (1-\gamma w_y)\Upsilon')\Sigma^{-1}\sigma dZ_t + \int_0^T(\dots)dt} \end{aligned}$$

- Mathing the two expressions, we guess $\eta = (1-\gamma)w_y$.

Risk Neutral Dynamics

- To find dynamics of R and Y under Q, recall Girsanov Theorem.
- If M_t has previous representation, dynamics under Q is:

$$dR_t = \sigma d\tilde{Z}_t$$
$$dY_t = (b - \Upsilon'\Sigma^{-1}\mu + (A - \Upsilon'\Sigma^{-1}\Upsilon)\eta)dt + ad\tilde{W}_t$$

- Since $\eta = (1-\gamma)w_y$, it follows that:

$$dR_t = \sigma d\tilde{Z}_t$$
$$dY_t = \left(b - \Upsilon'\Sigma^{-1}\mu + (A - \Upsilon'\Sigma^{-1}\Upsilon)(1-\gamma)w_y\right) dt + ad\tilde{W}_t$$

- Formula for (long-run) risk neutral measure for a given risk aversion.
- For $\gamma = 1$ (log utility) boils down to minimal martingale measure.
- Need to find w to obtain explicit solution.
- And need to check that above martingale problem has global solution.

Exponential Utility

- Instead of homoheneity, recall that wealth factors out of value function.
- Long-run guess: $V(x, y, t) = e^{-\alpha x + \alpha \beta t + w(y)}$.
  β is now equivalent annuity.
- Set $r = 0$, otherwise safe rate wipes out all other effects.
- The HJB equation becomes:

$$\left(-\beta + \frac{1}{2}\mu'\Sigma^{-1}\mu\right) + \left(b - \Upsilon'\Sigma^{-1}\mu\right) w_y + \frac{1}{2}\operatorname{tr}(w_{yy}A) - \frac{1}{2}w_y'\left(A - \Upsilon'\Sigma^{-1}\Upsilon\right) w_y = 0$$

- And the optimal portfolio:

$$x\pi = \frac{1}{\alpha}\Sigma^{-1}\mu - \Sigma^{-1}\Upsilon w_y$$

- Rule of thumb to obtain exponential HJB equation:
  write power HJB equation in terms of $\tilde{w} = \gamma w$, then send $\gamma \uparrow \infty$.
  Then remove the ˜
- Exponential utility like power utility with “∞” relative risk aversion.
- Risk-neutral dynamics is minimal entropy martingale measure:

$$dY_t = \left(b - \Upsilon'\Sigma^{-1}\mu - (A - \Upsilon'\Sigma^{-1}\Upsilon)w_y\right) dt + a d\tilde{W}_t$$

HJB Equation

Assumption

$w \in C^2(E, \mathbb{R})$ and $\beta \in \mathbb{R}$ solve the ergodic HJB equation:

$$r + \frac{1}{2\gamma}\mu'\bar{\Sigma}\mu + \frac{1-\gamma}{2}\nabla w'\left(A - (1-\frac{1}{\gamma})\Upsilon'\bar{\Sigma}\Upsilon\right)\nabla w + \nabla w'\left(b - (1-\frac{1}{\gamma})\Upsilon'\bar{\Sigma}\mu\right) + \frac{1}{2}\operatorname{tr}\left(AD^2 w\right) = \beta$$

- Solution must be guessed one way or another.
- PDE becomes ODE for a single state variable
- PDE becomes linear for logarithmic utility ($\gamma = 1$).
- Must find both w and β

Myopic Probability

Assumption

There exists unique solution $(\hat{P}^{r,y})_{r\in\mathbb{R}^n, y\in\mathbb{R}^k}$ to to martingale problem

$$\hat{L} = \frac{1}{2}\sum_{i,j=1}^{n+k}\tilde{A}^{i,j}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n+k}\hat{b}^i(x)\frac{\partial}{\partial x_i}$$

$$\hat{b} = \begin{pmatrix} \frac{1}{\gamma}\left(\mu + (1-\gamma)\Upsilon\nabla w\right) \\ b - (1-\frac{1}{\gamma})\Upsilon'\bar{\Sigma}\mu + \left(A - (1-\frac{1}{\gamma})\Upsilon'\bar{\Sigma}\Upsilon\right)(1-\gamma)\nabla w \end{pmatrix}$$

- Under $\hat{P}$, the diffusion has dynamics:

$$dR_t = \frac{1}{\gamma}\left(\mu + (1-\gamma)\Upsilon\nabla w\right)dt + \sigma d\hat{Z}_t$$

$$dY_t = \left(b - (1-\frac{1}{\gamma})\Upsilon'\bar{\Sigma}\mu + (A - (1-\frac{1}{\gamma})\Upsilon'\bar{\Sigma}\Upsilon)(1-\gamma)\nabla w\right)dt + a\,d\hat{W}_t$$

- Same optimal portfolio as a logarithmic investor living under $\hat{P}$.

Finite horizon bounds

Theorem

Under the previous assumptions:

$$\pi = \frac{1}{\gamma}\bar{\Sigma}\left(\mu + (1-\gamma)\Upsilon\nabla w\right), \qquad \eta = (1-\gamma)\nabla w$$

satisfy the equalities:

$$E_P^y\left[(X_T^\pi)^{1-\gamma}\right]^{\frac{1}{1-\gamma}} = e^{\beta T + w(y)} E_{\hat{P}}^y\left[e^{-(1-\gamma)w(Y_T)}\right]^{\frac{1}{1-\gamma}}$$

$$E_P^y\left[(M_T^\eta)^{\frac{\gamma-1}{\gamma}}\right]^{\frac{\gamma}{1-\gamma}} = e^{\beta T + w(y)} E_{\hat{P}}^y\left[e^{-\frac{1-\gamma}{\gamma}w(Y_T)}\right]^{\frac{\gamma}{1-\gamma}}$$

- Bounds are almost the same. Differ in L^p norm.
- Long run optimality if expectations grow less than exponentially.

Path to Long Run solution

- Find candidate pair w, β that solve HJB equation.
- Different β lead to to different solutions w.
- Must find w corresponding to the lowest β that has a solution. You look for the lowest certeinty equivalent rate.
- Using w, check that myopic probabiity is well defined. Y does not explode under dynamics of $\hat{P}$.
- Then finite horizon bounds hold.
- To obtain long run optimality, show that:

$$\limsup_{T\to\infty} \frac{1}{T} \log E^{y}_{\hat{P}}\left[e^{-\frac{1-\gamma}{\gamma} w(Y_T)}\right] = 0$$

$$\limsup_{T\to\infty} \frac{1}{T} \log E^{y}_{\hat{P}}\left[e^{-(1-\gamma) w(Y_T)}\right] = 0$$

Proof of wealth bound (1)

- Define $D_t = \frac{d\hat{P}}{dP}|_{\mathcal{F}_t}$, which equals to $\mathcal{E}(M)$, where:

$$M_t = \int_0^t \left(-q\Upsilon'\bar{\Sigma}\mu + \left(A - q\Upsilon'\bar{\Sigma}\Upsilon\right)\nabla v\right)'(a')^{-1}dW_t - \int_0^t q\left(\bar{\Sigma}\mu + \bar{\Sigma}\Upsilon\nabla v\right)'\sigma\bar{\rho}dB_t$$

- For the portfolio bound, it suffices to show that:

$$(X_T^\pi)^p = e^{p(\beta T + w(y) - w(Y_T))}D_T$$

which is the same as $\log X_T^\pi - \frac{1}{p}\log D_T = \beta T + w(y) - w(Y_T)$.

- The first term on the left-hand side is:

$$\log X_T^\pi = \int_0^T \left(r + \pi'\mu - \frac{1}{2}\pi'\Sigma\pi\right)dt + \int_0^T \pi'\sigma dZ_t$$

Proof of wealth bound (2)

- Set $\pi = \frac{1}{1-p}\bar{\Sigma}\left(\mu + p\Upsilon\nabla w\right)$, $Z = \rho W + \bar{\rho} B$. $\log X_T^\pi$ becomes:

$$\int_0^T \left(r + \frac{1-2p}{2(1-p)}\mu'\bar{\Sigma}\mu - \frac{p^2}{(1-p)^2}\mu'\bar{\Sigma}\Upsilon\nabla w - \frac{1}{2}\frac{p^2}{(1-p)^2}\nabla w'\Upsilon'\bar{\Sigma}\Upsilon\nabla w\right) dt$$
$$+ \frac{1}{1-p}\int_0^T \left(\mu + p\Upsilon\nabla w\right)' \bar{\Sigma}\sigma\rho dW_t - \frac{1}{1-p}\int_0^T \left(\mu + p\Upsilon\nabla w\right)' \bar{\Sigma}\sigma\bar{\rho} dB_t$$

- Similarly, $\log D_T/p$ becomes:

$$\int_0^T \left(-\frac{p}{2(1-p)^2}\mu'\bar{\Sigma}\mu - \frac{p}{(1-p)^2}\mu'\bar{\Sigma}\Upsilon\nabla w - \frac{p}{2}\nabla w'\left(A + \frac{p(2-p)}{(1-p)^2}\Upsilon'\bar{\Sigma}\Upsilon\right)\nabla w\right) dt +$$
$$\int_0^T \left(\nabla w' a + \frac{1}{1-p}\left(\mu + p\Upsilon\nabla w\right)' \bar{\Sigma}\sigma\rho\right) dW_t + \frac{1}{1-p}\int_0^T \left(\mu + p\Upsilon\nabla w\right)\bar{\Sigma}\sigma\bar{\rho} dB_t$$

- Subtracting yields for $\log X_T^\pi - \log D_T/p$

$$\int_0^T \left(r + \frac{1}{2(1-p)}\mu'\bar{\Sigma}\mu + \frac{p}{1-p}\mu'\bar{\Sigma}\Upsilon\nabla w + \frac{p}{2}\nabla w'\left(A + \frac{p}{1-p}\Upsilon'\bar{\Sigma}\Upsilon\right)\nabla w\right) dt$$
$$- \int_0^T \nabla w' a dW_t$$

Proof of wealth bound (3)

- Now, Itô's formula allows to substitute:

$$-\int_0^T \nabla w' a dW_t = w(y) - w(Y_T) + \int_0^T \nabla w' b dt + \frac{1}{2}\int_0^T \mathrm{tr}(AD^2 w) dt$$

- The resulting *dt* term matches the one in the HJB equation.
- $\log X_T^\pi - \log D_T/p$ equals to $\beta T + w(y) - w(Y_T)$.

Proof of martingale bound (1)

- For the discount factor bound, it suffices to show that:

$$\frac{1}{p-1}\log M_T^\eta - \frac{1}{p}\log D_T = \frac{1}{1-p}\left(\beta T + w(y) - w(Y_T)\right)$$

- The term $\frac{1}{p-1}\log M_T^\eta$ equals to:

$$\frac{1}{1-p}\int_0^T \left(r + \frac{1}{2}\mu'\bar{\Sigma}\mu + \frac{p^2}{2}\nabla w'\left(A - \Upsilon'\bar{\Sigma}\Upsilon\right)\nabla w\right)dt +$$
$$\frac{1}{p-1}\int_0^T \left(p\nabla w' a - (\mu + p\Upsilon\nabla w)'\bar{\Sigma}\sigma\rho\right)dW_t + \frac{1}{1-p}\int_0^T (\mu + p\Upsilon\nabla w)\bar{\Sigma}\sigma\bar{\rho}dB_t$$

- Subtracting $\frac{1}{p}\log D_T$ yields for $\frac{1}{p-1}\log M_T^\eta - \frac{1}{p}\log D_T$:

$$\frac{1}{1-p}\int_0^T \left(r + \frac{1}{2(1-p)}\mu'\bar{\Sigma}\mu + \frac{p}{1-p}\mu'\bar{\Sigma}\Upsilon\nabla w + \frac{p}{2}\nabla w'\left(A + \frac{p}{1-p}\Upsilon'\bar{\Sigma}\Upsilon\right)\nabla w\right)dt$$
$$-\frac{1}{1-p}\int_0^T \nabla w' a dW_t$$

Proof of martingale bound (2)

- Replacing again $\int_0^T \nabla w' a dW_t$ with Itô's formula yields:

$$\frac{1}{1-p}\int_0^T (r + \frac{1}{2(1-p)}\mu'\bar{\Sigma}\mu + (\frac{p}{1-p}\mu'\bar{\Sigma}\Upsilon + b')\nabla w +$$
$$\frac{1}{2}\operatorname{tr}(AD^2 w) + \frac{p}{2}\nabla w'\left(A + \frac{p}{1-p}\Upsilon'\bar{\Sigma}\Upsilon\right)\nabla w)dt$$
$$+ \frac{1}{1-p}(w(y) - w(Y_T))$$

- And the integral equals $\frac{1}{1-p}\beta T$ by the HJB equation.

Exponential Utility

Theorem

If $r = 0$ and w solves equation:

$$\frac{1}{2}\mu'\bar{\Sigma}\mu - \frac{1}{2}\nabla w'\left(A - \Upsilon'\bar{\Sigma}\Upsilon\right)\nabla w + \nabla w'\left(b - \Upsilon'\bar{\Sigma}\mu\right) + \frac{1}{2}\operatorname{tr}\left(AD^2 w\right) = \beta$$

and myopic dynamics is well posed:

$$\begin{aligned} dR_t &= \sigma d\hat{Z}_t \\ dY_t &= \left(b - \Upsilon'\bar{\Sigma}\mu - (A - \Upsilon'\bar{\Sigma}\Upsilon)\nabla w\right) dt + a d\hat{W}_t \end{aligned}$$

Then for the portfolio and risk premia (π, η) given by:

$$x\pi = \frac{1}{\alpha}\Sigma^{-1}\mu - \Sigma^{-1}\Upsilon\nabla w \qquad \eta = -\nabla w$$

finite-horizon bounds hold as:

$$\begin{aligned} -\frac{1}{\alpha}\log E_P^y\left[e^{-\alpha(X_T^\pi - x)}\right] &= \beta T + \frac{1}{\alpha}\log E_{\hat{P}}^y\left[e^{w(y) - w(Y_T)}\right] \\ \frac{1}{2}E_P^y\left[M^\eta \log M^\eta\right] &= \beta T + \frac{1}{\alpha}E_{\hat{P}}^y\left[w(y) - w(Y_T)\right] \end{aligned}$$

Long-Run Optimality

Theorem

If, in addition to the assumptions for finite-horizon bounds,

- *the random variables $(Y_t)_{t\geq 0}$ are $\hat{P}^y$-tight in E for each $y \in E$;*
- $\sup_{y\in E}(1-\gamma)F(y) < +\infty$*, where $F \in C(E, \mathbb{R})$ is defined as:*

$$F = \begin{cases} \left(r - \beta + \frac{\mu'\Sigma^{-1}\mu}{2\gamma} - \frac{(1-\gamma)^2}{2\gamma}\nabla w'\Upsilon'\Sigma^{-1}\Upsilon\nabla w\right) e^{-(1-\gamma)w} & \gamma > 1 \\ \left(r - \beta + \frac{\mu'\Sigma^{-1}\mu}{2\gamma} - \frac{(1-\gamma)^2}{2}\nabla w'\left(A - \Upsilon'\Sigma^{-1}\Upsilon\right)\nabla w\right) e^{-\frac{1-\gamma}{\gamma}w} & \gamma < 1 \end{cases}$$

Then long-run optimality holds.

- Straightforward to check, once w is known.
- Tightness checked with some moment condition.
- Does not require transition kernel for Y under any probability.

Proof of Long-Run Optimality (1)

- By the duality bound:

$$0 \leq \liminf_{T\to\infty} \frac{1}{p}\left(\frac{1}{T}\log E_P^y\left[(M_T^\eta)^q\right]^{1-p} - \frac{1}{T}\log E_P^y\left[(X_T^\pi)^p\right]\right)$$
$$\leq \limsup_{T\to\infty} \frac{1}{pT}\log E_P^y\left[(M_T^\eta)^q\right]^{1-p} - \liminf_{T\to\infty}\frac{1}{pT}\log E_P^y\left[(X_T^\pi)^p\right]$$
$$= \limsup_{T\to\infty}\frac{1-p}{pT}\log E_{\hat{P}}^y\left[e^{-\frac{1}{1-p}v(Y_T)}\right] - \liminf_{T\to\infty}\frac{1}{pT}\log E_{\hat{P}}^y\left[e^{-v(Y_T)}\right]$$

- For $p < 0$ enough to show lower bound

$$\liminf_{T\to\infty}\frac{1}{T}\log E_{\hat{P}}^y\left[\exp\left(-\tfrac{1}{1-p}v(Y_T)\right)\right] \geq 0$$

and upper bound:

$$\limsup_{T\to\infty}\tfrac{1}{T}\log E_{\hat{P}}^y\left[\exp\left(-v(Y_T)\right)\right] \leq 0$$

- Lower bound follows from tightness.

