HEDGING OF COVERED OPTIONS WITH LINEAR MARKET IMPACT AND GAMMA CONSTRAINT*

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Abstract. Within a financial model with linear price impact, we study the problem of hedging a covered European option under gamma constraint. Using stochastic target and partial differential equation smoothing techniques, we prove that the super-replication price is the viscosity solution of a fully non-linear parabolic equation. As a by-product, we show how ε-optimal strategies can be constructed. Finally, a numerical resolution scheme is proposed.

Key words. Hedging, Price impact, Stochastic target.

AMS subject classifications. 91G20; 93E20; 49L20

1. Introduction. Inspired by [1, 18], authors in [4] considered a financial market with permanent price impact, in which the impact function behaves as a linear function (around the origin) in the number of bought stocks. This class of models is dedicated to the pricing and hedging of derivatives under situations of non-negligible delta-hedging. In fact, the number of stocks required for hedging purpose becomes comparable to the average daily volume traded on the underlying asset. As a consequence, the delta-hedging strategy has an impact on the price dynamics, and also incurs liquidity costs. The linear impact models studied in [1, 4, 18] incorporate both effects into the pricing and hedging of the derivative, while maintaining the completeness of the market (up to a certain extent). These models in turn lead to exact replication strategies. As in perfect market models, this approach provides an approximation of the real market conditions and hence can be used by practitioners to design a suitable hedge in a systematic way. Thus, eliminating the need to rely on any ad hoc risk criterion.

In [4], the authors considered the hedging of a cash-settled European option: at inception the option seller builds the initial delta-hedge, and later liquidates the hedge at maturity to settle the final claim in cash. It is shown therein that the price function of the optimal super-replicating strategy no longer solves a linear parabolic equation, as in the classical case, rather a quasi-linear one. The hedging strategy in this case, essentially follows a modified delta-hedging rule where the delta is computed at the "unperturbed" value of the underlying, i.e., the one the underlying would have been if the trader's position were liquidated immediately.

The approach and the results obtained in [4] thus differ substantially from [1, 18]. While in [1, 18] the impact model considered is the same, the control problem is different in the sense that it is applied to the hedging of *covered options*. The hedging of covered options refers to situations where the buyer of the option delivers at inception the required initial delta position, and accepts a mix of stocks (at their current market price) and cash as payment of the final claim. The buyer's indifference between stock and cash eliminates the cost incurred by the initial and final hedge. Quite surprisingly, this is not a genuine approximation of the problem studied in

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[4]. The question of the initial and final hedge is fundamental, to the point that the structure of the pricing question is completely different: in [4] the equation is quasi-linear, while it is fully non-linear in [1, 18].

As opposed to [4], authors in [1, 18] use a verification argument to build an exact replication strategy. Due to the special form of the non-linearity, the equation is ill-posed when the solution does not satisfy a gamma-type constraint. The aim of the current paper is to provide a direct characterization via stochastic target techniques, and to incorporate right from the beginning a gamma constraint on the hedging strategy.

Note that, in [18], the author establishes, for a particular type of impact function (see f below), that the fully non-linear pricing equation has a smooth solution which provides an exact replication strategy. However it is not shown that this (exact replication) strategy is the cheapest way of super-replicating the final payoff. In the present paper, we assume a more general form for the market impact, and show that the weak (viscosity) solution to the pricing equation indeed provides the price of the cheapest super-replication strategy. Note also that the gamma-constraint is obtained in [18] as a by product of the regularity, as opposed to the present paper where it has to be imposed.

In our context, the super-solution property can be proved by essentially following the arguments of [8]. The sub-solution characterization is much more difficult to obtain. This is a second main difference with [4], in which classical geometric dynamic programming and viscosity solutions techniques could be used, once an appropriate change of variable was performed. In the current paper, however unlike in [8], we could not prove the required geometric dynamic programming principle. The underlying reason being the strong interaction between the hedging strategy and the underlying price process due to the market impact. Instead, we use the smoothing technique developed in [5]. We construct a sequence of smooth super-solutions which, by a verification argument, provide upper-bounds on the super-hedging price. As they converge to a solution of the targeted pricing equation, a comparison principle argument implies that their limit is the super-hedging price. A by-product of this construction is the explicit ε-optimal hedging strategies. We also provide the comparison principle and a numerical resolution scheme. To begin with, our analysis takes a simplified approach by restricting the models to only have permanent price impact. Later in Section 4, we show why adding a resilience effect does not affect our analysis. Note that this is because the resilience effect considered here has no quadratic variation. This is in contrast to [1], in which the resilience can break the parabolicity of the equation, and renders the exact replication non optimal.

We close this introduction by pointing out some related references. [6] incorporates liquidity costs but no price impact, the price curve is not affected by the trading strategy. It can be modified by adding restrictions on admissible strategies as in [7] and [23]. This leads to a modified pricing equation, which exhibits a quadratic term in the second order derivative of the solution, and renders the pricing equation fully non-linear, even not unconditionally parabolic. Other articles focus on the derivation of the price dynamics through clearing condition, see e.g., [12], [21], [20] in which the supply and demand curves arise from "reference" and "program" traders (i.e., option hedgers). This results in a modified price dynamics, but with no liquidity costs taken into account, see also [17]. Finally, the series of papers [22], [8], [23] addresses the liquidity issue indirectly by imposing bounds on the "gamma" of admissible trading strategies, no liquidity cost or price impact are modeled explicitly.

General notations. Throughout this paper, Ω is the canonical space of continuous functions on $\mathbb{R}_+$ starting at 0, $\mathbb{P}$ is the Wiener measure, W is the canonical process, and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the augmentation of its raw filtration $\mathbb{F}^\circ = (\mathcal{F}_t^\circ)_{t\geq 0}$. All random variables are defined on $(\Omega, \mathcal{F}_\infty, \mathbb{P})$. We denote by $|x|$ the Euclidean norm of $x \in \mathbb{R}^n$, the integer $n \geq 1$ is given by the context. Unless otherwise specified, inequalities involving random variables are taken in the $\mathbb{P}-$ a.s. sense. We use the convention $x/0 = \text{sign}(x) \times \infty$ with $\text{sign}(0) = +$.

2. Model and hedging problem.

This section is dedicated to the derivation of the dynamics and the description of the gamma constraint. We also explain in detail how the pricing equation can be obtained and state our main result.

2.1. Impact rule and discrete time trading dynamics.

We consider the framework studied in [4]. Namely, the impact of a strategy on the price process is modeled by an impact function f: the price variation due to buying a (infinitesimal) number $\delta \in \mathbb{R}$ of shares is $\delta f(x)$, given that the price of the asset is x before the trade. The cost of buying the additional δ units is

$$\delta x + \frac{1}{2}\delta^2 f(x) = \delta \int_0^\delta \frac{1}{\delta}(x + \iota f(x)) d\iota,$$

in which

$$\int_0^\delta \frac{1}{\delta}(x + \iota f(x)) d\iota$$

can be interpreted as the average cost for each additional unit.

Between two trading instances τ_1, τ_2 with $\tau_1 \leq \tau_2$, the dynamics of the stock is given by the strong solution of the stochastic differential equation

$$dX_t = \mu(X_t)dt + \sigma(X_t)dW_t.$$

Throughout this paper, we assume that

$$\begin{aligned} &f \in C_b^2 \text{ and } \inf f > 0, \\ &(\mu, \sigma) \text{ is Lipschitz and bounded, } \inf \sigma > 0. \end{aligned} \tag{1}$$

The above regularity assumptions are used in [4] to derive the dynamics of Proposition 2.2 below. The lower bound on σ is used later on, in particular to express the hedging policy in terms of a gamma, which is crucial for our analysis, see (8) and the equation just before. Relaxing these assumptions in the form of local conditions or by only assuming that f is C^1 with Lipschitz derivative should be feasible. This however would significantly increase the complexity of our proofs and we leave this to future researches.

As in [4], the number of shares the trader would like to hold is given by a continuous Itô process Y of the form

$$Y = Y_0 + \int_0^\cdot b_s ds + \int_0^\cdot a_s dW_s. \tag{2}$$

We say[1] that (a, b) belongs to $\mathcal{A}_k^\circ$ if (a, b) is continuous, $\mathbb{F}$-adapted,

$$a = a_0 + \int_0^\cdot \beta_s ds + \int_0^\cdot \alpha_s dW_s$$

[1] In [4], (a, b) is only required to be progressively measurable and essentially bounded. The additional restrictions imposed here will be necessary for our results in Section 3.2.

where (α,β) is continuous, $\mathbb{F}$-adapted, and $\zeta := (a,b,\alpha,\beta)$ is essentially bounded by k and such that

$$\mathbb{E}\left[\sup\left\{|\zeta_{s'}-\zeta_s|,\ t\le s\le s'\le s+\delta\le T\right\}|\mathcal{F}_t^\circ\right]\le k\delta$$

for all $0\le\delta\le 1$ and $t\in[0,T-\delta]$.

We then define

$$\mathcal{A}^\circ := \cup_k \mathcal{A}_k^\circ.$$

To derive the continuous time dynamics, we first consider a discrete time setting and then pass to the limit. In the discrete time setting, the position is re-balanced only at times

$$t_i^n := iT/n,\ i=0,\ldots,n,\ n\ge 1.$$

In other words, the trader keeps the position $Y_{t_i^n}$ in stocks over each time interval $[t_i^n, t_{i+1}^n)$. Hence, his position in stocks at t is

$$Y_t^n := \sum_{i=0}^{n-1} Y_{t_i^n}\mathbf{1}_{\{t_i^n\le t<t_{i+1}^n\}} + Y_T\mathbf{1}_{\{t=T\}}, \tag{3}$$

and the number of shares purchased at t_{i+1}^n is

$$\delta^n_{t^n_{i+1}} := Y_{t^n_{i+1}} - Y_{t^n_i}.$$

Given our impact rule, the corresponding dynamics for the stock price process is

$$X^n = X_0 + \int_0^\cdot \mu(X_s^n)ds + \int_0^\cdot \sigma(X_s^n)dW_s + \sum_{i=1}^n \mathbf{1}_{[t_i^n,T]}\delta^n_{t_i^n} f(X^n_{t_i^n-}), \tag{4}$$

in which X_0 is a constant.

The portfolio process is described as the sum V^n of the amount of cash held and the potential wealth Y^nX^n associated to the position in stocks:

$$V^n = \text{cash position } + Y^nX^n.$$

It does not correspond to the liquidation value of the portfolio, except when $Y^n=0$. This is due to the fact that the liquidation of Y^n stocks does not generate a gain equal to Y^nX^n, because of the price impact. However, one can infer the exact composition in cash and stocks of the portfolio from the knowledge of the couple (V^n,Y^n).

Throughout this paper, we assume that the risk-free interest rate is zero (for ease of notations). Then,

$$V^n = V_0 + \int_0^\cdot Y^n_{s-}dX^n_s + \sum_{i=1}^n \mathbf{1}_{[t_i^n,T]}\frac{1}{2}(\delta^n_{t_i^n})^2 f(X^n_{t_i^n-}). \tag{5}$$

This wealth equation is derived as in [4] following elementary calculations. The last term of the right-hand side comes from the fact that, at time t_i^n, $\delta^n_{t_i^n}$ shares are bought at the average execution price $X^n_{t_i^n-}+\frac{1}{2}\delta^n_{t_i^n}f(X^n_{t_i^n-})$, and the stock's price ends at $X^n_{t_i^n-}+\delta^n_{t_i^n}f(X^n_{t_i^n-})$, whence the additional profit term. However, one can check that a profitable round trip trade can not be built, see [4, Remark 3].

Remark 2.1. *Note that in this work we restrict ourselves to a permanent price impact, no resilience effect is modeled. We shall explain in Section 4 below why taking resilience into account does not affect our analysis. See in particular Proposition 4.1.*

2.2. Continuous time trading dynamics. The continuous time trading dynamics is obtained by passing to the limit $n \to \infty$, i.e., by considering strategies with increasing frequency of rebalancement.

PROPOSITION 2.2. *[4, Proposition 1] Let $Z := (X, Y, V)$ where Y is defined as in (2) for some $(a, b) \in \mathcal{A}^\circ$, and (X, V) solves*

$$X = X_0 + \int_0^\cdot \sigma(X_s)dW_s + \int_0^\cdot f(X_s)dY_s + \int_0^\cdot (\mu(X_s) + a_s(\sigma f')(X_s))ds$$

$$= X_0 + \int_0^\cdot \sigma_X^{a_s}(X_s)dW_s + \int_0^\cdot \mu_X^{a_s,b_s}(X_s)ds \quad (6)$$

with

$$\sigma_X^{a_s} := (\sigma + a_s f) \ , \ \mu_X^{a_s,b_s} := (\mu + b_s f + a_s \sigma f'),$$

and

$$V = V_0 + \int_0^\cdot Y_s dX_s + \frac{1}{2}\int_0^\cdot a_s^2 f(X_s)ds. \quad (7)$$

Let $Z^n := (X^n, Y^n, V^n)$ be defined as in (4)-(3)-(5). Then, there exists a constant $C > 0$ such that

$$\sup_{[0,T]} \mathbb{E}\left[|Z^n - Z|^2\right] \le Cn^{-1}$$

for all $n \ge 1$.

For the rest of the paper, we shall therefore consider (7)-(6) for the dynamics of the portfolio and price processes.

REMARK 2.3. *As explained in [4], the previous analysis could be extended to a non-linear impact rule in the size of the order. To this end, we note that the continuous time trading dynamics described above would be the same for a more general impact rule $\delta \mapsto F(x, \delta)$ whenever it satisfies $F(x, 0) = \partial^2_{\delta\delta}F(x, 0) = 0$ and $\partial_\delta F(x, 0) = f(x)$. For our analysis, we only need to consider the value and the slope of the impact function at the origin.*

2.3. Hedging equation and gamma constraint. Given $\phi = (y, a, b) \in \mathbb{R} \times \mathcal{A}^\circ$ and $(t, x, v) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, we now write $(X^{t,x,\phi}, Y^{t,\phi}, V^{t,x,v,\phi})$ for the solution of (6)-(2)-(7) associated to the control (a, b) with time-t initial condition (x, y, v).

