Dealing with market frictions: Some challenges for stochastic analysis and optimal control

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based on joint work with

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Frictions in financial modelling

- Classical Black-Scholes theory: dynamic trading of arbitrary amounts, arbitrarily fast without impact on (exogenously given) asset prices and without taxes, transaction fees, etc.
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- **Transaction fees:** often mixture of lump sum payments and (capped) proportional fees
  $\rightsquigarrow$ optimal control problems which received a lot of attention in recent years: asymptotic analysis for small fees, shadow prices, ...

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  $\rightsquigarrow$ stylized models of price impact to be considered first and then move to more sophisticated models

Price impact due to limited market liquidity

Kyle '85 identifies different notions to assess market liquidity:

Tightness: *The cost of turning around a position over a short period of time.*
Well captured by spread between bid- and ask-prices.

Depth: *The size of an order flow innovation required to change prices a given amount.*
Good proxy: volume available for trading on bid- and ask-side of limit order book.

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Liquidity model has to identify *relevant time-scale* for its purpose:

- high-frequency trading: limit order book model.
- option pricing, hedging, optimal investment: mesoscopic models like the ones of this course.

Course outline

This course will discuss:

- models with purely temporary price impact: [3], [4]
  - indifference pricing and asymptotics for small impact
  - quadratic hedging
  - (essentially) classical stochastic control
- tractable models with transient price impact: [1], [2]
  - super-replication duality
  - utility maximization
  - singular stochastic control
- equilibrium with frictions: [5]
- new approach to information modeling in optimal control problems: [6] (time permitting)

Outline

Temporary price impact

Consider an arbitrage-free stock price model:

$$P = (P_t)_{0 \leq t \leq T} \text{ with volatility } d\langle P \rangle_t = \sigma_t^2 dt$$

Temporary price impact when an investor changes her position X at speed $\dot{X}$ with market liquidity described by $\kappa > 0$:

$$P_t^\kappa = P_t + \dot{X}_t \kappa_t \quad (0 \leq t \leq T)$$

PnL at time T when starting and ending with a flat stock position:

$$V_T^\kappa(X) = -\int_0^T P_t^\kappa dX_t = \int_0^T X_t dP_t - \int_0^T \dot{X}_t^2 \kappa_t dt$$

How to invest optimally? How to price & hedge contingent claims?

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How to invest optimally? How to price & hedge contingent claims?

Asymptotic expansions about the frictionless case "$\kappa = 0$":

Guasoni, Weber ('15), Moreau, Muhle-Karbe, Soner ('15), ...

Heuristics for indifference price asymptotics

With ξ denoting the *frictionless* optimizer with endowment $-H+p$, we want to find for "small" κ an X which minimizes

$$\begin{aligned}
&\mathbb{E}u(V^0_T(\xi)-H+p)-\mathbb{E}u(V^\kappa_T(X)-H+p)\\
&\approx \mathbb{E}\Big[u'(V^0_T(\xi)-H+p)\ \underbrace{(\ V^0_T(\xi)\ -\ V^\kappa_T(X)\ )}_{=\int_0^T(\xi_t-X_t)dP_t+\int_0^T \dot{X}_t^2\kappa_t dt}\\
&\qquad +\frac{1}{2}u''(V^0_T(\xi)-H+p)(V^0_T(\xi)-V^\kappa_T(\xi))^2\Big]\\
&\approx \mathbb{E}^0\left[\int_0^T \dot{X}_t^2\kappa_t dt+\frac{1}{2}\alpha\int_0^T(\xi_t-X_t)^2 d\langle P\rangle_t\right]+\dots
\end{aligned}$$

where $d\mathbb{P}^0/d\mathbb{P}\propto u'(V^0_T(\xi)-H+p)$ is the density of a martingale measure for P and where $u(x)=-\exp(-\alpha x)$.

$\rightsquigarrow$ simplified quadratic optimization problem!

Quadratic tracking problem

Mathematical optimization problem

For a given predictable ξ and given $x \in \mathbb{R}$, find an absolutely continuous, adapted process $X_t = x + \int_0^t u_s ds$ with $u \in L^2(\mathbb{P} \otimes \kappa_s ds)$, which minimizes

$$J(u) \triangleq \mathbb{E}\left[\int_0^T (\xi_t - X_t)^2 \sigma_t^2 dt + \int_0^T u_t^2 \kappa_t dt\right]$$

for given progressively measurable, strictly positive processes σ, κ.

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for given progressively measurable, strictly positive processes σ, κ.
Possible additional constraint on terminal position:

$$X_T = \xi_T \text{ for some given } \xi_T \in \mathscr{F}_T.$$

Closely related references from Mathematical Finance

Rogers & Singh (2010) , Naujokat & Westray (2011), Frei & Westray (2013), Schied (2013), Horst & Naujokat (2014), Almgren & Li (2014), Cartea & Jaimungal (2015), Cai et al. (2015, 2016), . . .

Constant coefficients in the unconstrained case

Theorem

If σ and κ are constant and there is no constraint on the terminal position, it is optimal to always trade towards

$$\hat{\xi}_t = \mathbb{E}\left[\int_t^T \xi_s \frac{\cosh(\frac{T-s}{\sqrt{\lambda}})}{\sinh(\frac{T-t}{\sqrt{\lambda}})\sqrt{\lambda}} ds \,\middle|\, \mathscr{F}_t \right]$$

according to

$$dX_t^* = \frac{1}{\sqrt{\lambda}} \tanh(\frac{T-t}{\sqrt{\lambda}}) \left(\hat{\xi}_t - X_t^* \right) dt$$

where $\lambda \triangleq \kappa/\sigma^2$.

Rather than towards the current target ξ_t, one should trade towards its expected future $\hat{\xi}_t$; cf. Garleanu & Pedersen (2014).

Constant coefficients in the constrained case

Theorem

If σ and κ are constant and the terminal position has to be $X^*_T = \xi_T \in L^2(\mathbb{P})$, it is optimal to always trade towards

$$\hat{\xi}_t = \frac{1}{\cosh(\frac{T-t}{\sqrt{\lambda}})} \mathbb{E}\left[\xi_T \,|\, \mathscr{F}_t\right] + \left(1 - \frac{1}{\cosh(\frac{T-t}{\sqrt{\lambda}})}\right) \mathbb{E}\left[\int_t^T \xi_s \frac{\sinh(\frac{T-s}{\sqrt{\lambda}})}{(\cosh(\frac{T-t}{\sqrt{\lambda}}) - 1)\sqrt{\lambda}} \,\middle|\, \mathscr{F}_t\right]$$

according to

$$dX^*_t = \frac{1}{\sqrt{\lambda}} \coth(\frac{T-t}{\sqrt{\lambda}}) \left(\hat{\xi}_t - X^*_t\right) dt$$

where $\lambda \triangleq \kappa/\sigma^2$.

As $t \uparrow T$ we have to trade towards $\hat{\xi}$ (and thus towards ξ_T) with higher and higher urgency.

Illustration: Frictionless hedge with jump midway

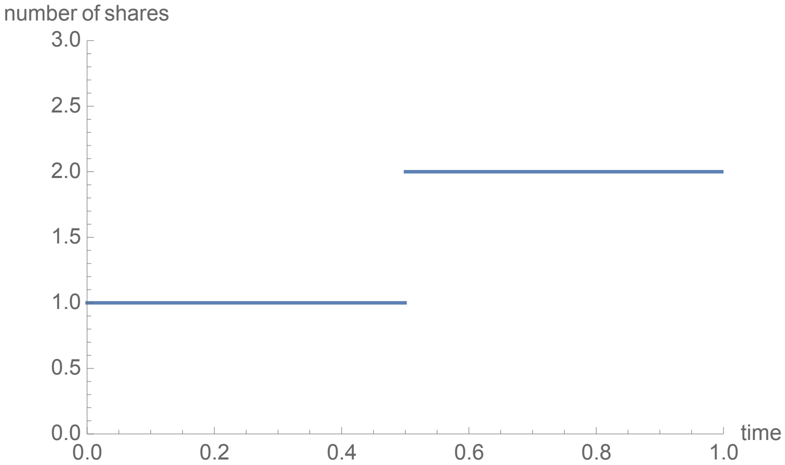

Figure: Target strategy ξ with a jump at $t = T/2$ (blue)

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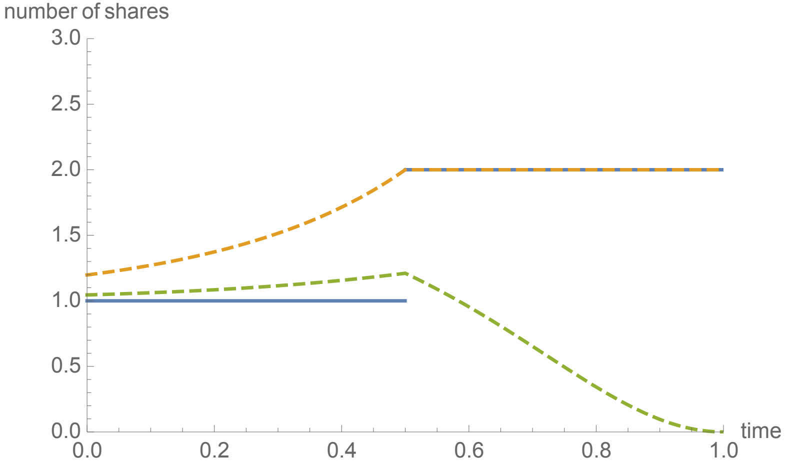

Figure: Target strategy ξ with a jump at $t = T/2$ (blue), unconstrained (orange, dashed) and constrained (green, dashed) target

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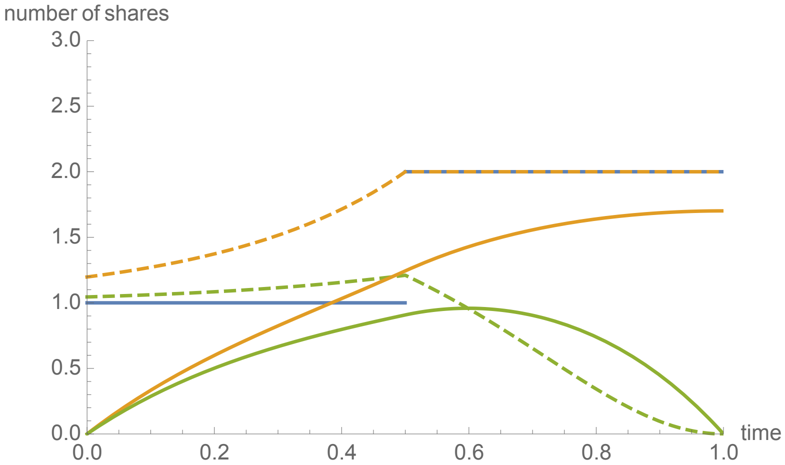

