Interest Rate Theory

An Introduction

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1. General theory. Arbitrage. Completeness. Martingale measures,

2. The bond market.

3. Short rate models Affine term structures Inverting the yield curve

4. Forward rate models. Heath-Jarrow-Morton, Musiela.

5. Change of numeraire

6. New directions. LIBOR models. Credit risk.

7. Current research.

I

General Arbitrage Theory

Pricing financial derivatives

Definition:
A **contingent claim** (derivative) with **delivery time** T, is a random variable

$$X \in \mathcal{F}_T.$$

"At $t = T$ the amount X is paid to the holder of the claim".

Example: (European Call Option)

$$X = \max\left[S_T - K, 0\right]$$

(S_T =stock price at time T)

Let X be a contingent T-claim.

Problem: What is an "reasonable" price process $\Pi\left[t; X\right]$ for X?

Philosophy

- The derivative is **defined in terms of** underlying.

- The derivative can be **priced in terms of** underlying price.

- **Consistent** pricing.

- **Relative** pricing.

- No **mispricing** between derivative and underlying.

- No **arbitrage possibilities**.

Financial Markets

Price Process:

$$S(t) = [S_0(t), ..., S_N(t)]$$

$S_i(t) =$ price of asset i at time t. ($S_0 > 0$)

Example: (Black-Scholes, $S_0 := B$, $S_1 := S$)

$$\begin{aligned} dS &= \alpha S dt + \sigma S dW, \\ dB &= rB dt. \end{aligned}$$

Portfolio:

$$h(t) = [h_0(t), ..., h_N(t)]$$

$h_i(t) =$ number of units of asset i at time t.

Value Process:

$$V_h(t) = \sum_{i=0}^{N} h_i(t) S_i(t) = h(t) S(t)$$

Self Financing Portfolios

Definition: (intuitive)
A portfolio is **self-financing** if there is no exogenous infusion or withdrawal of money. "The purchase of a new asset must be financed by the sale of an old one."

Definition: (mathematical)
A portfolio is **self-financing** if the value process satisfies

$$dV_h(t) = \sum_{i=0}^{N} h_i(t) dS_i(t)$$

Major insight:
If the price process S is a **martingale**, and if h is **self-financing**, then V_h is a **martingale**.

Arbitrage

A portfolio h is an **arbitrage strategy** if

- h is self financing
- $V_h(0) = 0$
- $P(V_h(T) > 0) = 1$

  or more precisely

$$P(V_h(T) \geq 0) = 1$$
$$P(V_h(T) > 0) > 0$$

Interpretation:
An arbitrage possibilty is a serious case of mis-pricing on the market.

Main Question: When is the market free of arbitrage?

Absence of Arbitrage

The market is arbitrage free

iff

There exists a probability measure $Q \sim P$ such that all normalized price processes are **Q-martingales**.

i.e.

$$Z(t) = \frac{S(t)}{S_0(t)} = [1, Z_1(t), ..., Z_N(t)]$$

is a Q martingale.

i.e.

$$E^Q\left[Z_i(s) | \mathcal{F}_t\right] = Z_i(t), \quad t \leq s$$

Choice of Numeraire

The **numeraire** price S_0 can be chosen arbitrarily. Typically we choose the **riskless asset**, i.e.

$$S_0(t) = B(t)$$

where

$$dB(t) = r(t)B(t)dt$$

$$B(t) = e^{\int_0^t r(s)ds}$$

B = The **money account** (a bank with short rate r).

In this case Q is called the "risk neutral" measure.

Pricing

Definition:
A **contingent claim** with **delivery time** T, is a random variable

$$X \in \mathcal{F}_T.$$

"At $t = T$ the amount X is paid to the holder of the claim".

Example: (European Call Option)

$$X = \max\left[S_T - K, 0\right]$$

Let X be a contingent T-claim.

Main Pricing Problem:
What is an arbitrage free price process $\Pi\left[t; X\right]$ for X?

Solution: The extended market

$$S_t, \quad \Pi\left[t;X\right]$$

must be free of arbitrage. In particular, the process $\frac{\Pi[t;X]}{B(t)}$ must be a martingale, under some martingale measure Q, i.e.

$$\frac{\Pi\left[t;X\right]}{B(t)} = E^Q\left[\left.\frac{\Pi\left[T;X\right]}{B(T)}\right|\mathcal{F}_t\right]$$

Pricing formula:

$$\Pi\left[t;X\right] = E^Q\left[\left.e^{-\int_t^T r(s)ds} \times X\right|\mathcal{F}_t\right]$$

Black-Scholes Model:

$$\Pi\left[t; X\right] = e^{-r(T-t)} E^{Q}\left[X \mid \mathcal{F}_t\right]$$

Q-dynamics:

$$dS = rSdt + \sigma S d\tilde{W}.$$

Simple claims:

$$X = \Phi(S_T),$$

$$\Pi\left[t; X\right] = e^{-r(T-t)} E^{Q}\left[\Phi(S_T) \mid \mathcal{F}_t\right]$$

Kolmogorov $\Rightarrow$

$$\Pi\left[t; X\right] = F(t, S_t).$$

$F(t, s)$ solves the Black-Scholes equation:

$$\begin{cases} \frac{\partial F}{\partial t} + rs\frac{\partial F}{\partial s} + \frac{1}{2}\sigma^2 s^2 \frac{\partial^2 F}{\partial s^2} - rF & = & 0, \\ F(T, s) & = & \Phi(s). \end{cases}$$

Risk neutral dynamics

- For every arbitrage free price process Π_t, the process
$$\frac{\Pi_t}{B_t}$$
is a Q-martingale.