In this paper, we consider covered options, in the sense that the trader is given at the initial time t the number of shares $Y_t = y$ required to launch his hedging strategy and can pay the option's payoff at T in cash and stocks (evaluated at their time-T value). Therefore, he does not exert any immediate impact at time t nor T due to the initial building or final liquidation of his position in stocks. Recalling that V stands for the sum of the position in cash and the number of held shares multiplied by their price, the super-hedging price at time t of the option with payoff $g(X_T^{t,x,\phi})$ is defined as

$$\mathrm{v}(t, x) := \inf\{v = c + yx \ : \ (c, y) \in \mathbb{R}^2 \text{ s.t. } \mathcal{G}(t, x, v, y) \neq \emptyset\},$$

in which $\mathcal{G}(t, x, v, y)$ is the set of elements $(a, b) \in \mathcal{A}^\circ$ such that $\phi := (y, a, b)$ satisfies

$$V_T^{t,x,v,\phi} \ge g(X_T^{t,x,\phi}).$$

In order to understand what the associated partial differential equation is, let us first rewrite the dynamics of Y in terms of X:

$$dY_t^{t,\phi} = \gamma_Y^{a_t}(X_t^{t,x,\phi})dX_t^{t,x,\phi} + \mu_Y^{a_t,b_t}(X_t^{t,x,\phi})dt$$

with

$$\gamma_Y^a := \frac{a}{\sigma + fa} \quad \text{and} \quad \mu_Y^{a,b} := b - \gamma_Y^a \mu_X^{a,b}. \tag{8}$$

Assuming that the hedging strategy is to track the super-hedging price, as in classical complete market models, then one should have $V^{t,x,v,\phi} = \mathrm{v}(\cdot, X^{t,x,\phi})$. If v is smooth, recalling (6)-(7) and applying Itô's lemma twice implies

$$Y^{t,\phi} = \partial_x \mathrm{v}(\cdot, X^{t,x,\phi}) \ \ , \ \ \gamma_Y^a(X^{t,x,\phi}) = \partial_{xx}^2 \mathrm{v}(\cdot, X^{t,x,\phi}), \tag{9}$$

and

$$\frac{1}{2}a^2 f(X^{t,x,\phi}) = \partial_t \mathrm{v}(\cdot, X^{t,x,\phi}) + \frac{1}{2}(\sigma_X^a)^2(X^{t,x,\phi})\partial_{xx}^2 \mathrm{v}(\cdot, X^{t,x,\phi}). \tag{10}$$

Then, the right-hand side of (9) combined with the definition of γ_Y^a leads to

$$a = \frac{\sigma \partial_{xx}^2 \mathrm{v}(\cdot, X^{t,x,\phi})}{1 - f\partial_{xx}^2 \mathrm{v}(\cdot, X^{t,x,\phi})} \ \ , \ \ \sigma_X^a = \frac{\sigma}{1 - f\partial_{xx}^2 \mathrm{v}(\cdot, X^{t,x,\phi})},$$

and (10) simplifies to

$$\left[-\partial_t \mathrm{v} - \frac{1}{2}\frac{\sigma^2}{(1 - f\partial_{xx}^2 \mathrm{v})}\partial_{xx}^2 \mathrm{v}\right](\cdot, X^{t,x,\phi}) = 0 \quad \text{on } [t,T). \tag{11}$$

This is precisely the pricing equation obtained in [1, 18].

Equation (11) needs to be considered with some precautions due to the singularity at $f\partial_{xx}^2 \mathrm{v} = 1$. Hence, one needs to enforce that $1 - f\partial_{xx}^2 \mathrm{v}$ does not change sign. We choose to restrict the solutions to satisfy $1 - f\partial_{xx}^2 \mathrm{v} > 0$, since having the opposite inequality would imply that a does not have the same sign as $\partial_{xx}^2 \mathrm{v}$, so that, having sold a convex payoff, one would sell when the stock goes up and buy when it goes down, a very counter-intuitive fact.

In the following, we impose that the constraint

$$-k \leq \gamma_Y^a(X^{t,x,\phi}) \leq \bar\gamma(X^{t,x,\phi}) \ , \ \ \text{on } [t,T] \ \ \mathbb{P}-\text{a.e.}, \tag{12}$$

should hold for some $k \geq 0$, in which $\bar\gamma$ is a bounded continuous map satisfying

$$\iota \leq \bar\gamma \leq 1/f - \iota, \quad \text{for some } \iota > 0. \tag{13}$$

We now denote by $\mathcal{A}_{k,\bar\gamma}(t,x)$ the collection of elements $(a,b) \in \mathcal{A}_k^\circ$ such that (12) holds. Define

$$\mathcal{A}_{\bar\gamma}(t,x) := \cup_{k\geq 0}\mathcal{A}_{k,\bar\gamma}(t,x),$$

and let $\mathrm{v}_{\bar\gamma}$ be defined as v but with

$$\mathcal{G}_{\bar\gamma}(t,x,v,y) := \mathcal{G}(t,x,v,y) \cap \mathcal{A}_{\bar\gamma}(t,x)$$

in place of $\mathcal{G}(t,x,v,y)$. More precisely,

$$\mathrm{v}_{\bar\gamma}(t,x) := \inf\{v = c + yx \ : \ (c,y) \in \mathbb{R}^2 \text{ s.t. } \mathcal{G}_{\bar\gamma}(t,x,v,y) \neq \emptyset\}. \tag{14}$$

Then, the equation (11) has to be modified to take the gamma constraint into account. This equation needs to impose that the second derivative is lower that the bound $\bar{\gamma}$. On the other hand, the above informal analysis shows that the pricing function $\mathrm{v}_{\bar{\gamma}}$ needs at least to be a super-solution of (11) to guarantee that a hedging strategy can be found. Then, the equation associated to the gamma constraint should read

$$(15)\quad F[\mathrm{v}_{\bar{\gamma}}] := \min\left\{-\partial_t \mathrm{v}_{\bar{\gamma}} - \frac{1}{2}\frac{\sigma^2}{1-f\partial^2_{xx}\mathrm{v}_{\bar{\gamma}}}\partial^2_{xx}\mathrm{v}_{\bar{\gamma}}\ ,\ \bar{\gamma} - \partial^2_{xx}\mathrm{v}_{\bar{\gamma}}\right\} = 0 \ \text{ on } [0,T)\times\mathbb{R}.$$

As for the T-boundary condition, we know that $\mathrm{v}_{\bar{\gamma}}(T,\cdot) = g$ by definition. However, as usual, the constraint on the gamma in (15) should propagate up to the boundary and g has to be replaced by its face-lifted version $\hat{g}$, defined as the smallest function above g that is a viscosity super-solution of the equation $\bar{\gamma} - \partial^2_{xx}\varphi \geq 0$. It is obtained by considering any twice continuously differentiable function $\bar{\Gamma}$ such that $\partial^2_{xx}\bar{\Gamma} = \bar{\gamma}$, and then setting

$$\hat{g} := (g - \bar{\Gamma})^{\mathrm{conc}} + \bar{\Gamma},$$

in which the superscript conc means concave envelope, cf. [22, Lemma 3.1].[2] Hence, we expect that

$$\mathrm{v}_{\bar{\gamma}}(T-,\cdot) = \hat{g} \ \text{ on } \mathbb{R}.$$

From now on, we assume that

$$(16)\quad \begin{array}{c} \hat{g} \text{ is uniformly continuous,} \\ g \text{ is lower-semicontinuous, } g^- \text{ is bounded and } g^+ \text{ has linear growth.} \end{array}$$

We are now in a position to state our main result. In the sequel,

$$\mathrm{v}_{\bar{\gamma}}(T,x) \ \text{ stands for } \lim_{\substack{(t',x')\to(T,x)\\ t'<T}} \mathrm{v}_{\bar{\gamma}}(t',x')$$

whenever it is well defined.

Theorem 2.4. *The value function $\mathrm{v}_{\bar{\gamma}}$ is continuous with linear growth. Moreover, $\mathrm{v}_{\bar{\gamma}}$ is the unique viscosity solution with linear growth of*

$$(17)\quad F[\varphi]\mathbf{1}_{[0,T)} + (\varphi - \hat{g})\mathbf{1}_{\{T\}} = 0 \quad \textit{on } [0,T]\times\mathbb{R}.$$

We conclude this section with additional remarks.

Remark 2.5. *Note that $\hat{g}$ can be uniformly continuous without g being continuous. Take for instance $g(x) = \mathbf{1}_{\{x\geq K\}}$ with $K\in\mathbb{R}$, and consider the case where $\bar{\gamma} > 0$ is a constant. Then, $\hat{g}(x) = [\mathbf{1}_{\{x\geq x_o\}}\frac{\bar{\gamma}}{2}(x-x_o)^2]\wedge 1$ with $x_o := K - (2/\bar{\gamma})^{\frac{1}{2}}$.*

Remark 2.6. *The map $\hat{g}$ inherits the linear growth of g. Indeed, let $c_0, c_1 \geq 0$ be constants such that $|g(x)| \leq w(x) := c_0 + c_1|x|$. Since $\hat{g} \geq g$ by construction, we have $\hat{g}^- \leq w$. On the other hand, since $\bar{\gamma} \geq \iota > 0$, by (13), it follows from the arguments in [22, Lemma 3.1] that $\hat{g} \leq (w - \tilde{\Gamma})^{\mathrm{conc}} + \tilde{\Gamma}$, in which $\tilde{\Gamma}(x) = \iota x^2/2$. Now, one can easily check by direct computations that*

$$(w-\tilde{\Gamma})^{\mathrm{conc}} = (w-\tilde{\Gamma})(x_o)\mathbf{1}_{[-x_o,x_o]} + (w-\tilde{\Gamma})\mathbf{1}_{[-x_o,x_o]^c}$$

with $x_o := c_1/\iota$. Hence, $(w-\tilde{\Gamma})^{\mathrm{conc}} + \tilde{\Gamma}$ has the same linear growth as w.

[2] Obviously, adding an affine map to $\bar{\Gamma}$ does not change the definition of $\hat{g}$.

REMARK 2.7. *As will appear in the rest of our analysis, one could very well introduce a time dependence in the impact function f and in $\bar{\gamma}$. Another interesting question studied by the second author in [18] concerns the smoothness of the solution and how the constraint on $\partial^2_{xx}\mathrm{v}$ gets naturally enforced by the fast diffusion arising when $1 - f\partial^2_{xx}\mathrm{v}$ is close to 0.*

REMARK 2.8 (Existence of a smooth solution to the original partial differential equation). *When the pricing equation (17) admits smooth solutions (cf. [18] that allow to use the verification theorem, then one can construct exact replication strategies from the classical solution. By the comparison principle of Theorem 3.11 below, this shows that the value function is the classical solution of the pricing equation, and that the optimal strategy exists and is an exact replication strategy of the option with payoff function $\hat{g}$. We will explain in Remark 3.18 below how almost optimal super-hedging strategies can be constructed explicitly even when no smooth solution exists.*

REMARK 2.9 (Monotonicity in the impact function). *Note that the map $\lambda \in \mathbb{R} \mapsto \frac{\sigma^2(x)M}{1-\lambda M}$ is non-decreasing on $\{\lambda : \lambda M < 1\}$, for all $(t,x,M) \in [0,T] \times \mathbb{R} \times \mathbb{R}$. Let us now write $\mathrm{v}_{\bar{\gamma}}$ as $\mathrm{v}^f_{\bar{\gamma}}$ to emphasize its dependence on f, and consider another impact function $\tilde{f}$ satisfying the same requirements as f. We denote by $\mathrm{v}^{\tilde{f}}_{\bar{\gamma}}$ the corresponding super-hedging price. Then, the above considerations combined with Theorem 2.4 and the comparison principle of Theorem 3.11 below imply that $\mathrm{v}^{\tilde{f}}_{\bar{\gamma}} \geq \mathrm{v}^f_{\bar{\gamma}}$ whenever $\tilde{f} \geq f$ on $\mathbb{R}$. The same implies that $\mathrm{v}^f_{\bar{\gamma}} \geq \mathrm{v}$ in which v solves the heat-type equation*

$$-\partial_t\varphi - \frac{1}{2}\sigma^2\partial^2_{xx}\varphi = 0 \quad \text{on } [0,T) \times \mathbb{R},$$

with terminal condition $\varphi(T,\cdot) = g$ (recall that $\hat{g} \geq g$). See Section 5.2 for a numerical illustration of this fact.

3. Viscosity solution characterization. In this section, we provide the proof of Theorem 2.4. Our strategy is the following.

1. First, we adapt the partial differential equation smoothing technique used in [5] to provide a smooth supersolutions $\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar{\gamma}}$ of (17) on $[\delta,T] \times \mathbb{R}$, with $\epsilon > 0$, from which super-hedging strategies can be constructed by a standard verification argument. In particular, $\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar{\gamma}} \geq \mathrm{v}_{\bar{\gamma}}$ on $[\delta,T] \times \mathbb{R}$. Moreover, this sequence has a uniform linear growth and converges to a viscosity solution $\bar{\mathrm{v}}_{\bar{\gamma}}$ of (17) as $\delta, \epsilon \to 0$ and $K \to \infty$. See Section 3.1.
2. Second, we construct a lower bound $\underline{\mathrm{v}}_{\bar{\gamma}}$ for $\mathrm{v}_{\bar{\gamma}}$ that is a supersolution of (17). It is obtained by considering a weak formulation of the super-hedging problem and following the arguments of [8, Section 5] based on one side of the geometric dynamic programming principle, see Section 3.2. It is shown that this function has linear growth as well.
3. We can then conclude by using the above and the comparison principle for (17) of Theorem 3.11 below: $\underline{\mathrm{v}}_{\bar{\gamma}} \geq \bar{\mathrm{v}}_{\bar{\gamma}}$ but $\underline{\mathrm{v}}_{\bar{\gamma}} \leq \mathrm{v}_{\bar{\gamma}} \leq \bar{\mathrm{v}}_{\bar{\gamma}}$ so that $\mathrm{v}_{\bar{\gamma}} = \bar{\mathrm{v}}_{\bar{\gamma}} = \underline{\mathrm{v}}_{\bar{\gamma}}$ and $\mathrm{v}_{\bar{\gamma}}$ is a viscosity solution of (17), and has linear growth.
4. Our comparison principle, Theorem 3.11 below, allows us to conclude that $\mathrm{v}_{\bar{\gamma}}$ is the unique solution of (17) with linear growth.

As already mentioned in the introduction, unlike [8], we could not prove the required geometric dynamic programming principle that should directly lead to a subsolution property (thus avoiding to use the smoothing technique mentioned in 1. above). This is due to the strong interaction between the hedging strategy and the underlying price process through the market impact. Such a feedback effect is not present in [8].

3.1. A sequence of smooth supersolutions. We first construct a sequence of smooth supersolutions $\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}$ of (17) which appears to be an upper bound on the super-hedging price $\mathrm{v}_{\bar{\gamma}}$, by a simple verification argument. For this, we adapt the methodology introduced in [5]: we first construct a viscosity solution of a version of (17) with shaken coefficients (in the terminology of [15]) and then smooth it out with a kernel. The main difficulty here is that our terminal condition $\hat{g}$ is unbounded, unlike [5]. This requires additional non trivial technical developments.

3.1.1. Construction of a solution for the operator with shaken coefficients. We start with the construction of the operator with shaken coefficients. Given $\epsilon > 0$ and a (uniformly) strictly positive continuous map κ with linear growth, that will be defined later on, let us introduce a family of perturbations of the operator appearing in (17):

$$F_\kappa^\epsilon(t,x,q,M) := \min_{x' \in D_\kappa^\epsilon(x)} \min \left\{ -q - \frac{\sigma^2(x')M}{2(1-f(x')M)}, \bar{\gamma}(x') - M \right\},$$

where

$$D_\kappa^\epsilon(x) := \{x' \in \mathbb{R} : (x-x')/\kappa(x') \in [-\epsilon,\epsilon]\}. \tag{18}$$

For ease of notation, we set

$$F_\kappa^\epsilon[\varphi](t,x) := F_\kappa^\epsilon(t,x,\partial_t\varphi(t,x),\partial_{xx}^2\varphi(t,x)),$$

whenever φ is smooth.

Remark 3.1. *For later use, note that the map $M \in (-\infty,\bar{\gamma}(x)] \mapsto \frac{\sigma^2(x)M}{2(1-f(x)M)}$ is non-decreasing and convex, for each $x \in \mathbb{R}$, recall (13). Hence, $(q,M) \in \mathbb{R} \times (-\infty,\bar{\gamma}(x)] \mapsto F_\kappa^\epsilon(\cdot,q,M)$ is concave and non-increasing in M, for all $\epsilon \geq 0$. This is fundamental for our smoothing approach to go through.*

We also modify the original terminal condition $\hat{g}$ by using an approximating sequence whose elements are affine for large values of $|x|$.

Lemma 3.2. *For all $K > 0$ there exists a uniformly continuous map $\hat{g}_K$ and $x_K \geq K$ such that*

- *$\hat{g}_K$ is affine on $[x_K,\infty)$ and on $(-\infty,-x_K]$*
- *$\hat{g}_K = \hat{g}$ on $[-K,K]$*
- *$\hat{g}_K \geq \hat{g}$*
- *$\hat{g}_K - \bar{\Gamma}$ is concave for any C^2 function $\bar{\Gamma}$ satisfying $\partial_{xx}^2\bar{\Gamma} = \bar{\gamma}$.*

Moreover, $(\hat{g}_K)_{K>0}$ is uniformly bounded by a map with linear growth and converges to $\hat{g}$ uniformly on compact sets.