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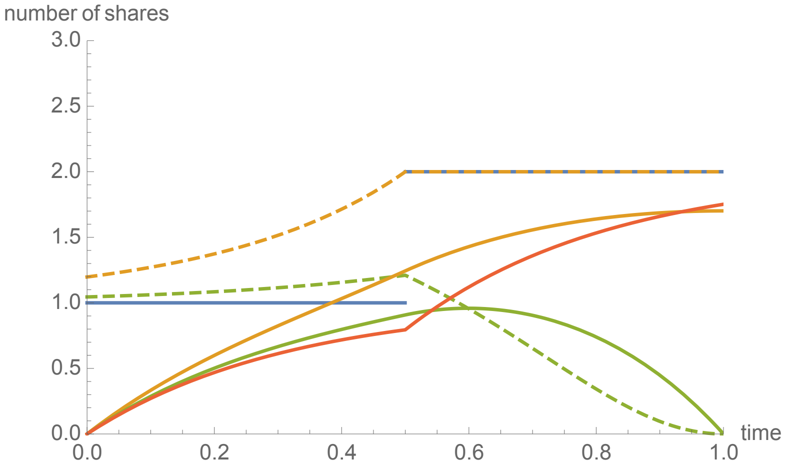

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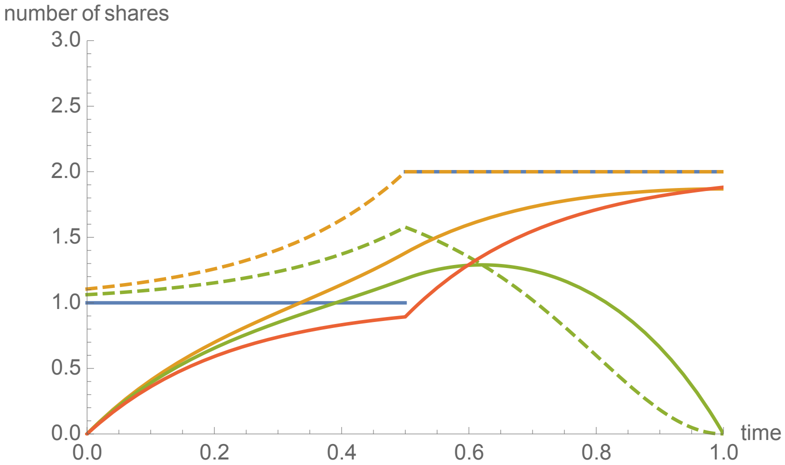

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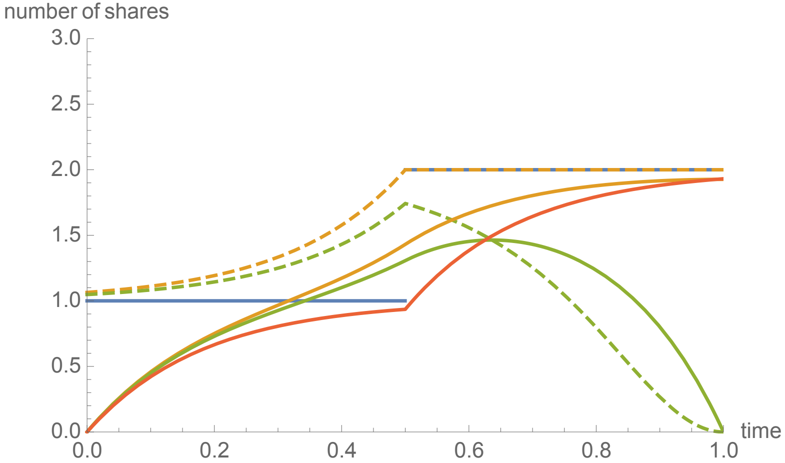

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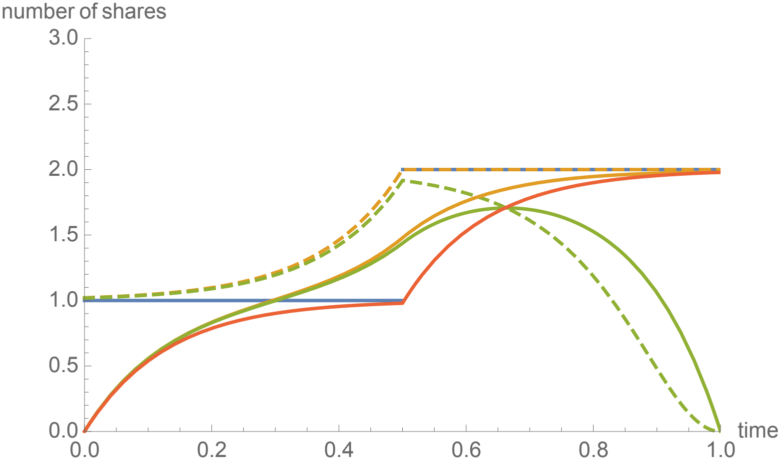

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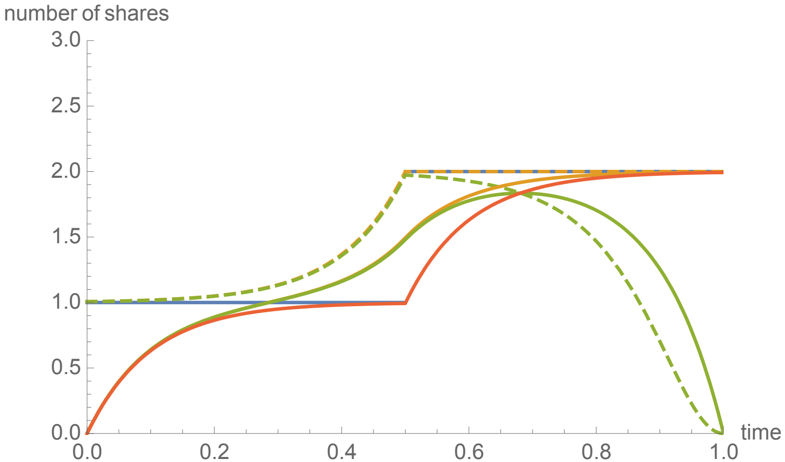

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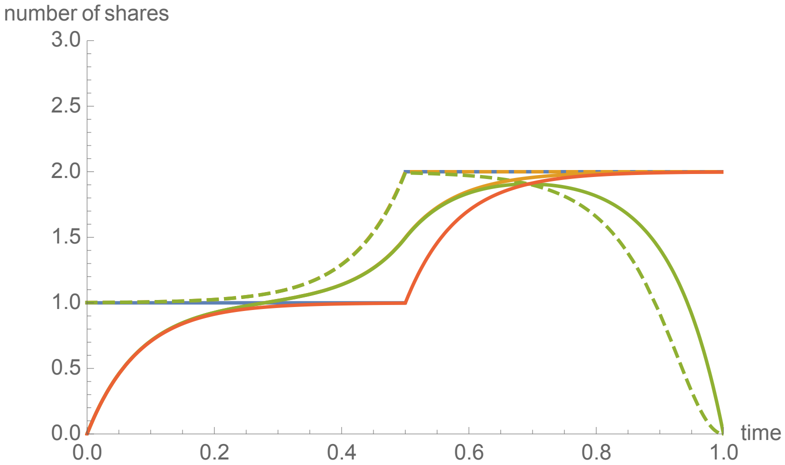

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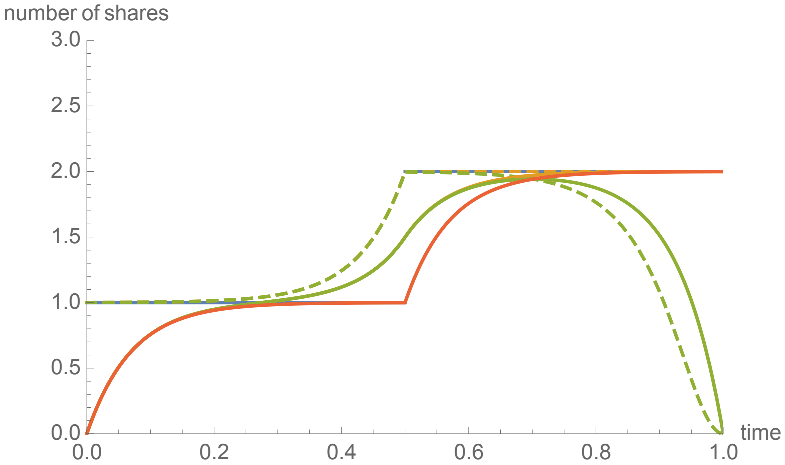

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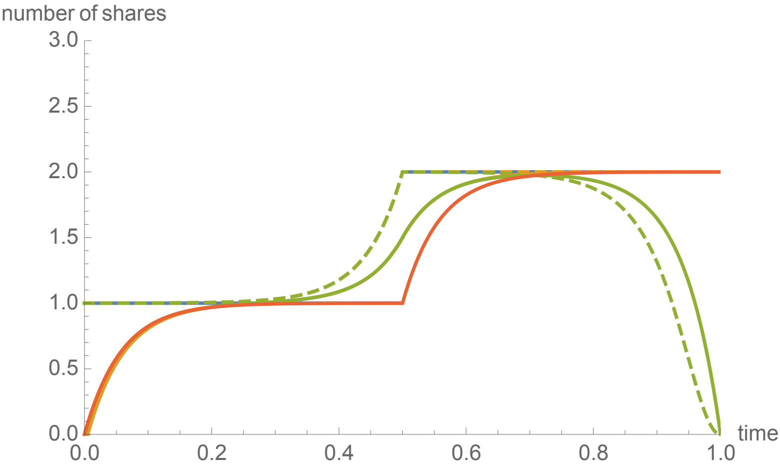

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Illustration: Discretely monitored Asian option

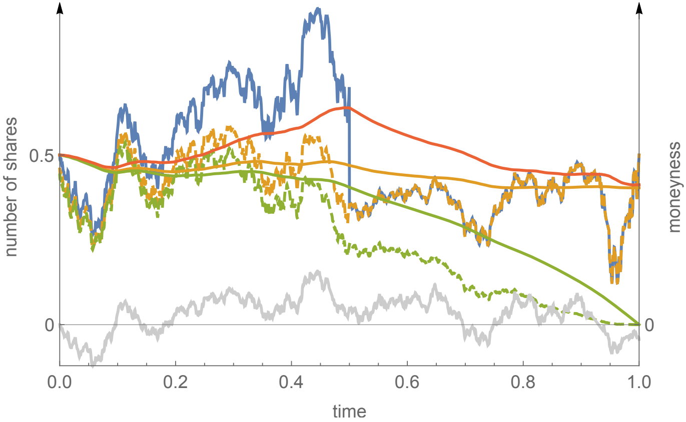


Figure: Target strategy ξ of "Asian option" $(\frac{1}{2}(S_{T/2} + S_T) - K)^+$ (blue), unconstrained (orange, dashed) and constrained (green, dashed) target, corresponding unconstrained (orange) and constrained (green) frictional hedge, and directly targeting strategy (red)

Illustration: Call option with physical delivery

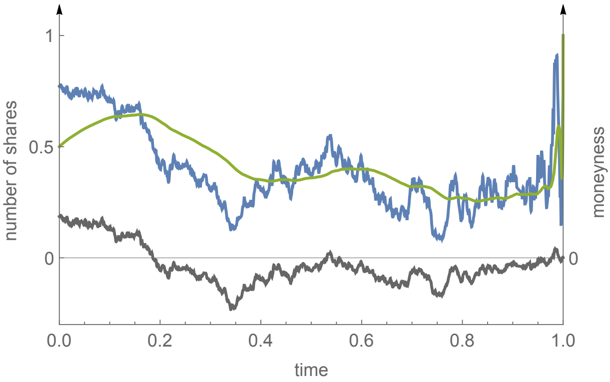
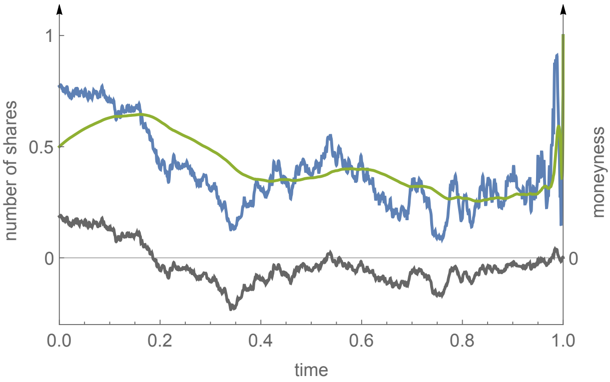


Illustration: Call option with physical delivery ???



