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Machine Learning for Derivative Pricing

Gaussian processes vs. Gradient boosting

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Joint work with Jesse Davis, Laurens Devos, Jan De Spiegeleer, Dilip Madan and Wim Schoutens

Derivative pricing is time-consuming...

- Sophisticated models
- Exotic products

  → we need to rely on numerical methods, e.g. Monte Carlo simulations

- Portfolio valuation
  - Sensitivity analysis
  - VaR/ES calculations

… and markets are moving!

time-consuming algorithms

$\updownarrow$

continuously moving markets

$\rightarrow$ prices are outdated when available, risk calculations cannot be completed in time, …

Let a machine learn the pricing function

machine learning

product, market and model parameters

model price

time-consuming method

Expensive pricing function is summarized with machine learning.

Let a machine learn the pricing function

machine learning

product, market and model parameters

model price

time-consuming method

When training is completed, prediction is extremely fast!

Literature

- Neural networks:
  - Ferguson & Green (2018), Deeply learning derivatives.
  - Liu, Oosterlee & Bohte (2019), Pricing options and computing implied volatilities using neural networks.
  - Horvath, Muguruza & Tomas (2019), Deep learning volatility.
  - ...

Literature

- Gaussian process regression:
  - De Spiegeleer, Madan, Reyners & Schoutens (2018), Machine learning for quantitative finance: fast derivative pricing, hedging and fitting.
  - Crépey & Dixon (2019), Gaussian process regression for derivative portfolio modeling and application to CVA computations.
  - Sousa, Esquível & Gaspar (2012), Machine Learning Vasicek Model Calibration with Gaussian Processes.
- Gradient boosting machines:
  - Davis, Devos, Reyners & Schoutens, Gradient Boosting for Quantitative Finance. Working paper.

Gaussian process regression

Gaussian process regression (GPR)

Consider a training set $(X, \boldsymbol{y}) = \{(\boldsymbol{x_i}, y_i) \mid i = 1, \dots, n\}$ and assume that

$$y_i = f(\boldsymbol{x_i}) + \varepsilon_i$$

- $f(\boldsymbol{x})$ is a Gaussian process, characterized by two functions:
  - mean function: $m(\boldsymbol{x}) = \mathbb{E}\big[f(\boldsymbol{x})\big]$
  - kernel function: $k(\boldsymbol{x}, \boldsymbol{x}') = \mathrm{Cov}(f(\boldsymbol{x}), f(\boldsymbol{x}'))$
- $\varepsilon_i \sim \mathcal{N}(0, \sigma_n^2)$ are i.i.d. random variables representing the noise in the data.

Gaussian process

- $f(\boldsymbol{x}) \sim GP(m(\boldsymbol{x}), k(\boldsymbol{x}, \boldsymbol{x}'))$
- If $(X, \boldsymbol{f}) = \{(\boldsymbol{x_i}, f_i) \mid i = 1, \ldots, n\}$ is a sample from $f(\boldsymbol{x})$, then

$$\boldsymbol{f} \sim \mathcal{N}\left(M(X), K(X, X)\right)$$

with

$$M(X) = \begin{bmatrix} m(\boldsymbol{x_1}) \\ \vdots \\ m(\boldsymbol{x_n}) \end{bmatrix}, \qquad K(X, X) = \begin{bmatrix} k(\boldsymbol{x_1}, \boldsymbol{x_1}) & \ldots & k(\boldsymbol{x_1}, \boldsymbol{x_n}) \\ \vdots & \ddots & \vdots \\ k(\boldsymbol{x_n}, \boldsymbol{x_1}) & \ldots & k(\boldsymbol{x_n}, \boldsymbol{x_n}) \end{bmatrix}.$$

GPR: a Bayesian method

- Don't model the relation as one function, but as a distribution over functions.
- Procedure:
  1. Start from a prior GP (with zero mean)
     - $\rightarrow$ prior knowledge: smooth function, periodic function, ...
     - $\rightarrow$ prior distribution over functions
  2. Include observed data points
  3. Compute a posterior GP

Posterior distribution

Only consider functions that agree with the data.