Proof. Fix a C^2 function $\bar{\Gamma}^\circ$ satisfying $\partial_{xx}^2\bar{\Gamma}^\circ = \bar{\gamma}$. By definition, $\hat{g} - \bar{\Gamma}^\circ$ is concave. Let us consider an element Δ^+ (resp. Δ^-) of its super-differential at K (resp. $-K$). Set

$$\begin{aligned}
\hat{g}_K^\circ(x) := {} & \hat{g}(x)\mathbf{1}_{[-K,K]}(x) \\
& + \left[\hat{g}(K) + (\Delta^+ + \partial_x\bar{\Gamma}^\circ(K))(x-K)\right]\mathbf{1}_{(K,\infty)}(x) \\
& + \left[\hat{g}(-K) + (\Delta^- + \partial_x\bar{\Gamma}^\circ(-K))(x+K)\right]\mathbf{1}_{(-\infty,-K)}(x).
\end{aligned}$$

Consider now another C^2 function $\bar\Gamma$ satisfying $\partial^2_{xx}\bar\Gamma = \bar\gamma$. Since $\bar\Gamma^\circ$ and $\bar\Gamma$ differ only by an affine map, the concavity of $\hat g^\circ_K - \bar\Gamma$ is equivalent to that of $\hat g^\circ_K - \bar\Gamma^\circ$. The concavity of the latter follows from the definition of $\hat g^\circ_K$, as the superdiffential of $\hat g^\circ_K - \bar\Gamma^\circ$ is non-increasing by construction. In particular, $\hat g^\circ_K - \bar\Gamma^\circ \ge \hat g - \bar\Gamma^\circ$ and therefore $\hat g^\circ_K \ge \hat g$.
We finally define $\hat g_K$ by

$$\hat g_K = \min\{\hat g^\circ_K, (2c_0 + c_1|\cdot| - \bar\Gamma^\circ)^{\rm conc} + \bar\Gamma^\circ\}, \tag{19}$$

with $c_0 > 0$ and $c_1 \ge 0$ such that

$$-c_0 \le \hat g(x) \le c_0 + c_1|x|,\ x \in \mathbb{R},$$

recall Remark 2.6. The function $\hat g_K$ has the same linear growth as $2c_0 + c_1|\cdot|$, by the same reasoning as in Remark 2.6. Since $2c_0 > c_0$, $\hat g_K = \hat g^\circ_K = \hat g$ on $[-K, K]$. Furthermore, as the minimum of two concave functions is concave, so is $\hat g_K - \bar\Gamma$ for any C^2 function $\bar\Gamma$ satisfying $\partial^2_{xx}\bar\Gamma = \bar\gamma$. The other assertions are immediate. □

We now set

$$\hat g^\epsilon_K := \hat g_K + \epsilon \tag{20}$$

and consider the equation

$$F^\epsilon_\kappa[\varphi]\mathbf{1}_{[0,T)} + (\varphi - \hat g^\epsilon_K)\mathbf{1}_{\{T\}} = 0. \tag{21}$$

We then choose κ and $\epsilon_\circ \in (0,1)$ such that

$$\begin{aligned} &\kappa \in C^\infty \text{ with bounded derivatives of all orders,}\\ &\inf\kappa > 0 \text{ and } \kappa = |\hat g_K| + 1 \text{ on } (-\infty, -x_K] \cup [x_K, \infty),\\ &-1/\epsilon_\circ < \partial_x\kappa < 1/\epsilon_\circ, \end{aligned} \tag{22}$$

in which $x_K \ge K$ is defined in Lemma 3.2. We omit the dependence of κ on K for ease of notations.

REMARK 3.3. *For later use, note that the condition $|\partial_x\kappa| < 1/\epsilon_\circ$ ensures that the map $x \mapsto x + \epsilon\kappa(x)$ and $x \mapsto x - \epsilon\kappa(x)$ are uniformly strictly increasing for all $0 \le \epsilon \le \epsilon_\circ$. Also observe that $x_n \to x$ and $x'_n \in D^\epsilon_\kappa(x_n)$, for all n, imply that x'_n converges to an element $x' \in D^\epsilon_\kappa(x)$, after possibly passing to a subsequence. In particular, F^ϵ_κ is continuous.*

When $\kappa \equiv 1$ and $\hat g^\epsilon_K \equiv \hat g + \epsilon$, (21) corresponds to the operator in (17) with shaken coefficients, in the traditional terminology of [15]. The function κ will be used below to handle the potential linear growth at infinity of $\hat g$. The introduction of the additional approximation $\hat g^\epsilon_K$ is motivated by the fact that the proof of Proposition 3.7 below requires an affine behavior at infinity. As already mentioned, these additional complications do not appear in [5] because their terminal condition is bounded.

We now prove that (21) admits a viscosity solution that remains above the terminal condition $\hat g$ on a small time interval $[T - c^K_\epsilon, T]$. As already mentioned, we will later smooth this solution out with a regular kernel, so as to provide a smooth supersolution of (17).

PROPOSITION 3.4. *For all $\epsilon \in [0, \epsilon_\circ]$ and $K > 0$, there exists a unique continuous viscosity solution $\bar{\rm v}^{\epsilon,K}_{\bar\gamma}$ of (21) that has linear growth. It satisfies*

$$\bar{\rm v}^{\epsilon,K}_{\bar\gamma} \ge \hat g_K + \epsilon/2, \qquad \text{on } [T - c^K_\epsilon, T] \times \mathbb{R}, \tag{23}$$

for some $c_\epsilon^K \in (0,T)$.
Moreover, $\{[\bar{\mathrm{v}}_{\bar\gamma}^{\epsilon,K}]^+, \epsilon \in [0,\epsilon_\circ], K>0\}$ *is bounded by a map with linear growth, and* $\{[\bar{\mathrm{v}}_{\bar\gamma}^{\epsilon,K}]^-, \epsilon \in [0,\epsilon_\circ], K>0\}$ *is bounded by* $\sup g^-$.

Proof. The proof is mainly a modification of the usual Perron's method, see [10, Section 4].

a. We first prove that there exists two continuous functions $\bar w$ and $\underline{w}$ with linear growth that are respectively super- and subsolution of (21) for any $\epsilon \in [0,\epsilon_\circ]$.

Since $\hat g_K^\epsilon = \hat g_K + \epsilon \ge g$ by Lemma 3.2, it suffices to set

$$\underline{w} := \inf g > -\infty,$$

see (16). To construct a supersolution $\bar w$, let us fix $\eta \in (0, \iota \wedge \inf f^{-1})$ with ι as in (13), set $\tilde\Gamma(x) = \eta x^2/2$ and define $\tilde g = (\hat g_K^{\epsilon_\circ} - \tilde\Gamma)^{\mathrm{conc}} + \tilde\Gamma$. Then, $\tilde g \ge \hat g_K^{\epsilon_\circ}$, while the same reasoning as in Remark 2.6 implies that $\tilde g$ shares the same linear growth as $\hat g_K^{\epsilon_\circ}$, see (20) and Lemma 3.2. We then define $\bar w$ by

$$\bar w(t,x) = \tilde g(x) + 1 + (T-t)A$$

in which

$$A := \sup \frac{\sigma^2\bar\gamma}{2(1-f\bar\gamma)}.$$

The constant A is finite, and $\bar w$ has the same linear growth as $\tilde g$, see (1)-(13). Since a concave function is a viscosity supersolution of $-\partial^2_{xx}\varphi \ge 0$, we deduce that $\tilde g$ is a viscosity supersolution of $\eta - \partial^2_{xx}\varphi \ge 0$. Then, $\bar w$ is a viscosity supersolution of

$$\min\left\{-\partial_t\varphi - A\ ,\ \eta - \partial^2_{xx}\varphi\right\} \ge 0.$$

Since $\bar\gamma \ge \iota \ge \eta$, it remains to use Remark 3.1 to conclude that $\bar w$ is a supersolution of (21).

b. We now express (21) as a single equation over the whole domain $[0,T]\times\mathbb{R}$ using the following definitions

$$F_{\kappa,+}^{\epsilon,K}(t,x,r,q,M) := F_\kappa^\epsilon(t,x,q,M)\mathbf{1}_{[0,T)} + \max\left\{F_\kappa^\epsilon(t,x,q,M), r - \hat g_K^\epsilon(x)\right\}\mathbf{1}_{\{T\}}$$

$$F_{\kappa,-}^{\epsilon,K}(t,x,r,q,M) := F_\kappa^\epsilon(t,x,q,M)\mathbf{1}_{[0,T)} + \min\left\{F_\kappa^\epsilon(t,x,q,M), r - \hat g_K^\epsilon(x)\right\}\mathbf{1}_{\{T\}}.$$

As usual $F_{\kappa,\pm}^{\epsilon,K}[\varphi](t,x) := F_{\kappa,\pm}^{\epsilon,K}(t,x,\varphi(t,x),\partial_t\varphi(t,x),\partial^2_{xx}\varphi(t,x))$. Recall that the formulations in terms of $F_{\kappa,\pm}^{\epsilon,K}$ lead to the same viscosity solutions as (21) (see Lemma 6.1 in the Appendix). This is the formulation to which we apply Perron's method. In view of a., the functions $\underline{w}$ and $\bar w$ are sub- and supersolution of $F_{\kappa,-}^{\epsilon,K}=0$ and $F_{\kappa,+}^{\epsilon,K}=0$. Define:

$$\bar{\mathrm{v}}_{\bar\gamma}^{\epsilon,K} := \sup\{v \in \mathrm{USC} : \underline{w} \le v \le \bar w \text{ and } v \text{ is a subsolution of } F_{\kappa,-}^{\epsilon,K}=0\},$$

in which USC denotes the class of upper-semicontinuous maps. Then, the upper- (resp. lower-)semicontinuous envelope $(\bar{\mathrm{v}}_{\bar\gamma}^{\epsilon,K})^*$ (resp. $(\bar{\mathrm{v}}_{\bar\gamma}^{\epsilon,K})_*$) of $\bar{\mathrm{v}}_{\bar\gamma}^{\epsilon,K}$ is a viscosity subsolution of $F_{\kappa,-}^{\epsilon,K}[\varphi]=0$ (resp. supersolution of $F_{\kappa,+}^{\epsilon,K}[\varphi]=0$) with linear growth, recall the continuity property of Remark 3.3 and see e.g. [10, Section 4]. The comparison result of Theorem 3.11 stated below implies that

$$(\bar{\mathrm{v}}_{\bar\gamma}^{\epsilon,K})^* = (\bar{\mathrm{v}}_{\bar\gamma}^{\epsilon,K})_*, \quad \text{on } [0,T]\times\mathbb{R}.$$

Hence, $\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K}$ is a continuous viscosity solution of (21), recall Lemma 6.1. By construction, it has linear growth. Uniqueness in this class follows from Theorem 3.11 again.

c. It remains to prove (23). For this, we need a control on the behavior of $\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K}$ as $t \to T$. It is enough to obtain it for a lower bound $v_{\epsilon,K}$ that we first construct. Let φ be a test function such that

$$\text{(strict)} \min_{[0,T)\times\mathbb{R}} (\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K} - \varphi) = (\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K} - \varphi)(t_0, x_0)$$

for some $(t_0, x_0) \in [0, T) \times \mathbb{R}$. By the supersolution property,

$$\min_{x' \in D_\kappa^\epsilon(x_0)} \{\bar{\gamma}(x') - \partial_{xx}^2 \varphi(t_0, x_0)\} \geq 0.$$

Recalling (1) and (13), this implies that, for $x' \in D_\kappa^\epsilon(x_0)$,

$$1 - f(x')\partial_{xx}^2 \varphi(t_0, x_0) \geq \iota f(x') \geq \iota \inf f =: \tilde{\iota} > 0.$$

Using the supersolution property and the above inequalities yields

$$\begin{aligned}
0 &\leq \min_{x' \in D_\kappa^\epsilon(x_0)} \left\{ -\partial_t \varphi(t_0, x_0) - \frac{\sigma^2(x')\partial_{xx}^2 \varphi(t_0, x_0)}{2(1 - f(x')\partial_{xx}^2 \varphi(t_0, x_0))} \right\} \\
&\leq \min_{x' \in D_\kappa^\epsilon(x_0)} \left\{ -\partial_t \varphi(t_0, x_0) - \frac{\sigma^2(x')\left[\partial_{xx}^2 \varphi(t_0, x_0) - \bar{\gamma}(x_0)\right]}{2(1 - f(x')\partial_{xx}^2 \varphi(t_0, x_0))} \right\} \\
&\leq -\partial_t \varphi(t_0, x_0) - \frac{\tilde{\sigma}^2 \partial_{xx}^2 \varphi(t_0, x_0)}{2\tilde{\iota}} + \frac{\tilde{\sigma}^2 \bar{\gamma}(x_0)}{2\tilde{\iota}}
\end{aligned}$$

where $\tilde{\sigma} := \sup \sigma$.

Denote by $v_{\epsilon,K}$ the unique viscosity solution of

$$\left\{ -\partial_t \varphi - \frac{\tilde{\sigma}^2 \partial_{xx}^2 \varphi}{2\tilde{\iota}} + \frac{\tilde{\sigma}^2 \bar{\gamma}}{2\tilde{\iota}} \right\} \mathbf{1}_{[0,T)} + (\varphi - \hat{g}_K^\epsilon)\mathbf{1}_{\{T\}} = 0. \tag{24}$$

The comparison principle for (24) and the Feynman-Kac formula imply that

$$\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K}(t, x) \geq v_{\epsilon,K}(t, x) = \mathbb{E}\left[-\int_0^{T-t} \frac{\tilde{\sigma}^2 \bar{\gamma}(S_r^x)}{2\tilde{\iota}}\, dr + \hat{g}_K^\epsilon(S_{T-t}^x) \right]$$

where

$$S^x = x + \frac{\tilde{\sigma}}{\sqrt{\tilde{\iota}}} W.$$

It remains to show that (23) holds for $v_{\epsilon,K}$ in place of $\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K}$. The argument is standard. Since $\hat{g}_K$ is uniformly continuous, see Lemma 3.2, we can find $B_\varepsilon^K > 0$ such that

$$\left|\hat{g}_K^\epsilon(S_{T-t}^x) - \hat{g}_K^\epsilon(x)\right| \mathbf{1}_{\{|S_{T-t}^x - x| \leq B_\varepsilon^K\}} \leq \varepsilon$$

for all $\varepsilon > 0$. We now consider the case $|S_{T-t}^x - x| > B_\varepsilon^K$. Let $C > 0$ denote a generic constant that does not depend on (t, x) but can change from line to line. Because $\hat{g}_K$ is affine on $[x_K, \infty)$ and on $(-\infty, -x_K]$, see Lemma 3.2,

$$\mathbb{E}\left[\left|\hat{g}_K^\epsilon(S_{T-t}^x) - \hat{g}_K^\epsilon(x)\right| \mathbf{1}_{\{S_{T-t}^x \geq x_K\}}\right] \leq C(T-t)^{\frac{1}{2}} \text{ if } x \geq x_K,$$

and

$$\mathbb{E}\left[\left|\hat g^{\epsilon}_K(S^x_{T-t})-\hat g^{\epsilon}_K(x)\right|\mathbf{1}_{\{S^x_{T-t}\le -x_K\}}\right]\le C(T-t)^{\frac12}\text{ if }x\le -x_K.$$

On the other hand, by linear growth of $\hat g^{\epsilon}_K$, if $x<x_K$, then

$$\begin{aligned}
&\mathbb{E}\left[\left|\hat g^{\epsilon}_K(S^x_{T-t})-\hat g^{\epsilon}_K(x)\right|\mathbf{1}_{\{S^x_{T-t}\ge x_K\}}\mathbf{1}_{\{|S^x_{T-t}-x|\ge B^K_\varepsilon\}}\right]\\
&\le \mathbb{E}\left[\left|\hat g^{\epsilon}_K(S^x_{T-t})-\hat g^{\epsilon}_K(x)\right|^2\right]^{\frac12}\mathbb{P}\left[|S^x_{T-t}-x|\ge |x_K-x|\vee B^K_\varepsilon\right]^{\frac12}\\
&\le C\frac{(1+|x|)(T-t)^{\frac12}}{|x_K-x|\vee B^K_\varepsilon}\le \frac{C}{B^K_\varepsilon}(T-t)^{\frac12}.
\end{aligned}$$

The (four) remaining cases are treated similarly, and we obtain

$$\mathbb{E}\left[\left|\hat g^{\epsilon}_K(S^x_{T-t})-\hat g^{\epsilon}_K(x)\right|\right]\le \frac{C}{B^K_\varepsilon}(T-t)^{\frac12}+\varepsilon.$$

Since $\bar\gamma$ is bounded, this shows that

$$|v_{\epsilon,K}(t,x)-\hat g^{\epsilon}_K(x)|\le \frac{C}{B^K_\varepsilon}(T-t)^{\frac12}+\varepsilon$$

for $t\in[T-1,T]$. Hence the required result for $v_{\epsilon,K}$. Since $\bar{\mathrm{v}}^{\epsilon,K}_{\bar\gamma}\ge v_{\epsilon,K}$, this concludes the proof of (23). ☐

For later use, note that, by stability, $\bar{\mathrm{v}}^{\epsilon,K}_{\bar\gamma}$ converges to a solution of (17) when $\epsilon\to 0$ and $K\to\infty$.