Lemma

A terminal position ξ_T can be attained at finite expected costs if and only if it becomes known sufficiently fast towards the end:

$$\int_0^T \frac{\mathbb{E}[(\xi_T - \mathbb{E}[\xi_T \mid \mathscr{F}_t])^2]}{(T-t)^2} dt < \infty.$$

General case with stochastic coefficients

For a given predictable target strategy ξ, a given terminal position ξ_T and a given initial position $x \in \mathbb{R}$, find an absolutely continuous, adapted process $X = x + \int_0^{\cdot} u_t dt$ which minimizes

$$\mathbb{E}\left[\int_0^T (\xi_t - X_t)^2 \sigma_t^2 dt + \int_0^T u_t^2 \kappa_t dt + \eta(\xi_T - X_T)^2\right]$$

with σ, κ progressively measurable, strictly positive, bounded processes, nonnegative $\eta \in \mathscr{F}_T$.

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with σ, κ progressively measurable, strictly positive, bounded processes, nonnegative $\eta \in \mathscr{F}_T$.

Also allow for $\eta = +\infty$ with positive probability:

- $\rightsquigarrow$ imposes implicitly the terminal state constraint $X_T = \xi_T$ on $\{\eta = +\infty\}$ (constrained problem)
- $\rightsquigarrow$ we have to be careful with $\eta(\xi_T - X_T)^2$ if $\eta = \infty$ and $\xi_T = X_T$: "truncation in space" vs. "truncation in time".

Bounded penalization

Kohlmann and Tang (2002): For $\eta \geq 0$ bounded, construct optimal control $u^* = (b - cX^*)/\kappa$ from solutions to BSRDE

$$dc_t = \left(\frac{c_t^2}{\kappa_t} - \sigma_t^2 \right) dt - dM_t \quad (0 \leq t \leq T), \quad c_T = \eta,$$

and linear BSDE

$$db_t = \left(\frac{c_t}{\kappa_t} b_t - \sigma_t^2 \xi_t \right) dt + dM'_t \quad (0 \leq t \leq T), \quad b_T = \eta \xi_T.$$

Problem: How to make sense of this when $\mathbb{P}[\eta = +\infty] > 0$?

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Idea: Use signal process $\hat{\xi}$ for "consistent truncation in time"!

General result

Suppose:

- **integrable coefficients**: $\int_0^T (\sigma_t^2 + \kappa_t^{-1}) dt < \infty$ a.s.
- **effective time horizon is indeed** T, i.e. penalization for deviations from targets remains conceivable throughout:

$$\mathbb{P}\left[\eta = 0, \int_t^T \sigma_s^2\, ds = 0 \,\middle|\, \mathscr{F}_t\right] < 1, \quad \text{for all } t < T$$

- **supersolution for BSRDE**: consider semimartingale $c = (c_t)_{0 \le t < T} > 0$ with dynamics

$$dc_t = \left(\frac{c_t^2}{\kappa_t} - \sigma_t^2\right) dt - dM_t \quad (0 \le t < T), \quad \liminf_{t \uparrow T} c_t \ge \eta$$

such that $(M_t)_{t<T}$ is a martingale and

$$\int_{[0,T)} \frac{d[c]_t}{c_{t-}^2} < \infty \text{ on } \{\eta = +\infty\}.$$

- **integrable targets**: $\xi_t \in L^1(\mathbb{P} \otimes \sigma_t^2 dt)$, $\xi_T L_T^c \in L^1(\mathbb{P})$

General result (ctd)

Then $L_t^c \triangleq c_t e^{-\int_0^t \frac{c_u}{\kappa_u} du} \geq 0$ is a supermartingale and ...

- ... the signal process

$$\hat{\xi}_t^c \triangleq \frac{1}{L_t^c} \mathbb{E}\left[\xi_T L_T^c + \int_t^T \xi_r e^{-\int_0^r \frac{c_u}{\kappa_u} du} \sigma_r^2 dr \,\middle|\, \mathscr{F}_t \right] \quad (0 \leq t < T)$$

is well defined and satisfies

$$\lim_{t \uparrow T} \hat{\xi}_t^c = \Xi_T \text{ on } \{L_T^c > 0\} \supset \{\eta > 0\},$$

- ... the target functional with "truncation in time"

$$J^c(u) \triangleq \limsup_{\tau \uparrow T} \mathbb{E}\left[\int_0^\tau (X_t^u - \xi_t)^2 \sigma_t^2 dt + \int_0^\tau \kappa_t u_t^2 dt + c_\tau (X_\tau^u - \hat{\xi}_\tau^c)^2 \right]$$

dominates J: $J(u) \leq J^c(u)$. Its domain $\{u \mid J^c(u) < \infty\}$ is nonempty iff

$$\mathbb{E}\left[\int_0^T (\hat{\xi}_t^c)^2 \sigma_t^2 dt \right] < +\infty \quad \text{and} \quad \mathbb{E}\left[\int_{[0,T)} c_t d[\hat{\xi}^c]_t \right] < +\infty,$$

General result (ctd)

- ...if $\{u \mid J^c(u) < \infty\} \neq \emptyset$, the optimal control u^c can be described in feedback form as

$$u_t^c = \frac{c_t}{\kappa_t}(\hat{\xi}_t^c - X_t^{u^c}), \quad 0 \leq t < T,$$

- ...the minimal costs decompose as

$$J(u^c) = c_0(x - \hat{\xi}_0^c)^2 + \mathbb{E}\left[\int_0^T (\xi_t - \hat{\xi}_t^c)^2 \sigma_t^2 dt\right] + \mathbb{E}\left[\int_{[0,T)} c_t d[\hat{\xi}^c]_t\right]$$

into costs due to suboptimal starting position, to the (lack of) regularity and compatibility of the targets ξ, ξ_T, and to the signal's variability given new information on problem data.

Key insights for proof

A lengthy calculation reveals that

$$\begin{aligned}&\int_0^\tau (X_t^u-\xi_t)^2\sigma_t^2 dt+\int_0^\tau \kappa_t u_t^2 dt + c_\tau (X_\tau^u-\hat{\xi}_\tau^c)^2\\ =&c_0(x-\hat{\xi}_0^c)^2+\int_0^\tau (\xi_t-\hat{\xi}_t^c)^2\sigma_t^2 dt+\int_0^\tau c_t d[\hat{\xi}^c]_t\\ &\quad +\int_0^\tau \left(u_t-\frac{c_t}{\kappa_t}\left(\hat{\xi}_t^c-X_t^u\right)\right)^2\kappa_t dt+\text{local martingale}_\tau\,.\end{aligned}$$

$\rightsquigarrow$ Consistency of optimization problems with different time horizons τ: same feedback policy optimal for all $\tau < T$

Letting $\tau \uparrow T$ and taking expectations reveals optimality of given u^c along with necessary and sufficient conditions for $\{u \mid J^c(u) < \infty\} \neq \emptyset$.

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A lengthy calculation reveals that

$$\int_0^\tau (X_t^u - \xi_t)^2 \sigma_t^2 dt + \int_0^\tau \kappa_t u_t^2 dt + c_\tau (X_\tau^u - \hat{\xi}_\tau^c)^2$$
$$= c_0 (x - \hat{\xi}_0^c)^2 + \int_0^\tau (\xi_t - \hat{\xi}_t^c)^2 \sigma_t^2 dt + \int_0^\tau c_t d[\hat{\xi}^c]_t$$
$$+ \int_0^\tau \left(u_t - \frac{c_t}{\kappa_t} \left(\hat{\xi}_t^c - X_t^u \right) \right)^2 \kappa_t dt + \text{local martingale}_\tau \,.$$

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Conjecture:

$\operatorname{argmin} J = \operatorname{argmin} J^{c_{\min}}$ for *minimal* supersolution $c_{\min}$ of BSRDE.

References

- B., H. M. Soner, M. Voß, Hedging with Temporary Price Impact. Mathematics and Financial Economics, 11(2), (2017), 215-239
- B., M. Voß, Linear quadratic stochastic control problems with singular stochastic terminal constraint. SIAM J. on Control and Optimization.

Conclusions

- quadratic hedging with quadratic transaction costs from temporary price impact
- explicit solution for constant coefficients: trade towards expected average future position of suitable frictionless optimum
- . . . possibly combined with weighted expectation of ultimate target position
- characterization of ultimate positions which are attainable with finite expected costs
- closed-form hedging recipes also for frictionless reference hedges which have singularities
- very general optimal control with stochastic coefficients solved in terms of (singular) backward stochastic Riccati equation under minimal assumptions
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Thank you very much!