- Take new inputs X_*, with corresponding (unknown) function values $\boldsymbol{f}_*$.
- Joint distribution of training outputs and function values:

$$\begin{bmatrix} \boldsymbol{y} \\ \boldsymbol{f}_* \end{bmatrix} \sim \mathcal{N}\left(\mathbf{0}\ ,\ \begin{bmatrix} K(X,X) + \sigma_n^2 I & K(X,X_*) \\ K(X_*,X) & K(X_*,X_*) \end{bmatrix}\right)$$

Posterior distribution

- Condition on the observations:

$$\boldsymbol{f}_*|X_*, X, \boldsymbol{y} \sim \mathcal{N}\Big(\mu, \Sigma\Big)$$

with

$$\mu = K(X_*, X)\big[K(X, X) + \sigma_n^2 I\big]^{-1}\boldsymbol{y}$$

$$\Sigma = K(X_*, X_*) - K(X_*, X)\big[K(X, X) + \sigma_n^2 I\big]^{-1}K(X, X_*)$$

- Point prediction: μ

Kernel function

Squared exponential kernel function (with ARD):

$$k(\boldsymbol{x}, \boldsymbol{x}') = \sigma_f^2 \exp\left(-\frac{1}{2}\sum_{j=1}^{d}\frac{|x_j - x_j'|^2}{\ell_j^2}\right)$$

with hyperparameters σ_f and $\boldsymbol{\ell}$:

- σ_f^2 : signal variance
- $\ell_1, \ldots, \ell_d$: characteristic length-scale parameters

$\rightarrow$ Hyperparameters (including σ_n) are estimated on the training set, usually with MLE.

Machine learning for derivative pricing

- Construct a training set:

$$\text{product, market and model parameters} \xrightarrow{\text{time-consuming method}} \text{model price}$$

Machine learning for derivative pricing

- Construct a training set:

product, market and model parameters $\xrightarrow{\text{time-consuming method}}$ model price

$\downarrow$

$$\boxed{\text{sample } n \text{ random combinations } \boldsymbol{x_i} \longrightarrow \text{compute } n \text{ corresponding prices } y_i}$$

Machine learning for derivative pricing

- Construct a training set:

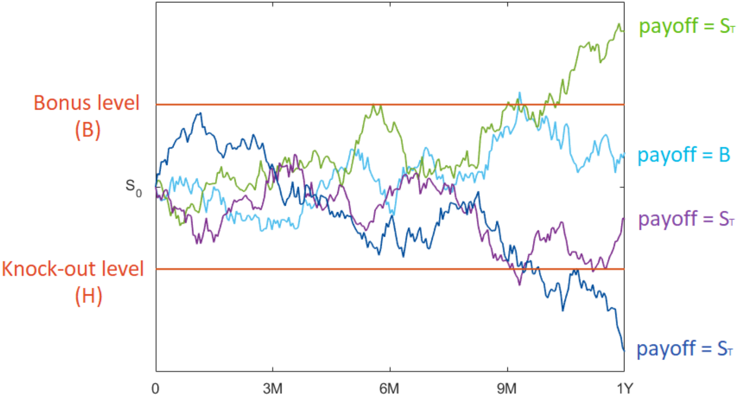
product, market and model parameters $\xrightarrow{\text{time-consuming method}}$ model price

- Fit a GPR model.

- Fast prediction of new model prices.

Case study:
pricing a structured product

Bonus certificate's payoffs



→ Price according to Heston model, using Monte Carlo simulation.

Build a training set

Parameter ranges:

Product/market	**Heston model** [(*)]
$B \in [105\%, 155\%]$	$\kappa \in [0.2, 1.6]$
$H \in [55\%, 95\%]$	$\rho \in [-0.95, -0.25]$
$T \in [11M, 1Y]$	$\theta \in [0.15, 0.65]$
$r \in [2\%, 3\%]$	$\eta \in [0.01, 0.25]$
$q \in [0\%, 5\%]$	$v_0 \in [0.01, 0.25]$

$\rightarrow$ sample parameter combinations + calculate corresponding prices.