PROPOSITION 3.5. *As $\epsilon\to0$ and $K\to\infty$, $\bar{\mathrm{v}}^{\epsilon,K}_{\bar\gamma}$ converges to a function $\bar{\mathrm{v}}_{\bar\gamma}$ that is the unique viscosity solution of (17) with linear growth.*

Proof. The family of functions $\{\bar{\mathrm{v}}^{\epsilon,K}_{\bar\gamma},\epsilon\in(0,\epsilon_\circ],K>0\}$ is uniformly bounded by a map with linear growth, see Proposition 3.4. In view of the comparison result of Theorem 3.11 below, it suffices to apply [2, Theorem 4.1]. ☐

REMARK 3.6. *The bounds on $\bar{\mathrm{v}}_{\bar\gamma}$ can be made explicit, which can be useful to design a numerical scheme, see Section 5.1 below. First, as a by-product of the proof of Proposition 3.4, $\bar{\mathrm{v}}^{\epsilon,K}_{\bar\gamma}\ge\inf g$. Passing to the limit as $\epsilon\to0$ and $K\to\infty$ leads to*

$$\bar{\mathrm{v}}_{\bar\gamma}\ge \inf g=:\underline{w}.$$

We have also obtained that

$$\bar{\mathrm{v}}^{\epsilon,K}_{\bar\gamma}\le(\hat g^{\epsilon_\circ}_K-\tilde\Gamma)^{\mathrm{conc}}+\tilde\Gamma+1+A$$

in which $x\mapsto\tilde\Gamma(x)=\eta x^2/2$ for some $\eta\in(0,\iota\wedge\inf f^{-1})$ with ι as in (13), and $A:=T\sup(\sigma^2\bar\gamma/[2(1-f\bar\gamma)])$. On the other hand, (19) implies

$$\hat g^{\epsilon_\circ}_K\le 1+(2c_0+c_1|\cdot|-\bar\Gamma^\circ)^{\mathrm{conc}}+\bar\Gamma^\circ$$

for $\bar\Gamma^\circ$ such that $\partial^2_{xx}\bar\Gamma^\circ=\bar\gamma$. Then,

$$\begin{aligned}
\bar{\mathrm{v}}^{\epsilon,K}_{\bar\gamma}&\le\left(1+(2c_0+c_1|\cdot|-\bar\Gamma^\circ)^{\mathrm{conc}}+\bar\Gamma^\circ-\tilde\Gamma\right)^{\mathrm{conc}}+\tilde\Gamma+1+A\\
&\le\left(1+(2c_0+c_1|\cdot|-\tilde\Gamma)^{\mathrm{conc}}+\tilde\Gamma-\tilde\Gamma\right)^{\mathrm{conc}}+\tilde\Gamma+1+A\\
&=\left(1+2c_0+c_1|\cdot|-\tilde\Gamma\right)^{\mathrm{conc}}+\tilde\Gamma+1+A=:\bar w
\end{aligned}$$

and

$$\bar{\mathrm{v}}_{\bar{\gamma}} \le \bar{w}.$$

The function $\bar{w}$ defined above can be computed explicitly by arguing as in Remark 2.6.

Also note that (19) and the arguments of Remark 2.6 imply that there exists a constant $C > 0$ such that

$$\limsup_{|x|\to\infty} |\bar{\mathrm{v}}^{\epsilon,K}_{\bar{\gamma}}(x)|/(1+|\hat{g}_K(x)|) \le C, \text{ for all } \epsilon \in [0,\epsilon_\circ] \text{ and } K > 0. \quad (25)$$

3.1.2. Regularization and verification. Prior to applying our verification argument, it remains to smooth out the function $\bar{\mathrm{v}}^{\epsilon,K}_{\bar{\gamma}}$. This is similar to [5, Section 3], but here again the fact that $\hat{g}$ may not be bounded incurs additional difficulties. In particular, we need to use a kernel with a space dependent window.

We first fix a smooth kernel

$$\psi_\delta := \delta^{-2}\psi(\cdot/\delta)$$

in which $\delta > 0$ and $\psi \in C^\infty_b$ is a non-negative function with the closure of its support $[-1,0]\times[-1,1]$ that integrates to 1, and such that

$$\int y\psi(\cdot,y)dy = 0. \quad (26)$$

Let us set

$$\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar{\gamma}}(t,x) := \int_{\mathbb{R}\times\mathbb{R}} \bar{\mathrm{v}}^{\epsilon,K}_{\bar{\gamma}}([t']^+,x')\frac{1}{\kappa(x)}\psi_\delta\left(t'-t, \frac{x'-x}{\kappa(x)}\right)dt'dx'. \quad (27)$$

We recall that κ enters into the definition of F^ϵ_κ and satisfies (22).

The following shows that $\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar{\gamma}}$ is a smooth supersolution of (17) with a space gradient admitting bounded derivatives. This is due to the space dependent rescaling of the window by κ and will be crucial for our verification arguments.

PROPOSITION 3.7. *For all $0 < \epsilon < \epsilon_\circ$ and $K > 0$ large enough, there exists $\delta_\circ > 0$ such that $\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar{\gamma}}$ is a C^∞ supersolution of (17) for all $0 < \delta < \delta_\circ$. It has linear growth and $\partial_x\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar{\gamma}}$ has bounded derivatives of any order.*

Proof. a. It follows from (22) and (25) that

$$\limsup_{|x|\to\infty} |\bar{\mathrm{v}}^{\epsilon,K}_{\bar{\gamma}}(x)|/(1+|\kappa(x)|) < \infty.$$

Direct computations and (22) then show that $\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar{\gamma}}$ has linear growth and that all derivatives of $\partial_x\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar{\gamma}}$ are uniformly bounded.
b. We now prove the supersolution property inside the parabolic domain. Since the proof is very close to that of [5, Theorem 3.3], we only provide the arguments that require to be adapted, and refer to their proof for other elementary details. Fix $\ell > 0$ and set

$$v_\ell(t,x) := \bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar{\gamma}}(t,(-\ell)\vee x\wedge\ell).$$

We omit the superscripts that are superfluous in this proof. Given $k \ge 1$, set

$$v_{\ell,k}(z) := \inf_{z'\in[0,T]\times\mathbb{R}} \left(v_\ell(z') + k|z-z'|^2\right).$$

Since v_ℓ is bounded and continuous, the infimum in the above is achieved by a point $\hat{z}_{\ell,k}(z) = (\hat{t}_{\ell,k}(z), \hat{x}_{\ell,k}(z))$, and $v_{\ell,k}$ is bounded, uniformly in $k \geq 1$. This implies that we can find $C_\ell > 0$, independent of k, such that

$$|z - \hat{z}_{\ell,k}(z)|^2 \leq C_\ell/k =: (\rho_{\ell,k})^2. \tag{28}$$

Moreover, a simple change of variables argument shows that, if φ is a smooth function such that $v_{\ell,k} - \varphi$ achieves a minimum at $z \in [0,T) \times (-\ell, \ell)$, then

$$(\partial_t \varphi, \partial_x \varphi, \partial^2_{xx}\varphi)(z) \in \bar{\mathcal{P}}^- v_\ell(\hat{z}_{\ell,k}(z)),$$

where $\bar{\mathcal{P}}^- v_\ell(\hat{z}_{\ell,k}(z))$ denotes the *closed* parabolic subjet of v_ℓ at $\hat{z}_{\ell,k}(z)$; see e.g. [10] for the definition. Then, Proposition 3.4 implies that $v_{\ell,k}$ is a supersolution of

$$\min_{x' \in D^\epsilon_\kappa(\hat{x}_{\ell,k}(z))} \min \left\{ -\partial_t \varphi(z) - \frac{\sigma^2(x')\partial^2_{xx}\varphi(z)}{2(1 - f(x')\partial^2_{xx}\varphi(z))}, \bar{\gamma}(x') - \partial^2_{xx}\varphi(z) \right\} \geq 0,$$

$z \in [\rho_{\ell,k}, T - \rho_{\ell,k}) \times (-\ell + \rho_{\ell,k}, \ell - \rho_{\ell,k})$. We next deduce from (28) that $x' \in D^{\epsilon/2}_\kappa(x)$ implies

$$-\frac{\epsilon}{2}\kappa(x') - C_\ell/k^{\frac{1}{2}} \leq \hat{x}_{\ell,k}(t,x) - x' \leq \frac{\epsilon}{2}\kappa(x') + C_\ell/k^{\frac{1}{2}}.$$

Since $\inf \kappa > 0$, this shows that $x' \in D^\epsilon_\kappa(\hat{x}_{\ell,k}(t,x))$ for k large enough with respect to ℓ. Hence, $v_{\ell,k}$ is a supersolution of

$$\min_{x' \in D^{\epsilon/2}_\kappa} \min \left\{ -\partial_t \varphi - \frac{\sigma^2(x')\partial^2_{xx}\varphi}{2(1 - f(x')\partial^2_{xx}\varphi)}, \bar{\gamma}(x') - \partial^2_{xx}\varphi \right\} \geq 0$$

on $[\rho_{\ell,k}, T - \rho_{\ell,k}) \times (-\ell + \rho_{\ell,k}, \ell - \rho_{\ell,k})$.

We now argue as in [13]. Since $v_{\ell,k}$ is semi-concave, there exist $\partial^{2,abs}_{xx} v_{\ell,k} \in L^1$ and a Lebesgue-singular negative Radon measure $\partial^{2,sing}_{xx} v_{\ell,k}$ such that

$$\partial^2_{xx} v_{\ell,k}(dz) = \partial^{2,abs}_{xx} v_{\ell,k}(z)dz + \partial^{2,sing}_{xx} v_{\ell,k}(dz) \text{ in the distribution sense}$$

and

$$(\partial_t v_{\ell,k}, \partial_x v_{\ell,k}, \partial^{2,abs}_{xx} v_{\ell,k}) \in \bar{\mathcal{P}}^- v_{\ell,k} \text{ a.e. on } [\rho_k, T - \rho_k] \times (-\ell + \rho_{\ell,k}, \ell - \rho_{\ell,k}),$$

see [14, Section 3]. Hence, the above implies that

$$\min_{x' \in D^{\epsilon/2}_\kappa} \min \left\{ -\partial_t v_{\ell,k} - \frac{\sigma^2(x')\partial^{2,abs}_{xx} v_{\ell,k}}{2(1 - f(x')\partial^{2,abs}_{xx} v_{\ell,k})}, \bar{\gamma}(x') - \partial^{2,abs}_{xx} v_{\ell,k} \right\} \geq 0$$

a.e. on $[\rho_{\ell,k}, T - \rho_{\ell,k}) \times (-\ell + \rho_{\ell,k}, \ell - \rho_{\ell,k})$, or equivalently, by (18),

$$\min \left\{ -\partial_t v_{\ell,k} - \frac{\sigma^2(x)\partial^{2,abs}_{xx} v_{\ell,k}}{2(1 - f(x)\partial^{2,abs}_{xx} v_{\ell,k})}, \bar{\gamma}(x) - \partial^{2,abs}_{xx} v_{\ell,k} \right\} (t', x') \geq 0$$

for all x and for a.e. $(t', x') \in [\rho_{\ell,k}, T - \rho_{\ell,k}) \times (-\ell + \rho_{\ell,k}, \ell - \rho_{\ell,k})$ such that $2|x' - x| \leq \epsilon\kappa(x)$. Take $0 < \delta < \varepsilon/2$. Integrating the previous inequality with respect to (t', x')

with the kernel function $\psi_\delta(\cdot, \cdot/\kappa)/\kappa$, using the concavity and monotonicity property of Remark 3.1 and the fact that $\partial^{2,sing}_{xx} v_{\ell,k}$ is non-positive, we obtain

$$\min\left\{-\partial_t v^\delta_{\ell,k} - \frac{\sigma^2 \partial^2_{xx} v^\delta_{\ell,k}}{2(1 - f\partial^2_{xx} v^\delta_{\ell,k})}, \bar\gamma - \partial^2_{xx} v^\delta_{\ell,k}\right\} \geq 0 \tag{29}$$

on $[\rho_{\ell,k} + \delta, T - \rho_{\ell,k}) \times (-x^-_{\ell,k}, x^+_{\ell,k})$, in which

$$v^\delta_{\ell,k}(t,x) := \int_{\mathbb{R}\times\mathbb{R}} v_{\ell,k}([t']^+, x')\frac{1}{\kappa(x)}\psi_\delta\left(t' - \cdot, \frac{x' - \cdot}{\kappa(x)}\right) dt'dx'$$

and

$$x^+_{\ell,k} + \frac{\delta}{2}\kappa(x^+_{\ell,k}) = \ell - \rho_{\ell,k} \text{ and } -x^-_{\ell,k} - \frac{\delta}{2}\kappa(-x^-_{\ell,k}) = -\ell + \rho_{\ell,k}.$$

The above are well defined, see Remark 3.3. By Remark 3.3 and (28), $\pm x^\pm_{\ell,k} \to \pm\infty$ and $\rho_{\ell,k} \to 0$ as $k \to \infty$ and then $\ell \to \infty$. Moreover, $v^\delta_{\ell,k} \to \bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}$ as $k \to \infty$ and then $\ell \to \infty$, and the derivatives also converge. Hence, (29) implies that $\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}$ is a supersolution of (17) on $[\delta, T) \times \mathbb{R}$.

c. We conclude by discussing the boundary condition at T. We know from Proposition 3.4 that

$$\bar{\mathrm{v}}^{\epsilon,K}_{\bar\gamma} \geq \hat g_K + \epsilon/2, \qquad \text{on } [T - c^K_\epsilon, T] \times \mathbb{R}.$$

Since $\hat g$ is uniformly continuous, see (16), so is $\hat g_K$, and therefore $\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}(T, \cdot) \geq \hat g_K$ on the compact set $[-2x_K, 2x_K]$ for $\delta > 0$ small enough with respect to ϵ, see Lemma 3.2 for the definition of $x_K \geq K$. Now observe that $x \geq 2x_K$ and $|x' - x| \leq \delta\kappa(x)$ imply that $x' \geq 2x_K(1 - \delta c^K_1) - \delta c^K_0$ in which c^K_1 and c^K_0 are constants. This actually follows from the affine behavior of κ on $[x_K, \infty)$, see (22) and Lemma 3.2. For δ small enough, we then obtain $x' \geq x_K$. Since $\hat g_K$ is affine on $[x_K, \infty)$, and since ψ is symmetric in its second argument, see (26), it follows that

$$\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}(T, x) \geq \int_{\mathbb{R}\times\mathbb{R}} \hat g_K(x')\frac{1}{\kappa(x)}\psi_\delta\left(t' - T, \frac{x' - x}{\kappa(x)}\right) dt'dx' = \hat g_K(x)$$

for all $x \geq 2x_K$. This also holds for $x \leq -2x_K$, by the same arguments. □

We can now use a verification argument and provide the main result of this section.