Proof of Long-Run Optimality (2)

- For upper bound, set:

$$Lf = \nabla f' \left(b - q\Upsilon'\Sigma^{-1}\mu + \left(A - q\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla v\right) + \frac{1}{2}\operatorname{tr}\left(AD^2 f\right)$$

- Then, for $\alpha \in \mathbb{R}$, the HJB equation implies that $L\left(e^{\alpha v}\right)$ equals to:

$$\alpha e^{\alpha v}\left(\nabla v' \left(b - q\Upsilon'\Sigma^{-1}\mu + \left(A - q\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla v\right) + \frac{1}{2}\operatorname{tr}\left(AD^2 v\right) + \frac{1}{2}\alpha\nabla v' A\nabla v\right.$$
$$= \alpha e^{\alpha v}\left(\frac{1}{2}\nabla v'\left((1+\alpha)A - q\Upsilon'\Sigma^{-1}\Upsilon\right)\nabla v + p\beta - pr + \frac{q}{2}\mu'\Sigma^{-1}\mu\right)$$

- Set $\alpha = -1$, to obtain that:

$$L\left(e^{-v}\right) = e^{-v}\left(\frac{q}{2}\nabla v'\Upsilon'\Sigma^{-1}\Upsilon\nabla v - \lambda + pr - \frac{q}{2}\mu'\Sigma^{-1}\mu\right)$$

- The boundedness hypothesis on F allows to conclude that:

$$E^y_{\hat{P}}\left[e^{-v(Y_T)}\right] \leq e^{-v(y)} + (K \vee 0)\, T$$

whence

$$\limsup_{T\to\infty} \frac{1}{T}\log E^y_{\hat{P}}\left[e^{-v(Y_T)}\right] \leq 0$$

- $0 < p < 1$ similar. Reverse inequalities for upper and lower bounds. Use $\alpha = -\frac{1}{1-p}$ for upper bound.

Two

Shortfall Aversion

Outline

- Motivation:
  Endowment Management. Universities, sovereign funds, trust funds.
  Retirement planning?
- Model:
  Constant investment opportunities.
  Constant Relative Risk and **Shortfall** Aversion.
- Result:
  Optimal spending and investment.

Tobin (1974)

- "The trustees of an endowed institution are the guardians of the future against the claims of the present.
- "Their task is to preserve equity among generations.
- "The trustees of an endowed university like my own assume the institution to be immortal.
- "They want to know, therefore, the rate of consumption from endowment which can be sustained indefinitely.
- "**Sustainable consumption** is their conception of permanent endowment income.
- "In formal terms, **the trustees are supposed to have a zero subjective rate of time preference**."

Endowment Management

- How to invest? How much to withdraw?
- Major goal: keep spending level.
- Minor goal: increasing it.
- Increasing and then decreasing spending is worse than not increasing in the first place.
- How to capture this feature?
- Heuristic rule among endowments and private foundations: spend a constant proportion of the moving average of assets.

Theory

- Classical Merton model: maximize

$$E\left[\int_0^\infty e^{-\beta t}\frac{c_t^{1-\gamma}}{1-\gamma}dt\right]$$

over spending rate c_t and wealth in stocks H_t.

- Optimal c_t and H_t are constant fractions of current wealth X_t:

$$\frac{c}{X}=\frac{1}{\gamma}\beta+\left(1-\frac{1}{\gamma}\right)\left(r+\frac{\mu^2}{2\gamma\sigma^2}\right)\qquad\frac{H}{X}=\frac{\mu}{\gamma\sigma^2}$$

with interest rate r, equity premium μ, and stocks volatility σ.

- Spending as volatile as wealth.
- Want stable spending? Hold nearly no stocks. But then $c_t\approx rX_t$.
- Not easy with interest rates near zero. Not used by financial planners.

The Four-Percent Rule

- Bengen (1994). Popular with financial planners.
- When you retire, spend in the first year 4% of your savings.
  Keep same amount for future years. Invest 50% to 75% in stocks.
- Historically, savings will last at least 30 years.
- Inconsistent with theory.
  Spending-wealth ratio variable, but portfolio weight fixed.
  Bankruptcy possible.
- Time-inconsistent.
  With equal initial capital, consume more if retired in Dec 2007 than if retired in Dec 2008. But savings will always be lower.
- How to embed preference for stable spending in objective?

Related Literature

- Consumption ratcheting: allow only increasing spending rates...
  Dybvig (1995), Bayraktar and Young (2008), Riedel (2009).
- ...or force maximum drawdown on consumption from its peak.
  Thillaisundaram (2012).
- Consume less than interest, or bankruptcy possible.
  No solvency with zero interest.
- Habit formation: utility from consumption *minus* habit.
  Sundaresan (1989), Constantinides (1990), Detemple and Zapatero (1992), Campbell and Cochrane (1999), Detemple and Karatzas (2003).

$$\max E\left[\int_0^\infty e^{-\beta t}\frac{(c_t - x_t)^{1-\gamma}}{1-\gamma}dt\right] \quad \text{where} \quad dx_t = (bc_t - ax_t)dt$$

- Infinite marginal utility at habit level. Like zero consumption.
  No solution if habit too high relative to wealth ($x_0 > X_0(r + a - b)$).
- Habit transitory. Time heals.

This Model

- Inputs
  - Risky asset follows geometric Brownian motion. Constant interest rate.
  - Constant relative risk aversion.
    Constant relative shortfall aversion by new parameter α.
- Outputs
  - Spending constant between endogenous boundaries, bliss and gloom.
    Increases in small amounts at bliss.
    Declines smoothly at gloom.
  - Portfolio weight varies between high and low bounds.
- Features
  - Solution solvent for any initial capital. Even with zero interest.
  - Spending target permanent. You never forget.

Shortfall Aversion and Prospect Theory

- Kanheman and Tversky (1979,1991). Precursor: Markowitz (1952).
- Reference dependence:
  utility is from gains and losses relative to reference level.
- Loss Aversion:
  marginal utility lower for gains than for losses.
- Prospect theory focuses on wealth gambles.
- We bring these features to spending instead.
  Shortfall aversion as loss aversion for spending.

Model

- Safe rate r. Risky asset price follows geometric Brownian motion:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

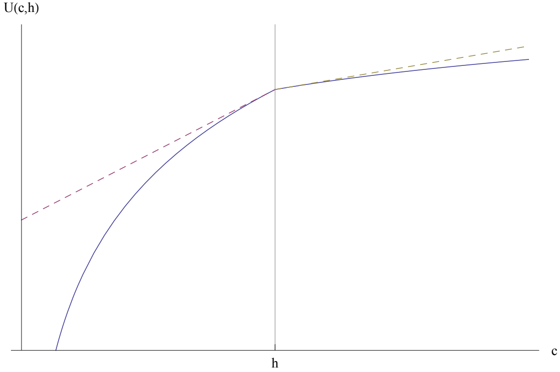
  for Brownian Motion $(W_t)_{t\geq 0}$ with natural filtration $(\mathcal{F}_t)_{t\geq 0}$.
- Utility from consumption rate c_t:

$$E\left[\int_0^\infty \frac{(c_t/h_t^\alpha)^{1-\gamma}}{1-\gamma} dt\right] \qquad \text{where} \qquad h_t = \sup_{0\leq s\leq t} c_s$$

- Risk aversion $\gamma > 1$ to make the problem well posed.
- $\alpha \in [0, 1)$ relative loss aversion.
  Strict monotonicity in c implies that $\alpha < 1$ and strict concavity that $\alpha \geq 0$.
- If you could forget the past you would be happier. But you can't.
- *More* is better today... but makes *less* worse tomorrow.
- No loss aversion with c_t increasing... or decreasing.

Sliding Kink

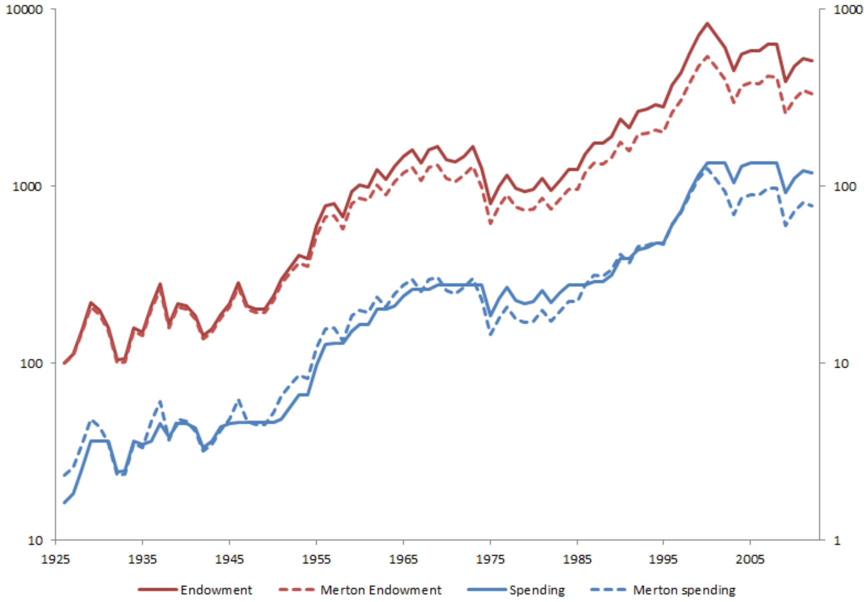
- Because $c_t \leq h_t$, utility effectively has a kink at h_t:



- Optimality: marginal utilities of spending and wealth equal.
- Spending h_t optimal for marginal utility of wealth between left and right derivative at kink.
- State variable: ratio h_t/X_t between reference spending and wealth.
- Investor rich when h_t/X_t low, and poor when h_t/X_t high.

The Merton Solution in 1926-2012 with $\gamma = 2$

Spending, Saving with Shortfall Aversion, 1926-2012



Control Argument

- Value function $V(x, h)$ depends on wealth $x = X_t$ and target $h = h_t$.
- Set $J_t = \int_0^t U(c_s, h_s)\mathrm{d}s + V(X_t, h_t)$. By Itô's formula:

$$\mathrm{d}J_t = L(X_t, \pi_t, c_t, h_t)\mathrm{d}t + V_h(X_t, h_t)\mathrm{d}h_t + V_x(X_t, h_t)X_t\pi_t^\top \sigma \mathrm{d}W_t$$

where drift $L(X_t, \pi_t, c_t, h_t)$ equals

$$U(c_t, h_t) + (X_t r_t - c_t + X_t\pi_t^\top \mu)V_x(X_t, h_t) + \frac{V_{xx}(X_t, h_t)}{2} X_t^2 \pi_t^\top \Sigma \pi_t$$

- J_t supermartingale for any c, and martingale for optimizer $\hat{c}$. Hence:

$$\max(\sup_{\pi, c} L(x, \pi, c, h), V_h(x, h)) = 0$$

- That is, either $\sup_{\pi,c} L(x, \pi, c, h) = 0$, or $V_h(x, h) = 0$.
- Optimal portfolio has usual expression $\hat{\pi} = -\frac{V_x}{xV_{xx}}\Sigma^{-1}\mu$.
- Free-boundaries:
  When do you increase spending? When do you cut it below h_t?

Duality

- Setting $\tilde{U}(y,h) = \sup_{c\geq 0}[U(c,h) - cy]$, HJB equation becomes

$$\tilde{U}(V_x, h) + xrV_x(x,h) - \frac{V_x^2(x,h)}{2V_{xx}(x,h)}\mu^\top\Sigma^{-1}\mu = 0$$

- Nonlinear equation. Linear in dual $\tilde{V}(y,h) = \sup_{x\geq 0}[V(x,h) - xy]$

$$\tilde{U}(y,h) - ry\tilde{V}_y + \frac{\mu^\top\Sigma^{-1}\mu}{2}y^2\tilde{V}_{yy} = 0$$

- Plug $U(c,h) = \frac{(c/h^\alpha)^{1-\gamma}}{1-\gamma}$. Kink spawns two cases:

$$\tilde{U}(y,h) = \begin{cases} \dfrac{h^{1-\gamma^*}}{1-\gamma} - hy & \text{if } (1-\alpha)h^{-\gamma^*} \leq y \leq h^{-\gamma^*} \\ \frac{(yh^\alpha)^{1-1/\gamma}}{1-1/\gamma} & \text{if } y > h^{-\gamma^*} \end{cases}$$

where $\gamma^* = \alpha + (1-\alpha)\gamma$.

Homogeneity

- $V(\lambda x, \lambda h) = \lambda^{1-\gamma^*} V(x,h)$ implies that $\tilde{V}(y,h) = h^{1-\gamma^*} q(z)$, where $z = yh^{\gamma^*}$.
- HJB equation reduces to:

$$\frac{\mu^\top \Sigma^{-1} \mu}{2} z^2 q''(z) - rzq'(z) = \begin{cases} z - \frac{1}{1-\gamma} & 1-\alpha \le z \le 1 \\ \frac{z^{1-1/\gamma}}{1-1/\gamma} & z > 1 \end{cases}$$

- Condition $V_h(x,h) = 0$ holds when desired spending must increase:

$$(1-\gamma^*)q(z) + \gamma^* zq'(z) = 0 \qquad \text{for } z \le 1-\alpha$$

- Agent "rich" for $z \in [1-\alpha, 1]$.
  Consumes at desired level, and increases it at bliss point $1-\alpha$.
- Agent becomes "poor" at gloom point 1.
  For $z > 1$, consumes below desired level.

Solving it

- For $r \neq 0$, $q(z)$ has solution:

$$q(z) = \begin{cases} C_{21} + C_{22} z^{1+\frac{2r}{\mu^\top \Sigma^{-1} \mu}} - \frac{z}{r} + \frac{2 \log z}{(1-\gamma)(2r+\mu^\top \Sigma^{-1} \mu)} & \text{if } 0 < z \leq 1, \\ C_{31} + \frac{\gamma}{(1-\gamma)\delta_0} z^{1-1/\gamma} & \text{if } z > 1, \end{cases}$$

  where δ_α defined shortly.
- Neumann condition at $z = 1 - \alpha$: $(1 - \gamma^*) q(z) + \gamma^* z q'(z) = 0$.
- Value matching at $z = 1$: $q(z-) = q(z+)$.
- Smooth-pasting at $z = 1$: $q'(z-) = q'(z+)$.
- Fourth condition?
- Intuitively, it should be at $z = \infty$.
- Marginal utility z infinite when wealth x is zero.
- Since $q'(z) = -x/h$, the condition is $\lim_{z \to \infty} q'(z) = 0$.

Main Quantities

- Ratio between safe rate and half squared Sharpe ratio:

$$\rho = \frac{2r}{(\mu/\sigma)^2}$$

  In practice, ρ is small $\approx 8\%$.
- Gloom ratio g (as wealth/target). $1/g$ coincides with Merton ratio:

$$\frac{1}{g} = \left(1 - \frac{1}{\gamma}\right)\left(r + \frac{\mu^2}{2\gamma\sigma^2}\right)$$

- Bliss ratio:

$$b = g\frac{(\gamma-1)(1-\alpha)^{\rho+1} + (\gamma\rho+1)(\alpha(\gamma-1)(\rho+1) - \gamma(\rho+1)+1)}{(\alpha-1)\gamma^2\rho(\rho+1)}$$

- For ρ small, limit:

$$\frac{g}{b} \approx \frac{1-\alpha}{1-\alpha+\frac{\alpha}{\gamma^2} - \frac{1}{\gamma}(1-\frac{1}{\gamma})(1-\alpha)\log(1-\alpha)}$$

  Ratio insensitive to market parameters, for typical investment opportunities.

