Capital Reserve Management for a Multi-Dimensional Risk Model

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Overview

1. Capital Risk Management
2. An insurance risk model
   - Multi-dimensional model
   - Dependence by environmental process
   - Capital calculation
3. Time-independent environmental process
   - Detection of the environmental process
   - Results
4. Markov environmental process
   - Approximations: diffusion and single-switch
   - Results

Credit Risk

Potential that a borrower or counterparty will fail to meet its obligations in accordance with agreed terms.

$$Loss_i := Exposure_i \times LGD_i \times \mathbb{1}_{D_i},$$

- $Exposure_i$ is the exposure
- LGD_i is the loss given default
- $\mathbb{1}_{D_i}$ is the default event of counterparty i often interpreted as a stochastic assets process dropping below zero, i.e. $\{D_i\} = \{Assets_i(t) < Debt(t)\}$

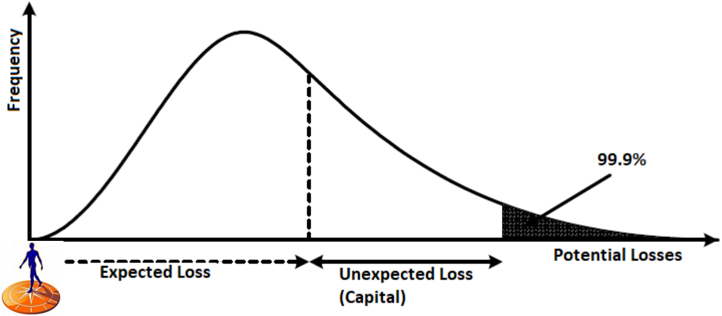
$$Loss_{portfolio} := \sum_{i=1}^{n} Exposure_i \times LGD_i \times \mathbb{1}_{D_i}.$$

Capital

A rainy day fund, so when bad things happen, there is money to cover it.

— IAA Solvency Working Party (2004)

Capital amounts are calculated so that the firm can cover the losses over a one year horizon with $> 99\%$ probability.



Credit risk capital model requirements

- ✔ Best estimate of credit risk.
- ✔ Model with economical interpretation.
- ✔ Dependence/correlation between clients.
- ✗ Time-dependent correlation.
- ✗ Efficient (quick) calculation.
- ✗ Easy allocation of capital.

Overview

An insurance risk model

Tranditional approach:
: Consider the losses of each individual client of the bank and determine the portfolio loss distribution. Quantile of the loss distribution determines the capital amount.

Insurance risk approach:
: Model the capital reserves of the bank itself and limit the probability of these reserves dropping below zero by setting the initial capital reserve level (at time 0) high enough.

An insurance risk model: Ruin Theory

Cramér-Lundberg model

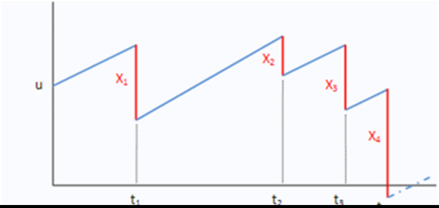
For business line i

- Starting cash reserves u_i.
- Fees r_i.
- Stochastic claims C_i with Poisson arrival $N_i(t)$ of intensity λ_i and claim size distribution F_i with mean m_i and variance σ_i^2,

$$X_i(t) = u_i + r_i t - \sum_{k=1}^{N_i(t)} C_{i,k}.$$

Ruin/Default: $D_i := \{\inf_{t\in[0,T]} X_i(t) < 0\}, \quad \psi_i(u_i, T) := P(D_i).$

Survival: $\bar{\psi}_i(u_i, T) := 1 - \psi_i(u_i, T).$



Ruin Theory

Multi-dimensional model

- Cash reserves entire firm

$$X_1 + X_2 + ... + X_n,$$

where X_i is the Cramér-Lundberg model for subline i.

- Subsets $S_1, ..., S_M$ of the n business lines.

Multivariate ruin probability

$$\psi_{S_m}(\mathbf{u}, T) := \mathbb{P}\left(\sup_{i \in S_m} \inf_{t \in [0,T]} X_i(t) < 0 \middle| \mathbf{X}(0) = \mathbf{u}\right).$$

Ruin Theory

Dependence

- Unobservable environmental state process $J := \{J(t) : t \geq 0\}$ with state space $\{1, ..., I\}$ influences the claim process.
- Claim arrival rate $\lambda_{i,j}$ and claim size distribution $F_{i,j}$ with mean $m_{i,j}$ and variance $\sigma_{i,j}$ for business line i when J is in state j.
- Conditional on the environmental state process, the capital reserve process of the business lines are assumed independent.
- Will consider a time-independent and Markov environmental state process.