THEOREM 3.8. *Let $\bar{\mathrm{v}}_{\bar\gamma}$ be defined as in Proposition 3.5. It has linear growth. Moreover, $\bar{\mathrm{v}}_{\bar\gamma} \geq \mathrm{v}_{\bar\gamma}$ on $[0, T] \times \mathbb{R}$.*

Proof. The linear growth property has already been stated in Proposition 3.5. We now show that $\bar{\mathrm{v}}_{\bar\gamma} \geq \mathrm{v}_{\bar\gamma}$ by applying a verification argument to $\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}$. From now on $0 < \epsilon \leq \epsilon_\circ$ in which $\epsilon_\circ$ is as in (22). The parameters $K, \delta > 0$ are chosen as in Proposition 3.7.

Fix $(t, x) \in (0, T) \times \mathbb{R}$ and $\delta \in (0, t \wedge \epsilon)$. Let (X, Y, V) be defined as in (6)-(2)-(7) with $(x, \partial_x \bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}(t, x), \bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}(t, x) - \partial_x \bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}(t, x)x)$ as initial condition at t, and for the Markovian controls

$$\hat a = \left(\frac{\sigma\partial^2_{xx}\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}}{1 - f\partial^2_{xx}\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}}\right)(\cdot, X)$$

$$\hat b = \left(\frac{\partial^2_{tx}\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma} + \partial^2_{xx}\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}(\mu + \hat a\sigma f') + \frac{1}{2}\partial^3_{xxx}\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}(\sigma + \hat a f)^2}{1 - f\partial^2_{xx}\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar\gamma}}\right)(\cdot, X).$$

By definition of F, (13) and (1), the above is well-defined as the denominators are always bigger than $\inf f\iota > 0$. All the involved functions being bounded and Lipschitz, see Proposition 3.7, it is easy to check that a solution to the corresponding stochastic differential equation exists, and that $(\hat{a}, \hat{b}) \in \mathcal{A}^{\circ}$. Direct computations then show that $Y = \partial_x \bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}(\cdot, X)$. Moreover, the fact that $\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}$ is a supersolution of $F[\varphi] = 0$ on $[t,T] \times \mathbb{R}$ ensures that the gamma constraint (12) holds, for some $k \geq 1$, and that

$$-\partial_t \bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}(\cdot, X) - \frac{1}{2}\sigma(X)\hat{a} \geq 0 \quad \text{on } [t,T).$$

The last inequality combined with the definition of $\hat{a}$ implies

$$\begin{aligned}
\frac{1}{2} f(X)\hat{a}^2 &\geq \partial_t \bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}(\cdot, X) + \frac{1}{2}(\sigma(X) + f(X)\hat{a})\hat{a} \\
&= \partial_t \bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}(\cdot, X) + \frac{1}{2}(\sigma_X^{\hat{a}}(X))^2 \partial_{xx}^2 \bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}(\cdot, X) \quad \text{on } [t,T).
\end{aligned}$$

Hence,

$$\begin{aligned}
V_T &= \bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}(t,x) + \frac{1}{2}\int_t^T f(X_u)\hat{a}_u^2 \, du + \int_t^T \partial_x \bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}(u, X_u)\, dX_u \\
&\geq \bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}(t,x) + \int_t^T d\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}(u, X_u) \\
&= \bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}(T, X_T) \geq g(X_T),
\end{aligned}$$

in which the last inequality follows from Proposition 3.7 again.

It remains to pass to the limit $\delta, \epsilon \to 0$. By Proposition 3.4, $\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K}$ is continuous, so that $\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}$ converges pointwise to $\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K}$ as $\delta \to 0$. By Proposition 3.5, $\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K}$ converges pointwise to $\bar{\mathrm{v}}_{\bar{\gamma}}$ as $\epsilon \to 0$ and $K \to \infty$. In view of the above this implies the required result: $\bar{\mathrm{v}}_{\bar{\gamma}} \geq \mathrm{v}_{\bar{\gamma}}$. □

REMARK 3.9. *Note that, in the above proof, we have constructed a super-hedging strategy in $\mathcal{A}_{k,\bar{\gamma}}(t,x)$ and starting with $|Y_t| \leq k$, for some $k \geq 1$ which can be chosen in a uniform way with respect to (t,x), while $\bar{\mathrm{v}}_{\bar{\gamma}}^{\epsilon,K,\delta}$ has linear growth.*

3.1.3. Comparison principle. We provide here the comparison principle that was used several times in the above. Before stating it, let us make the following observation, based on direct computations. Recall (1) and (13).

PROPOSITION 3.10. *Fix $\rho > 0$. Consider the map*

$$(t,x,M) \in [0,T] \times \mathbb{R} \times \mathbb{R} \mapsto \Psi(t,x,M) = \frac{\sigma^2(x)M}{2(e^{\rho t} - f(x)M)}.$$

Then, $M \mapsto \Psi(t,x,M)$ is continuous, uniformly in (t,x), on

$$O := \{(t,x,M) \in [0,T] \times \mathbb{R} \times \mathbb{R} : M \leq e^{\rho t}\bar{\gamma}(x)\}.$$

Moreover, there exists $L > 0$ such that $x \mapsto \Psi(t,x,M)$ is L-Lipschitz on O.

THEOREM 3.11. *Fix $\epsilon \in [0, \epsilon_\circ]$. Let U (resp. V) be a upper semicontinuous viscosity subsolution (resp. lower semicontinuous supersolution) of $F_\kappa^\epsilon = 0$ on $[0,T) \times \mathbb{R}$. Assume that U and V have linear growth and that $U \leq V$ on $\{T\} \times \mathbb{R}$, then $U \leq V$ on $[0,T] \times \mathbb{R}$.*

Proof. Set $\hat U(t,x) := e^{\rho t}U(t,x)$, $\hat V(t,x) := e^{\rho t}V(t,x)$. Then, $\hat U$ and $\hat V$ are respectively sub- and supersolution of

$$\min_{x'\in D^\epsilon_\kappa} \min\left\{\rho\varphi - \partial_t\varphi - \frac{\sigma^2(x')\partial_{xx}\varphi}{2(e^{\rho t} - f(x')\partial_{xx}\varphi)}, e^{\rho t}\bar\gamma(x') - \partial_{xx}\varphi\right\} = 0 \tag{30}$$

on $[0,T)\times\mathbb{R}$. For later use, note that the infimum over D^ϵ_κ is achieved in the above, by the continuity of the involved functions.

If $\sup_{[0,T]\times\mathbb{R}}(\hat U - \hat V) > 0$, then we can find $\lambda\in(0,1)$ such that $\sup_{[0,T]\times\mathbb{R}}(\hat U - \hat V_\lambda) > 0$ with $\hat V_\lambda := \lambda\hat V + (1-\lambda)w$, in which

$$w(t,x) := (T-t)A + (c^U_0 + c^U_1|\cdot| - \frac{\iota}{4}|\cdot|^2)^{\mathrm{conc}}(x) + \frac{\iota}{4}|x|^2$$

with c^U_0, c^U_1 two constants such that $e^{\rho T}|U| \le c^U_0 + c^U_1|\cdot|$ and

$$A := \frac{1}{2}\sup\frac{\sigma^2}{1-\frac{\iota}{2}f}\frac{\iota}{2},$$

where $\iota > 0$ is as in (13). Note that

$$\hat V_\lambda(T,\cdot) \ge \hat U(T,\cdot), \tag{31}$$

and that

$$\begin{array}{c} w \text{ is a viscosity supersolution of (30)} \\ \hat V_\lambda \text{ is a viscosity supersolution of } \lambda\bar\gamma + (1-\lambda)\frac{\iota}{2} - \partial^2_{xx}\varphi \ge 0. \end{array} \tag{32}$$

Moreover, by Remark 3.1, $\hat V_\lambda$ is a supersolution of (30). Define for $\varepsilon > 0$ and $n \ge 1$

$$\Theta^\varepsilon_n := \sup_{(t,x,y)\in[0,T]\times\mathbb{R}^2}\left[\hat U(t,x) - \hat V_\lambda(t,y) - \left(\frac{\varepsilon}{2}|x|^2 + \frac{n}{2}|x-y|^2\right)\right] =: \eta > 0, \tag{33}$$

in which the last inequality holds for $n > 0$ large enough and $\varepsilon > 0$ small enough. Denote by $(t^\varepsilon_n, x^\varepsilon_n, y^\varepsilon_n)$ the point at which this supremum is achieved. By (31), it must hold that $t^\varepsilon_n < T$, and, by standard arguments, see e.g., [10, Proposition 3.7],

$$\lim_{n\to\infty} n|x^\varepsilon_n - y^\varepsilon_n|^2 = 0. \tag{34}$$

Moreover, Ishii's lemma implies the existence of $(a^\varepsilon_n, M^\varepsilon_n, N^\varepsilon_n)\in\mathbb{R}^3$ such that

$$(a^\varepsilon_n, \varepsilon x^\varepsilon + n(x^\varepsilon_n - y^\varepsilon_n), M^\varepsilon_n) \in \bar{\mathcal{P}}^{2,+}\hat U(t^\varepsilon_n, x^\varepsilon_n)$$
$$(a^\varepsilon_n, -n(x^\varepsilon_n - y^\varepsilon_n), N^\varepsilon_n) \in \bar{\mathcal{P}}^{2,-}\hat V_\lambda(t^\varepsilon_n, y^\varepsilon_n),$$

in which $\bar{\mathcal{P}}^{2,+}$ and $\bar{\mathcal{P}}^{2,-}$ denote as usual the *closed* parabolic super- and subjets, see [10], and

$$\begin{pmatrix} M^\varepsilon_n & 0 \\ 0 & -N^\varepsilon_n \end{pmatrix} \le R^\varepsilon_n + \frac{1}{n}(R^\varepsilon_n)^2 = 3n\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \begin{pmatrix} 3\varepsilon + \frac{\varepsilon^2}{n} & -\varepsilon \\ -\varepsilon & 0 \end{pmatrix}$$

with

$$R^\varepsilon_n := n\begin{pmatrix} 1+\frac{\varepsilon}{n} & -1 \\ -1 & 1 \end{pmatrix}.$$

In particular,

$$M_n^\varepsilon - N_n^\varepsilon \le \delta_n^\varepsilon \text{ with } \delta_n^\varepsilon := \varepsilon + \frac{\varepsilon^2}{n}. \tag{35}$$

Then, by (32) and (13),

$$0 < (1-\lambda)\frac{\iota}{2} \le e^{\rho t_n^\varepsilon}\bar\gamma(\hat y_n^\varepsilon) - N_n^\varepsilon \le e^{\rho t_n^\varepsilon}\bar\gamma(\hat y_n^\varepsilon) - M_n^\varepsilon + \delta_n^\varepsilon, \tag{36}$$

in which $\hat y_n^\varepsilon \in D_\kappa^\epsilon(y_n^\varepsilon)$. In view of Remark 3.3, this shows that $e^{\rho t_n^\varepsilon}\bar\gamma(\hat x_n^\varepsilon) - M_n^\varepsilon > 0$ for some $\hat x_n^\varepsilon \in D_\kappa^\epsilon(x_n^\varepsilon)$, for n large enough and ε small enough, recall (34). Hence, the super- and subsolution properties of $\hat V_\lambda$ and $\hat U$ imply that we can find $u_n^\varepsilon \in [-\epsilon, \epsilon]$ together with $\hat y_n^\varepsilon$ and $\hat x_n^\varepsilon$ such that

$$\hat y_n^\varepsilon + u_n^\varepsilon \kappa(\hat y_n^\varepsilon) = y_n^\varepsilon \ , \ \hat x_n^\varepsilon + u_n^\varepsilon \kappa(\hat x_n^\varepsilon) = x_n^\varepsilon \tag{37}$$

None

and

$$\rho(\hat U(t_n^\varepsilon, x_n^\varepsilon) - \hat V_\lambda(t_n^\varepsilon, y_n^\varepsilon)) \le \frac{\sigma^2(\hat x_n^\varepsilon)M_n^\varepsilon}{2(e^{\rho t_n^\varepsilon} - f(\hat x_n^\varepsilon)M_n^\varepsilon)} - \frac{\sigma^2(\hat y_n^\varepsilon)N_n^\varepsilon}{2(e^{\rho t_n^\varepsilon} - f(\hat y_n^\varepsilon)N_n^\varepsilon)}.$$

By Remark 3.1 and (35), this shows that

$$\begin{aligned}&\rho(\hat U(t_n^\varepsilon, x_n^\varepsilon) - \hat V_\lambda(t_n^\varepsilon, y_n^\varepsilon))\\ &\le \frac{\sigma^2(\hat x_n^\varepsilon)(N_n^\varepsilon + \delta_n^\varepsilon)}{2(e^{\rho t_n^\varepsilon} - f(\hat x_n^\varepsilon)(N_n^\varepsilon + \delta_n^\varepsilon))} - \frac{\sigma^2(\hat y_n^\varepsilon)N_n^\varepsilon}{2(e^{\rho t_n^\varepsilon} - f(\hat y_n^\varepsilon)N_n^\varepsilon)}.\end{aligned}$$

It remains to apply Proposition 3.10 together with (36) for n large enough and ε small enough to obtain

$$\begin{aligned}&\rho(\hat U(t_n^\varepsilon, x_n^\varepsilon) - \hat V_\lambda(t_n^\varepsilon, y_n^\varepsilon))\\ &\le \frac{\sigma^2(\hat x_n^\varepsilon)N_n^\varepsilon}{2(e^{\rho t_n^\varepsilon} - f(\hat x_n^\varepsilon)N_n^\varepsilon)} - \frac{\sigma^2(\hat y_n^\varepsilon)N_n^\varepsilon}{2(e^{\rho t_n^\varepsilon} - f(\hat y_n^\varepsilon)N_n^\varepsilon)} + O_n^\varepsilon(1)\\ &\le L\,|\hat x_n^\varepsilon - \hat y_n^\varepsilon| + O_n^\varepsilon(1)\end{aligned}$$

for some $L > 0$ and where $O_n^\varepsilon(1) \to 0$ as $n \to \infty$ and then $\varepsilon \to 0$. By continuity and (34) combined with Remark 3.3 and (37), this contradicts (33) for n large enough. □

3.2. Supersolution property for the weak formulation. In this part, we provide a lower bound $\underline{\mathrm{v}}_{\bar\gamma}$ for $\mathrm{v}_{\bar\gamma}$ that is a supersolution of (17). It is constructed by considering a weak formulation of the stochastic target problem (14) in the spirit of [8, Section 5]. Since our methodology is slightly different, we provide the main arguments.