Main Result

Theorem

For $r \neq 0$, the optimal spending policy is:

$$\hat{c}_t = \begin{cases} X_t/b & \text{if} \quad h_t \leq X_t/b \\ h_t & \text{if} \quad X_t/b \leq h_t \leq X_t/g \\ X_t/g & \text{if} \quad h_t \geq X_t/g \end{cases} \tag{1}$$

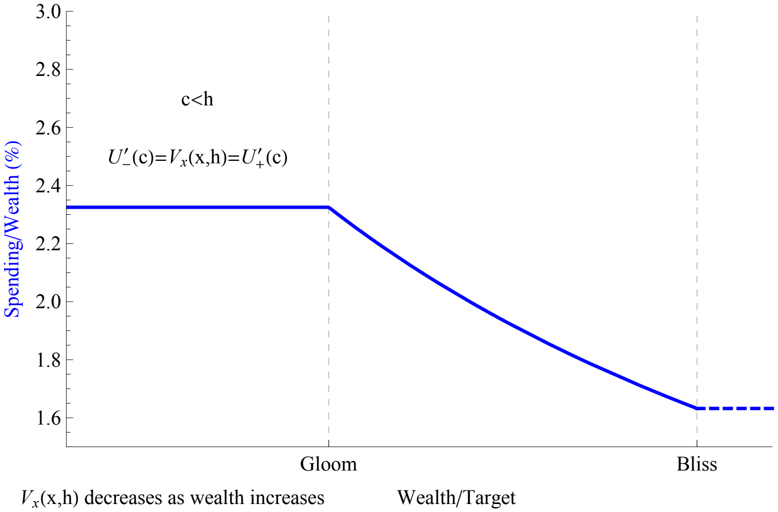
The optimal weight of the risky asset is the Merton risky asset weight when the wealth to target ratio is lower than the gloom point, $X_t/h_t \leq g$. Otherwise, i.e., when $X_t/h_t \geq g$ the weight of the risky asset is

$$\hat{\pi}_t = \frac{\rho\left(\gamma\rho + (\gamma-1)z^{\rho+1} + 1\right)}{(\gamma\rho+1)(\rho+(\gamma-1)(\rho+1)z) - (\gamma-1)z^{\rho+1}} \frac{\mu}{\sigma^2} \tag{2}$$

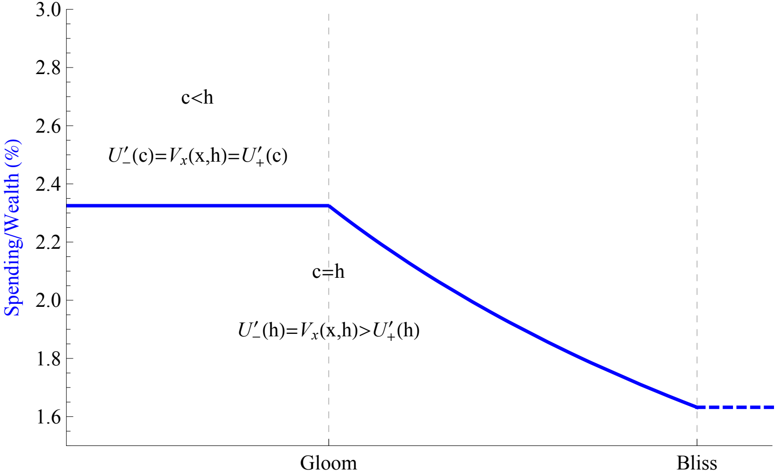
where the variable z satisfies the equation:

$$\frac{(\gamma\rho+1)(\rho+(\gamma-1)(\rho+1)z) - (\gamma-1)z^{\rho+1}}{(\gamma-1)(\rho+1)rz(\gamma\rho+1)} = \frac{x}{h} \tag{3}$$

Spending Region 1 - Merton

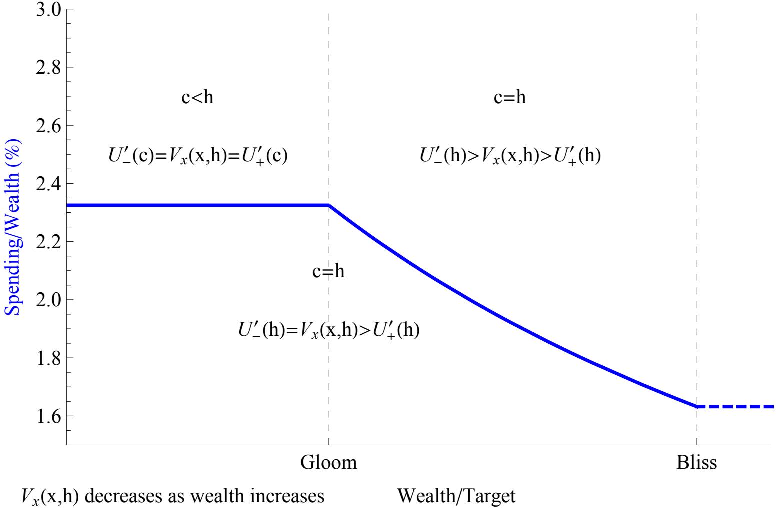


Spending Region 2 - Gloom Point



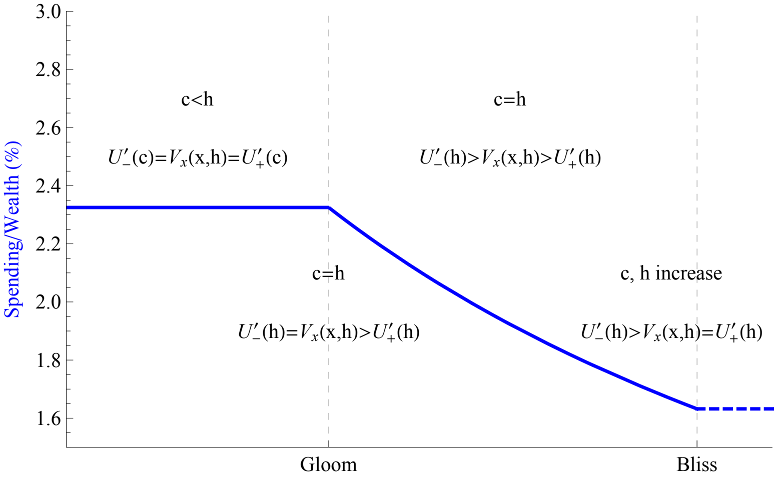
$V_x(x,h)$ decreases as wealth increases

Spending Region 3 - Target



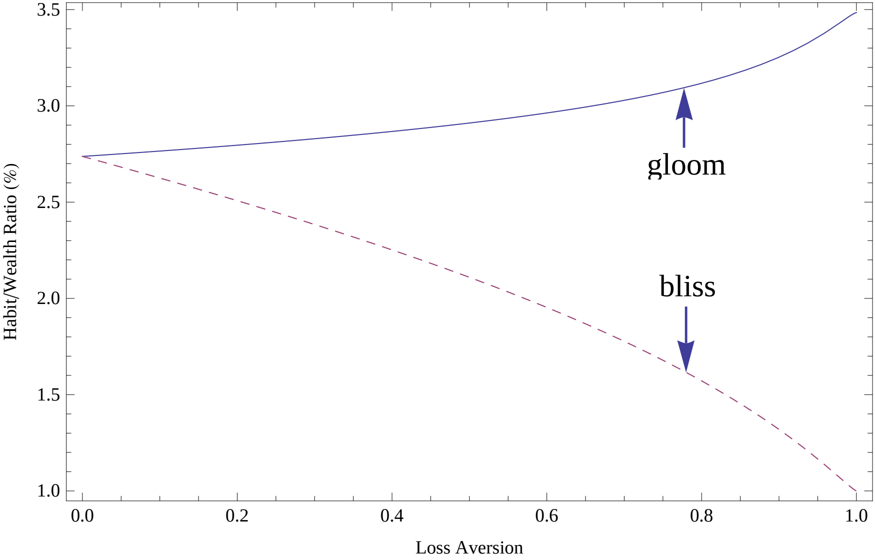
V_x(x,h) decreases as wealth increases

Spending Region 4 - Bliss Point



$V_x(x,h)$ decreases as wealth increases

Bliss and Gloom – α

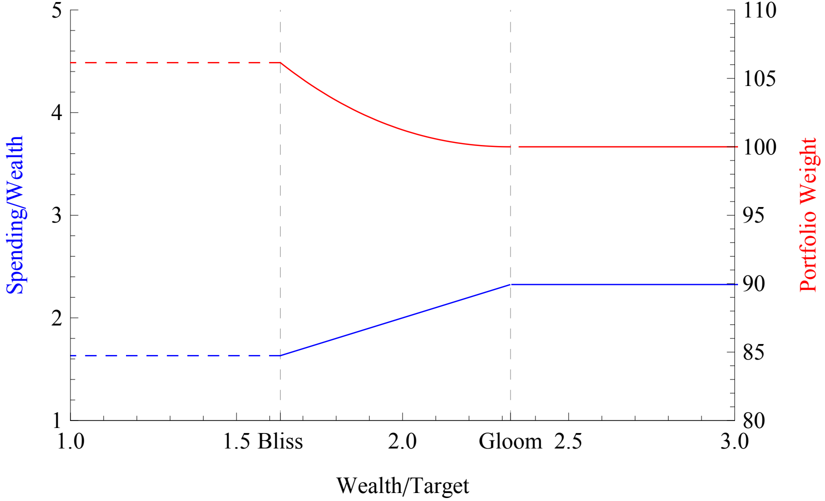


$\mu/\sigma = 0.4$, $\gamma = 2$, $r = 1\%$

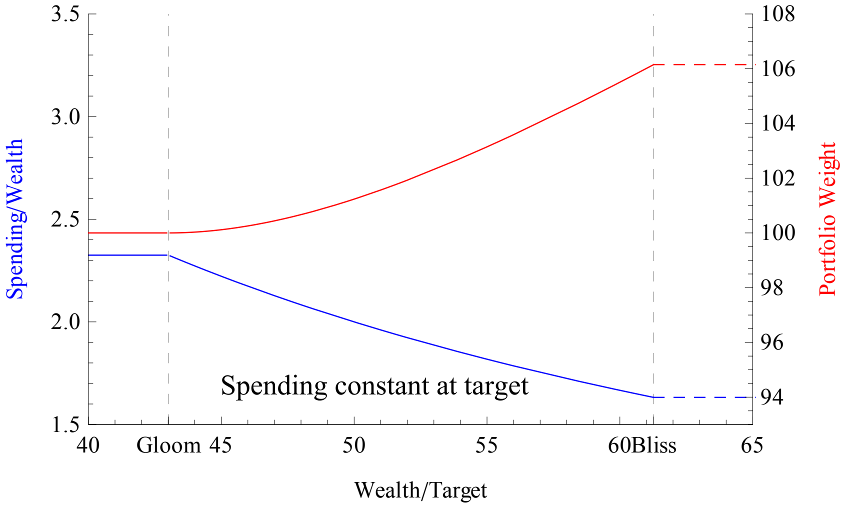
Bliss and Gloom

- Gloom ratio independent of the shortfall aversion α, and its inverse equals the Merton consumption rate.
- Bliss ratio increases as the shortfall aversion α increases.
- Within the target region, the optimal investment policy is independent of the shortfall aversion α.
- At $\alpha = 0$ the model degenerates to the Merton model and $b = g$, i.e., the bliss and the gloom points coincide. At $\alpha = 1$ the bliss point is infinity, i.e., shortfall aversion is so strong that the solution calls for no spending increases at all.

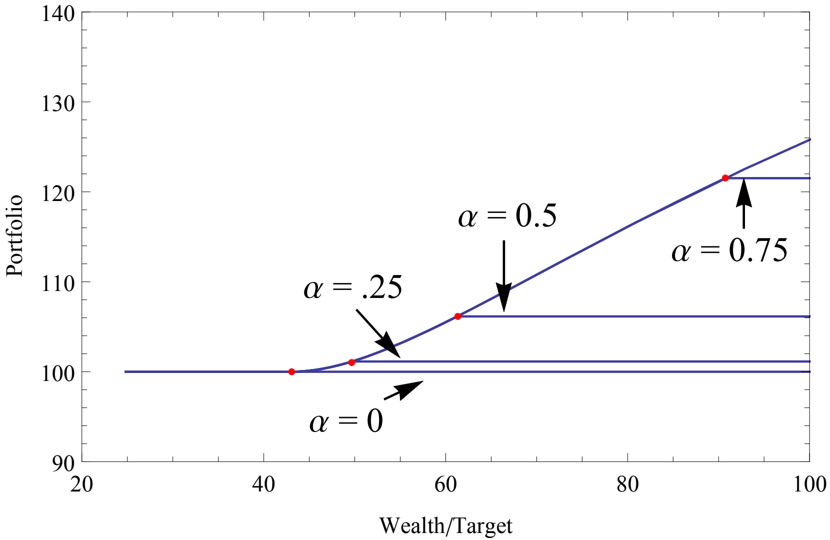
Spending and Investment as Target/Wealth Varies



Spending and Investment as Wealth/Target Varies



Investment – α



Steady State

Theorem

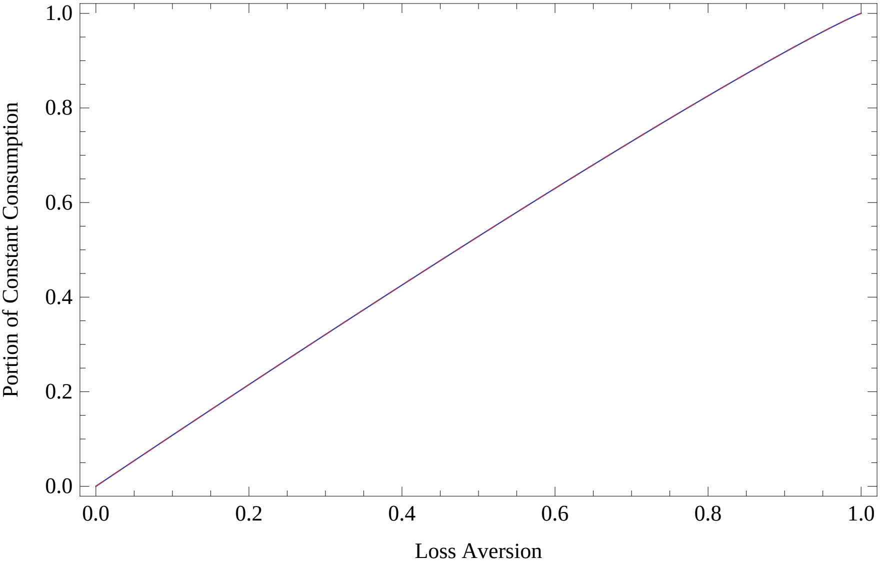
- *The long-run average time spent in the target zone is a fraction* $1-(1-\alpha)^{1+\rho}$ *of the total time. This fraction is approximately* α *because reasonable values of* ρ *are close to zero.*
- *Starting from a point* $z_0 \in [0,1]$ *in the target region, the expected time before reaching gloom is*

$$\mathbb{E}_{x,h}[\tau_{gloom}] = \frac{\rho}{(\rho+1)r}\left(\log(z_0) - \frac{(1-\alpha)^{-\rho-1}\left(z_0^{\rho+1}-1\right)}{\rho+1}\right)$$

In particular, starting from bliss ($z_0 = 1-\alpha$*) and for small* ρ*,*

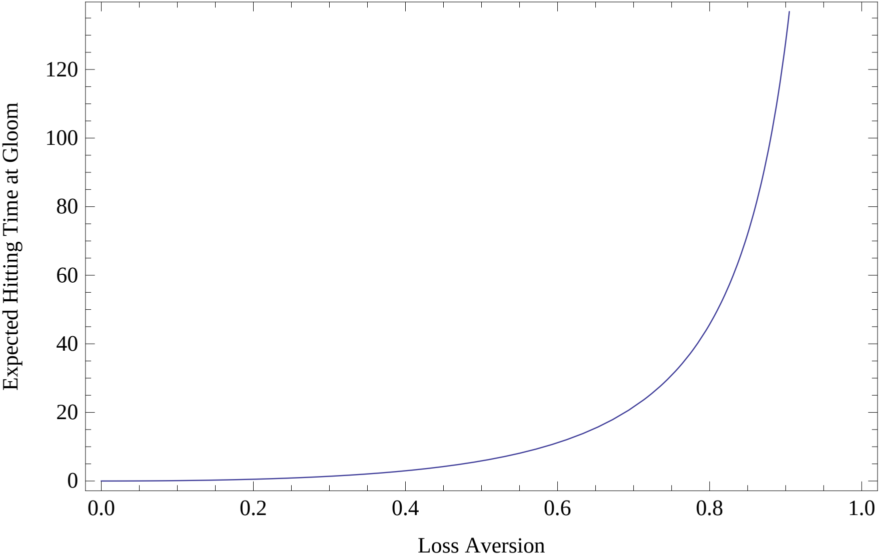
$$\mathbb{E}_{x,h}[\tau_{gloom}] = \frac{\rho}{r}\left(\frac{\alpha}{1-\alpha} + \log(1-\alpha)\right) + O(\rho^2)$$

Time Spent at Target – α



$\mu/\sigma = 0.35$, $\gamma = 2$, $r = 1\%$

Time from Bliss to Gloom – α



$\mu/\sigma = 0.35$, $\gamma = 2$, $r = 1\%$

Under the Hood

- With $\int_0^\infty U(c_t, h_t)dt$, first-order condition is:

$$U_c(c_t, h_t) = yM_t$$

where $M_t = e^{-(r+\mu^\top \Sigma^{-1}\mu/2)t-\mu^\top \Sigma^{-1} W_t}$ is stochastic discount factor.
- Candidate $c_t = I(yM_t, h_t)$ with $I(y, h) = U_c^{-1}(y, h)$. But what is h_t?
- h_t increases only at bliss. And at bliss U_c independent of past maximum:

$$h_t = c_0 \vee y^{-1/\gamma} \left(\inf_{s \le t} M_s \right)^{-1/\gamma}$$

Duality Bound

- For any spending plan c_t:

$$E\left[\int_0^\infty \frac{(c_t/{h_t}^\alpha)^{1-\gamma}}{1-\gamma}dt\right] \le \frac{x^{1-\gamma}}{1-\gamma}E\left[\int_0^\infty {Z_*}_t^{1-1/\gamma}\tilde{u}\left(\frac{Z_t}{{Z_*}_t}\right)dt\right]^\gamma$$

where $Z_t = M_t$ and ${Z_*}_t = \inf_{s\le t} Z_s$, and $\tilde{U}(y,h) = -\frac{h^{1-1/\gamma}}{1-1/\gamma}\tilde{u}(yh^\gamma)$.