Outline

Models with temporary price impact

Models with transient price impact

Equilibrium with market frictions

Modeling information flow in stochastic control

Continuous-time model with transient price impact

- no interest; unaffected, "fundamental" asset price: continuous adapted process $P = (P_t)$
- investment strategy of large investor: number of assets held $X_t = x_0 + X_t^{\uparrow} - X_t^{\downarrow},\ t \geq 0, \quad X_{0-} = x_0, \quad X_{0-}^{\uparrow} = X_{0-}^{\downarrow} = 0,$ right-continuous, adapted, of bounded total variation
- permanent impact on midquote price: $P_t^X = P_t + \iota X_t,\ t \geq 0, \quad P_{0-}^X = P_0 + \iota x_0$ (cf. Huberman-Stanzl '04)
- half-spread: $d\zeta_t^X = \frac{1}{\delta_t}(dX_t^{\uparrow} + dX_t^{\downarrow}) - r_t \zeta_t^X dt, \quad \zeta_{0-}^X = \zeta_0 \geq 0$
  - market depth: $\delta = (\delta_t)$ continuous adapted, bounded away from 0 and ∞.
  - resilience rate: $r = (r_t) \geq 0$ predictable, $\int_0^T r_t dt$ bounded
- bid-price: $P_t^X - \zeta_t^X$; ask-price: $P_t^X + \zeta_t^X$ (cf. Roch-Soner '13)

Wealth dynamics

Holding X_t assets at time $t \in [0, T]$ yields ultimate cash position:

$$V_T^X = v_0 - \int_{[0,T]} P_t^X \circ dX_t - \int_{[0,T]} \zeta_t^X \circ d(X_t^\uparrow + X_t^\downarrow).$$

where $\int_{[0,T]} Y_t \circ dX_t = \int_{[0,T]} \frac{1}{2}(Y_{t-} + Y_{t+}) dX_t$
(Stratonovich/Marcus-integral; cf. Becherer et al. '17)

Crucial observation: For $X \in \mathscr{X}$ with $X_T = 0$,

$$V_T^X = v_0 + \frac{1}{2}(\iota x_0^2 + \delta_0 \zeta_0^2) - \int_0^T P_t dX_t - \frac{1}{2}\int_0^T (\rho_t \zeta_t^X)^2 |d\kappa_t|$$

if $\kappa_t = \delta_t / \rho_t^2 1_{[0,T)}(t)$ is strictly decreasing (assumed henceforth)
with $\rho_t = \exp\left(\int_0^t r_s \, ds\right)$.
$\rightsquigarrow$ convex transaction costs, convex analytic methods apply

Super-replication of non-manipulable claims

Consider contingent claim with $\mathscr{F}_T$-measurable payoff $H \geq 0$.
Super-replication costs:

$$\pi(H) \triangleq \inf\{v_0 \in \mathbb{R} \;:\; V_T^X \geq H \text{ for some } X \in \mathscr{X} \text{ with } X_T = 0\}$$

$\rightsquigarrow$ convex functional of H

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$\rightsquigarrow$ convex functional of H

Question: What is its dual description?

Notice: payoff of H *not* affected by strategy X; this is no issue if $H = h(P_T^X)$ because $P_T^X = P_T$ when $X_T = 0$.

(See Frey '96, Bouchard et al '17, Becherer-Bilarev '17 for PDE-approach with manipulable claims.)

Super-replication duality

Additional technical assumption:

All $(\mathscr{F}_t)$-martingales have a continuous version.

Theorem

The super-replication costs of a contingent claim $H \geq 0$ have the dual description

$$\pi(H) = \sup_{(\mathbb{Q},M,\alpha)} \left\{ \mathbb{E}_{\mathbb{Q}}[H] - \frac{1}{2}\|\alpha - \zeta_0\|^2_{L^2(\mathbb{Q}\otimes|d\kappa|)} - M_0 x_0 - \frac{1}{2}\iota x_0^2 \right\} > -\infty$$

where the supremum is taken over all triples $(\mathbb{Q}, M, \alpha)$ of probability measures $\mathbb{Q} \ll \mathbb{P}$ on $\mathscr{F}_T$, martingales $M \in \mathscr{M}^2(\mathbb{Q})$ and all optional $\alpha \in L^2(\mathbb{Q} \otimes |d\kappa|)$ which control the fluctuations of P in the sense that

$$|P_t - M_t| \leq \frac{\rho_t}{\delta_t} \mathbb{E}_{\mathbb{Q}} \left[\int_{[t,T]} \alpha_u \, |d\kappa_u| \middle| \mathscr{F}_t \right], \quad 0 \leq t \leq T.$$

Connections with other duality formulae

$$\pi(H) = \sup_{(\mathbb{Q},M,\alpha)} \left\{ \mathbb{E}_{\mathbb{Q}}[H] - \frac{1}{2}\|\alpha - \zeta_0\|^2_{L^2(\mathbb{Q}\otimes|d\kappa|)} - M_0 x_0 - \frac{1}{2}\iota x_0^2 \right\} > -\infty$$

subject to $|P_t - M_t| \leq \frac{\rho_t}{\delta_t}\mathbb{E}_{\mathbb{Q}}\left[\int_{[t,T]} \alpha_u \, |d\kappa_u| \middle| \mathscr{F}_t\right], \quad 0 \leq t \leq T.$

- If $\mathbb{Q} \ll \mathbb{P}$ martingale measure for P, $\pi(H) \geq \mathbb{E}_{\mathbb{Q}}[H]$.
- Classical transaction cost models correspond roughly to $r = 0$ and $\delta = \infty$, without permanent impact: $\iota = 0$. Then we have a constant spread $\zeta^X \equiv \zeta_0$ and can choose $\alpha \equiv \zeta_0$ for any *consistent price system*, ($\mathbb{Q} \ll \mathbb{P}$ with $M \in \mathscr{M}^2(\mathbb{Q})$) to ensure closeness constraint and thus obtain $\pi(H) \geq \mathbb{E}_{\mathbb{Q}}[H]$.
- Classical transaction cost models *not* a special case, though, since they require admissibility notion because of linear scaling. **No** notion of admissibility required for our "quadratic" price impact model.
- With temporary transaction costs $\int_0^T G_t(\dot{X}_t)dt$ for suitable convex G_t (like $G_t(x) = x^2$), Dolinsky-Soner '13 and Guasoni-Rasonyi'15 also get convex risk measure description.

Proof

lower bound: not too hard given convex form of wealth dynamics
upper bound:

- construct $\hat{\mathbb{Q}}$ by separation argument: standard because of convex wealth dynamics; need to understand

$$\inf_{X \in \mathscr{X}^2,\, X_T = 0} \mathbb{E}_{\hat{\mathbb{Q}}} \left[\int_0^T P_t dX_t + \frac{1}{2} \int_0^T (\rho_t \zeta_t^X)^2 |d\kappa_t| \right] = ?$$

- construct $\hat{M}$ as a Lagrange multiplier for $X_T = 0$; solve

$$\inf_{X \in \mathscr{X} \text{ bdd.}} \mathbb{E}_{\hat{\mathbb{Q}}} \left[\int_0^T (P_t - \hat{M}_t) dX_t - \hat{M}_0 x_0 + \frac{1}{2} \int_0^T (\rho_t \zeta_t^X)^2 |d\kappa_t| \right] = ?$$

- construct $\hat{\alpha}$ for which $\dots = -\frac{1}{2} \|\hat{\alpha} - \zeta_0\|^2_{L^2(\hat{\mathbb{Q}} \otimes |d\kappa|)} - \hat{M}_0 x_0 + \frac{1}{2} \zeta_0^2 \delta_0$: need continuity of filtration for representation theorem in B. & El Karoui '04

Optimal investment

Question: How to determine optimal investment strategies with transient price impact?

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Corollary

Consider a strictly concave, increasing and differentiable utility function u *for which* $\sup_{X \in \mathscr{X},\, X_T = 0} \mathbb{E}[u(V_T^X) \vee 0] < \infty$. *Suppose* $\hat{X} \in \mathscr{X}$ *with* $\hat{X}_T = 0$ *yields via* $\frac{d\hat{\mathbb{Q}}}{d\mathbb{P}} = \frac{u'(V_T^{\hat{X}})}{\mathbb{E}[u'(V_T^{\hat{X}})]}$ *a probability measure* $\hat{\mathbb{Q}} \ll \mathbb{P}$ *with a shadow price* $\hat{M}$ *for spread dynamics*

$$\hat{\lambda}_t = \frac{\rho_t}{\delta_t} \mathbb{E}_{\hat{\mathbb{Q}}} \left[\int_{[t,T]} \hat{\alpha}_u \mu(du) \middle| \mathscr{F}_t \right], \quad 0 \leq t \leq T,$$

with $\hat{\alpha} = \rho \zeta^{\hat{X}} \in L^2(\hat{\mathbb{Q}} \otimes \mu)$.
Then $\hat{X}$ *yields the highest expected utility* $\mathbb{E}[u(V_T^X)]$ *among all strategies* $X \in \mathscr{X}$ *with* $X_T = 0$.

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Then $\hat{X}$ *yields the highest expected utility* $\mathbb{E}[u(V_T^X)]$ *among all strategies* $X \in \mathscr{X}$ *with* $X_T = 0$.

$\rightsquigarrow$ *not* constructive, but suitable for verification of optimality

Exponential utility maximization in Bachelier model with constant transient impact

Choose:

- utility function: $u(x) = -\exp(-\alpha x)$
- Bachelier reference model: $P_t = \mu t + \sigma B_t$
- constant coefficients: $r_t \equiv r \geq 0$, $\delta_t \equiv \delta > 0$; $\iota = 1/\delta$

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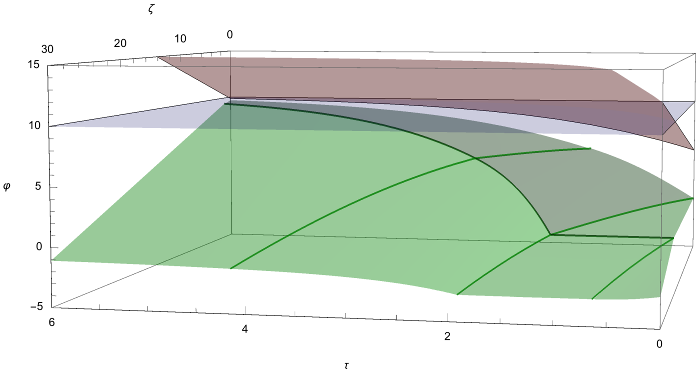

Figure: Buying region below green surface, selling region above red surface, holding region in between.

Illustration: Phenomenology of optimal trading trajectories

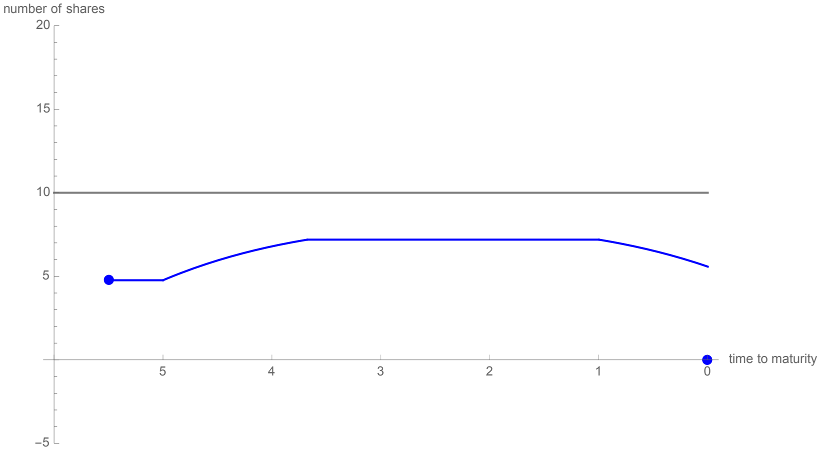

Figure: After waiting for spread to recover, one buys towards Merton's optimal holdings (grey) and holds this position before unwinding it as time for investing elapses (blue).