On $C(\mathbb{R}_+)^5$, let us now denote by $(\tilde\zeta := (\tilde a, \tilde b, \tilde\alpha, \tilde\beta), \tilde W)$ the coordinate process and let $\tilde{\mathbb{F}}^\circ = (\tilde{\mathcal{F}}_s^\circ)_{s\le T}$ be its raw filtration. We say that a probability measure $\tilde{\mathbb{P}}$ belongs to $\tilde{\mathcal{A}}_k$ if $\tilde W$ is a $\tilde{\mathbb{P}}$-Brownian motion and if for all $0 \le \delta \le 1$ and $r \ge 0$ it holds $\tilde{\mathbb{P}}$-a.s. that

$$\tilde a = \tilde a_0 + \int_0^\cdot \tilde\beta_s ds + \int_0^\cdot \tilde\alpha_s d\tilde W_s \text{ for some } \tilde a_0 \in \mathbb{R}, \tag{38}$$

$$\sup_{\mathbb{R}_+} |\tilde\zeta| \le k\ , \tag{39}$$

and

$$\mathbb{E}^{\tilde{\mathbb{P}}}\left[\sup\left\{|\tilde{\zeta}_{s'}-\tilde{\zeta}_s|,\ r\le s\le s'\le s+\delta\right\}|\tilde{\mathcal{F}}^\circ_r\right]\le k\delta. \tag{40}$$

For $\tilde{\phi}:=(y,\tilde{a},\tilde{b})$, $y\in\mathbb{R}$, we define $(\tilde{X}^{x,\tilde{\phi}},\tilde{Y}^{\tilde{\phi}},\tilde{V}^{x,v,\tilde{\phi}})$ as in (6)-(2)-(7) associated to the control $(\tilde{a},\tilde{b})$ with time-0 initial condition (x,y,v), and with $\tilde{W}$ in place of W. For $t\le T$ and $k\ge 1$, we say that $\tilde{\mathbb{P}}\in\tilde{\mathcal{G}}_{k,\bar{\gamma}}(t,x,v,y)$ if

$$\left[\tilde{V}^{x,v,\tilde{\phi}}_{T-t}\ge g(\tilde{X}^{x,\tilde{\phi}}_{T-t})\ \text{ and }\ -k\le\gamma^{\tilde{a}}_Y(\tilde{X}^{x,\tilde{\phi}})\le\bar{\gamma}(\tilde{X}^{x,\tilde{\phi}})\text{ on }\mathbb{R}_+\right]\ \ \tilde{\mathbb{P}}-\text{a.s.} \tag{41}$$

We finally define

$$\underline{\mathrm{v}}^k_{\bar{\gamma}}(t,x):=\inf\{v=c+yx\ :\ (c,y)\in\mathbb{R}\times[-k,k]\text{ s.t. }\tilde{\mathcal{A}}_k\cap\tilde{\mathcal{G}}_{k,\bar{\gamma}}(t,x,v,y)\neq\emptyset\},$$

and

$$\underline{\mathrm{v}}_{\bar{\gamma}}(t,x):=\liminf_{\substack{(k,t',x')\to(\infty,t,x)\\(t',x')\in[0,T)\times\mathbb{R}}}\underline{\mathrm{v}}^k_{\bar{\gamma}}(t',x'),\quad (t,x)\in[0,T]\times\mathbb{R}. \tag{42}$$

The following is an immediate consequence of our definitions.

Proposition 3.12. *$\mathrm{v}_{\bar{\gamma}}\ge\underline{\mathrm{v}}_{\bar{\gamma}}$ on $[0,T)\times\mathbb{R}$.*

In the rest of this section, we show that $\underline{\mathrm{v}}_{\bar{\gamma}}$ is a viscosity supersolution of (17). We start with an easy remark.

Remark 3.13. *Observe that the gamma constraint in (41) implies that we can find $\varepsilon>0$ such that*

$$\frac{\varepsilon}{1+k\varepsilon^{-1}}\le\sigma^{\tilde{a}}_X(\tilde{X}^{x,\tilde{\phi}})\le\varepsilon^{-1}+\varepsilon^{-2}\ \ \text{and } |\tilde{a}|\le\varepsilon^{-1}\ \ \tilde{\mathbb{P}}-\text{a.s.},$$

for all $\tilde{\mathbb{P}}\in\tilde{\mathcal{A}}_k\cap\tilde{\mathcal{G}}_{k,\bar{\gamma}}(t,x,v,y)$ and $k\ge1$. Indeed, if $\tilde{a}\ge-\sigma/f$ then $-k\le\gamma^{\tilde{a}}_Y\le\bar{\gamma}$ implies

$$(-\frac{k\sigma}{1+kf})\vee(-\frac{\sigma}{f})\le\tilde{a}\le\frac{\bar{\gamma}\sigma}{1-\bar{\gamma}f}\ \ \text{and }\ \tilde{a}f+\sigma\ge\sigma/(1+kf).$$

Then our claim follows from (1)-(13). On the other hand, if $\sigma+\tilde{a}f<0$, then $\gamma^{\tilde{a}}_Y\le\bar{\gamma}$ implies $\tilde{a}\ge\bar{\gamma}\sigma/(1-f\bar{\gamma})\ge0$, see (13), while $\tilde{a}<-f/\sigma<0$, a contradiction.

We then show that $\underline{\mathrm{v}}^k_{\bar{\gamma}}$ has linear growth, for k large enough.

Proposition 3.14. *There exists $k_o\ge1$ such that $\{|\underline{\mathrm{v}}^k_{\bar{\gamma}}|,k\ge k_o\}$ is uniformly bounded from above by a continuous map with linear growth.*

Proof. **a.** First note that Remark 3.9 implies that $\{(\underline{\mathrm{v}}^k_{\bar{\gamma}})^+,\ k\ge k_o\}$ is uniformly bounded from above by a map with linear growth, for some k_o large enough.
b. Let us now fix $\tilde{\mathbb{P}}\in\tilde{\mathcal{A}}_k\cap\tilde{\mathcal{G}}_{k,\bar{\gamma}}(t,x,v,y)$. Using Remark 3.13 combined with (1) and the condition that $(\tilde{a},\tilde{b},\tilde{\alpha},\tilde{\beta})$ is $\tilde{\mathbb{P}}$-essentially bounded, one can find $\check{\mathbb{P}}\sim\tilde{\mathbb{P}}$ under which $\int_0^\cdot\tilde{Y}^{\tilde{\phi}}_s d\tilde{X}^{x,\tilde{\phi}}_s$ is a martingale on $[0,T-t]$. Then, the condition $\tilde{V}^{x,v,\tilde{\phi}}_{T-t}\ge g(\tilde{X}^{x,\tilde{\phi}}_{T-t})$ $\tilde{\mathbb{P}}$-a.s. implies $v+\mathbb{E}^{\check{\mathbb{P}}}[\frac12\int_0^{T-t}\tilde{a}^2_s f(\tilde{X}^{x,\tilde{\phi}}_s)ds]\ge\inf g>-\infty$, recall (16). By Remark 3.13 and (1), $v\ge\inf g-C>-\infty$, for some constant C independent of $\tilde{\mathbb{P}}\in\cup_k(\tilde{\mathcal{A}}_k\cap\tilde{\mathcal{G}}_{k,\bar{\gamma}}(t,x,v,y))$. Hence $\{(\underline{\mathrm{v}}^k_{\bar{\gamma}})^-,\ k\ge k_o\}$ is bounded by a constant. □

We now prove that existence holds in the problem defining $\underline{\mathrm{v}}^k_{\bar{\gamma}}$ and that it is lower-semicontinuous.

Proposition 3.15. *For all $(t,x)\in[0,T]\times\mathbb{R}$ and $k\ge 1$ large enough, there exists $(c,y)\in\mathbb{R}\times[-k,k]$ such that $\underline{\mathrm{v}}^k_{\bar\gamma}(t,x)=c+yx$ and $\tilde{\mathcal{A}}_k\cap\tilde{\mathcal{G}}_{k,\bar\gamma}(t,c+xy,y)\neq\emptyset$. Moreover, $\underline{\mathrm{v}}^k_{\bar\gamma}$ is lower-semicontinuous for each $k\ge 1$ large enough.*

Proof. By [19, Proposition XIII.1.5] and the condition (40) taken for $r=0$, the set $\tilde{\mathcal{A}}_k$ is weakly relatively compact. Moreover, [16, Theorem 7.10 and Theorem 8.1] implies that any limit point $(\mathbb{P}_*,t_*,x_*,c_*,y_*)$ of a sequence $(\mathbb{P}_n,t_n,x_n,c_n,y_n)_{n\ge1}$ such that $\mathbb{P}_n\in\tilde{\mathcal{A}}_k\cap\tilde{\mathcal{G}}_{k,\bar\gamma}(t_n,x_n,c_n+x_ny_n,y_n)$ for each $n\ge1$ satisfies $\mathbb{P}_*\in\tilde{\mathcal{A}}_k\cap\tilde{\mathcal{G}}_{k,\bar\gamma}(t_*,x_*,c_*+x_*y_*,y_*)$. Since $\underline{\mathrm{v}}^k_{\bar\gamma}$ is locally bounded, by Proposition 3.14 when $k\ge k_o$, the announced existence and lower-semicontinuity readily follow. □

We can finally prove the main result of this section.

Theorem 3.16. *The function $\underline{\mathrm{v}}_{\bar\gamma}$ is a viscosity supersolution of (17). It has linear growth.*

Proof. The linear growth property is an immediate consequence of the uniform linear growth of $\{|\underline{\mathrm{v}}^k_{\bar\gamma}|,k\ge k_o\}$ stated in Proposition 3.14. To prove the supersolution property, it suffices to show that it holds for each $\underline{\mathrm{v}}^k_{\bar\gamma}$, with $k\ge k_o$, and then to apply standard stability results, see e.g. [2].
a. We first prove the supersolution property on $[0,T)\times\mathbb{R}$. We adapt the arguments of [8] to our context. Let us consider a C^∞_b test function φ and $(t_0,x_0)\in[0,T)\times\mathbb{R}$ such that

$$\text{(strict)}\min_{[0,T)\times\mathbb{R}}(\underline{\mathrm{v}}^k_{\bar\gamma}-\varphi)=(\underline{\mathrm{v}}^k_{\bar\gamma}-\varphi)(t_0,x_0)=0.$$

Recall that $\underline{\mathrm{v}}^k_{\bar\gamma}$ is lower-semicontinuous by Proposition 3.15.

Because the infimum is achieved in the definition of $\underline{\mathrm{v}}^k_{\bar\gamma}$, by the afore-mentioned proposition, there exists $|y_0|\le k$ and $\tilde{\mathbb{P}}\in\tilde{\mathcal{A}}_k\cap\tilde{\mathcal{G}}_k(t_0,x_0,v_0,y_0)$, such that $v_0:=c_0+y_0x_0=\underline{\mathrm{v}}^k_{\bar\gamma}(t_0,x_0)$ for some $c_0\in\mathbb{R}$. Let us set $(\tilde X,\tilde Y,\tilde V):=(\tilde X^{x_0,\tilde\phi},\tilde Y^{\tilde\phi},\tilde V^{x_0,v_0,\tilde\phi})$ where $\tilde\phi=(y_0,\tilde a,\tilde b)$. Let θ_o be a stopping time for the augmentation of the raw filtration $\tilde{\mathbb{F}}^\circ$, and define

$$\theta:=\theta_o\wedge\theta_1\text{ with }\theta_1:=\inf\{s:|\tilde X_s-x_0|\ge1\}.$$

Then, it follows from Proposition 3.17 below that

$$\tilde V_{\theta_o}\ge\underline{\mathrm{v}}^k_{\bar\gamma}(t_0+\theta_o,\tilde X_{\theta_o})\ge\varphi(t_0+\theta_o,\tilde X_{\theta_o}),$$

in which here and hereafter inequalities are taken in the $\tilde{\mathbb{P}}$-a.s. sense. After applying Itô's formula twice, the above inequality reads:

$$(43)\qquad \int_0^\theta \ell_s\,ds+\int_0^\theta\left(y_0-\partial_x\varphi(t_0,x_0)+\int_0^s m_r dr+\int_0^s n_r d\tilde X_r\right)d\tilde X_s\ge0.$$

where

$$\ell:=\frac12\tilde a^2 f(\tilde X)-\mathcal{L}^{\tilde a}\varphi(t_0+\cdot,\tilde X_\cdot)\ ,\ m:=\mu_Y^{\tilde a,\tilde b}(\tilde X)-\mathcal{L}^{\tilde a}\partial_x\varphi(t_0+\cdot,\tilde X_\cdot)$$
$$n:=\gamma_Y^{\tilde a}(\tilde X)-\partial^2_{xx}\varphi(t_0+\cdot,\tilde X_\cdot),$$

with

$$\mathcal{L}^{\tilde a}:=\partial_t+\frac12(\sigma_X^{\tilde a})^2\partial^2_{xx}$$

For the rest of the proof, we recall (39). Together with (1) and Remark 3.13, this implies that $\sigma_X^{\tilde a}(\tilde X)$, $\sigma_X^{\tilde a}(\tilde X)^{-1}$ and $\mu_X^{\tilde a,\tilde b}(\tilde X)$ are $\tilde{\mathbb{P}}$-essentially bounded. After performing an equivalent change of measure, we can thus find $\check{\mathbb{P}} \sim \tilde{\mathbb{P}}$ and a $\check{\mathbb{P}}$-Brownian motion $\check W$ such that:

$$\tilde X = \int_0^\cdot \sigma_X^{\tilde a_s}(\tilde X_s)d\check W_s. \tag{44}$$

Clearly, both $\check{\mathbb{P}}$ and $\check W$ depend on $(\tilde a, \tilde b, y_0)$.

1. We first show that $y_0 = \partial_x\varphi(t_0,x_0)$, and therefore

$$\int_0^\theta \ell_s\,ds + \int_0^\theta\int_0^s m_r dr d\tilde X_s + \int_0^\theta\int_0^s n_r d\tilde X_r d\tilde X_s \ge 0. \tag{45}$$

Let $\check{\mathbb{P}}^\lambda \sim \check{\mathbb{P}}$ be the measure under which

$$\check W^\lambda := \check W + \int_0^\cdot \lambda[\sigma_X^{\tilde a_s}(\tilde X_s)]^{-1}(y_0 - \partial_x\varphi(t_0,x_0))ds$$

is a $\check{\mathbb{P}}^\lambda$-Brownian motion. Consider the case $\theta_o := \eta > 0$. Since all the coefficients are bounded, taking expectation under $\check{\mathbb{P}}^\lambda$ and using (43) imply

$$\begin{aligned} C'\eta &\ge \lambda(y_0 - \partial_x\varphi(t_0,x_0))^2\mathbb{E}^{\check{\mathbb{P}}^\lambda}[\theta] \\ &+ \mathbb{E}^{\check{\mathbb{P}}^\lambda}\left[\int_0^\theta\left(\int_0^s m_r dr + \int_0^s n_r d\tilde X_r\right)\lambda(y_0 - \partial_x\varphi(t_0,x_0))ds\right] \end{aligned}$$

for some $C' > 0$. We now divide both sides by η and use the fact that $(\eta\wedge\theta_1)/\eta \to 1$ $\check{\mathbb{P}}^\lambda$-a.s. as $\eta \to 0$ to obtain

$$C' \ge \lambda(y_0 - \partial_x\varphi(t_0,x_0))^2.$$

Then, we send $\lambda \to \infty$ to deduce that $y_0 = \partial_x\varphi(t_0,x_0)$.