(∗) κ = rate of mean reversion, ρ = correlation stock - vol, θ = vol-of-vol,
η = long run variance, v_0 = initial variance.

Pricing a bonus certificate

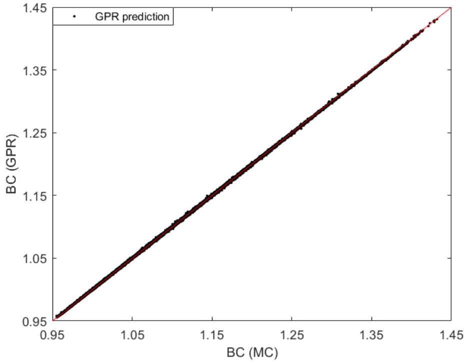
- Train a GPR model

$$\begin{matrix} B,\, H,\, T,\, r,\, q, \\ \kappa,\, \rho,\, \theta,\, \eta,\, v_0 \end{matrix} \xrightarrow{\text{learn this relation}} \text{BC price}$$

$\rightarrow$ matrix inversion + hyperparameter optimization

- Construct a test set (100 000 instances):
  - similarly to training set construction
  - slightly smaller parameter intervals

GPR's predictions



Maximal error

absolute:	0.0058
relative:	0.51%

Prediction:	84 seconds
Speed-up	$\times$ 2900

$\rightarrow$ GPR model trained on 10 000 samples, tested on 100 000 samples.

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Gradient Boosting Machines

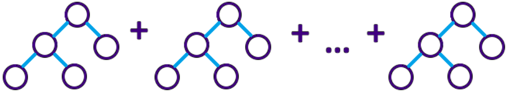
Gradient boosting machine (GBM)

= ensemble of regression trees h

$$\hat{y}_i \;=\; F_K(\boldsymbol{x_i}) \;=\; \sum_{k=0}^{K} f_k(\boldsymbol{x_i}) \;=\; c + \sum_{k=1}^{K} \nu h_k(\boldsymbol{x_i})$$

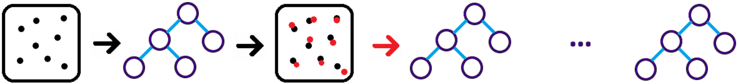
with $0 < \nu \leq 1$ a shrinkage parameter.

→ Many simple models f_k are combined into one strong model.



Gradient BOOSTING machine

- GBM is trained **stage-wise**:

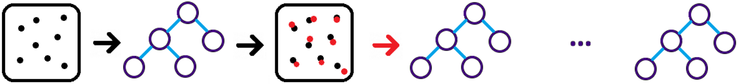


$\rightarrow$ Trees are added sequentially:

f_k tries to correct for the errors made by model F_{k-1}.

$\rightarrow$ Previously fitted trees are not readjusted.

GRADIENT boosting machine



How to measure the errors?

- Use pseudo-residuals

$$g_{i,k} = -\left[\frac{\partial \mathcal{L}(y_i, \hat{y})}{\partial \hat{y}}\right]_{\hat{y}=F_{k-1}(\boldsymbol{x_i})} \qquad i = 1, \ldots, n$$

where $\mathcal{L}$ is a loss function that measures how well the data is captured by the model.

Case study:
pricing a structured product

Boosting derivative pricing

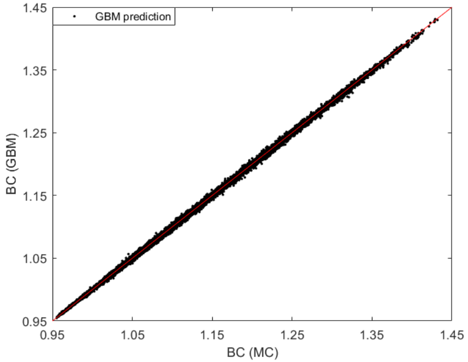
- Construct a training set.
- Train a GBM model:

$$\begin{matrix} B,\ H,\ T,\ r,\ q, \\ \kappa,\ \rho,\ \theta,\ \eta,\ v_0 \end{matrix} \xrightarrow{\text{learn this relation}} \text{BC price}$$

$\rightarrow$ LightGBM implementation.