Illustration: Phenomenology of optimal trading trajectories

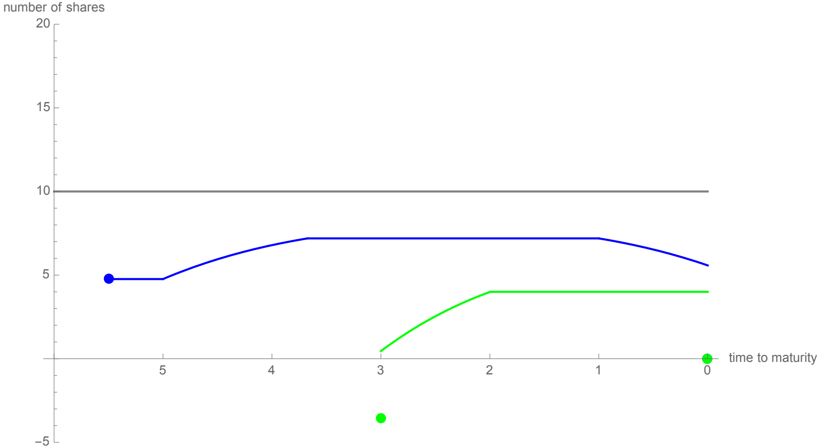

Figure: When the trading period starts with a "small" spread, one should do an initial block trade and then gradually build up a position which is held until liquidation at the end.

Illustration: Phenomenology of optimal trading trajectories

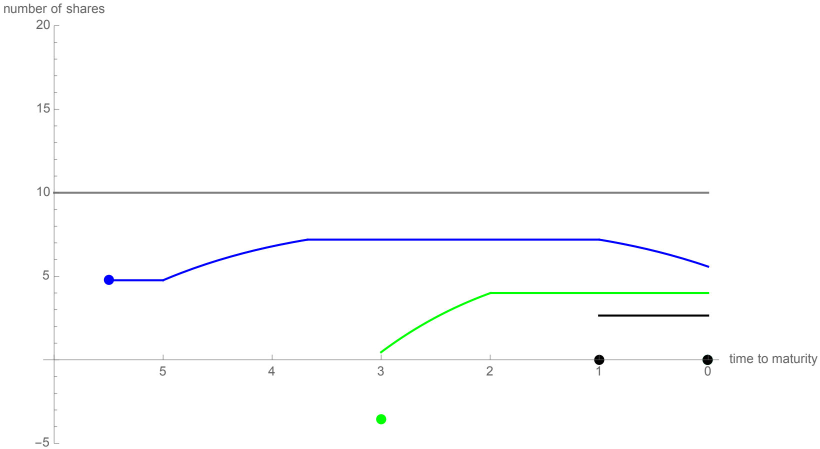

Figure: But, maybe somewhat counterintuitively, it may also be optimal to do an initial block trade, hold the position and then liquidate everything with a final block trade.

Illustration: Phenomenology of optimal trading trajectories

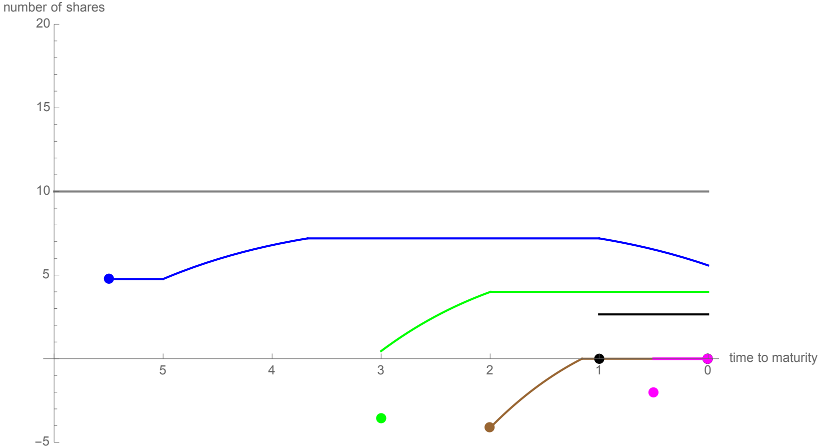

Figure: Starting with a short position, it may be optimal to clear this position and go away then, even though there is still time for a more moderate unwinding of short position or for even holding stock.

Illustration: Phenomenology of optimal trading trajectories

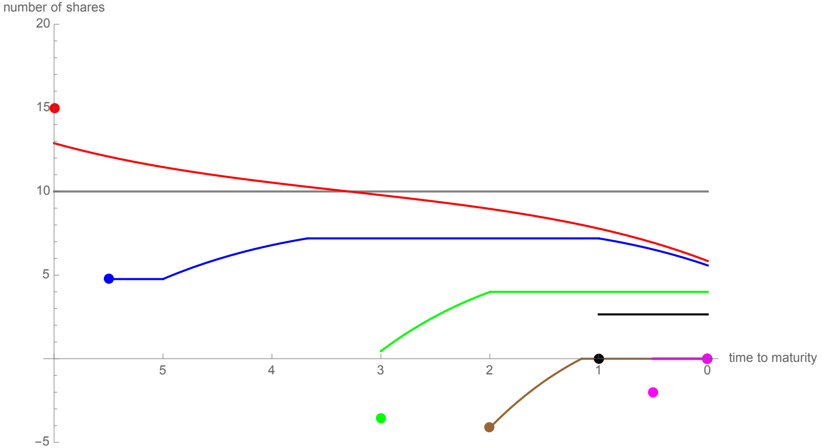

Figure: Starting with a long position beyond the Merton position, an initial block sell is followed by a smooth unwinding at varying speed depending on time to go; cf. Obizhaeva & Wang (2013).

Illustration: Phenomenology of optimal trading trajectories

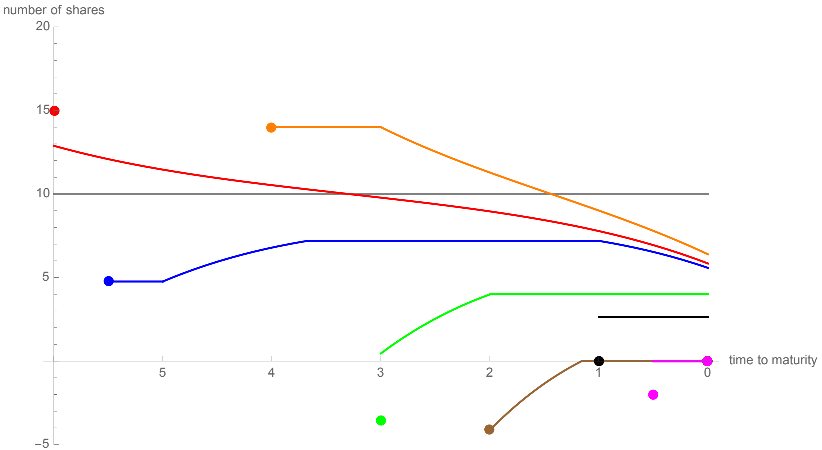

Figure: When the initial spread is very large, a starting position beyond the Merton position will be unwound only after an initial waiting period.

Illustration: Trading trajectories embedded in state space

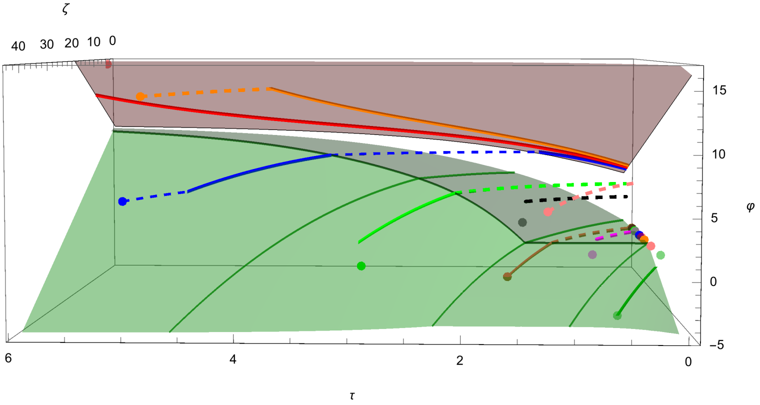

Figure: The different optimal trading trajectories as they move through the buying region, the selling region, and the holding region in state space; dashed lines indicated holding periods.

Papers for this talk

- with **Yan Dolinsky**:
  Super-replication with Transient Price Impact,
  to appear in The Annals of Applied Probability
  arXiv:1808.09807

  Scaling Limits for Super–replication with Transient Price Impact
  submitted
  arXiv:1810.07832
- with **Moritz Voss**:
  Optimal Investment with Transient Price Impact
  SIAM J. Finan. Math. 10-3 (2019), pp. 723-768
  https://doi.org/10.1137/18M1182267

Conclusion and Outlook

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  What if convexity condition on depth/resiliency fails?
  cf. B. & Fruth '14 for order execution result then

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Equilibrium with market frictions

Modeling information flow in stochastic control

Dramatis personae

- **Dealers**: in perfect competition for their clients' business; can manage their inventory risk by trading with "end-users" at fundamental prices, but they incur search costs
- **Clients**: demand immediacy for their trades from dealers; no direct access to "end-user" market
- **"End-users"**: accept positions at exogenous, fundamental prices; dealers can only find them incurring search costs

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- **"End-users"**: accept positions at exogenous, fundamental prices; dealers can only find them incurring search costs

Questions:

- How do the **dealers**' prices match demand with supply? How are they related to fundamentals? What role is played by the dealers' search costs and holding costs?
- How should **clients** choose their demand to manage their exogenously given risk? What if they internalize their impact? Do they benefit from the dealers' presence?

The dealers' problem

For dealer market prices (S_t) and fundamental prices (P_t), the dealers servicing their clients' requested positions (K_t) and cumulatively transferring $U_t = \int_0^t u_s \, ds$ to the end-users at costs $\frac{\lambda}{2} u_t^2 dt$ in $t \in [0, T]$, will generate proceeds

$$\int_0^T (-K_t) dS_t - (P_T - S_T) K_T + \int_0^T U_t dV_t - \frac{\lambda}{2} \int_0^T u_t^2 \, dt.$$

Assuming P is a martingale, i.e., ruling out speculation by the dealers, we get the **dealers' expected proceeds** to be

$$\mathbb{E}\left[\int_0^T (-K_t) dS_t - (P_T - S_T) K_T - \frac{\lambda}{2} \int_0^T u_t^2 \, dt\right].$$

The **dealers' inventory risk** is determined by $U - K$:

$$\frac{1}{2}\mathbb{E}\left[\int_0^T (K_t - U_t)^2 \, dt\right]$$

The dealers' problem

Dealers' target functional with holding costs $\gamma_d > 0$:

$$J_d(K, u; S) \triangleq \mathbb{E}\left[\int_0^T (-K_t) dS_t - (P_T - S_T) K_T - \frac{\lambda}{2} \int_0^T u_t^2 \, dt\right] - \frac{\gamma_d}{2} \mathbb{E}\left[\int_0^T (K_t - U_t)^2 \, dt\right] \to \max_{K,u}$$

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Observe: Problem can be addressed in two stages.

Stage 1: *Given* K, maximization over u is a quadratic tracking problem

$$\mathbb{E}\left[\frac{\gamma_d}{2}\int_0^T (K_t - U_t)^2\, dt + \frac{\lambda}{2}\int_0^T u_t^2\, dt\right] \to \min_u$$

as solved explicitly in B., Soner, Voß'17.

Stage 2: *Given* the optimal transfer policy u^K for *any* K, optimize over K.