- Show that equality holds for candidate optimizer.

Lemma

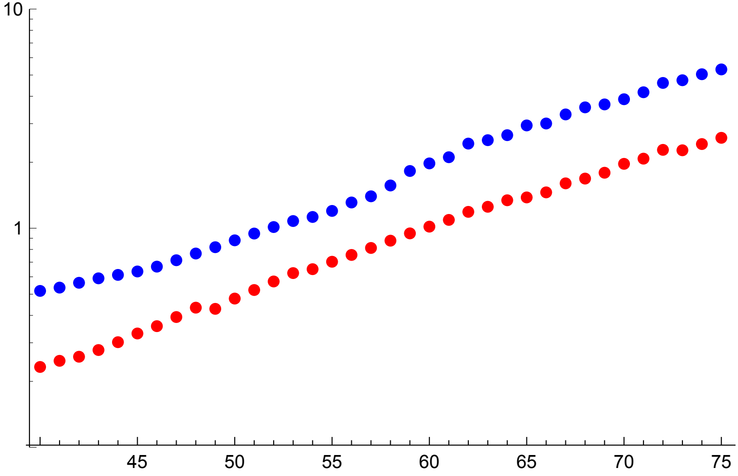
For $\gamma > 1$, $\lim_{T\uparrow\infty} \mathbb{E}^{x,h}\left[\tilde{V}(yM_T, \hat{h}_t(y))\right] = 0$ *for all* $y \ge 0$.

- Estimates on Brownian motion and its running maximum.

Three

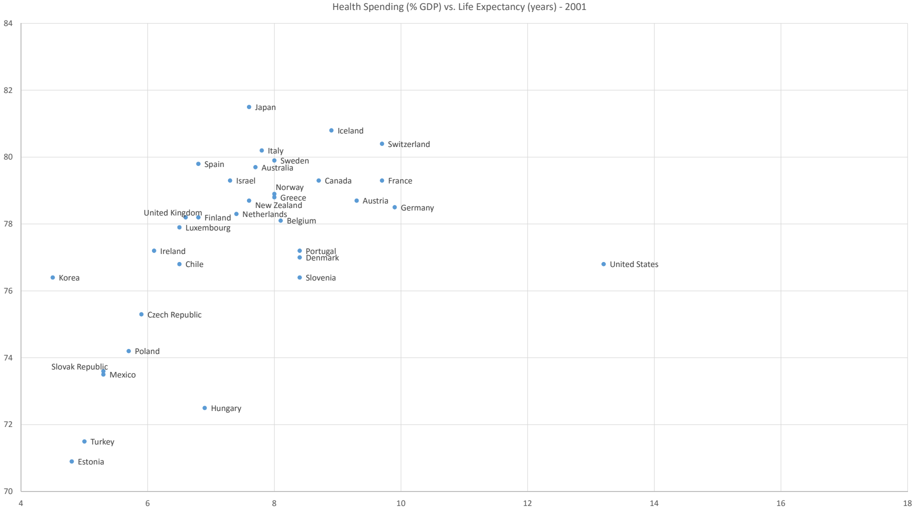
Consumption, Investment, and Healthcare

Mortality Increases with Age, Decreases with Time

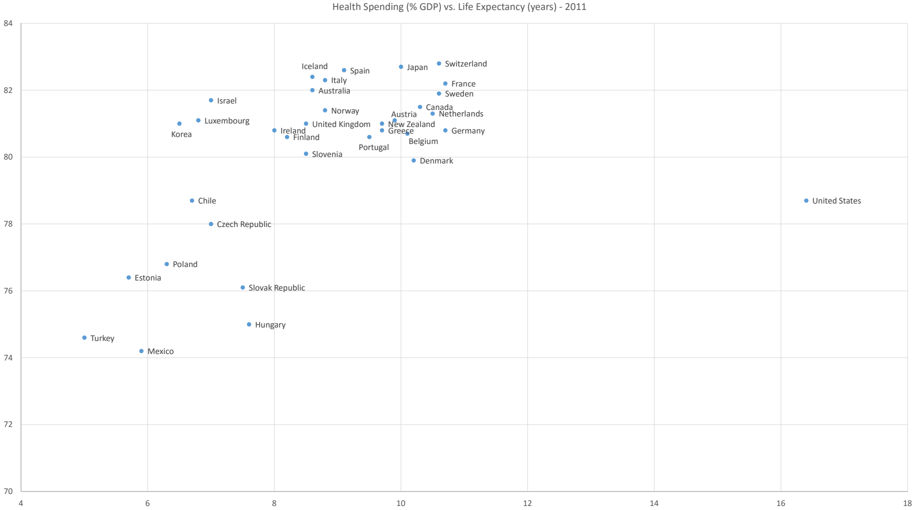


- Approximate exponential increase in age (Gompertz' law). Then as now.
- Secular decline across adult ages. More income? Better healthcare? Deaton (2003), Cutler et al. (2006)

Longer Life with More Healthcare...



...across Countries and over Time



Literature

- Mortality risk as higher discount rate (Yaari, 1965).
  High annuitization even with incomplete markets (Davidoff et al., 2005).
  Medical costs?
- Exogenous Mortality (Richard, 1975). Healthcare?
- Health as Capital, Healthcare as Investment (Grossman, 1972).
  Demand for Longevity (Ehrlich and Chuma, 1990).
  Predictable death?
- Mortality rates that decline with health capital. (Ehrlich, 2000), (Hall and Jones, 2007), (Yogo, 2009), (Hugonnier et al., 2012).
- Gompertz' law?
- Challenge:
  Combine endogenous mortality and healthcare with Gompertz' law. Does healthcare availability explain decline in mortality rates?

This Model

- Idea
  - Household maximizes utility from lifetime consumption.
  - Using initial wealth only. (Wealth includes value of future income.)
  - Constant risk-free rate. No risky assets.
  - Without healthcare, mortality increases exponentially.
  - Money can buy...
  - ...consumption, which generates utility...
  - ...or healthcare, which reduces mortality growth...
  - ...thereby buying time for more consumption.
- Assumptions
  - Constant Relative Risk Aversion.
  - Constant Relative Loss at Death.
  - Gompertz' Mortality without Healthcare.
  - Isoelastic Efficacy in Relative Healthcare Spending.
- Questions
  - Mortality law under optimal behavior?
  - Consistent with evidence?

To be, or not to be?

- Maximize expected utility from future consumption. Naïve approach:

$$\mathbb{E}\left[\int_0^\tau e^{-\delta t} U(c_t X_t) dt\right]$$

 where τ is lifetime, X_t wealth, and c_t consumption-wealth ratio.
- Not so fast. Result not invariant to utility translation. $U + k$ yields

$$\mathbb{E}\left[\int_0^\tau e^{-\delta t} U(c_t) dt\right] + kE\left[\frac{1 - e^{-\delta\tau}}{\delta}\right]$$

 Irrelevant if τ exogenous. Problematic if endogenous. (Shepard and Zeckhauser, 1984; Rosen, 1988; Bommier and Rochet, 2006)
- U negative? Preference for death!
- Quick fix: add constant to make U positive.
- Works only with U bounded from below and...
- ...results are still sensitive to translation.
- Death as preference change? (From $x \mapsto U(x)$ alive to $x \mapsto 0$ dead.)

Household Utility

- Our approach: death scales household wealth by factor $\zeta \in [0, 1]$.
  Estate and inheritance tax, pension and annuity loss, foregone income...
- After death, household carries on with same mortality as before.

$$\mathbb{E}\left[\sum_{n=1}^{\infty} \int_{\tau_{n-1}}^{\tau_n} e^{-\delta t} U(\zeta^n \bar{X}_t c_t) dt\right] \qquad \text{where } \tau_0 = 0.$$

  where $\bar{X}_t$ is wealth without accounting for losses.
- Surviving spouse in similar age group.
  Indefinite household size simplifies problem.
  Most weight carried by first two lifetimes.
- $\zeta = 1$: Immortality.
  $\zeta = 0$: 0 consumption and $U(0)$ utility in afterlife.
- Translation Invariant.
- Isoelastic utility:

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma} \qquad 0 < \gamma \neq 1$$

Mortality Dynamics

- Without healthcare, mortality M_t grows exponentially. Gompertz' law:

$$dM_t = \beta M_t dt$$

- Healthcare slows down mortality growth

$$dM_t = (\beta M_t - g(h_t))dt$$

  where h_t is the healthcare-wealth ratio, and $g(h)$ measures its *efficacy*.
- $g(0) = 0$, g positive, increasing, and concave.
- Diminishing returns from healthcare spending.
- Simplification: effect only depends on healthcare-wealth ratio.
- Lost income: proportional to wealth if proxy for future income.
- Means-tested subsidies.
- Life-expectancy correlated with health behaviors but not with access to care. (Chetty et al, 2016)
- Isoelastic efficacy:

$$g(h) = \frac{a}{q} h^q \qquad a > 0, q \in (0, 1)$$

Wealth Dynamics

- Household wealth grows at rate r, minus consumption and health spending, and death losses:

$$\frac{dX_t}{X_t} = (r - c_t - h_t)dt - (1 - \zeta)dN_t$$

- N_t counting process for number of deaths. $N_0 = 0$, and jumps at rate M_t:

$$P(N_{t+dt} - N_t = 1) = M_t dt$$

- Household chooses processes c, h to maximize expected utility.
- Bequest motive embedded in preferences.

Four Settings

- Two new features: Aging and Healthcare.
- To understand effects, consider four settings.
  1. Immortality.
  2. Neither Aging nor Healthcare (exponential death).
  3. Aging without Healthcare (Gompertz death).
  4. Aging with Healthcare.
- Sample parameters:
  r = 1% , δ = 1% , γ = 0.67 , β = 7.7% , m_0 = 0.019% ,
  ζ = 50% , q = 0.46 , a = 0.1

Immortality ($\beta = 0$, $g = 0$, $M_0 = 0$)

- Special case of Merton model.
- Optimal consumption-wealth ratio constant:

$$c = \frac{1}{\gamma}\delta + \left(1 - \frac{1}{\gamma}\right) r \qquad \approx 1\%$$

- No randomness. Risk aversion irrelevant.
- But $\psi = 1/\gamma$ is elasticity of intertemporal substitution.
- Consumption increases with time preference δ.
  Increases with r for $\gamma > 1$ (income), decreases for $\gamma < 1$ (substitution).
- With logarithmic utility $\gamma = 1$, $c_t = \delta$ for any rate r.

Neither Aging nor Healthcare ($\beta = 0$, $g = 0$)

- Deaths arrive at exponential times. Poisson process with rate $m = M_0$. Forever young, but not younger.
- Optimal consumption-wealth ratio constant:

$$c = \frac{1}{\gamma}\left(\delta + (1-\zeta)^{1-\gamma} m\right) + \left(1 - \frac{1}{\gamma}\right) r \qquad \approx 1\% + 31\% \cdot m$$

- With total loss ($\zeta = 0$), mortality m adds one-to-one to time preference δ.
- Partial loss adds less than the mortality rate for $\gamma < 1$, more for $\gamma > 1$.
- Income and substitution again.
- Death brings lower wealth and lower consumption.
- Before the loss, more wealth can be spent.
- After the loss, remaining wealth is more valuable.
- $\gamma > 1$: reduce present consumption to smooth it over time. (Income: if you expect to be poor tomorrow, start saving today.)
- $\gamma < 1$: increase present consumption to enjoy wealth before it vanishes. (Substitution: if you expect to be poor tomorrow, spend while you can.)

Aging without Healthcare ($\beta > 0$, $g = 0$)

- This is non-standard (cf. Huang, Milevsky, and Salisbury, 2012).
- Optimal consumption-wealth ratio depends on age t through mortality m_t:

$$c_\beta(m_t) = \left(\int_0^\infty e^{-\frac{(1-\zeta^{1-\gamma})v}{\gamma} m_t} (\beta v + 1)^{-\left(1 + \frac{\delta + (\gamma - 1)r}{\beta\gamma}\right)} dv \right)^{-1}$$

- As $\beta \downarrow 0$, the previous case recovers:

$$c_0(m_t) = \frac{1}{\gamma}\left(\delta + (1-\zeta)^{1-\gamma} m_t\right) + \left(1 - \frac{1}{\gamma}\right) r$$

- Asymptotics for small β:

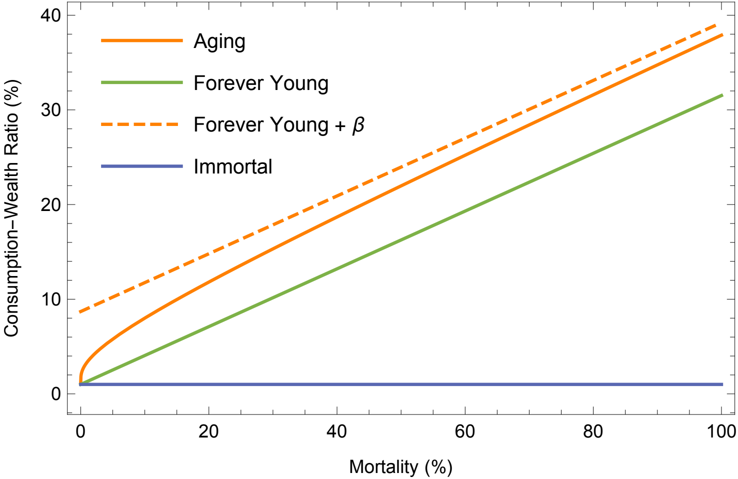
$$c_\beta(m_t) = c_0(m_t) + \frac{m_t}{c_0(m_t)} \frac{1-\zeta^{1-\gamma}}{\gamma} \beta + O(\beta^2)$$

- Asymptotics for old age (large m):

$$c_\beta(m_t) = \frac{1}{\gamma}\left(\delta + (1-\zeta)^{1-\gamma} m_t\right) + \left(1 - \frac{1}{\gamma}\right) r + \beta + O\left(\frac{1}{m}\right)$$

- Correction term large. Aging matters.

Immortal, Forever Young, and Aging



- Mortality and aging have large impacts on consumption-wealth ratios.
- β upper bound on consumption increase from aging.

Healthcare

- Solve control problem

$$\max_{c,h} \mathbb{E}\left[\int_0^\infty e^{-\delta t} U(X_t c_t) dt\right]$$

subject to state dynamics

$$dX_t = X_t(r - c_t - h_t)dt - (1 - \zeta)X_t dN_t$$
$$dm_t = \left(\beta m_t - \frac{a}{q} h_t^q\right) dt$$

- Value function $V(x, m)$ depends on wealth x and mortality m.
- Isoelastic preferences imply solution of the type

$$V(x, m) = \frac{x^{1-\gamma}}{1-\gamma} u(m)^{-\gamma}$$

for some function $u(m)$ of mortality alone.

Main Result

Theorem

Let $\gamma \in (0,1)$, $\bar{c} := \frac{\delta}{\gamma} + \left(1 - \frac{1}{\gamma}\right) r > 0$, *and let* $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *be concave,* $g(0) = 0$, *with*

$$g\left(I\left(\frac{1-\gamma}{\gamma}\right)\right) < \beta \quad \text{with} \quad I := (g')^{-1},$$

then the value function satisfies $V(x,m) = \frac{x^{1-\gamma}}{1-\gamma} u^*(m)^{-\gamma}$ *where* $u^* : \mathbb{R}_+ \mapsto \mathbb{R}_+$ *is the unique nonnegative, strictly increasing solution to the equation*

$$u^2(m) - c_0(m)u(m) + mu'(m)\left(\sup_{h \geq 0}\left\{g(h) - \frac{1-\gamma}{\gamma}\frac{u(m)}{mu'(m)}h\right\} - \beta\right) = 0.$$

Furthermore, u^* *is strictly concave, and* $(\hat{c}, \hat{h})$ *defined by*

$$\hat{c}_t := u^*(M_t) \quad \text{and} \quad \hat{h}_t := I\left(\frac{1-\gamma}{\gamma}\frac{u^*(M_t)}{M_t (u^*)'(M_t)}\right), \quad \text{for all } t \geq 0,$$

is optimal.