- Fast prediction of new model prices.

GBM's predictions



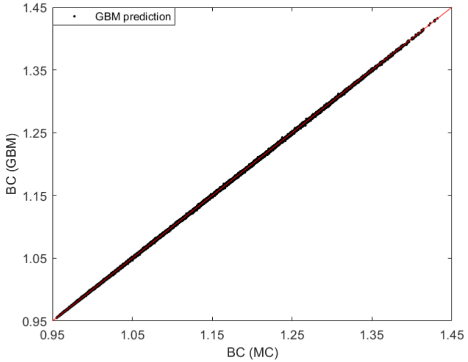
Maximal error

absolute:	0.0146
relative:	1.11%

Prediction:	42 seconds
Speed-up	$\times$ 5800

$\rightarrow$ GBM model trained on 10 000 samples, tested on 100 000 samples.

GBM's predictions



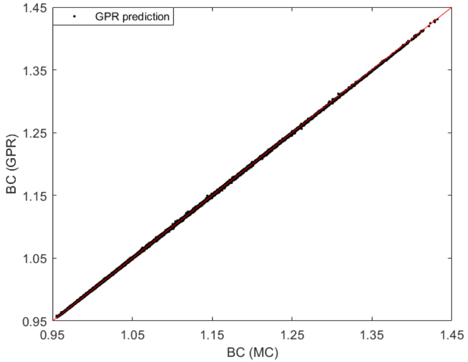
Maximal error

absolute: 0.0052
relative: 0.41%

Prediction: 42 seconds
Speed-up $\times$ 5800

$\rightarrow$ GBM model trained on 100 000 samples, tested on 100 000 samples.

GPR's predictions



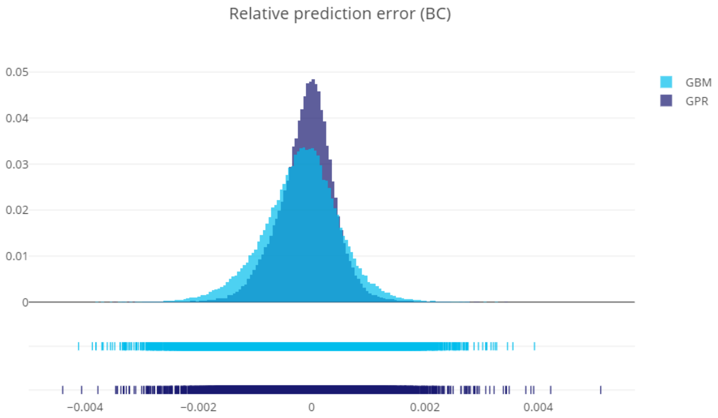
Maximal error

absolute:	0.0058
relative:	0.51%

Prediction:	84 seconds
Speed-up	$\times$ 2900

$\rightarrow$ GPR model trained on 10 000 samples, tested on 100 000 samples.

GBM vs. GPR



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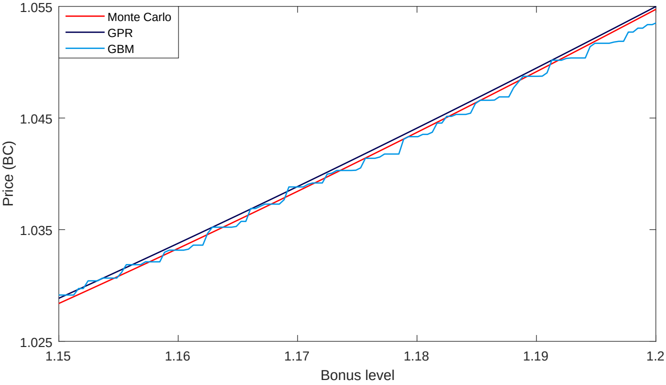
GPR predictions are smoother



Fixed parameters: $H = 80\%$, $T = 1Y$, $r = 2.5\%$, $q = 3\%$, $\kappa = 1$, $\eta = 0.05$, $\theta = 0.5$, $\rho = -0.7$, $v_0 = 0.05$.