Quadratic tracking problem

Theorem (B., Soner, Voß'17)

The dealers' optimal trading rate minimizing

$$\mathbb{E}\left[\frac{\gamma_d}{2}\int_0^T (K_t - U_t)^2\,dt + \frac{\lambda}{2}\int_0^T u_t^2\,dt\right]$$

is

$$u_t^K \triangleq \frac{d}{dt}U_t^K = \frac{\tanh((T-t)/\sqrt{\kappa})}{\sqrt{\kappa}}(\hat{K}_t - U_t^K)$$

where

$$\kappa \triangleq \lambda/\gamma_d \text{ and } \hat{K}_t \triangleq \mathbb{E}\left[\int_t^T K_u \frac{\cosh((T-u)/\sqrt{\kappa})}{\sqrt{\kappa}\sinh((T-t)/\sqrt{\kappa})}\,du \,\middle|\, \mathscr{F}_t\right]$$

$\rightsquigarrow$ Dealers form a view $\hat{K}$ on expected future demand and trade with the end-users towards this ideal position.

Back to our equilibrium considerations . . .

Back to our equilibrium considerations . . .

Stage 2: Dealers' target functional with holding costs $\gamma_d > 0$:

$$
\begin{aligned}
J_d(K; S) \triangleq & \mathbb{E}\left[\int_0^T (-K_t) dS_t - (P_T - S_T) K_T\right] \\
& - \mathbb{E}\left[\frac{\gamma_d}{2} \int_0^T (K_t - U_t^K)^2 \, dt + \frac{\lambda}{2} \int_0^T (u_t^K)^2 \, dt\right] \to \max_K
\end{aligned}
$$

Back to our equilibrium considerations ...

Stage 2: Dealers' target functional with holding costs $\gamma_d > 0$:

$$J_d(K;S) \triangleq \mathbb{E}\left[\int_0^T (-K_t)dS_t - (P_T - S_T)K_T\right] - \mathbb{E}\left[\frac{\gamma_d}{2}\int_0^T (K_t - U_t^K)^2\,dt + \frac{\lambda}{2}\int_0^T (u_t^K)^2\,dt\right] \to \max_K$$

Dealer market prices (S_t) will generate an **equilibrium** if at these quotes the dealers' optimal supply matches their clients' demand:

$$\mathscr{K} \in \arg\max_K J_d(K;S)$$

Back to our equilibrium considerations ...

Stage 2: Dealers' target functional with holding costs $\gamma_d > 0$:

$$J_d(K;S) \triangleq \mathbb{E}\left[\int_0^T (-K_t)dS_t - (P_T - S_T)K_T\right] - \mathbb{E}\left[\frac{\gamma_d}{2}\int_0^T (K_t - U_t^K)^2\,dt + \frac{\lambda}{2}\int_0^T (u_t^K)^2\,dt\right] \to \max_K$$

Dealer market prices (S_t) will generate an **equilibrium** if at these quotes the dealers' optimal supply matches their clients' demand:

$$\mathscr{K} \in \arg\max_K J_d(K;S)$$

Theorem

Given clients' demand $\mathscr{K}$, the unique equilibrium quotes $S^{\mathscr{K}}$ are

$$S_t^{\mathscr{K}} \triangleq P_t + \gamma_d \mathbb{E}\left[\int_t^T (\mathscr{K}_s - U_s^{\mathscr{K}})\,ds \,\middle|\, \mathscr{F}_t\right], \quad 0 \leq t \leq T,$$

where $U^{\mathscr{K}}$ describes the dealers' optimal cumulative transfers to the end-users as determined by B., Soner, Voß '17.

Equilibrium

$$S_t^{\mathscr{K}} = P_t + \gamma_d \mathbb{E}\left[\int_t^T (\mathscr{K}_s - U_s^{\mathscr{K}})\, ds \,\middle|\, \mathscr{F}_t\right], \quad 0 \le t \le T,$$

- fundamental value P adjusted for dealers' effective risk
- adjustment in line with asymptotic expansion for small dealer risk aversion in exponential utility setting by Kramkov-Pulido '16 (who do not consider end-users)
- small search costs asymptotics of dealers' surcharge depend on demand regularity:
  - absolutely continuous demand $\mathscr{K} = \int_0^{\cdot} \mu_t^{\mathscr{K}}\, dt$:

$$\int_0^T K_t d(P_t - S_t^{\mathscr{K}}) = \lambda \int_0^T (\mu_t^{\mathscr{K}})^2 dt + o(\lambda) \text{ in } L^1 \text{ as } \lambda \downarrow 0$$

  - diffusive demand $\mathscr{K} = \int_0^{\cdot} (\mu_t^{\mathscr{K}}\, dt + \sigma_t^{\mathscr{K}}\, dW_t)$:

$$\int_0^T \mathscr{K}_t d(P_t - S_t^{\mathscr{K}}) = \sqrt{\lambda \gamma_d} \int_0^T (\sigma_t^{\mathscr{K}})^2 dt + o(\sqrt{\lambda}) \text{ in } L^1 \text{ as } \lambda \downarrow 0$$

- endogenous price impact model *with resilience*, in contrast to B.-Kramkov '15

The clients' problem

How should the clients choose their demand $\mathscr{K}$ given quotes (S_t)?

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$$J_c(\mathscr{K}\,; S) \triangleq \mathbb{E}\left[\int_0^T \mathscr{K}_t \, dS_t\right] - \frac{\gamma_c}{2}\mathbb{E}\left[\int_0^T (\zeta_t - \mathscr{K}_t)^2 dt\right] \to \max_{\mathscr{K}}$$

If (S_t) has drift (μ_t), this amounts to

$$\mathbb{E}\left[\int_0^T \left(\mathscr{K}_t \mu_t - \frac{\gamma_c}{2}(\zeta_t - \mathscr{K}_t)^2\right) dt\right] \to \max_{\mathscr{K}}, \text{ i.e. } \mathscr{K}_t^* = \zeta_t - \mu_t / \gamma_c$$

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Given demand $\mathscr{K}^*$, the equilibrium quotes' $S^{\mathscr{K}^*}$ drift is

$$\mu_t^{\mathscr{K}^*} = -\gamma_d(\mathscr{K}_t^* - U_t^{\mathscr{K}^*})$$

which yields the **equilibrium demand equation**:

$$\mathscr{K}_t^* = \frac{\gamma_d}{\gamma_d + \gamma_c} U_t^{\mathscr{K}^*} + \frac{\gamma_c}{\gamma_d + \gamma_c}\zeta_t, \quad t \in [0, T],$$

where, again, $U^{\mathscr{K}^*}$ is as in B., Soner, Voß '17.

Equilibrium demand

The **equilibrium demand equation**:

$$\mathscr{K}_t^* = \frac{\gamma_d}{\gamma_d + \gamma_c} U_t^{\mathscr{K}^*} + \frac{\gamma_c}{\gamma_d + \gamma_c} \zeta_t, \quad t \in [0, T],$$

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is an integral equation for $\mathscr{K}^*$. With

$$k_t \triangleq \mathscr{K}_t^* - \frac{\gamma_c}{\gamma_d + \gamma_c} \zeta_t \text{ and } K_t \triangleq \mathbb{E}\left[\int_t^T \mathscr{K}_u^* \frac{\cosh((T-u)/\sqrt{\kappa})}{\sqrt{\kappa}\cosh((T-t)/\sqrt{\kappa})} du \,\middle|\, \mathscr{F}_t \right]$$

it is equivalent to the *linear forward backward stochastic differential equation* (FBSDE):

$$k_0 = 0, \; dk_t = \left(\frac{\gamma_d}{\gamma_d + \gamma_c} K_t - \frac{\tanh((T-t)/\sqrt{\kappa})}{\sqrt{\kappa}} k_t \right) dt,$$

$$K_T = 0, \; dK_t = \left(\frac{\tanh((T-t)/\sqrt{\kappa})}{\sqrt{\kappa}} K_t - \frac{1}{\kappa}\left(k_t + \frac{\gamma_c}{\gamma_d + \gamma_c} \zeta_t\right) \right) dt + dM_t^K$$

for a suitable martingale M^K determined uniquely by the FBSDE.

Equilibrium demand

Theorem

The unique equilibrium demand is given explicitly by

$$\mathscr{K}_t^* = \frac{\gamma_c}{\gamma_d + \gamma_c}\zeta_t + \tilde{U}_t^{\frac{\gamma_d}{\gamma_d+\gamma_c}\zeta}, \quad t \in [0, T]$$

where $\tilde{U}^{\frac{\gamma_d}{\gamma_d+\gamma_c}\zeta}$ *denotes the tracking portfolio from B., Soner, Voß:*

$$\frac{d}{dt}\tilde{U}_t^{\frac{\gamma_d}{\gamma_d+\gamma_c}\zeta} = \frac{\tanh((T-t)/\sqrt{\tilde{\kappa}})}{\sqrt{\tilde{\kappa}}}\left(\frac{\gamma_d}{\gamma_d+\gamma_c}\zeta_t - \tilde{U}_t^{\frac{\gamma_d}{\gamma_d+\gamma_c}\zeta}\right),$$

for the **aggregate holding costs** $\tilde{\gamma} = (1/\gamma_d + 1/\gamma_c)^{-1}$, *i.e.,*

$$\tilde{\kappa} \triangleq \lambda/\tilde{\gamma} \text{ and } \tilde{\zeta}_t \triangleq \mathbb{E}\left[\int_t^T \zeta_u \frac{\cosh((T-u)/\sqrt{\tilde{\kappa}})}{\sqrt{\tilde{\kappa}}\sinh((T-t)/\sqrt{\tilde{\kappa}})}\, du \middle| \mathscr{F}_t\right].$$

This balances the clients' demand for immediacy with their holding costs, taking into account also their dealers' holding costs and their ability of transferring risk to end-users: $\tilde{U}^{\zeta} = U^{\mathscr{K}^*}$.

When do the clients really need their dealers?

Example: Constant target position

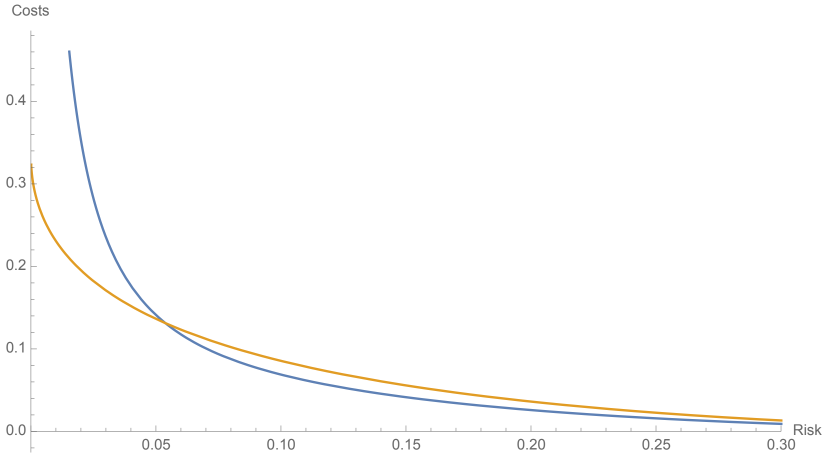

Figure: Risk or holding costs vs. search costs when clients are trading through their dealers' or are searching end-users themselves.