Estimates

Theorem

Assume $0 < \gamma < 1$, $\frac{\delta}{\gamma} + \left(1 - \frac{1}{\gamma}\right) r > 0$, *and set*

$$\beta_g := \beta - \sup_{h \geq 0} \left\{ g(h) - \frac{1-\gamma}{\gamma} h \right\} \in (0, \beta],$$

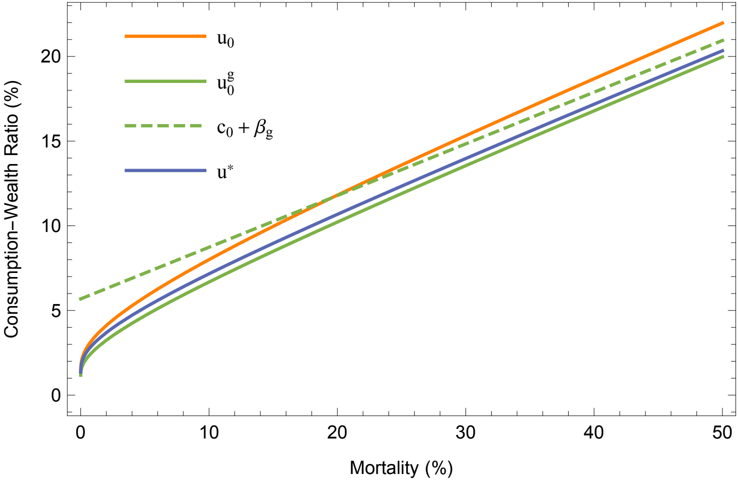
Defines $u_0^g(m)$ *analogously with* β_g *in place of* β. *Then, for any* $m > 0$,

$$u_0^g(m) \leq u^*(m) \leq \min\{u_0(m), c_0(m) + \beta_g\}$$

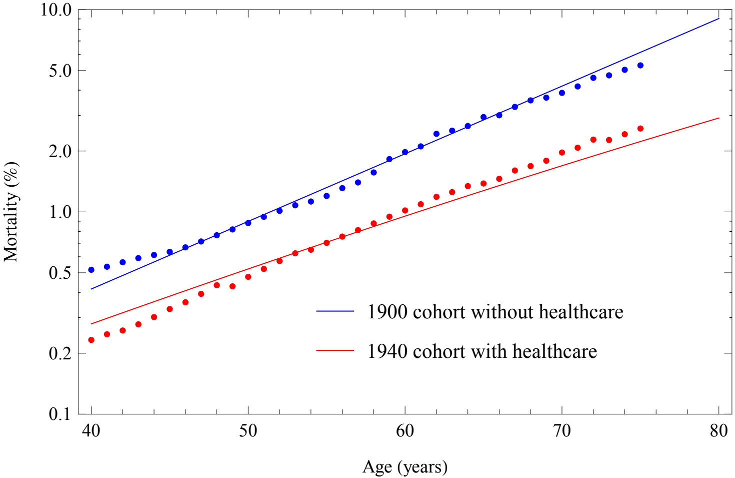
and

$$\lim_{m \to \infty} (c_0(m) - u^*(m)) = \beta_g$$

Aging and Healthcare

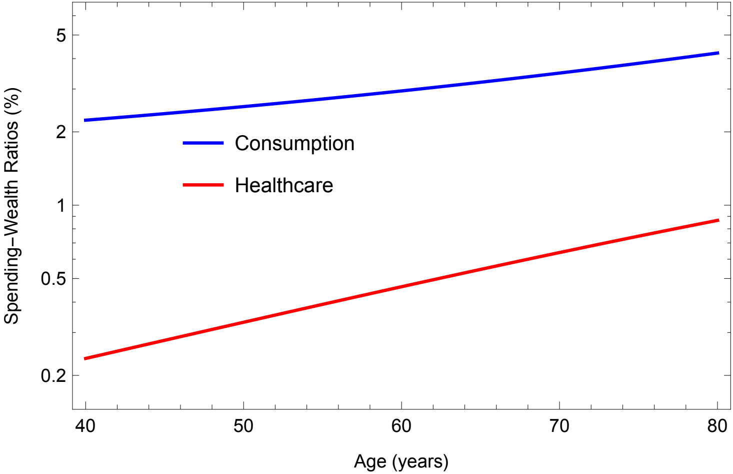


Longer Lives



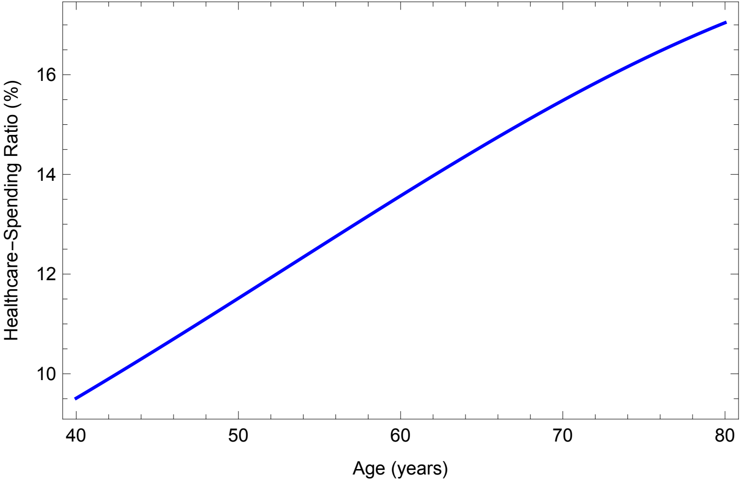
- Model explains in part decline in mortality at old ages.

Senectus Ipsa Morbus



- Healthcare negligible in youth.
- Increases faster than consumption. (In log scale!)

Healthcare as Fraction of Spending



- Convex, then concave.
- Rises quickly to contain mortality.
- Slows down when cost-benefit declines.

Risky Assets

- Risky assets S following geometric Brownian motion

$$dS_t^i = S_t^i(\mu^i + r)dt + S_t^i \sum_{j=1}^{d} \sigma^{ij} dW_t^j,$$

with $\mu \in \mathbb{R}$, $\sigma\sigma' =: \Sigma \in \mathbb{R}^{d \times d}$ positive definite.
- W standard Brownian motion independent of deaths $\{Z_n\}_{n \in \mathbb{N}}$.
- Constant optimal portfolio:

$$\pi = \frac{1}{\gamma} \Sigma^{-1} \mu$$

- Mortality does not explain lower stock allocations in old age.
- Same solution as before, with r replaced by $r + \frac{\mu \Sigma^{-1} \mu'}{2\gamma}$ in consumption formula.
- Risky assets equivalent to increase in equivalent safe rate.

HJB Equation

- Usual control arguments yield the HJB equation for u

$$u(m)^2 - c_0(m)u(m) - \beta m u'(m) + \left(\frac{1}{\gamma} - 1\right)\left(\frac{1}{q} - 1\right) a^{\frac{1}{1-q}} u(m)^{\frac{q}{1-q}} u'(m)^{\frac{1}{1-q}} = 0$$

- First-order ODE.
- $a = 0$ recovers aging without healthcare (Gompertz law).
- Factor ζ embedded in $c_0(m)$.
- Local condition with jumps?
- Boundary condition?

Derivation

- Evolution of value function:

$$\begin{aligned} dV(X_t, m_t) = & (-\delta V(X_t, m_t) + U(c_t X_t) + V_x(X_t, m_t) X_t (r - c_t - h_t)) dt \\ & + (V(\zeta X_t, m_t) - V(X_t, m_t)) dN_t \\ & + V_m(X_t, m_t)(\beta - g(h_t)) m_t dt \end{aligned}$$

- Process N_t jumps at rate m_t. Martingale condition:

$$\sup_c (U(cx) - hxV_x(x,m)) + \sup_h (-g(h) m V_m(x,m) - hxV_x(x,m))$$
$$-\delta V(x,m) + rxV_x(x,m) + (V(\zeta x, m) - V(x,m))m + \beta m V_m(x,m) = 0$$

- Includes value function before $V(x,m)$ and after $V(\zeta x, m)$ jump. Non-local condition.
- Homogeneity with isoelastic U. $V(x,m) = \frac{x^{1-\gamma}}{1-\gamma} v(m)$.

$$\sup_c \left(\frac{c^{1-\gamma}}{1-\gamma} - hv(m) \right) + \sup_h \left(-g(h) \frac{mv'(m)}{1-\gamma} - hv(m) \right)$$
$$-\delta \frac{v(m)}{1-\gamma} + r \frac{v(m)}{1-\gamma} + (\zeta^{1-\gamma} - 1) \frac{v(m)}{1-\gamma} m + \beta \frac{mv'(m)}{1-\gamma} = 0$$

Derivation (2)

- Calculating suprema with $g(h) = ah^q/q$ and substituting $v(m) = u(m)^{-\gamma}$ yields HJB equation

$$u(m)^2 - c_0(m)u(m) - \beta m u'(m) + \left(\frac{1}{\gamma} - 1\right)\left(\frac{1}{q} - 1\right) a^{\frac{1}{1-q}} u(m)^{\frac{q}{1-q}} u'(m)^{\frac{1}{1-q}} = 0$$

- Optimal policies:

$$\hat{c} = \frac{V_x(x,m)^{-\frac{1}{\gamma}}}{x} = u(m) \qquad \hat{h} = \left(\frac{xV_x(x,m)}{amV_m(x,m)}\right)^{\frac{1}{q-1}} = \left(\frac{a\gamma m u'(m)}{(1-\gamma)u(m)}\right)^{\frac{1}{1-q}}$$

- Unknown $u(m)$ is consumption-wealth ratio itself.

Probability Setting

- $(\Omega, \mathcal{F}, \mathbb{P})$ probability space.
- $\{Z_n\}_{n\in\mathbb{N}}$ IID exponential: $\mathbb{P}(Z_n > z) = e^{-z}$ for all $z \geq 0$ and $n \in \mathbb{N}$.
- $\mathcal{G}_0 := \{\emptyset, \Omega\}$ and $\mathcal{G}_n := \sigma(Z_1, \cdots, Z_n)$ for all $n \in \mathbb{N}$.
- $g : \mathbb{R}_+ \mapsto \mathbb{R}_+$ nonnegative, nondecreasing, and concave. $g(0) = 0$.
- $M^{t,x,h}$ deterministic process satisfying dynamics

$$dM_s^{t,m,h} = M_s^{t,m,h}\left[\beta - g(h(s))\right] ds, \quad M_t^{t,m,h} = m, \tag{1}$$

- Set $\theta_n = (m, h_0, h_1, \cdots, h_{n-1})$ for $n \in \mathbb{N}$ and $\theta = (m, \mathfrak{h})$.
- Define recursively a sequence $\{\tau^{\theta_n}\}_{n\geq 0}$ of random times.
- Set $\tau^{\theta_0} := 0$ and $m_0 := m$.
- For each $n \in \mathbb{N}$, define

$$\tau^{\theta_n} := \inf\left\{ t \geq \tau^{\theta_{n-1}} \;\middle|\; \int_{\tau^{\theta_{n-1}}}^t M_s^{\tau^{\theta_{n-1}}, m_{n-1}, h_{n-1}} ds \geq Z_n \right\}, m_n := M_{\tau^{\theta_n}}^{\tau^{\theta_{n-1}}, m_{n-1}, h_{n-1}}$$

Probability Setting (2)

- Set $\mathbb{F}^\theta = \{\mathcal{F}_t^\theta\}_{t\geq 0}$ as $\mathbb{P}$-augmentation of the filtration

$$\left\{ \bigvee_{n\in\mathbb{N}} \sigma\left(1_{\{\tau^{\theta_n}\leq s\}} \mid 0 \leq s \leq t\right) \right\}_{t\geq 0}.$$

- Introduce counting process $\{N_t\}_{t\geq 0}$:

$$N_t^\theta := n \quad \text{for } t \in [\tau^{\theta_n}, \tau^{\theta_{n+1}}),$$

- By construction of $\{\tau^{\theta_n}\}_{n\geq 0}$,

$$\mathbb{P}\left(N_t^\theta = n \mid \mathcal{F}_{\tau^{\theta_n}}^\theta\right) =$$
$$\mathbb{P}\left(\tau^{\theta_n} \leq t < \tau^{\theta_{n+1}} \mid \mathcal{F}_{\tau^{\theta_n}}^\theta\right) = \exp\left(-\int_{\tau^{\theta_n}}^t M_s^{\tau^{\theta_n}, m_n, h_n} ds\right) 1_{\{t\geq \tau^{\theta_n}\}}.$$

Verification

Theorem

Let $w \in C^{1,1}(\mathbb{R}_+ \times \mathbb{R}_+)$ satisfy the HJB equation. If, for any (x, m) and $(\mathfrak{c}, \mathfrak{h})$,

$$\lim_{t\to\infty} \mathbb{E}\left[\exp\left(-\int_{\tau^{\theta_n}}^t (\delta + M_s^{0,m,\mathfrak{h}^{\theta_n}})ds\right) w\left(X_t^{0,x,\mathfrak{c}^{\theta_n},\mathfrak{h}^{\theta_n}}, M_t^{0,m,\mathfrak{h}^{\theta_n}}\right) \,\middle|\, \mathcal{F}^{\theta}_{\tau^{\theta_n}}\right] = 0 \; \forall n \geq$$

$$\lim_{n\to\infty} \mathbb{E}\left[e^{-\delta\tau^{\theta_n}} w\left(\zeta^n X_{\tau^{\theta_n}}^{0,x,\mathfrak{c}^{\theta},\mathfrak{h}^{\theta}}, M_{\tau^{\theta_n}}^{0,m,\mathfrak{h}^{\theta}}\right)\right] = 0.$$

(i) *$w(x, m) \geq V(x, m)$ on $\mathbb{R}_+ \times \mathbb{R}_+$.*

(ii) *If $\hat{c}$, $\hat{h} : \mathbb{R}_+^2 \mapsto \mathbb{R}_+$ such that $\hat{c}(x, m)$ and $\hat{h}(x, m)$ maximize*

$$\sup_{c\geq 0} \{U(cx) - cxw_x(x, m)\} \quad \textit{and} \quad \sup_{h\geq 0} \{-w_m(x, m)g(h) - hxw_x(x, m)\},$$

Let $\hat{X}$ and $\hat{M}$ denote the solutions to the ODEs

$$dX_s = X_s[r - (\hat{c}(X_s, M_s) + \hat{h}(X_s, M_s))]ds \quad X_0 = x,$$
$$dM_s = \left[\beta M_s - g(\hat{h}(X_s, M_s))\right] ds \quad M_0 = m.$$

Then $w(x, m) = V(x, m)$ on $\mathbb{R}_+ \times \mathbb{R}_+$, and the policy $(\hat{\mathfrak{c}}, \hat{\mathfrak{h}})$ is optimal.

Forever Young

Proposition

Suppose

$$\delta + (1 - \zeta^{1-\gamma})m - (1-\gamma)^{+} r > 0.$$

Then, $V(x, m) = \frac{x^{1-\gamma}}{1-\gamma}\hat{c}_0(m)^{-\gamma}$ *for all* $x \geq 0$*, and* $\hat{\mathfrak{c}} := \{\hat{c}_0(m)\}_{n\geq 0}$ *is optimal.*

- Check two conditions of verification theorem.
- Works also for $\gamma > 1$. Up to a point.
- Parametric restrictions!

Aging without Healthcare

Proposition

Assume either one of the conditions:

(i) $\gamma, \zeta \in (0,1)$ *and* $\delta + m + (\gamma - 1)r > 0$.

(ii) $\gamma, \zeta > 1$.

Then, for any $(x,m) \in \mathbb{R}_+^2$, $V(x,m) = \frac{x^{1-\gamma}}{1-\gamma} u_0(m)^{-\gamma}$, *where*

$$u_0(m) := \left[\frac{1}{\beta}\int_0^\infty e^{-\frac{(1-\zeta^{1-\gamma})mu}{\beta\gamma}} (u+1)^{-\left(1+\frac{\delta+(\gamma-1)r}{\beta\gamma}\right)} du\right]^{-1} > 0.$$

Moreover, $\hat{\mathfrak{c}} := \{u_0(me^{\beta t})\}_{n\geq 0}$ *is optimal.*

- Works with $\gamma > 1$... if $\zeta > 1$!
- With $\gamma > 1$ and $\zeta < 1$, household worries too much.
- Extreme savings. Problem ill-posed.