Ruin Theory

Capital calculation

Objective is to minimize the total initial capital of the firm $\sum_{i=1}^{n} u_i$ subject to constraints on the default probabilities of the S_m.

Let $\delta_m > 0$, capital can be determined by various optimizations, e.g.

$$\min_{\mathbf{u} \succ 0} \sum_{i=1}^{n} u_i, \ s.t. \ \psi_{S_m}(\mathbf{u}, T) \leq \delta_m.$$

Why this model?

- ✔ Model with economical interpretation in insurance and banking.
- ✔ Multivariate risk model with intricate correlation structure (scarce in literature).
- ✔ Time-dependent correlation.
- ✔ No simulation, only optimization: very time efficient.
- ✔ Allocated capital is a direct result of the model.
- ✔ Few model parameters.

Overview

Time-independent environmental state process

During the entire time interval $[0, T]$ the environmental state process J is fixed.

$$\mathbb{P}\left(\lambda_i = \lambda_{i,j}\right) = \mathbb{P}\left(F_i = F_{i,j}\right) = \mathbb{P}\left(J = j\right) = p_j$$

Multivariate ruin probability:

$$\psi_{S_m}(\mathbf{u}, T) = \sum_{j=1}^{I} p_j \prod_{i \in S_m} \mathbb{P}\left(\inf_{t \in [0,T]} X_i(t) < 0 | J = j, X_i(0) = u_i\right)$$

Time-independent environmental state process

Calibration

Observations for all business lines during time interval t_m:

- Number of claims $Y^m := (Y_1^m, \cdots, Y_n^m)$ with $Y_i^m := N_i(t_m) - N_i(t_{m-1})$.
- Claim sizes $Z^m := (Z_1^m, \cdots, Z_n^m)$ with $Z_i^m := (C_{i,1}, ..., C_{i,Y_i^m})$.
- Density f.

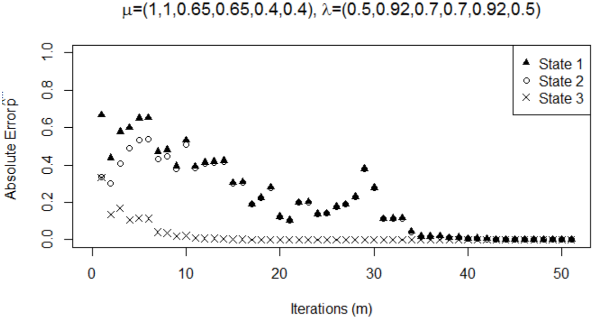
Bayesian calibration of unobservable environmental process:

$$\begin{aligned}\hat{p}_j^m :&= \mathbb{P}(J = j | Y^1, \cdots, Y^m, Z^1, \cdots, \mathcal{Z}^m) \\ &= \hat{p}_j^{m-1} \frac{f(Y^m, Z^m | J = j)}{\sum_{k=1}^{I} \hat{p}_k^{m-1} f(Y^m, Z^m | J = k)} \\ &= \frac{\hat{p}_j^0 f(Y^1, \cdots, Y^m, Z^1, \cdots, \mathcal{Z}^m | J = j)}{\sum_{k=1}^{I} \hat{p}_k^0 f(Y^1, \cdots, Y^m, Z^1, \cdots, \mathcal{Z}^m | J = k)}.\end{aligned}$$

Time-independent environmental state process

Results

- Two lines of business ($n = 2$) & three environmental states ($I = 3$).
- Fees $r_1 = r_2 = 1$ & $Exp(\mu)$ claims with intensity λ.
- True environmental state 1.
- Starting probabilities $\hat{p}^0 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.



(c)

Time-independent environmental state process

Advantages

- ✔ Conditional on the random variable J, the risk processes are independent.
- ✔ Allows for an easy-to-implement Bayesian detection of the process J based on observed claims.
- ✔ The Bayesian calibration approach is robust in terms of the distribution of J as well as allocated capital even when the environmental process changes.
- ✗ Assumes the environmental state process does not change (often) over time.

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Markov environmental state process

Environmental state process J is Markov with intensity matrix $Q = (q_{i,j})_{i,j \in \{1,\dots,I\}}$. Assume $\pi := (\pi_1, \dots, \pi_I)$ is the (unique) stationary distribution of J.