GPR has built-in analytic derivatives

- GPR's point prediction for $\boldsymbol{x}_*$:

$$f^*(\boldsymbol{x}_*) = K(\boldsymbol{x}_*, X)\underbrace{\left[K(X,X) + \sigma_n^2 I\right]^{-1}\boldsymbol{y}}_{\alpha}$$

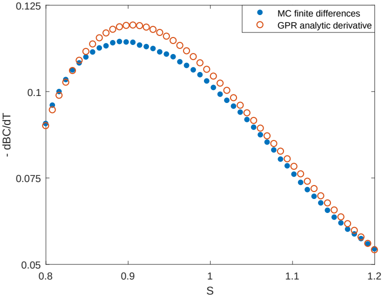
- Derivatives w.r.t. $\boldsymbol{x}_*$:

$$\frac{\partial f^*(\boldsymbol{x}_*)}{\partial \boldsymbol{x}_*} = \frac{\partial K(\boldsymbol{x}_*, X)}{\partial \boldsymbol{x}_*}\boldsymbol{\alpha} = -\Lambda^{-1}\tilde{X}_*{}^T\left[K(\boldsymbol{x}_*, X)^T \circ \boldsymbol{\alpha}\right]$$

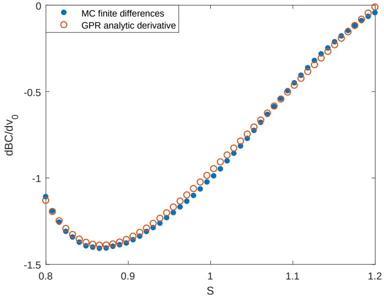
with

$$\tilde{X}_* = [\boldsymbol{x}_* - \boldsymbol{x}_1, \ldots, \boldsymbol{x}_* - \boldsymbol{x}_n]^T, \qquad \Lambda = \begin{bmatrix} \ell_1^2 & & 0 \\ & \ddots & \\ 0 & & \ell_d^2 \end{bmatrix}$$

Smooth Greek profiles for free!



(a) GPR's Theta



(b) GPR's Vega

$\rightarrow$ Monte Carlo simulations based on 100 million price paths.

Conclusion

- Time-consuming pricing methods
- Machine learning methods
  - Gaussian process regression (GPR)
  - Gradient boosting machines (GBM)
- Pricing a structured product:
  - Speed-ups of several orders of magnitude.
  - GBM prediction is faster, training scales more easily.
  - GPR provides smoother price predictions and Greeks.

Thank you!

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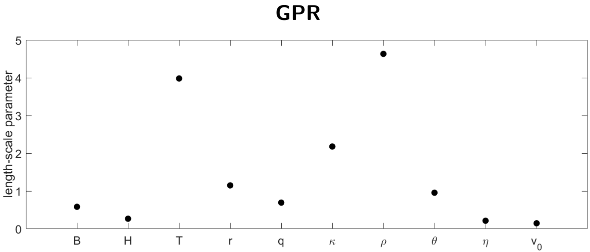
Hyperparameters

ntrees = 10 000	bagging_freq = 1
learning_rate = 0.2	bagging_fraction = 0.5
num_leaves = 16	L2_regularization = 0.8
min_obs_in_leaf = 24	booster = DART
max_bin = 255	drop_rate = 0.1



$$\sigma_f = 0.4580$$

$$\sigma_n = 0.0011$$