Aging with Healthcare

- Aging without healthcare policy c_β supersolution.
- Forever young policy c_0 subsolution.
- $\mathcal{S}$ denotes collection of $f : [0, \infty) \mapsto \mathbb{R}$ such that
    1. $c_0 \le f \le c_\beta$.
    2. f is a continuous viscosity supersolution on $(0, \infty)$.
    3. f is concave and nondecreasing.
- Define $u^* : \mathbb{R}_+ \mapsto \mathbb{R}$ by

$$u^*(m) := \inf_{f \in \mathcal{S}} f(m).$$

Proposition

The function u^* belongs to $\mathcal{S}$. Moreover, if

$$\sup_{h \ge 0} \left\{ g(h) - \tfrac{1-\gamma}{\gamma} h \right\} \le \beta,$$

then u^* is continuously differentiable on $(0, \infty)$.

- Well-posedeness if healthcare cannot defeat aging.

Conclusion

- Model for optimal consumption and healthcare spending.
- Natural mortality follows Gompertz' law.
- Isoelastic utility and efficacy.
- Reduced mortality growth under optimal policy.
- Share of spending for healthcare rises with age.

Four

Commodities and Stationary Risks

Outline

- Motivation:
  Commodity Futures as an Asset Class.
  Mean Reversion. Fund Separation?
- Model:
  Portfolio Choice in Commodity Indexes.
  Constant relative risk aversion.
- Results:
  Policy and Performance: Index vs. Price Observations.

Commodity Futures Investing

- Commodity Futures:
  "No trade deserves more the full protection of the law, and no trade requires it so much; because no trade is so much exposed to popular odium." (Adam Smith, 1776)
- Commodity Futures as an Asset Class:
  Inflation hedges (Bodie, 1983)
  Similar return as equities (Bodie and Rosansky, 1980)
  Negative correlation with equities (Gorton and Rouwenhorst, 2006)
  Individual commodities uncorrelated (Erb and Harvery, 2006)
  Positive, predictable returns (Levine, Ooi, and Richardson, 2016)
- Commodity Futures in Practice:
  Rising popularity since 2004 (Singleton, 2014)
  Financialization? (Tang and Xiong, 2012)
  Investment through commodity index ETFs (Basak and Pavlova, 2016)
- Fund separation?

No Fund Separation

- $(k+2)$-fund separation (Merton, 1973).
  If k predictors are available, $k+2$ funds span optimal portfolios.
  Usual two-fund separation with constant returns ($k=0$).
- Commodities returns mean-reverting *and* uncorrelated?
  Forget two-fund separation.
- A priori, prices of *all* commodities are states.
- But investment is in index only.
- Prices of individual commodities worth observing?

Related Models

- Enlargement of filtrations.
- Logarithmic utility:
  Karatzas and Pikovsky (1996), Grorud and Pontier (1998), Amendinger, Imkeller, Schweizer (1998), Corcuera et al. (2004), Guasoni (2006)
- Power and exponential utilities in complete markets:
  Amendinger, Becherer, Schweizer (2003)
- Filtering theory.
- Portfolio choice with partial information.
  Lakner (1995, 1998), Brennan (1998), Brennan and Xia (2001), Rogers (2001), Brendle (2006), Cvitanic et al (2006).
- Asset Pricing with learning:
  Detemple (1986), Dothan and Feldman (1986), Veronesi (2000).

This Model

- Portfolio choice for a commodity index
- With or Without observing commodities' prices.
- Power utility and long horizon.
- Commodities: transitory price shocks
- Myopic policies far from optimal. Large intertemporal demand.
- Additional price information large even for risk-averse investors.
- Gains in equivalent safe rate of about 0.5%.

Commodity Futures

- P_t spot price of commodity at time t. Cannot be held like financial asset.
- F_t^T futures price at time t for expiration T. Zero cost.
- At time t, buy contracts expiring at $t + \Delta t$ equal to portfolio amount at t.
- At time $t + \Delta t$, liquidate contract (and buy new contracts expiring at $t + 2\Delta t$ equal to portfolio amount at $t + \Delta t$)
- Return on $[t, t + \Delta t]$, assuming zero safe rate:

$$\frac{F_{t+\Delta t}^{t+\Delta t} - F_t^{t+\Delta t}}{F_t^{t+\Delta t}} = Q_t^{t+\Delta t} \frac{P_{t+\Delta t}}{P_t} - 1 = Q_t^{t+\Delta t} \underbrace{\frac{P_{t+\Delta t} - P_t}{P_t}}_{\text{spot return}} + \underbrace{(Q_t^{t+\Delta t} - 1)}_{\text{roll return}}$$

  where $Q_t^{t+\Delta t} = P_t / F_t^{t+\Delta t}$ is the spot-futures ratio at time t.
- With roll-return of order dt, dynamics for rolled-over futures portfolio S_t is:

$$\frac{dS_t}{S_t} = \mu_t dt + \frac{dP_t}{P_t}$$

- Key difference: P_t is stationary.
  Empirically and theoretically.

Commodity Index Model

- n commodities. Return on futures portfolio of i-th commodity:

$$\frac{dS_t^i}{S_t^i} = \mu^i dt + \sigma^i dU_t^i \qquad dU_t^i = -\lambda^i U_t^i dt + dW_t^i$$

  W_t^i independent Brownian motions.

- Commodity index with weights w^i:

$$\frac{dS_t}{S_t} = \sum_{i=1}^{n} w^i \frac{dS_t^i}{S_t^i} = \left(\mu - \sum_{i=1}^{n} w^i \sigma^i \lambda^i U_t^i\right) dt + \sigma d\tilde{W}_t$$

  where $\mu = \sum_{i=1}^{n} w^i \mu^i$ and $\sigma \tilde{W}_t = \sum_{i=1}^{n} w^i \sigma^i W_t^i$ (W Brownian motion).

- Spot returns depend on spot prices $P_t^i = P_0^i e^{\sigma^i U_t^i}$...
- ...and so do optimal investment strategies. Notation:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dY_t \quad Y_t = \sum_{i=1}^{n} p_i U_t^i \quad p_i = w^i \sigma^i / \sigma \quad \sum_{i=1}^{n} p_i^2 = 1$$

One Asset Prelude

- One commodity only, $n = 1$. Compare Föllmer and Schachermayer (2008)
- Constant relative risk aversion. Utility $U(x) = x^{1-R}/(1-R)$.
- Wealth X_t satisfies budget equation $\frac{dX_t}{X_t} = \pi_t \frac{dS_t}{S_t}$. π_t portfolio weight.
- Maximize equivalent safe rate $\lim_{T\to\infty} \frac{1}{T} \log E[X_T^{1-R}]^{\frac{1}{1-R}}$
- Optimal policy:

$$\pi_t = \frac{\mu}{\sigma^2} - \frac{\lambda_1}{\sigma\sqrt{R}} U_t^1$$

- No R in the denominator! Interpretation?
- Equivalent safe rate:

$$\delta = \frac{\mu^2}{2\sigma^2} + \frac{\lambda_1}{2(1+\sqrt{R})}$$

- Risk-premium, plus market timing. Risk premium without risk aversion!
- Why?

Intertemporal Balance

- Myopic and intertemporal hedging decomposition:

$$\pi_t = \underbrace{\frac{\mu - \sigma\lambda_1 U_t^1}{R\sigma^2}}_{\textit{myopic}} + \underbrace{\frac{(R-1)\mu + (1-\sqrt{R})\sigma\lambda_1 U_t^1}{R\sigma^2}}_{\textit{intertemporal}}$$

$$= \frac{\mu}{\sigma^2} - \frac{\lambda_1}{\sigma\sqrt{R}} U_t^1$$

- Myopic demand offset by terms in intertemporal component.
- Transitory risks are not like permanent risks.
- They are “less risky”, so more risk averse investors take more such risks (than it they were permanent).
- Discontinuity for $\lambda \downarrow 0$, as transitory becomes permanent in the limit.
- Strategic vs. tactical exposures.
  Strategic independent of risk aversion and state: captures risk premium.
  Tactical independent of risk premium, captures imbalance in state.

Dynamics and Information

- Dynamics with observation of **commodities** prices:

$$\frac{dS_t}{S_t} = \left(\mu - \sum_{i=1}^{n} w^i \sigma^i \lambda^i U_t^i\right) dt + \sigma d\tilde{W}_t$$
$$dU_t^i = -\lambda^i U_t^i dt + dW_t^i$$

- Dynamics with observation of **index** price only (Kalman filter):

$$\frac{dS_t}{S_t} = \left(\mu - \sum_{i=1}^{n} w^i \sigma^i \lambda^i \tilde{U}_t^i\right) dt + \sigma d\tilde{W}_t$$
$$d\tilde{U}_t^i = -\lambda^i \tilde{U}_t^i dt + (p' - \gamma_t b') d\tilde{W}_t$$
$$\frac{d\gamma_t}{dt} = -\lambda\gamma_t - \gamma_t\lambda + I - (p' - \gamma_t b')(p' - \gamma_t b')'$$

  $p = (p_1, \ldots, p_n)$, λ as diagonal matrix, γ_t $n \times n$ matrix.

- Time-dependent Kalman filter. A non-starter for (interesting) formulas.

Long Term Filter

Proposition

For any initial γ_0, the solution γ_t to the Riccati differential equation converges:

$$\lim_{t\to+\infty} \gamma_t = \gamma,$$

where the matrix γ satisfies the Riccati algebraic equation

$$-\lambda\gamma - \gamma\lambda' + I - [p' - \gamma b'][p' - \gamma b']' = 0.$$

The dynamics of the filters $\tilde{U}_t^i$ becomes

$$d\tilde{U}_t^i = -\lambda_i \tilde{U}_t^i dt + \alpha_i d\tilde{W}_t \qquad \text{where} \qquad \alpha_i' = p_i - \sum_{k=1}^{n} p_k \lambda_k \gamma_{ik}.$$

- Convergence relies on results on controllable and stabilizable systems.
- Bad news: Riccati matrix equation has no explicit solution in general.

Observing Commodities

Theorem

If an investor trades the index by observing the prices of all commodities:

$$\pi_t^{C*} = \frac{\mu}{\sigma^2} - \frac{p\left(\lambda - \mathbf{A}^C\right)U_t}{R\sigma} \qquad \textit{(Optimal Portfolio)}$$

$$\mathrm{EsR}^C = \frac{\mu^2}{2\sigma^2} + \frac{\mathrm{tr}\left(\mathbf{A}^C\right)}{2(1-R)} \qquad \textit{(Equivalent Safe Rate)}$$

$$\mathbf{A}^C = \lambda - \mathbf{C}^{-\frac{1}{2}}\left(\mathbf{C}^{\frac{1}{2}}\frac{\lambda^2}{2}\mathbf{C}^{\frac{1}{2}}\right)^{\frac{1}{2}}\mathbf{C}^{-\frac{1}{2}} \qquad \textit{and} \qquad \mathbf{C} = \frac{I}{2} + \frac{(1-R)p'p}{2R}$$

- Explicit solution to Riccati equation.
- Strategic vs. Tactical as with one-asset.
- Long-run verification theorem.

Observing Index

Theorem

If an investor trades the index observing only the index:

$$\pi_t^{I*} = \frac{\mu}{R\sigma^2} + \frac{(1-R)\beta^{I\prime}\alpha}{R\sigma} - \frac{\left(p\lambda - \alpha'\mathbf{A}^I\right)\tilde{U}_t}{R\sigma}$$

$$\mathrm{EsR}^I = \frac{\mu^2}{2R\sigma^2} + \frac{(1-R)^2(\beta^I\alpha)^2}{2R} + \frac{(1-R)\mu\beta^I\alpha}{R\sigma} + \frac{\mathrm{tr}(\alpha\alpha'\mathbf{A}^I)}{2(1-R)} + \frac{(1-R)\,\mathrm{tr}(\alpha\alpha'\beta^{I\prime}\beta^I)}{2}$$

where

$$\beta^I = -\frac{\mu\left(p\lambda - \alpha'\mathbf{A}^I\right)}{R\sigma}\left(\lambda + \frac{(1-R)\alpha p\lambda}{R} - \frac{\alpha\alpha'\mathbf{A}^I}{R}\right)^{-1},$$

and $\mathbf{A}^I$ is the symmetric, definite-positive solution of the matrix Riccati equation

$$\frac{\mathbf{A}^I\lambda+\lambda\mathbf{A}^I}{2} + \frac{1}{2R}\lambda p' p\lambda + \frac{(1-R)}{2R}\mathbf{A}^I\alpha\alpha'\mathbf{A}^I - \frac{(1-R)}{R}\frac{\lambda p'\alpha'\mathbf{A}^I+\mathbf{A}^I\alpha p\lambda}{2} = 0$$

- No explicit solution. Easy to solve numerically.
- Qualitative structure similar. Quantitative differences?

Example

Filtered Shocks Correlation (%)

n	2		3			4			
	100		100			100			
	-100	100	-66.92	100		-54.41	100		
			-50.01	-30.88	100	-40.41	-25.42	100	
						-33.18	-21.74	-18.46	100

- $\lambda_i = i$, $\sigma_i = 1$, $p_i = 1/\sqrt{n}$
- With two states, one filter is perfectly negatively correlated with the other. Their sum is observed.
- With more states, more shock ascribed to more persistent states (lower λ).
- Imperfect correlations, higher among more persistent states.

Commodities

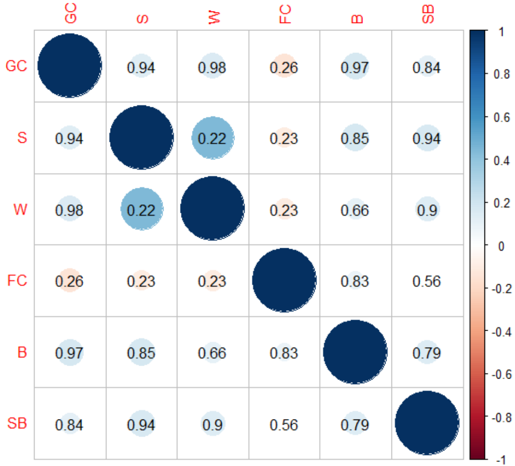
- S&P GSCI Index.
- 6 commodities explain about 85% of index return variance.
- Rolled-over commodity futures:
  Each month, invest in two-month contract. Sell month afterwards.
- Understand optimal portfolios and equivalent safe rates.
  With or without observing commodities.

Commodities: Mean-Reverting

Calibrated Parameters					
Commodity	**Symbol**	p_i	λ_i	$\omega_i(\%)$	$\sigma_i(\%)$
Wheat	**W**	0.07	0.11	9.2%	31.6%
Soybeans	**S**	0.08	0.21	6.8%	24.9%
Sugar	**SB**	0.04	0.12	2.5%	33.0%
Feeder Cattle	**FC**	0.05	0.17	7.6%	14.2%
Brent Crude Oil	**B**	0.85	0.12	58.9%	31.1%
Gold	**GC**	0.06	0.12	8.4%	16.0%
Residual	**RE**	0.51	0.01		
GSCI		$\mu(\%)$	$\sigma(\%)$		
		2.4	19.84		

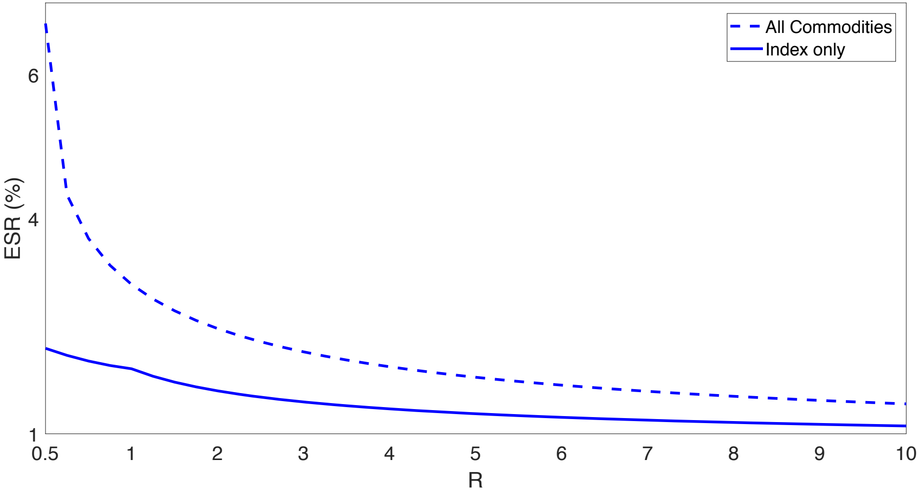
- Monthly Returns 1993:05-2018:02
- Weights estimated from sensitivities. Do not have to add to one.