2. We now prove that

$$\partial^2_{xx}\varphi(t_0,x_0) \le \gamma_Y^{\tilde a_0}(x_0) \le \bar\gamma(x_0). \tag{46}$$

We first consider the time change

$$h(t) = \inf\{r \ge 0 : \int_0^r \left[(\sigma_X^{\tilde a_s}(\tilde X_s))^2\mathbf{1}_{[0,\theta]}(s) + \mathbf{1}_{[0,\theta]^c}(s)\right]ds \ge t\}.$$

Again, $\sigma_X^{\tilde a}(\tilde X)$ and $\sigma_X^{\tilde a}(\tilde X)^{-1}$ are essentially bounded by Remark 3.13, so that h is absolutely continuous and its density $\mathfrak{h}$ satisfies

$$0 < \underline{\mathfrak{h}}t \le \mathfrak{h}(t) := \left[(\sigma_X^{\tilde a}(\tilde X))^2\mathbf{1}_{[0,\theta]}(t) + \mathbf{1}_{[0,\theta]^c}(t)\right]^{-1} \le \bar{\mathfrak{h}}t \tag{47}$$

for some constants $\underline{\mathfrak{h}}$ and $\bar{\mathfrak{h}}$, for all $t \ge 0$. Moreover, $\hat W := \tilde X_h$ is a Brownian motion in the time changed filtration. Let us now take $\theta_o := h^{-1}(\eta)$ for some $0 < \eta < 1$. Then, (45) reads

$$\begin{aligned} 0 &\le \int_0^{\eta\wedge h^{-1}(\theta_1)} \ell_{h(s)}\mathfrak{h}(s)\,ds + \int_0^{\eta\wedge h^{-1}(\theta_1)}\int_0^s m_{h(r)}\mathfrak{h}(r)drd\hat W_s \\ &+ \int_0^{\eta\wedge h^{-1}(\theta_1)}\int_0^s n_{h(r)}d\hat W_r d\hat W_s. \end{aligned} \tag{48}$$

Since all the involved processes are continuous and bounded, and since $(\eta \wedge h^{-1}(\theta_1))/\eta \to 1$ a.s. as $\eta \to 0$, the above combined with [8, Theorem A.1 b. and Proposition A.3] implies that

$$\gamma_Y^{\tilde a_0}(x_0) - \partial^2_{xx}\varphi(t_0,x_0) = \lim_{r\downarrow 0} n_{h(r)} = \lim_{r\downarrow 0} n_r \geq 0.$$

Since $\gamma_Y^{\tilde a}(\tilde X) \leq \bar\gamma(\tilde X)$, this proves (46).

3. It remains to show that the first term in the definition of $F[\varphi](t_0,x_0)$ is also non-negative, recall (15). Again, let us take $\theta_o := h^{-1}(\eta)$ and recall from 2. that $\lim_{\eta\to 0}(\eta\wedge h^{-1}(\theta_1))/\eta = 1$ $\check{\mathbb{P}}$-a.s. Note that $\tilde a$ being of the form (38) with the condition (39), it satisfies [8, Condition (A.2)], and so does n. Using [8, Theorem A.2 and Proposition A.3] and (48), we then deduce that $\ell_0\mathfrak{h}(0) - \frac{1}{2}n_0 \geq 0$. Hence, (47) and direct computations based on (8) imply

$$\begin{aligned}
0 &\leq \frac{1}{2}\tilde a_0^2 f(x_0) - \mathcal{L}^{\tilde a_0}\varphi(t_0,x_0) - \frac{1}{2}\left(\gamma_Y^{\tilde a_0}(x_0) - \partial^2_{xx}\varphi(t_0,x_0)\right)(\sigma_X^{\tilde a_0}(x_0))^2 \\
&= \frac{1}{2}\tilde a_0^2 f(x_0) - \partial_t\varphi(t_0,x_0) - \frac{1}{2}\gamma_Y^{\tilde a_0}(x_0)(\sigma_X^{\tilde a_0}(x_0))^2 \\
&= -\partial_t\varphi(t_0,x_0) - \frac{1}{2}\frac{\sigma^2(x_0)}{1-f(x_0)\gamma_Y^{\tilde a_0}(x_0)}\gamma_Y^{\tilde a_0}(x_0) \\
&\leq -\partial_t\varphi(t_0,x_0) - \frac{1}{2}\frac{\sigma^2(x_0)}{1-f(x_0)\partial^2_{xx}\varphi(t_0,x_0)}\partial^2_{xx}\varphi(t_0,x_0),
\end{aligned}$$

in which we use the facts that $\partial^2_{xx}\varphi(t_0,x_0) \leq \gamma_Y^{\tilde a_0}(x_0) \leq \bar\gamma(x_0)$ and $z \mapsto z/(1-f(x_0)z)$ in non-decreasing on $(-\infty,\bar\gamma(x_0)] \subset (-\infty, 1/f(x_0))$, for the last inequality.

b. We now consider the boundary condition at T. Since $\underline{\mathrm{v}}^k_{\bar\gamma}$ is a supersolution of $\bar\gamma - \partial^2_{xx}\varphi \geq 0$ on $[0,T)\times\mathbb{R}$, the same arguments as in [11, Lemma 5.1] imply that $\underline{\mathrm{v}}^k_{\bar\gamma} - \bar\Gamma$ is concave for any twice differentiable function $\bar\Gamma$ such that $\partial^2_{xx}\bar\Gamma = \bar\gamma$. The function $\underline{\mathrm{v}}^k_{\bar\gamma}$ being lower-semicontinuous, the map

$$x \mapsto G(x) := \liminf_{\substack{t' \to T,\, x' \to x \\ t' < T}} \underline{\mathrm{v}}^k_{\bar\gamma}(t',x')$$

is such that $G \geq g$ and $G - \bar\Gamma$ is concave. Hence, $G = (G-\bar\Gamma)^{\mathrm{conc}} + \bar\Gamma \geq (g-\bar\Gamma)^{\mathrm{conc}} + \bar\Gamma = \hat g$. □

It remains to state the dynamic programming principle used in the above proof.

PROPOSITION 3.17. *Fix* $(t,x,v,y) \in [0,T]\times\mathbb{R}^2\times[-k,k]$ *and let* θ *be a stopping time for the* $\tilde{\mathbb{P}}$*-augmentation of* $\tilde{\mathbb{F}}^\circ$ *that takes* $\tilde{\mathbb{P}}$*-a.s. values in* $[0,T-t]$*. Assume that* $\tilde{\mathbb{P}} \in \tilde{\mathcal{A}}_k \cap \tilde{\mathcal{G}}_{k,\bar\gamma}(t,x,v,y)$*. Then,*

$$\tilde V_\theta^{x,v,\tilde\phi} \geq \underline{\mathrm{v}}^k_{\bar\gamma}(t+\theta, \tilde X_\theta^{x,\tilde\phi}) \;\; \tilde{\mathbb{P}}-\text{a.s.},$$

in which $\tilde\phi := (y,\tilde a,\tilde b)$*.*

Proof. Since $\underline{\mathrm{v}}^k_{\bar\gamma}$ is lower-semicontinuous and all the involved processes have continuous paths, up to approximating θ by a sequence of stopping times valued in finite time grids, it suffices to prove our claim in the case $\theta \equiv r \in [0,T-t]$. Let $\tilde{\mathbb{P}}_\omega$ be a regular conditional probability given $\tilde{\mathcal{F}}_r^\circ$ for $\tilde{\mathbb{P}}$. It coincides with $\tilde{\mathbb{P}}[\cdot|\tilde{\mathcal{F}}_r^\circ](\omega)$ outside a set N of $\tilde{\mathbb{P}}$-measure zero. Then, for all $\omega \notin N$, $0 \leq \delta \leq 1$ and $r \geq 0$ the conditions (38)-(39)-(40) hold for $\tilde{\mathbb{P}}^r_\omega$ defined on $C(\mathbb{R}_+)^5$ by

$$\tilde{\mathbb{P}}^r_\omega[\omega' \in A] = \tilde{\mathbb{P}}_\omega[\omega'_{r+\cdot} \in A].$$

Moreover, [9, Theorem 3.3] ensures that, after possibly modifying N,

$$\tilde{\mathbb{P}}^r_\omega\left[\tilde{V}^{\xi_r(\omega),\vartheta_r(\omega),\hat{\phi}(\omega)}_{T-(t+r)} \geq g(\tilde{X}^{\xi_r(\omega),\hat{\phi}(\omega)}_{T-(t+r)})\right] = 1$$

$$\text{and } \tilde{\mathbb{P}}^r_\omega\left[\gamma^{\tilde{a}}_Y(\tilde{X}^{\xi_r(\omega),\hat{\phi}(\omega)}) \leq \bar{\gamma}(\tilde{X}^{\xi_r(\omega),\hat{\phi}(\omega)}) \text{ on } \mathbb{R}_+\right] = 1,$$

for $\omega \notin N$, in which

$$(\xi_r, \vartheta_r, \hat{\phi}) := (\tilde{X}^{x,\tilde{\phi}}_r, \tilde{V}^{x,v,\tilde{\phi}}_r, (\tilde{Y}^{x,\tilde{\phi}}_r, \tilde{a}, \tilde{b})).$$

This shows that $\vartheta_r(\omega) \geq \underline{\mathrm{v}}^k_{\bar{\gamma}}(t+r, \xi_r(\omega))$ outside the null set N, which is the required result. □

3.3. Conclusion of the proof and construction of almost optimal strategies. We first conclude the proof of Theorem 2.4.

Proof of Theorem 2.4. Proposition 3.5 and Theorem 3.8 imply that $\bar{\mathrm{v}}_{\bar{\gamma}} \geq \mathrm{v}_{\bar{\gamma}}$ in which $\bar{\mathrm{v}}_{\bar{\gamma}}$ has linear growth and is a continuous viscosity solution of (17). On the other hand, Proposition 3.12 and Theorem 3.16 imply that $\underline{\mathrm{v}}_{\bar{\gamma}} \leq \mathrm{v}_{\bar{\gamma}}$ on $[0, T) \times \mathbb{R}$ in which $\underline{\mathrm{v}}_{\bar{\gamma}}$ has linear growth and is a viscosity supersolution of (17). By the comparison result of Theorem 3.11 applied with $\epsilon = 0$, $\underline{\mathrm{v}}_{\bar{\gamma}} \geq \bar{\mathrm{v}}_{\bar{\gamma}}$. Hence,

$$\mathrm{v}_{\bar{\gamma}} = \underline{\mathrm{v}}_{\bar{\gamma}} = \bar{\mathrm{v}}_{\bar{\gamma}} \;\text{ on } [0, T) \times \mathbb{R} \text{ and } \underline{\mathrm{v}}_{\bar{\gamma}} = \bar{\mathrm{v}}_{\bar{\gamma}} \text{ on } [0, T] \times \mathbb{R} \tag{49}$$

Since $\bar{\mathrm{v}}_{\bar{\gamma}}$ is continuous, this shows that

$$\lim_{\substack{(t',x') \to (T,x) \\ t' < T}} \mathrm{v}_{\bar{\gamma}}(t', x') = \bar{\mathrm{v}}_{\bar{\gamma}}(T, x) = \underline{\mathrm{v}}_{\bar{\gamma}}(T, x).$$

Hence, $\mathrm{v}_{\bar{\gamma}}$ is a viscosity solution of (17), with linear growth. □

REMARK 3.18 (Almost optimal controls). *In the proof of Theorem 3.8, we have constructed a super-hedging strategy starting from* $\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar{\gamma}}(t,x)$. *Since* $\bar{\mathrm{v}}^{\epsilon,K,\delta}_{\bar{\gamma}}(t,x) \to \bar{\mathrm{v}}_{\bar{\gamma}}(t,x) = \mathrm{v}_{\bar{\gamma}}(t,x)$ *as* $\delta, \epsilon \to 0$ *and* $K \to \infty$, *this provides a way to construct super-hedging strategies associated to any initial wealth* $v > \mathrm{v}_{\bar{\gamma}}(t,x)$.

4. Adding a resilience effect. In this section, we explain how a resilience effect can be added to our model. In the discrete rebalancement setting, we replace the dynamics (4) by

$$X^n = X_0 + \int_0^{\cdot} \mu(X^n_s)ds + \int_0^{\cdot} \sigma(X^n_s)dW_s + R^n,$$

in which R^n is defined by

$$R^n = R_0 + \sum_{i=1}^{n} \mathbf{1}_{[t^n_i,T]}\delta^n_{t^n_i} f(X^n_{t^n_i-}) - \int_0^{\cdot} \rho R^n_s ds,$$

for some $\rho > 0$ and $R_0 \in \mathbb{R}$. The process R^n models the impact of past trades on the price, the last term in its dynamics is the resilience effect. Then, the continuous time dynamics becomes

$$X = X_0 + \int_0^{\cdot} \sigma(X_s)dW_s + \int_0^{\cdot} f(X_s)dY_s + \int_0^{\cdot} (\mu(X_s) + a_s(\sigma f')(X_s) - \rho R_s)ds$$

$$R = R_0 + \int_0^{\cdot} f(X_s)dY_s + \int_0^{\cdot} (a_s(\sigma f')(X_s) - \rho R_s)ds$$

$$V = V_0 + \int_0^{\cdot} Y_s dX_s + \frac{1}{2}\int_0^{\cdot} a^2_s f(X_s)ds.$$

This is obtained as a straightforward extension of [4, Proposition 1.1].

Let $\mathrm{v}^R_{\bar\gamma}(t,x)$ be defined as the super-hedging price $\mathrm{v}_{\bar\gamma}(t,x)$ but for these new dynamics and for $R_t = 0$. The following states that $\mathrm{v}^R_{\bar\gamma} = \mathrm{v}_{\bar\gamma}$, i.e. adding a resilience effect does not affect the super-hedging price.

PROPOSITION 4.1. *$\mathrm{v}_{\bar\gamma} = \mathrm{v}^R_{\bar\gamma}$ on $[0,T]\times\mathbb{R}$.*

Proof. 1. To show that $\mathrm{v}_{\bar\gamma} \ge \mathrm{v}^R_{\bar\gamma}$, it suffices to reproduce the arguments of the proof of Theorem 3.8 in which the drift part of the dynamics of X does not play any role. More precisely, these arguments show that $\bar{\mathrm{v}}_{\bar\gamma} \ge \mathrm{v}^R_{\bar\gamma}$. Then, one uses the fact that $\mathrm{v}_{\bar\gamma} = \bar{\mathrm{v}}_{\bar\gamma}$, by (49).
2. As for the opposite inequality, we use the weak formulation of Section 3.2 and a simple Girsanov's transformation. For ease of notations, we restrict to $t = 0$. Fix $v > \mathrm{v}^R_{\bar\gamma}(0,x)$, for some $x \in \mathbb{R}$. Then, one can find $k \ge 1$, $(c,y) \in \mathbb{R}\times[-k,k]$ satisfying $v = c + yx$, and $(a,b) \in \mathcal{A}_{k,\bar\gamma}(0,x)$ such that $V_T \ge g(X_T)$, with (V,X,Y,R) defined by the corresponding initial data and controls. We let

$$a = a_0 + \int_0^\cdot \beta_s ds + \int_0^\cdot \alpha_s dW_s$$

be the decomposition of a into an Itô process, see Section 2.1. Let $\mathbb{Q}^R \sim \mathbb{P}$ be the probability measure under which $W^R := W - \int_0^\cdot (\rho R_s/\sigma(X_s))ds$ is a $\mathbb{Q}^R$-Brownian motion, recall (1). Then,

$$\begin{aligned}
X &= X_0 + \int_0^\cdot \sigma(X_s)dW^R_s + \int_0^\cdot f(X_s)dY_s + \int_0^\cdot (\mu(X_s) + a_s(\sigma f')(X_s))ds\\
Y &= Y_0 + \int_0^\cdot (b_s + a_s\rho R_s/\sigma(X_s))ds + \int_0^\cdot a_s dW^R_s\\
a &= a_0 + \int_0^\cdot (\beta_s + \alpha_s\rho R_s/\sigma(X_s))ds + \int_0^\cdot \alpha_s dW^R_s\\
V &= V_0 + \int_0^\cdot Y_s dX_s + \frac{1}{2}\int_0^\cdot a_s^2 f(X_s)ds.
\end{aligned}$$

Upon seeing $(a, b + a\rho R/\sigma(X), \alpha, \beta + \alpha\rho R/\sigma(X), W^R)$ as a generic element of the canonical space $C([0,T])^5$ introduced in Section 3.2, then $\mathbb{Q}^R$ belongs to $\tilde{\mathcal{A}}_k \cap \tilde{\mathcal{G}}_{k,\bar\gamma}(t, x, v, y)$, and therefore $v > \underline{\mathrm{v}}_{\bar\gamma}(0,x)$. Hence, $\mathrm{v}^R_{\bar\gamma}(0,x) \ge \underline{\mathrm{v}}_{\bar\gamma}(0,x)$, and thus $\mathrm{v}^R_{\bar\gamma}(0,x) \ge \mathrm{v}_{\bar\gamma}(0,x)$ by (49). □

5. Numerical approximation and examples.

In this section, we provide an example of numerical schemes that converges towards the unique continuous viscosity solution of (17) with linear growth. The scheme is then exemplified using two numerical applications in the case of constant market impact and gamma constraint.