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In other words: What if the dealers are facing a **large trader**?

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Quadratic criterion: Facing exogenous FX cash flow (ζ_t), the large investor seeks to maximize

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This is still **concave** in $\mathscr{K}$ since $\mathscr{K} \mapsto -\mathbb{E}\left[\int_0^T \mathscr{K}_t \, dS_t^{\mathscr{K}}\right]$ is the dealers' expected profit in equilibrium and thus nonnegative.

$\rightsquigarrow$ **no statistical arbitrage** in this model with **endogenously derived market impact**.

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i.e. the **same equilibrium demand equation** as before, albeit with **half** the clients' holding costs.

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"Price of anarchy": $J_c(\mathscr{K}^*) \geq J_c(\mathscr{K}^*) = J_c(\mathscr{K}^*; S^{\mathscr{K}^*})$

Conclusions

- analyzed dealer market with clients and end-users
- quadratic setting allows for explicit computations following previous optimal tracking results
- equilibrium quotes for arbitrary demand take into account legacy position and expected future positions
- optimization of demand with and without impact awareness
- dealers will be used if their search and holding costs are small compared to those of their clients
- harder to serve sophisticated clients aware of their impact
- endogenously derived impact model ruling out statistical arbitrage
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Thank you very much!

Outline

Information flow and optimal control

In many financial optimal control problems, there are moments known in advance when significant new information will become available:

- interest rate decisions by central banks, elections, referendums
- publication of data on GDP growth, job market statistics, trade balances
- price jumps, e.g., at opening of exchanges, due to earning announcements, ...
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Before these moments, investors will form an opinion and take precautionary actions: *proactive trading*.
Afterwards, when the news are fully revealed, further measures may have to be taken: *reactive trading*.

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How to describe such information flows mathematically?
How to do optimal control with them?

Illustration: Optimal investment with a twist

- asset price fluctuations modeled by symmetric compound Poisson process

$$\tilde{P}_t = \tilde{p} + \sum_{k=1}^{N_t} Y_k \text{ with i.i.d. } Y_k \sim U[-1,1]$$

- strategy $C = (C_t)_{0 \leq t \leq 1}$ with $|C| \leq 1$ yields expected P&L

$$\mathbb{E} \int_0^1 C_t d\tilde{P}_t = \mathbb{E} \sum_{k=1}^{N_1} C_{T_k} Y_k$$

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$$\mathbb{E} \int_0^1 C_t d\tilde{P}_t \equiv 0$$

for any control

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with "$=$" for $C_t^{\mathscr{O}} = \operatorname{sign}(\Delta \tilde{P}_t)$

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- If controller alerted only by large enough jumps $|\Delta \tilde{P}_t| \geq \eta$:

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with "$=$" for $C^\eta_t = \mathrm{sign}(\Delta \tilde{P}_t 1_{\{|\Delta \tilde{P}_t| \geq \eta\}})$ in which argmax?

Meyer σ-fields Λ: Definition and some properties

A σ-field on $\Omega \times [0, \infty)$ is called a **Meyer σ-field** if

- it is generated by càdlàg processes;
- it contains all deterministic Borel-measurable events;
- it is stable with respect to stopping: with Z also $(Z_{s \wedge t})_{s \geq 0}$ is Λ-measurable for any $t \geq 0$.

Examples:

$\mathscr{O}$, $\mathscr{P}$,

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Theorem

$$C^\eta = \operatorname{sign}\left(\Delta\tilde{P} 1_{\{|\Delta\tilde{P}| \geq \eta\}}\right) \in \operatorname*{arg\,max}_{C \in \Lambda^\eta,\ |C| \leq 1} \mathbb{E} \int_0^1 C_t \, dP_t$$

Meyer σ-fields Λ: Definition and some properties

Proof: To argue:

$$\mathbb{E}\int_0^1 C_t d\tilde{P}_t \leq \mathbb{E}\sum_{k=1}^{N_1} |C_{T_k}||Y_k| 1_{\{|Y_{T_k} \geq \eta\}} \leq \mathbb{E}\sum_{k=1}^{N_1} |Y_k| 1_{\{|Y_{T_k} \geq \eta\}}$$

Observe decomposition of jump times

$$T_k = \underbrace{(T_k)_{\{|Y_k| \geq \eta\}}}_{\Lambda^\eta-\text{st.time}} \wedge \underbrace{(T_k)_{\{|Y_k| < \eta\}}}_{\text{tot.inacc.}}$$

yields for Λ^η-measurable C (with $C_\infty := 0$):

$$C_{T_k} = C_{(T_k)_{\{|Y_k| \geq \eta\}}} + ({}^{\mathscr{P}}C)_{(T_k)_{\{|Y_k| < \eta\}}}$$

So:

$$\mathbb{E}\int_0^1 C_t d\tilde{P}_t = \mathbb{E}\sum_{k=1}^{N_1} C_{T_k} 1_{\{|Y_k| \geq \eta\}} Y_k + \underbrace{\mathbb{E}\sum_{k=1}^{N_1} ({}^{\mathscr{P}}C)_{(T_k)_{\{|Y_k| < \eta\}}} Y_k}_{=0 \text{ as pred.stoch.int. wrt. mart.}}$$

Illustrative control problem: Irreversible investment

- Classic problem: Dixit and Pindyck (1994), Bertola (1998), Merhi and Zervos (2007), Riedel and Su (2011), Ferrari (2015), Al Motairi and Zervos (2017), De Angelis et al. (2017)...
- Consider target functional:

$$\tilde{V}(C) = \mathbb{E}\left[\int_{[0,\infty)} P_t \, dC_t - \int_{[0,\infty)} \rho_t(C_t) \, dR_t\right] \to \max_{C \geq c_0 \nearrow \text{càd, adapted}} .$$

  P discounted reward process, $\rho_t(c)$ risk penalty convex in c, R risk assessment clock
- *Standard* assumptions: $P_t = e^{-rt}\tilde{P}_t$ for compound Poisson $\tilde{P}_t = \tilde{p} + \sum_{k=1}^{N_t} Y_k$; $\rho_t(c) = c^2/2$;$dR_t = e^{-rt}dt$:

$$\mathbb{E}\left[\int_{[0,\infty)} e^{-rt}\tilde{P}_t \, dC_t - \int_{[0,\infty)} \frac{1}{2}(C_t)^2 e^{-rt} \, dt\right] \to \max_C$$

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- *Alternative* assumptions: $P_t = e^{-rt}\tilde{P}_t$ for compound Poisson $\tilde{P}_t = \tilde{p} + \sum_{k=1}^{N_t} Y_k$; $\rho_t(c) = c^2/2$; $dR_t = e^{-rt} dN_t$:

$$\mathbb{E}\left[\int_{[0,\infty)} e^{-rt}\tilde{P}_t \, dC_t - \int_{[0,\infty)} \frac{1}{2}(C_t)^2 e^{-rt} \, dN_t\right] \to \max_C$$

- New issues: C right- or left-continuous or just làdlàg? What is known about $\tilde{P}_t$ at time of decision on dC_t?

Relaxation of the problem

Theorem

$$\sup_{\Lambda \ni C \geq c_0 \nearrow,\ \textit{càd}} \tilde{V}(C) = \max_{\Lambda \ni C \geq c_0 \nearrow} V(C)$$

where

$$V(C) = \mathbb{E}\left[\int_{[0,\infty)} {}^{\Lambda}P_t \, {}^{*}dC_t - \int_{[0,\infty)} \rho_t(C_t)\, dR_t\right]$$

with ${}^{\Lambda}P$ *the Meyer-projection of* P *and* ${}^{*}d$*-integral defined by*

$$\int_{[0,\infty)} Q_t \, {}^{*}dC_t = \int_{[0,\infty)} Q_t \, dC_t^c + \sum_{t \geq 0} Q_t(C_t - C_{t-}) + \sum_{t \geq 0} Q_t^*(C_{t+} - C_t)$$

for $Q_t^* = \limsup_{u \downarrow t} Q_u$

Heuristics from first order conditions

First order conditions for optimality of C^*:

$$^{\wedge}P_S \leq \mathbb{E}\left[\int_{[S,\infty)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_t)dR_t \middle| \mathscr{F}_S\right]$$

with "$=$" holding true whenever it is optimal to intervene: $dC^*_S > 0$

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with "$=$" holding true whenever it is optimal to intervene: $dC^*_S > 0$
If optimal to intervene at S, then for any T with $T > S$:

$$ \begin{aligned} \mathbb{E}\left[{}^{\Lambda}P_S - {}^{\Lambda}P_T \middle| \mathscr{F}^{\Lambda}_S\right] &\geq \mathbb{E}\left[\int_{[S,T)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_t)dR_t \middle| \mathscr{F}^{\Lambda}_S\right] \\ &\geq \mathbb{E}\left[\int_{[S,T)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_S)dR_t \middle| \mathscr{F}^{\Lambda}_S\right] \end{aligned} $$

$\rightsquigarrow$ $\widehat{C}_S \leq \operatorname{ess\,inf}_T \ell_{S,T} =: L^{\Lambda}_S$

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with "$=$" holding true whenever it is optimal to intervene: $dC^*_S > 0$
If not optimal to intervene a S, then for next time T_S that $dC > 0$

$$\mathbb{E}\left[{}^{\Lambda}P_S - {}^{\Lambda}P_{T_S} \middle| \mathscr{F}^{\Lambda}_S\right] \leq \mathbb{E}\left[\int_{[S,T_S)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_t)dR_t \middle| \mathscr{F}^{\Lambda}_S\right]$$

$$= \mathbb{E}\left[\int_{[S,T_S)} \frac{\partial}{\partial c}\rho_t(\widehat{C}_S)dR_t \middle| \mathscr{F}^{\Lambda}_S\right]$$

$\rightsquigarrow\ \widehat{C}_S \geq \ell_{S,T_S} \geq \operatorname{ess\,inf}_T \ell_{S,T} =: L^{\Lambda}_S$
$\rightsquigarrow \widehat{C}_S = c_0 \vee \sup_{v\in[0,S]} L^{\Lambda}_S$

A stochastic representation theorem

Theorem (B. & El Karoui (2004), B. & Besslich (2019))

Under suitable integrability and upper-semicontinuity assumptions, there exists $L^\Lambda \in \Lambda$ *such that*

$$^\Lambda P_S = \mathbb{E}\left[\int_{[S,\infty)} \frac{\partial}{\partial c}\rho_t\left(\sup_{v\in[S,t]} L_v^\Lambda\right) dR_t \middle| \mathscr{F}_S^\Lambda\right], \quad S \in \mathscr{S}^\Lambda.$$

The maximal such L^Λ *is uniquely determined by*

$$L_S^\Lambda = \operatorname{essinf}_{T\in\mathscr{S}^\Lambda, T>S} \ell_{S,T}, \quad S \in \mathscr{S}^\Lambda,$$

where for $S < T$, $\ell_{S,T} \in \mathscr{F}_S^\Lambda$ *is defined by*

$$\mathbb{E}\left[P_S - P_T \middle| \mathscr{F}_S^\Lambda\right] = \mathbb{E}\left[\int_{[S,T)} \frac{\partial}{\partial c}\rho_t(\ell_{S,T}) dR_t \middle| \mathscr{F}_S^\Lambda\right]$$

on $\{\mathbb{P}\left(R_{T-} - R_{S-} > 0 \middle| \mathscr{F}_S^\Lambda\right) > 0\}$ *and* $\ell_{S,T} := \infty$ *elsewhere.*