Commodities: Uncorrelated Returns



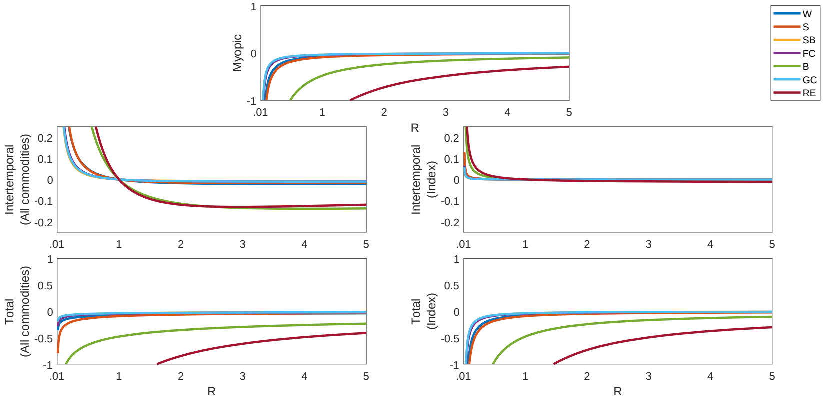
Colors = Correlations. Numbers = p-values.

Commodities: Equivalent Safe Rate



- Difference minimal near logarithmic utility ($R = 1$)

Commodities: Optimal Portfolio



- Sensitivities of optimal portfolios with respect to each commodity.
- Sensitivity is driven by...
- ...intertemporal component for with full information.
- ...myopic component with partial information

HJB Equation

- Denote by Z vector of state variables.
  $Z_t = U_t$ with full information, $Z_t = \tilde{U}_t$ with partial information.
- Write HJB equation for finite-horizon problem

$$V_t - V_Z \lambda Z_t + \frac{\mathrm{tr}(V_{ZZ})}{2} + \sup_{\pi} \left[V_X \pi_t X_t \left(\mu - \sigma p \lambda Z_t \right) + \frac{V_{XX} (\pi_t)^2 X_t^2 \sigma^2}{2} + V_{XZ} \pi_t X_t \sigma p' \right] = 0$$

- Use exponential-quadratic ansatz

$$V(x,t,z) = \frac{x^{1-R}}{1-R} e^{(1-R)[\delta(T-t) + \beta(t) z + \frac{1}{2} z' \mathbf{A}(t) z]}$$

- With full information, obtain system of equations for $\delta, \beta, \mathbf{A}$

$$-\delta + \frac{1}{2}\,\mathrm{tr}\,(\mathbf{A}) + \frac{\mu^2}{2R\sigma^2} + \frac{(1-R)}{2} tr\,(\beta'\beta) + \frac{(1-R)^2}{2R} (\beta p')^2 + \frac{\mu(1-R)}{R\sigma} \beta p' = 0$$

$$-\beta\lambda + (1-R)\beta\mathbf{A} - \frac{\mu}{R\sigma} p\lambda + \frac{\mu(1-R)}{R\sigma} p\mathbf{A} - \frac{1-R}{R} \beta p' p \lambda + \frac{(1-R)^2}{R} \beta p' p \mathbf{A} = 0$$

$$-\frac{\mathbf{A}\lambda + \lambda\mathbf{A}}{2} + \frac{1-R}{2}\mathbf{A}^2 + \frac{1}{2R} \lambda p' p \lambda + \frac{(1-R)^2}{2R} \mathbf{A} p' p \mathbf{A} - \frac{1-R}{R} \frac{\mathbf{A} p' p \lambda + \lambda p' p \mathbf{A}}{2} = 0$$

- Bottom-up solution. Find matrix $\mathbf{A}$, vector β, then scalar δ.

Explicit Solution with Full Information

- With full information, matrix equation if of the form

$$\mathbf{A}^C\mathbf{C}\mathbf{A}^C - \mathbf{A}^C\mathbf{C}\mathbf{D} - \mathbf{D}\mathbf{C}\mathbf{A}^C + \mathbf{F} = 0$$

- Set $\tilde{\mathbf{A}^C} = \mathbf{C}^{\frac{1}{2}}\mathbf{A}^C\mathbf{C}^{\frac{1}{2}}$, $\tilde{\mathbf{D}} = \mathbf{C}^{\frac{1}{2}}\mathbf{D}\mathbf{C}^{\frac{1}{2}}$, $\tilde{\mathbf{F}} = \mathbf{C}^{\frac{1}{2}}\mathbf{F}\mathbf{C}^{\frac{1}{2}}$, which yields

$$\tilde{\mathbf{A}^C}\tilde{\mathbf{A}^C} - \tilde{\mathbf{A}^C}\tilde{\mathbf{D}} - \tilde{\mathbf{D}}\tilde{\mathbf{A}^C} + \tilde{\mathbf{F}} = 0$$

- whence

$$\tilde{\mathbf{A}^C} = \tilde{\mathbf{D}} + (\tilde{\mathbf{D}}^2 - \tilde{\mathbf{F}})^{\frac{1}{2}}$$

- and thus

$$\mathbf{A}^C = \frac{\lambda}{1-R} + \frac{1}{|1-R|}\mathbf{C}^{-\frac{1}{2}}\left(\mathbf{C}^{\frac{1}{2}}\frac{\lambda^2}{2}\mathbf{C}^{\frac{1}{2}}\right)^{\frac{1}{2}}\mathbf{C}^{-\frac{1}{2}}$$

- Resulting $\mathbf{A}^C$ is symmetric and definite-positive.

Verification

- Find Lyapunov-type function G.
- Define new measure $\hat{P}$ under which

$$\frac{dS_t}{S_t} = \frac{1}{R}\left(\mu - \sigma p\lambda + \sigma\nabla\upsilon\right) + \sigma d\hat{W}_t^I$$

$$dZ_t = \left(\lambda + \frac{1-R}{R\sigma}\lambda + \frac{1}{R}\left(-\frac{\mu}{\sigma}p + z'\mathbf{A}^C\right)\right) dt + d\hat{W}_t$$

- $(Z_t)_{t\geq 0}$ is $\hat{P}$-tight.
- Z is a multivariate Ornstein-Uhlenbeck also under $\hat{P}$.
- Under $\hat{P}$ finite-horizon duality bounds hold.
- Estimate transitory terms using Gaussian distribution and conclude.
- Similar argument for partial information, but no explicit matrix **A**.

Conclusion

- Should Commodity Investors Follow Commodities' Prices?
- Long term investors should, even the more risk averse.
- Mean exposure to commodities insensitive to risk aversion...
- ...and optimal strategies benefit from the extra information.
- Gains similar to earning an extra risk-free 0.5% on wealth.

Five

Leveraged Funds: Robust Replication and Performance Evaluation

Outline

- Motivation:
  Leveraged Exchange-Traded Funds: Tracking Error vs. Excess Return.
- Model:
  Optimal Tracking with Trading Costs.
  Volatility process stationary.
- Results:
  Optimal Tracking Policy.
  Optimal Tradeoff between Tracking Error and Excess Return.

Leveraged and Inverse ETFs

- Funds that attempt to replicate multiple of *daily* return on an index.
- Multiple either positive (leveraged) or negative (inverse).
- On equities, bonds, commodities, currencies, real estate, volatility.
- In the United States, since 2006 multiples of -2, -1, 2.
- Since 2009, multiples of -3 and 3.
- What next?

6, 8, 10, 15 since 2015 in Germany

Deutsche Bank
X-markets

Deutsche AM | db X-trackers | db-X ETC | db-X funds | Imprint | Deutsch | Contact

Warrants, Certificates & Bonds

Home | Prices & markets | Tools | Services | Knowledge & Academy

Welcome to X-markets. Please logon to use MyX-markets.

Login | Register | Advantages

WKN, ISIN, Name or Underlying

Product search > Faktor Certificates

Product list | Product filter

Product Type
- Warrants
- Faktor Certificates

Underlying
- Underlying Type
- Underlying

Status
- All active products

Strike Price

Product search

75 Products found

Product Type Faktor Certificates

Select action | 20 Products per page

Name/ WKN	Bid	Ask	Factor	FactorType	Maturity
Faktor 15x long Euro Bund Future Index-Certificate relating to the Euro Bund Future Long Index Faktor 15					
DX8B15	1,061.70	1,069.10	15	Long	Open End
Faktor 10x long Euro Bund Future Index-Certificate relating to the Euro Bund Future Long Index Faktor 10					
DX8B10	586.85	589.55	10	Long	Open End
Faktor 10x long DAX® Index-Certificate relating to the LevDAX® x10 Index					
XM9L10	0.24	0.25	10	Long	Open End
Faktor 8x Long Silber Index-Certificate relating to the Silber Long-Index Faktor 8					
DL8XAG	8.93	9.33	8	Long	Open End
Faktor 8x long DAX® Index-Zertifikat bezogen auf den LevDAX x8® Index					

Simple – In principle

- If index follows

$$\frac{dS_t}{S_t} - r_t dt = \mu_t dt + \sigma_t dW_t$$

- Then leveraged or inverse ETF is portfolio with constant proportion Λ

$$\frac{dw_t}{w_t} - r_t dt = \Lambda \mu_t dt + \Lambda \sigma_t dW_t$$

- Fund price

$$w_t = w_0 \exp\left(\int_0^t r_s ds + \Lambda \int_0^t \mu_s ds - \frac{\Lambda^2}{2} \int_0^t \sigma_s^2 ds + \Lambda \int_0^t \sigma_s dW_s \right)$$

- Compounding implies Volatility decay: when the multiple Λ is high, the fund loses value regardless of the index's return.
- High expected value, but almost surely going to zero.
- Annual or monthly returns are not multiplied.
- From a frictionless viewpoint, risky but simple products.
- But when the multiple is high also rebalancing costs are high...

Literature

- Compounding:
  Avellaneda and Zhang (2010), Jarrow (2010), Lu, Wang, Zhang (2009)
- Rebalancing and Market Volatility:
  Cheng and Madhavan (2009), Charupat and Miu (2011).
- Underexposure:
  Tang and Xu (2013): "LETFs show an underexposure to the index that they seek to track."
- Underperformance:
  Jiang and Yan (2012), Avellaneda and Dobi (2012), Guo and Leung (2014), Wagalath (2014).
- Many authors attribute deviations to frictions. Model?

Dilemma

- With costless rebalancing, keep constant leverage. Perfect tracking:

$$dD_t = \frac{dw_t}{w_t} - \Lambda\frac{dS_t}{S_t} + (\Lambda - 1)r_t dt = 0$$

- Excess return D_T/T and tracking error $\sqrt{\langle D\rangle_T/T}$ both zero.
- With costly rebalancing, dream is broken.
- Rebalancing reduces tracking error but makes deviation more negative.
- **Questions**:
- What are the optimal rebalancing policies?
- How to compare funds differing in excess return and tracking error?
- **Implications**:
- Underexposure consistent with optimality?
- Underperformance significant?

This Model

- Model of Optimal Tracking with Trading Costs.
- **Excess Return** vs. **Tracking Error**.
- Underlying index follows Itô process. Zero drift in basic model.
- Robust tracking policy: independent of volatility process at first order.
- Excess return and tracking error depend only on average volatility.
- Explains underexposure puzzle.

Basic Model

- Filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ with Brownian motion W and its augmented natural filtration.
- Safe asset with adapted, integrable rate r_t.
- Index with ask (buying) price S_t:

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dW_t,$$

  where σ_t^2 is adapted and integrable, i.e. S_t is well defined.
- Proportional costs: bid price equals $(1 - \varepsilon)S_t$.
- Zero excess return assumption:
  no incentive to outperform index through extra exposure.
- Main results robust to typical risk premia.
- Stationary volatility: for some $\bar{\sigma} > 0$,

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T \sigma_t^2 dt = \bar{\sigma}^2$$

Objective

- Trading strategy:
  Number of shares $\varphi_t = \varphi_t^{\uparrow} - \varphi_t^{\downarrow}$ as purchases minus sales. Fund value:

$$dX_t = r_t X_t dt - S_t d\varphi_t^{\uparrow} + (1-\varepsilon) S_t d\varphi_t^{\downarrow} \qquad \text{(cash)}$$

$$dY_t = S_t d\varphi_t^{\uparrow} - S_t d\varphi_t^{\downarrow} + \varphi_t dS_t \qquad \text{(index)}$$

  $w_t = X_t + Y_t$. Admissibility: $w_t \geq 0$ a.s. for all t.

- (Annual) Excess Return:

$$\text{ExR} = \frac{1}{T} \int_0^T \left(\frac{dw_t}{w_t} - \Lambda \frac{dS_t}{S_t} + (\Lambda - 1) r_t dt \right) = \frac{D_T}{T}$$

- (Annual) Tracking Error:

$$\text{TrE} = \sqrt{\frac{\langle D \rangle_T}{T}}$$

- Maximize long term excess return given tracking error:

$$\max_{\varphi} \limsup_{T \to \infty} \frac{1}{T} \left(D_T - \frac{\gamma}{2} \langle D \rangle_T \right)$$

Main Result

Theorem (Exact)

Assume $\Lambda \neq 0, 1$*.*

i) *For any* $\gamma > 0$ *there exists* $\varepsilon_0 > 0$ *such that for all* $\varepsilon < \varepsilon_0$*, the system*

$$\tfrac{1}{2}\zeta^2 W''(\zeta) + \zeta W'(\zeta) - \tfrac{\gamma}{(1+\zeta)^2}\left(\Lambda - \tfrac{\zeta}{1+\zeta}\right) = 0,$$
$$W(\zeta_-) = 0, \quad W'(\zeta_-) = 0,$$
$$W(\zeta_+) = \tfrac{\varepsilon}{(1+\zeta_+)(1+(1-\varepsilon)\zeta_+)}, \quad W'(\zeta_+) = \tfrac{\varepsilon(\varepsilon-2(1-\varepsilon)\zeta_+-2)}{(1+\zeta_+)^2(1+(1-\varepsilon)\zeta_+)^2}$$

has a unique solution (W, ζ_-, ζ_+) *for which* $\zeta_- < \zeta_+$*.*

ii) *The optimal policy is to buy at* $\pi_- := \zeta_-/(1+\zeta_-)$ *and sell at* $\pi_+ := \zeta_+/(1+\zeta_+)$ *to keep* $\pi_t = \zeta_t/(1+\zeta_t)$ *within the interval* $[\pi_-, \pi_+]$*.*

iii) *The maximum performance is*

$$\limsup_{T\to\infty}\left(D_T - \frac{\gamma}{2}\langle D\rangle_T\right) = -\frac{\gamma\overline{\sigma}^2}{2}(\pi_- - \Lambda)^2,$$

Main Result (continued)

Theorem (Exact)

iv) *Excess return and tracking error:*

$$\mathrm{ExR} = \frac{\overline{\sigma}^2}{2}\frac{\pi_-\pi_+(\pi_+ - 1)^2}{(\pi_+ - \pi_-)(1/\varepsilon - \pi_+)} \qquad \mathrm{TrE} = \overline{\sigma}\sqrt{\pi_-\pi_+ + \Lambda(\Lambda - 2\bar{\beta})}$$

where $\bar{\beta}$ is the average exposure

$$\bar{\beta} := \lim_{T\to\infty}\frac{\langle \int^{\cdot} \frac{dw}{w}, \int^{\cdot} \frac{dS}{S}\rangle_T}{\langle \int^{\cdot} \frac{dS}{S}\rangle_T} = \lim_{T\to\infty}\frac{\int_0^T \sigma_t^2 \pi_t dt}{\int_0^T \sigma_t^2 dt} = \log(\pi_+/\pi_-)\frac{\pi_+\pi_-}{\pi_+ - \pi_-}$$

- ODE depends only on multiple Λ and trading cost ε. So do π_-, π_+.
- ExR and TrE depend also on $\bar{\sigma}$.

Main Result

Theorem (Robust Approximation)

- *Trading boundaries:*

$$\pi_{\pm} = \Lambda \pm \left(\frac{3}{4\gamma}\Lambda^2(\Lambda-1)^2\right)^{1/3}\varepsilon^{1/3} + O(\varepsilon^{2/3})$$

- *Excess return:*

$$\mathrm{ExR} = -\frac{3\overline{\sigma}^2}{\gamma}\left(\frac{\gamma\Lambda(\Lambda-1)}{6}\right)^{4/3}\varepsilon^{2/3} + O(\varepsilon)$$

- *Tracking error:*

$$\mathrm{TrE} = \overline{\sigma}\sqrt{3}\left(\frac{\Lambda(\Lambda-1)}{6\sqrt{\gamma}}\right)^{2/3}\varepsilon^{1/3} + O(\varepsilon)$$

The Tradeoff

- Previous formulas imply that

$$\mathrm{ExR} = -\frac{3^{1/2}}{12}\overline{\sigma}^3\Lambda^2(\Lambda - 1)^2\frac{\varepsilon}{\mathrm{TrE}} + O(\varepsilon^{4/3})$$

- Maximum excess return for given tracking error.
- Equality for optimal policy, otherwise lower excess return. In general:

$$\mathrm{ExR}\cdot\mathrm{TrE} \leq -\frac{3^{1/2}}{12}\overline{\sigma}^3\Lambda^2(\Lambda - 1)^2\varepsilon + O(\varepsilon^{4/3})$$

- Robust formula. Depends on model only through average volatility. Dynamic irrelevant.
- If ε is observed, theoretical upper bound on replication performance.
- Want less negative excess return? Accept more tracking error.
- In practice, ε hard to observe. Swaps, futures...