- ✔ Environmental state changes over time.
- ✔ Distribution of claim sizes and claim intensities can change over time.
- ✗ Much more difficult to calculate finite-time multivariate ruin probabilities as business lines are dependent on the entire path of the environmental state process.

Diffusion approximation

Theorem

For the Markov-modulated compound Poisson process $\mathbf{Y}(t) := (Y_1(t), ..., Y_n(t))$ with $Y_i(t) := \sum_{k=1}^{N_i(t)} C_{i,k}$:

$$\frac{1}{\sqrt{c}}\left(\mathbf{Y}(ct) - ct\sum_{j=1}^{I} \boldsymbol{\lambda}_j \mathbf{m}_j \pi_j\right) \Rightarrow \mathbf{W}, \text{ as c} \to \infty,$$

where $\mathbf{W}$ is a n-dimensional Brownian Motion (BM) with mean zero and some covariance matrix Σ.

Using the joint distribution of BM extrema, a diffusion approximation of the multivariate ruin/survival probability is found.

Single-switch approximation

Let the environmental process start in state j at time 0. Assuming there is only 1 switch in environmental state in the interval $[0, T]$:

- The environmental state switches to state k after an exponentially distributed time T^j with intensity $q_{j,k}$.
- The ruin probability of business line i is given by

$$\psi_i^{jk}(\mathbf{u}, T^j, T) := \mathbb{P}\left(\inf_{t\in[0,T]} X_i(t) \leq 0 \middle| \mathbf{X}(0) = \mathbf{u}, \{J(t)\}_{0\leq t< T^j} = j, \{J(t)\}_{T^j\leq t\leq T} = k\right).$$

- The $\psi_i^{jk}(\mathbf{u}, T^j, T)$ are independent.

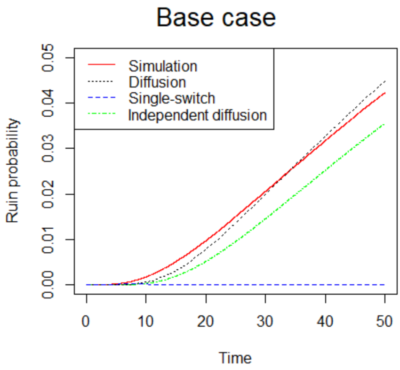
Summary Approximations

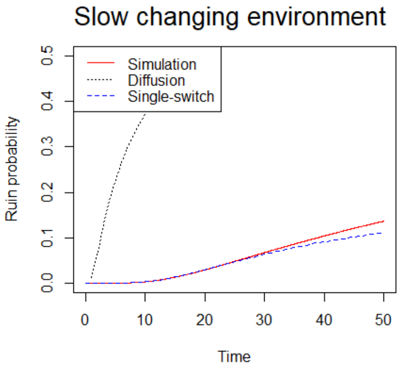
- For low environmental transition rates Q, we recommend the single-switch approximation.
- For high environmental transition rates Q and high arrival intensity λ, we recommend the diffusion approximation.

Numerical Setting

- 2 identical lines of business ($n = 2$),
- Fees $r_1 = r_2 = 1$.
- $Exp(1)$ claims with intensity $\lambda = \begin{pmatrix} 0.45 & 1.8 \\ 0.45 & 1.8 \end{pmatrix}$.
- 2 environmental states, $Q = \gamma \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$
- In the base case $\gamma = 1$ and $\gamma = \frac{1}{64}$ in a slow changing environment.
- Environmental state 1 is “up”, less risky, state 2 is “down”.

Numerical Results





Summary

- Capital modelling by considering the capital reserves process of the firm using risk theory.
- Set up a multi-dimensional risk process for firms with multiple business lines.
- Dependence through a common environmental state process, time-independent or Markov.
- Fast approximations, no simulation necessary.
- Capital calculation by optimization which results in allocated capital figures for each business line directly.

References

1. G. Delsing, M. Mandjes, P. Spreij, E. Winands (2019). Asymptotics and approximations of ruin probabilities for multivariate risk processes in a Markovian environment. *Methodology and Computing in Applied Probability*, *to appear.*
2. G. Delsing, M. Mandjes, P. Spreij, E. Winands (2019). An optimization approach to adaptive multi-dimensional capital management. *Insurance: Mathematics and Economics*, **84**, pp. 87-97.

Thank you for your attention.