Solutions via a representation theorem

Theorem

Suppose C^{L^Λ} *given by*

$$C^{L^\Lambda}_{0-} := c_0, \quad C^{L^\Lambda}_t := c_0 \vee \sup_{v \in [0,t]} L^\Lambda_v, \quad t \in [0, \infty),$$

satisfies

$$\mathbb{E}\left[\int_{[0,\infty)} \left\{ \frac{\partial}{\partial c} \rho_t \left(C^{L^\Lambda}_t \right) (C^{L^\Lambda}_t - c_0) \right\} \vee 0 \; dR_t \right] < \infty.$$

Then C^{L^Λ} *is optimal for the relaxed problem whose value is*

$$V(C^{L^\Lambda}) = \mathbb{E}\left[\int_{[0,\infty)} \left\{ \frac{\partial}{\partial c} \rho_t \left(C^{L^\Lambda}_t \right) (C^{L^\Lambda}_t - c_0) - \rho_t \left(C^{L^\Lambda}_t \right) \right\} dR_t \right] < \infty.$$

Explicit solution in the compound Poisson example

Let $P_t = e^{-rt}\tilde{P}_t$ with $\tilde{P}_t = \tilde{p} + \sum_{k=1}^{N_t} Y_k$ for Poisson N with param. λ, i.i.d. $Y_k \in L^2$, $\mathbb{E}Y_k = m$; $dR_t = e^{-rt}dN_t$; $\rho_t(c) = \frac{1}{2}c^2$;

$$\Lambda = \Lambda^\eta := \mathscr{P} \vee \sigma\left(\sum_{k=1}^{N_\cdot} Y_k 1_{\{|Y_k|\geq\eta\}}\right) \quad \rightsquigarrow \text{Large jump alerts}$$

Probability of failure to alert: $p(\eta) = \mathbb{P}[|Y_k| \leq \eta]$.

- $p(\eta) = 1$: no alerts, predictable case $\Lambda^\eta = \mathscr{P}$

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- $p(\eta) = 1$: no alerts, predictable case $\Lambda^\eta = \mathscr{P}$
- $p(\eta) = 0$: alerts for all jumps, optional case $\Lambda^\eta = \mathscr{O}$
- $p(\eta) \in (0,1)$: Meyer case $\mathscr{P} \subsetneq \Lambda^\eta \subsetneq \mathscr{O}$

Solution in the predictable case

In the case $p(\eta) = 1$, i.e. without alerts::

$$L_t^{\mathscr{P}} = a(\tilde{P}_{t-} - b), \quad t \in [0, \infty),$$

where the constants a, b are given by

$$a := \frac{1}{\mathbb{E}[R_{\infty-}]} = \frac{r}{\lambda},$$

$$b := \sup_{0 < T \text{ pred.}} \frac{\mathbb{E}\left[\mathrm{e}^{-rT} \sum_{k=1}^{N_T} Y_k\right]}{1 - \mathbb{E}\left[\mathrm{e}^{-rT}\right]} = \frac{\mathbb{E}\left[\int_{[0,\infty)} \left(\sup_{v \in [0,t]} \tilde{P}_{v-} - \tilde{p}\right) dR_t\right]}{\mathbb{E}[R_{\infty-}]}.$$

$\leadsto$ $C^{\mathscr{P}} = c_0 \vee \sup_{0 \le v \le .} L_v^{\mathscr{P}}$ left-continuous with exclusively reactive jumps because jump times are totally inaccessible to controller

Solution in the Meyer case

In the case $p(\eta) \in (0,1)$ with alerts for some, but not all jumps:

$$L_t^{\Lambda^\eta} = \begin{cases} 0, & \tilde{P}_t^\eta \geq b,\ |\Delta \tilde{P}_t^\eta| \geq \eta, \\ \frac{r}{\lambda}(\tilde{P}_t^\eta - b), & \tilde{P}_t^\eta \geq b,\ |\Delta \tilde{P}_t^\eta| < \eta, \\ \inf\limits_{\gamma^0 \in (0, B_0^\eta \cdot (b - \tilde{P}_t^\eta))} f_1^\eta(\gamma^0, 0, \tilde{P}_t^\eta) < 0, & \tilde{P}_t^\eta < b,\ |\Delta \tilde{P}_t^\eta| \geq \eta, \\ \inf\limits_{\gamma^1 \in (-B_1^\eta \cdot (b - \tilde{P}_t^\eta), 0)} f_0^\eta(0, \gamma^1, \tilde{P}_t^\eta) < 0, & \tilde{P}_t^\eta < b,\ |\Delta \tilde{P}_t^\eta| < \eta \end{cases}$$

where $\tilde{P}^\eta := {}^{\Lambda^\eta}\tilde{P} = (\tilde{P}_{t-} + \Delta \tilde{P}_t 1_{\{|\Delta \tilde{P}_t| \geq \eta\}})_{t \geq 0}$ a Meyer-projection,

$$f_\Delta^\eta = \frac{\left(1 - \mathbb{E}\left[e^{-rT^\eta(\gamma^0,\gamma^1)}\right]\right) p - \mathbb{E}\left[e^{-rT^\eta(\gamma^0,\gamma^1)} \sum_{k=1}^{N_{T^\eta(\gamma^0,\gamma^1)}} Y_k\right]}{\frac{\lambda}{r}\left(1 - \mathbb{E}\left[e^{-rT^\eta(\gamma^0,\gamma^1)}\right]\right) - \mathbb{E}\left[e^{-rT^\eta(\gamma^0,\gamma^1)} 1_{\{|\Delta \tilde{P}_{T^\eta(\gamma^0,\gamma^1)}| \geq \eta\}}\right] + \Delta},$$

$$T^\eta(\gamma^0,\gamma^1) = \inf\Big\{ t \in \{{}^{\Lambda^\eta}N > 0\} \,\Big|\, \Big(|\Delta \tilde{P}_t| < \eta \text{ and } \tilde{P}_{t-} - \tilde{p} \geq \gamma^0\Big) \text{ or } \Big(|\Delta \tilde{P}_t| \geq \eta \text{ and } \tilde{P}_t - \tilde{p} \geq \gamma^1\Big)\Big\}.$$

Solution in the optional case

In the optional case with complete alerts $p(\eta) = 0$:

$$L_t^{\mathscr{O}} = \begin{cases} 0, & \tilde{P}_t \geq b,\ |\Delta \tilde{P}_t| > 0, \\ \frac{r}{\lambda}(\tilde{P}_t - b), & \tilde{P}_t \geq b,\ \Delta \tilde{P}_t = 0, \\ \frac{r}{\lambda + r}(b - \tilde{P}_t), & \tilde{P}_t < b,\ |\Delta \tilde{P}_t| > 0, \\ \inf\limits_{\gamma \in (-\infty, 0)} f(\gamma, \tilde{P}_t) < 0, & m\frac{\lambda}{r} \leq \tilde{P}_t < b,\ \Delta \tilde{P}_t = 0, \\ -\infty, & \tilde{P}_t < m\frac{\lambda}{r},\ \Delta \tilde{P}_t = 0. \end{cases}$$

where

$$f(\gamma, p) := \frac{\left(1 - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)}\right]\right) p - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)} \sum_{k=1}^{N_{T(\gamma)}} Y_k\right]}{\frac{\lambda}{r}\left(1 - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)}\right]\right) - \mathbb{E}\left[\mathrm{e}^{-rT(\gamma)}\right]},$$

$$T(\gamma) := \inf\left\{t \in \{N > 0\} \,\middle|\, |\Delta \tilde{P}_t| > 0 \text{ and } \tilde{P}_t - \tilde{p} \geq \gamma\right\}.$$

Solution in the optional case

In the optional case with complete alerts $p(\eta) = 0$:

$$L_t^{\mathscr{O}} = \begin{cases} 0, & \tilde{P}_t \geq b,\ |\Delta\tilde{P}_t| > 0, \\ \frac{r}{\lambda}(\tilde{P}_t - b), & \tilde{P}_t \geq b,\ \Delta\tilde{P}_t = 0, \\ \frac{r}{\lambda+r}(b - \tilde{P}_t), & \tilde{P}_t < b,\ |\Delta\tilde{P}_t| > 0, \\ \inf\limits_{\gamma\in(-\infty,0)} f(\gamma, \tilde{P}_t) < 0, & m\frac{\lambda}{r} \leq \tilde{P}_t < b,\ \Delta\tilde{P}_t = 0, \\ -\infty, & \tilde{P}_t < m\frac{\lambda}{r},\ \Delta\tilde{P}_t = 0. \end{cases}$$

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Observation: $L_t^{\mathscr{P}} \xleftarrow[p(\eta)\to 1]{} L^{\Lambda^\eta} \xrightarrow[p(\eta)\to 0]{} L_t^{\mathscr{O}}$

Illustration

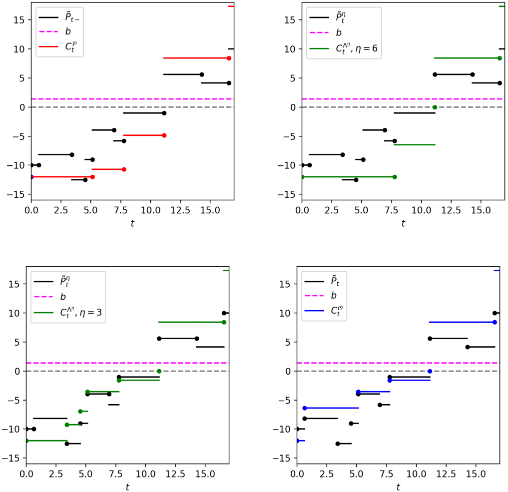

Figure: $\tilde{P}^\eta$ (black), b (Magenta) and optimal controls for $\eta = 0$ (blue, optional), $\eta = 3$, $\eta = 6$ (green) and $\eta = \infty$ (red, predictable). The dots indicate the processes' value at their jump times.

Papers forming the basis of this talk

- Modelling information flows by Meyer-σ-fields in the singular stochastic control problem of irreversible investment, B. & Besslich 2019, arxiv:1810.08495
- On a Stochastic Representation Theorem for Meyer-measurable Processes and its Applications in Stochastic Optimal Control and Optimal Stopping, B. & Besslich 2019, arxiv: 1810.08491
- On Lenglart's Theory of Meyer-sigma-fields and El Karoui's Theory of Optimal Stopping, B. & Besslich 2019, arxiv: 1810.08485

Conclusion and Outlook

- continuous-time information modeling most flexible via Meyer σ-fields
- rich toolbox for mathematically rigorous treatment
- allows for modeling instant signals on jumps
- general solution to irreversible investment problem
- explicit solution in compound Poisson setting with jump size dependent alerts
- làdlàg controls in general
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Thank you very much!