Implied Spread

- Instead, use equation to derive the *implied spread*

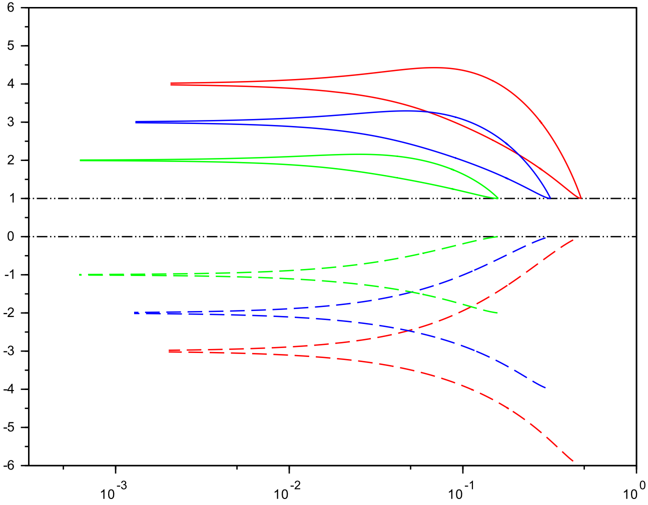
$$\tilde{\varepsilon} := \frac{12}{3^{1/2}} \frac{(-\,\mathrm{ExR}) \cdot \mathrm{TrE}}{\overline{\sigma}^3 \Lambda^2 (\Lambda - 1)^2}$$

- Scalar summary of fund performance.
- Compares funds with different ExR, TrE, and factor.
- Similar to using Black-Scholes formula to find implied volatility.
- Interpretation: suppose investor could swap F_t for $\tilde{F}_t$ which satisfies

$$\frac{d\tilde{F}_t}{\tilde{F}_t} - r_t dt = \Lambda \left(\frac{dS_t}{S_t} - r_t dt \right) - \phi dt,$$

- No tracking error or trading cost, but fee ϕ. Better F_t or $\tilde{F}_t$?
- $\tilde{F}_t$ better if $\phi < \tilde{\varepsilon}$. Indifference level.

Trading Boundaries



- Trading boundaries (vertical) vs.tracking error (horizontal) for leveraged (solid) and inverse (dashed) funds, for 4 (top), 3, 2, -1, -2, -3 (bottom).

Selected American ETFs

	Ticker	X	Track. Error (bp)	Excess Return (%)	Implied Spread (bp)	Beta	T-stat (Beta)	R Squared (%)	Volatility (%)
S&P	SPXU	-3	8.43	-1.96	2.02	-2.99	2.39	99.52	47.32
500	SDS	-2	3.74	-1.42	2.58	-2.00	-0.92	99.79	31.68
	SH	-1	2.38	-1.18	12.31	-1.00	-1.85	99.66	15.87
(SPY)	SSO	2	3.79	-1.00	16.65	2.00	-0.57	99.78	31.62
	UPRO	3	8.32	-1.22	4.95	2.99	-2.83	99.53	47.29
MSCI	EDZ	-3	8.11	-4.59	1.51	-2.97	7.94	99.78	67.90
Emerging	EEV	-2	5.28	-3.40	2.91	-1.99	4.21	99.80	45.47
Markets	EUM	-1	4.66	-2.06	14.00	-1.00	1.63	99.37	22.82
(EEM)	EET	2	17.95	-1.44	37.70	1.95	-6.67	97.60	45.00
	EDC	3	11.52	-3.65	6.80	2.93	-13.73	99.55	67.04
Nasdaq	SQQQ	-3	8.60	-3.47	2.70	-2.97	6.89	99.59	51.65
100	QID	-2	3.61	-2.41	3.16	-1.98	8.00	99.84	34.64
	PSQ	-1	2.54	-1.60	13.25	-1.00	2.57	99.68	17.41
(QQQ)	QLD	2	4.27	-0.66	9.20	1.98	-7.72	99.77	34.61
	TQQQ	3	7.16	-0.53	1.37	2.96	-10.16	99.71	51.48
Russell	TZA	-3	6.71	-6.90	2.40	-2.98	6.30	99.83	62.73
2000	TWM	-2	4.90	-3.63	3.68	-2.00	1.43	99.80	42.04
	RWM	-1	3.82	-2.19	15.54	-1.00	0.30	99.51	21.07
(IWM)	UWM	2	5.29	-0.69	6.84	1.99	-4.69	99.76	41.87
	TNA	3	6.36	-1.45	1.91	2.97	-8.55	99.84	62.60

German DAX Certificates

Index	X	Track. Error (bp)	Excess Return (%)	Implied Spread (bp)	Beta	T-stat (Beta)	R^2 (%)	Volat. (%)	Years Data
	-12	196.27	-47.76	3.96	-11.67	3.93	97.90	273.87	1.62
	-10	96.27	-11.54	0.94	-9.82	3.89	98.65	207.55	2.50
	-8	69.05	-19.58	2.68	-7.71	8.32	98.40	154.41	3.17
	-6	43.71	-5.30	1.35	-5.90	4.15	98.50	112.90	3.99
	-5	32.46	-14.84	5.50	-5.07	-4.81	99.47	108.88	2.36
	-4	21.24	-4.39	2.40	-4.03	-2.48	99.09	76.72	4.74
	-3	16.02	-4.07	4.65	-3.03	-3.82	99.63	64.68	2.38
DAX	-2	15.18	-1.58	6.85	-1.99	0.65	98.14	38.27	4.70
	3	20.75	-1.53	9.05	3.02	2.19	99.39	64.73	2.37
	4	21.71	-1.64	2.54	4.02	2.14	99.04	76.78	4.75
	5	43.40	1.45	-1.62	5.09	4.67	99.06	109.39	2.36
	6	36.07	-7.81	3.22	6.10	5.00	99.00	115.36	4.05
	8	61.74	-19.15	3.88	8.11	3.39	98.78	160.06	3.26
	10	101.16	-24.55	3.15	9.90	-1.99	98.52	208.81	2.51
	12	210.88	-30.98	3.86	11.64	-3.89	96.99	266.39	1.91

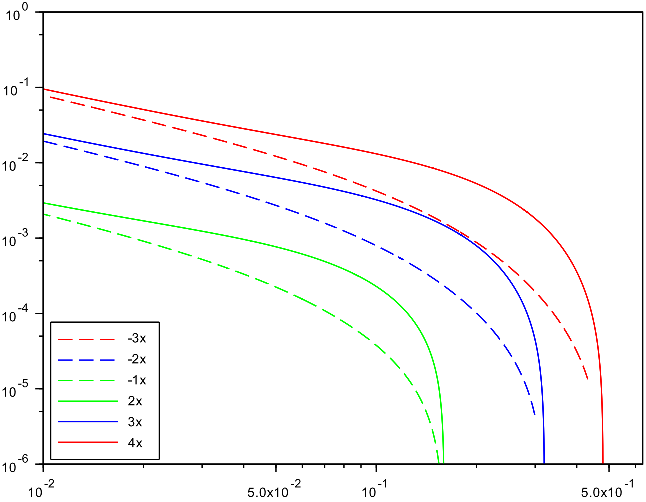
Underexposure Explained

- Average exposure:

$$\bar{\beta} = \Lambda - \frac{2\Lambda - 1}{\gamma} \left(\frac{\gamma \Lambda (\Lambda - 1)}{6} \right)^{1/3} \varepsilon^{2/3} + O(\varepsilon),$$

- $0 < \bar{\beta} < \Lambda$ for leveraged funds.
- $\Lambda < \bar{\beta} < 0$ for inverse funds.
- Underexposure results from optimal rebalancing.
- Effect increases with multiple and illiquidity.
- Decreases with tracking error.

Excess Return vs. Tracking Error



- – ExR (vertical) against TrE (horizontal) for leveraged (solid) and inverse (dashed) funds, for -3, +4 (top), -2, +3 (middle), -1, +2 (bottom).

Robustness to Risk Premium

- Basic model assumes asset with zero risk premium. Does it matter?
- Not at the first order. Not for typical risk premia.
- Extended model. Index price

$$\frac{dS_t}{S_t} = (r_t + \kappa\sigma_t^2)dt + \sigma_t dW_t,$$

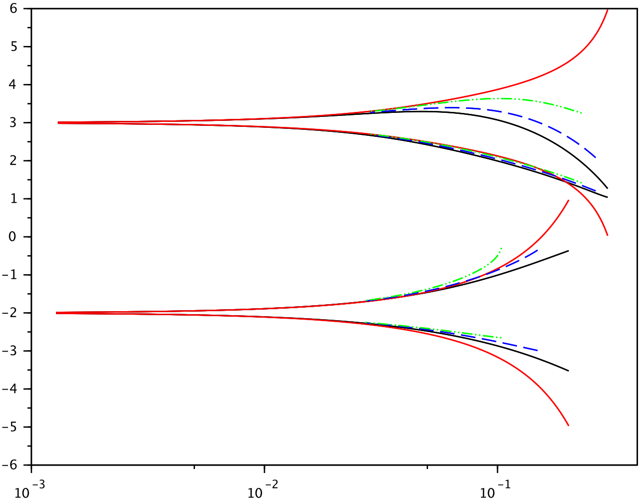
- Manager can generate positive excess return through overexposure.
- But investor observes and controls for overexposure. Objective

$$\limsup_{T\to\infty} \frac{1}{T}\left(D_T - \kappa\bar{\beta}\left\langle \int_0^\cdot \frac{dS}{S}\right\rangle_T - \frac{\gamma}{2}\langle D\rangle_T\right)$$

- Effect of κ remains only in state dynamics, not in objective.
- Trading boundaries:

$$\pi_\pm = \Lambda \pm \left(\frac{3}{4\gamma}\Lambda^2(\Lambda-1)^2\right)^{1/3}\varepsilon^{1/3} - \frac{(\Lambda - \kappa/2)}{\gamma}\left(\frac{\gamma\Lambda(\Lambda-1)}{6}\right)^{1/3}\varepsilon^{2/3} + O(\varepsilon).$$

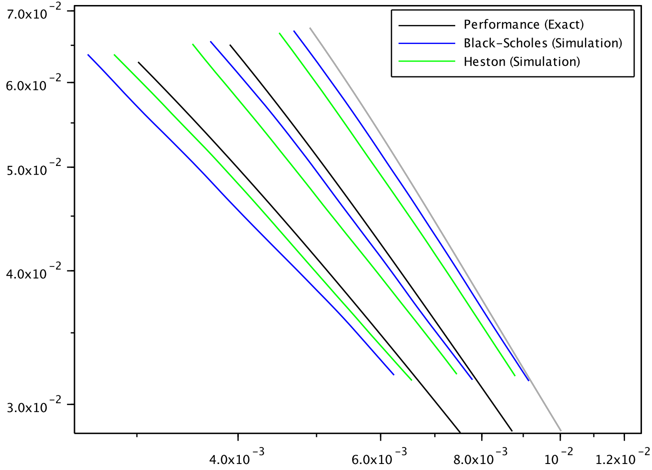
Robust Boundaries



- Trading boundaries for 3 and -2. Robust approximation (Red).
- Exact for $\kappa = 0$ (Black), $\kappa = 1.5625$ (Blue), $\kappa = 3.125$ (Green).

Robustness to Finite Horizon

- Basic model assumes long horizon. Does it matter?
- Compare by simulation excess returns and tracking errors to Black-Scholes and Heston models. (μ = 0 (top), 4%, 8% bottom.)



Value Function

- $$\max_{\varphi\in\Phi}\mathbb{E}\left[\int_0^T\left(\gamma\sigma_t^2\Lambda\pi_t-\frac{\gamma\sigma_t^2}{2}\pi_t^2\right)dt-\varepsilon\int_0^T\pi_t\frac{d\varphi_t^{\downarrow}}{\varphi_t}\right]$$
- $$F^{\varphi}(t)=\int_0^t\left(\gamma\sigma_s^2\Lambda\tfrac{\zeta_s}{1+\zeta_s}-\tfrac{\gamma\sigma_s^2}{2}\tfrac{\zeta_s^2}{(1+\zeta_s)^2}\right)ds-\varepsilon\int_0^t\tfrac{\zeta_s}{1+\zeta_s}\tfrac{d\varphi_s^{\downarrow}}{\varphi_s}+V(t,\zeta_t)$$
- Dynamics of F^{φ} by Itô's formula

$$dF^{\varphi}(t)=\left(\gamma\Lambda\sigma_t^2\frac{\zeta_t}{1+\zeta_t}-\frac{\gamma\sigma_t^2}{2}\frac{\zeta_t^2}{(1+\zeta_t)^2}\right)dt-\varepsilon\frac{\zeta_t}{1+\zeta_t}\frac{d\varphi_t^{\downarrow}}{\varphi_t}$$
$$+V_t(t,\zeta_t)dt+V_{\zeta}(t,\zeta_t)d\zeta_t+\frac{1}{2}V_{\zeta\zeta}(t,\zeta_t)d\langle\zeta\rangle_t,$$

- Self-financing condition yields

$$\frac{d\zeta_t}{\zeta_t}=\sigma_t dW_t+(1+\zeta_t)\frac{d\varphi_t}{\varphi_t}+\varepsilon\zeta_t\frac{d\varphi_t^{\downarrow}}{\varphi_t},$$

- Whence

$$dF^{\varphi}(t)=\left(\gamma\Lambda\sigma_t^2\frac{\zeta_t}{1+\zeta_t}-\frac{\gamma\sigma_t^2}{2}\frac{\zeta_t^2}{(1+\zeta_t)^2}+V_t+\frac{\sigma_t^2}{2}\zeta_t^2V_{\zeta\zeta}\right)dt$$

Control Argument

- $F^{\varphi}(t)$ supermartingale for any policy φ, martingale for optimal policy.
- $\varphi^{\uparrow}$ and $\varphi^{\downarrow}$ increasing processes. Supermartingale condition implies

$$-\frac{\varepsilon}{(1+\zeta)(1+(1-\varepsilon)\zeta)} \leq V_{\zeta} \leq 0,$$

- Likewise,

$$\gamma\sigma_t^2\Lambda\frac{\zeta}{1+\zeta} - \frac{\gamma\sigma_t^2}{2}\frac{\zeta^2}{(1+\zeta)^2} + V_t + \frac{\sigma_t^2}{2}\zeta^2 V_{\zeta\zeta} \leq 0$$

- For stationary solution, suppose residual value function

$$V(t,\zeta) = \lambda\int_t^T \sigma_s^2 ds - \int^{\zeta} W(z)dz$$

- λ to be determined, represents optimal performance over a long horizon

$$\frac{\lambda}{T}\int_0^T \sigma_t^2 dt \approx \lambda \times \overline{\sigma}^2$$

Identifying System

- Above inequalities become

$$0 \le W(\zeta) \le \frac{\varepsilon}{(1+\zeta)(1+(1-\varepsilon)\zeta)},$$

$$\gamma\Lambda\frac{\zeta}{1+\zeta} - \frac{\gamma}{2}\frac{\zeta^2}{(1+\zeta)^2} - \lambda - \frac{1}{2}\zeta^2 W'(\zeta) \le 0,$$

- Optimality conditions

$$\begin{aligned} \frac{1}{2}\zeta^2 W'(\zeta) - \gamma\Lambda\frac{\zeta}{1+\zeta} + \frac{\gamma}{2}\frac{\zeta^2}{(1+\zeta)^2} + \lambda &= 0 \qquad \text{for } \zeta \in [\zeta_-, \zeta_+], \\ W(\zeta_-) &= 0, \\ W(\zeta_+) &= \frac{\varepsilon}{(\zeta_+ + 1)(1+(1-\varepsilon)\zeta_+)}, \end{aligned}$$

- Boundaries identified by the smooth-pasting conditions

$$\begin{aligned} W'(\zeta_-) &= 0, \\ W'(\zeta_+) &= \frac{\varepsilon(\varepsilon - 2(1-\varepsilon)\zeta_+ - 2)}{(1+\zeta_+)^2(1+(1-\varepsilon)\zeta_+)^2}. \end{aligned}$$

- Four unknowns and four equations.

Conclusion

- Optimal tracking of leveraged and inverse funds.
- Excess Return vs. Tracking Error with Frictions.
- Optimal tracking policy independent of volatility dynamics.
- Robust to risk premia and finite horizons.
- Performance depends on volatility only through its average value.
- Sufficient performance statistic: excess return times tracking error.