5.1. Finite difference scheme.

Given a map ϕ and $h := (h_t, h_x) \in (0,1)^2$, define

$$\begin{aligned}
L^h_1(t,x,y,\phi) &:= -\frac{\phi(t+h_t,x) - y}{h_t} - \frac{\sigma^2(x)G^h(t,x,y,\phi)}{2(1 - f(x)G^h(t,x,y,\phi))}\\
L^h_2(t,x,y,\phi) &:= \bar\gamma(x) - G^h(t,x,y,\phi)
\end{aligned}$$

where

$$G^h(t,x,y,\phi) := \frac{\phi(t+h_t,x+h_x) + \phi(t+h_t,x-h_x) - 2y}{h_x^2}.$$

The numerical scheme is set on the grid $\pi_h := \{(t_i, x_j) = (ih_t, \underline{x} + jh_x) : i \leq n_t, j \leq n_x\}$, with $n_t h_t = T$ for some $n_t \in \mathbb{N}$, and $n_x h_x = \overline{x} - \underline{x}$, for some real numbers $\underline{x} < \overline{x}$. To paraphrase, $\mathrm{v}^h_{\bar\gamma}$ is defined on π_h as the solution of

$$
\begin{aligned}
(50) \qquad S(h, t_i, x_j, \mathrm{v}^h_{\bar\gamma}(t_i, x_j), \mathrm{v}^h_{\bar\gamma}) = 0 \quad &\text{for} \quad i < n_t, 1 \leq j \leq n_x - 1 \\
\mathrm{v}^h_{\bar\gamma} = \hat g \quad &\text{on } \pi_h \cap \{(\{T\} \times \mathbb{R}) \cup ([0,T] \cap \{\underline{x}, \overline{x}\})\}
\end{aligned}
$$

where

$$S(h, t, x, y, \phi) := (\bar w - y) \vee (y - \underline{w}) \wedge \min_{l=1,2} \left\{ L^h_l(t, x, y, \phi) \right\}$$

with $\bar w$ and $\underline{w}$ as in Remark 3.6.

Theorem 5.1. *The equation* (50) *admits a unique solution* $\mathrm{v}^h_{\bar\gamma}$*, for all* $h := (h_t, h_x) \in (0,1)^2$*. Moreover, if* $h_t/h_x^2 \to 0$ *and* $h_x^2 \to 0$*, then* $\mathrm{v}^h_{\bar\gamma}$ *converges locally uniformly to the unique continuous viscosity solution of* (17) *that has linear growth.*

Proof. The existence of a solution, that is bounded by the map with linear growth $|\bar w| + |\underline{w}|$, is obvious. We now prove uniqueness. First observe that L^h_2 is strictly increasing in its y-component, and that

$$\frac{\partial L^h_1}{\partial y}(t, x, y, \phi) = \frac{1}{h_t} + \frac{\sigma^2(x)}{h_x^2 (1 - f(x) G^h(t, x, y, \phi))^2} > 0$$

on the domain $\{y : L^h_2(t_i, x_j, y, \phi) \geq 0\}$. Uniqueness of the solution follows.

It is easy to see that $\phi \mapsto S(\cdot, \phi)$ is non-decreasing, so that our scheme is monotone. Consistency is clear. Moreover, it is not difficult to check that the comparison result of Theorem 3.11 extends to this equation (there is an equivalence of the notions of super- and subsolutions in the class of functions w such that $\underline{w} \leq w \leq \bar w$). It then follows from [3, Theorem 2.1] that $\mathrm{v}^h_{\bar\gamma}$ converges locally uniformly to the unique continuous viscosity solution with linear growth of

$$\Big[(\bar w - \varphi) \vee (\varphi - \underline{w}) \wedge F[\varphi] \Big] \mathbf{1}_{[0,T)} + (\varphi - \hat g) \mathbf{1}_{\{T\}} = 0.$$

In view of (49), Remark 3.6 and Theorem 2.4, $\mathrm{v}_{\bar\gamma}$ is the unique viscosity solution of the above equation. □

5.2. Numerical examples: the fixed impact case. To illustrate the above numerical scheme, we place ourselves in the simpler case where $f \equiv \lambda > 0$ and $\bar\gamma > 0$ are constant. The dynamics of the stock is given by the Bachelier model

$$dX_t = \sigma \, dW_t,$$

with $\sigma := 0.2$. In the following, $T = 2$.

First, we consider a European Butterfly option with three strikes $K_1 = -1 < K_2 = 0 < K_3 = 1$, where $K_1 + 1/(2\bar\gamma) \leq K_2 \leq K_3 - 1/(2\bar\gamma)$. Its pay-off is

$$g(x) = (x - K_1)^+ - 2(x - K_2)^+ + (x - K_3)^+,$$

and the corresponding face-lifted function $\hat g$ can be computed explicitly:

$$
\begin{aligned}
\hat g(x) = &\frac{\bar\gamma}{2}(x - x_1^-)^2 \mathbf{1}_{[x_1^-, x_1^+)} + (x - K_1) \mathbf{1}_{[x_1^+, K_2)} \\
&+ (x - K_1 - 2(x - K_2)) \mathbf{1}_{[K_2, x_2^-)} \\
&+ \left(\frac{\bar\gamma}{2}(x - x_2^+)^2 + 2K_2 - (K_1 + K_3) \right) \mathbf{1}_{[x_2^-, x_2^+)} \\
&+ (2K_2 - (K_1 + K_3)) \mathbf{1}_{[x_2^+, +\infty)},
\end{aligned}
$$

where $x_1^{\pm} = K_1 \pm 1/(2\bar{\gamma})$ and $x_2^{\pm} = K_3 \pm 1/(2\bar{\gamma})$.

In Figure 1, we separately show the effect of the gamma constraint and of the market impact. As observed in Remark 2.9, the price is non-decreasing with respect to the impact parameter λ and bounded from below by the hedging price obtained in the model without impact nor gamma constraint. On the left and right tails of the curves, we observe the effect of the gamma constraint. It does not operate around $x = 0$ where the gamma is non-positive. The effect of the market impact operates only in areas of high convexity (around $x = -1.5$ and $x = 1.5$) or of high concavity (around $x = 0$).

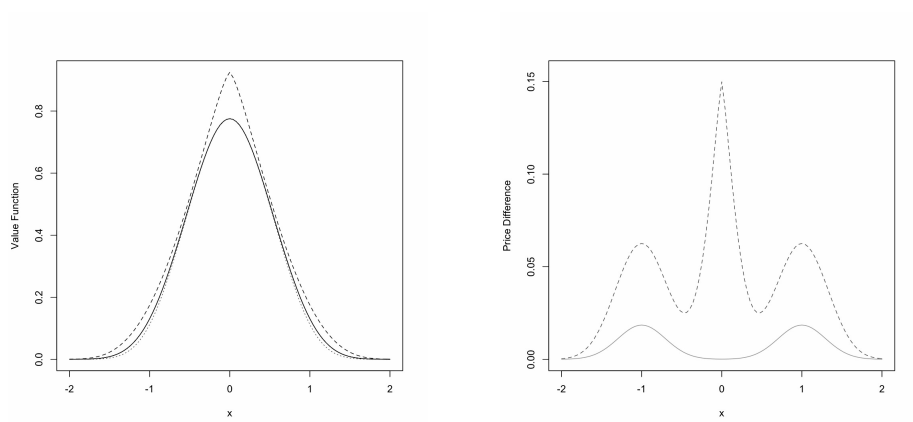

FIG. 1. *Left: Super-hedging price of the Butterfly option. Dashed line:* $\lambda = 0.5$, $\bar{\gamma} = 1.75$*; solid line:* $\lambda = 0$, $\bar{\gamma} = 1.75$*; dotted line:* $\lambda = 0$, $\bar{\gamma} = +\infty$*. Right: Difference with the price associated to* $\lambda = 0$, $\bar{\gamma} = +\infty$*. Dashed line:* $\lambda = 0.5$, $\bar{\gamma} = 1.75$*; solid line:* $\lambda = 0$, $\bar{\gamma} = 1.75$ *.*

In Figure 2, we perform similar computations but for a call spread option, where

$$g(x) = (x - K_1)^+ - (x - K_2)^+,$$

with $K_1 = -1 < K_2 = 1$ such that $K_1 + 1/(2\bar{\gamma}) \le K_2$. The face-lifted function $\hat{g}$ is given by

$$\hat{g}(x) = \frac{\bar{\gamma}}{2}(x - x^-)^2 \mathbf{1}_{[x^-, x^+)} + (x - K_1)\mathbf{1}_{[x^+, K_2)} + (K_2 - K_1)\mathbf{1}_{[K_2, +\infty)}$$

with $x^{\pm} = K_1 \pm 1/(2\bar{\gamma})$.

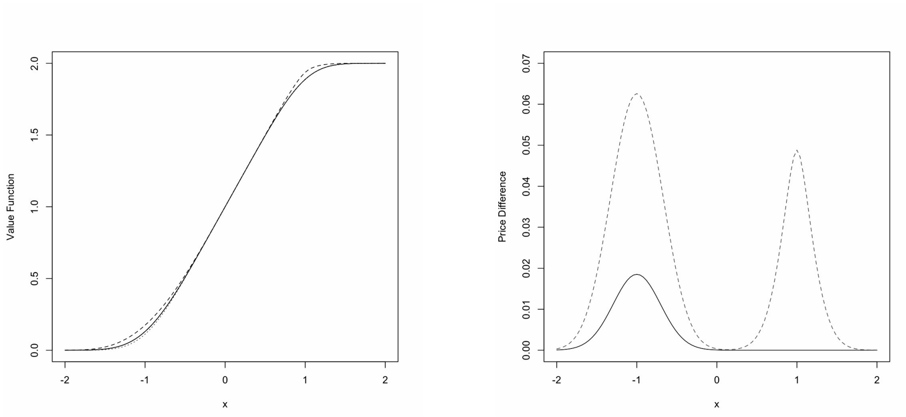

Fig. 2. *Left: Super-hedging price of the Call Spread option. Dashed line: $\lambda = 0.5$, $\bar{\gamma} = 1.75$; solid line: $\lambda = 0$, $\bar{\gamma} = 1.75$; dotted line: $\lambda = 0$, $\bar{\gamma} = +\infty$. Right: Difference with the price associated to $\lambda = 0$, $\bar{\gamma} = +\infty$. Dashed line: $\lambda = 0.5$, $\bar{\gamma} = 1.75$; solid line: $\lambda = 0$, $\bar{\gamma} = 1.75$.*

6. Appendix.

The following is very standard, we prove it for completeness.

Lemma 6.1. *A upper-semicontinuous (resp. lower-semicontinuous) map is a viscosity subsolution (resp. supersolution) of*

$$F^{\epsilon}_{\kappa}[\varphi]\mathbf{1}_{[0,T)} + (\varphi - \hat{g}^{\epsilon}_{K})\mathbf{1}_{\{T\}} = 0$$

if and only if it is a viscosity subsolution (resp. supersolution) of $F^{\epsilon,K}_{\kappa,-}[\varphi] = 0$ (resp. $F^{\epsilon,K}_{\kappa,+}[\varphi] = 0$).

Proof. The equivalence on $[0, T)$ is evident, we only consider the parabolic boundary $\{T\} \times \mathbb{R}$. Since $F^{\epsilon,K}_{\kappa,+} \geq F^{\epsilon}_{\kappa}$ and $F^{\epsilon,K}_{\kappa,-} \leq F^{\epsilon}_{\kappa}$, only one implication is not completely trivial.

a. Let v be a viscosity supersolution of $F^{\epsilon,K}_{\kappa,+}[\varphi] = 0$, and $\varphi \in C^2$ be a test function such that

$$(\text{strict}) \min_{[0,T]\times\mathbb{R}} (v - \varphi) = (v - \varphi)(T, x_0) = 0,$$

for some $x_0 \in \mathbb{R}$. We define a new test function $\phi \in C^2$,

$$\phi(t, x) := \varphi(t, x) - C(T - t),$$

so that $\partial_t \phi = \partial_t \varphi + C$. For $C > 0$ large enough,

$$\min_{x' \in D^{\epsilon}_{\kappa}} \min \left\{ -\partial_t \phi - \frac{\sigma^2(x')\partial_{xx}\phi}{2(1 - f(x')\partial_{xx}\phi)}, \bar{\gamma}(x') - \partial_{xx}\phi \right\} < 0$$

at (T, x_0). Since,

$$(\text{strict}) \min_{[0,T]\times\mathbb{R}} (v - \phi) = (v - \phi)(T, x_0) = 0,$$

it must hold that $F^{\epsilon,K}_{\kappa,+}[\phi](T, x_0) \geq 0$, and therefore

$$v(T, x_0) - \hat{g}^{\epsilon}_{K}(x_0) = \varphi(T, x_0) - \hat{g}^{\epsilon}_{K}(x_0) = \phi(T, x_0) - \hat{g}^{\epsilon}_{K}(x_0) \geq 0.$$

b. Let now v be a viscosity subsolution of $F^{\epsilon,K}_{\kappa,-}[\varphi]=0$, and $\varphi\in C^2$ be a test function such that

$$\text{(strict)}\max_{[0,T]\times\mathbb{R}}(v-\varphi)=(u-\varphi)(T,x_0),$$

for some $x_0\in\mathbb{R}$. Then, $F^{\epsilon,K}_{\kappa,-}[\varphi](T,x_0)\le 0$. By replacing φ by ϕ, defined for $\alpha>0$ as

$$\phi(t,x):=\varphi(t,x_0+\alpha(x-x_0))+C(T-t),$$

we obtain a new test function at (T,x_0). Since $\inf\bar\gamma>0$, recall (1), we can take α small enough so that

$$\min_{x'\in D^\epsilon_\kappa}\{\bar\gamma(x')-\partial_{xx}\phi(T,x_0)\}>0.$$

As in the previous step, we can now choose $C>0$ such that

$$\min_{x'\in D^\epsilon_\kappa}\left\{-\partial_t\phi-\frac{\sigma^2(x')\partial_{xx}\phi}{2(1-f(x')\partial_{xx}\phi)}\right\}>0$$

at (T,x_0). Since $F^{\epsilon,K}_{\kappa,-}[\phi](T,x_0)\le 0$, we conclude that $v(T,x_0)=\phi(T,x_0)\le\hat g^\epsilon_K(x_0)$. □

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