- The Q-dynamics of Π_t are of the form:
$$d\Pi_t = r_t \Pi_t dt + dM_t$$
where M is a Q-martingale

Problem: What if there are several different martingale measures Q?

Hedging

Def: A portfolio is a **hedge** against X ("replicates X") if

- h is self financing
- $V_h(T) = X, \quad P - a.s.$

Def: The market is **complete** if every X can be hedged.

Pricing Formula:
If h replicates X, then a natural way of pricing X is

$$\Pi[t; X] = V_h(t)$$

When can we hedge?

Existence of hedge

$$\Updownarrow$$

Existence of stochastic integral representation

Fix T-claim X.

If h is a hedge for X then

- $V^Z(T) = \frac{X}{B(T)}$

- h is self financing, i.e.

$$dV^Z = \sum_1^K h^i dZ_i$$

Thus V^Z is a Q-**martingale**.

$$V^Z(T) = E^Q \left[\frac{X}{B(T)} \middle| \mathcal{F}_t \right]$$

Lemma:
Fix T-claim X. Define martingale M by

$$M(t) = E^Q \left[\frac{X}{B(T)} \middle| \mathcal{F}_t \right]$$

Suppose that there exist predictable processes $h^1, \cdots, h^K$ such that

$$M(t) = x + \sum_{i=1}^{K} \int_0^t h^i(s) dZ_i(s),$$

Then X is attainable.

Proof: Easy.

Theorem:

The market is complete

iff

the martingale measure Q is unique.

Main Results:

- The market is arbitrage free $\Leftrightarrow$ There exists a martingale measure Q

- The market is complete $\Leftrightarrow$ Q is unique.

- Every X must be priced by the formula
$$\Pi\left[t;X\right] = E^Q\left[e^{-\int_t^T r(s)ds} \times X \middle| \mathcal{F}_t\right]$$
for some choice of Q.

- In a non-complete market, different choices of Q will produce different prices for X.

- For a hedgeable claim X, all choices of Q will produce the same price for X:
$$\Pi\left[t;X\right] = V_h(t) = E^Q\left[e^{-\int_t^T r(s)ds} \times X \middle| \mathcal{F}_t\right]$$

Metatheorem:

Assume that
N = Number of risky assets.

R = Number of independent sources of randomness.

Then the following hold.

- The market is arbitrage free **iff** $R \geq N$.

- The market is complete **iff** $R \leq N$.

- The market is arbitrage free and complete **iff** $R = N$.

Example: Black-Scholes Model:

$$\begin{aligned} dS &= \alpha S dt + \sigma S dW, \\ dB &= rB dt. \end{aligned}$$

For B-S we have $N = R = 1$. The market is arbitrage free and complete.

II

Interest Rate Theory

The Bond Market

Bonds: T-bond = Zero coupon bond, which pays 1 \$ at time of maturity T.

$$\begin{aligned} p(t,T) &= \text{price, at time } t\text{, of a } T\text{-bond.} \\ p(T,T) &= 1. \end{aligned}$$

Main problem:
Determine the **term structure**, i.e. the structure of $\{p(t,T);\ 0 \le t \le T,\ T \ge 0\}$ on an arbitrage free bond market.
Determine arbitrage free prices of other interest rate derivatives (interest rate options, swap rates, caps, floors etc.)

Interest Rate Options

Problem:
We want to price, at t, a European Call, with exercise date S, and strike price K, on an underlying T-bond. $(t < S < T)$.

Naive approach: Use Black-Scholes' formula.

$$F(t,p) = pN\left[d_1\right] - e^{-r(T-t)}KN\left[d_2\right].$$

$$\begin{aligned} d_1 &= \frac{1}{\sigma\sqrt{T-t}}\left\{\ln\left(\frac{p}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)\right\}, \\ d_2 &= d_1 - \sigma\sqrt{T-t}. \end{aligned}$$

where

$$p = p(t,T)$$

Is this allowed?

- p shall be the price of a traded asset. OK!

- The volatility of p must be constant. Here we have a problem because of **pull-to-par**, i.e. the fact that $p(T,T) = 1$. Bond volatilities will tend to zero as the bond approaches the time of maturity.

- The short rate must be **constant** and **deterministic**. Here the approach collapses completely, since the whole point of studying bond prices lies in the fact that interest rates are stochastic.

Deeply felt need

A consistent **arbitrage free** model for the bond market

Short Rate Models

Model: (Under the objective measure.)

P:

$$
\begin{aligned}
dr &= \mu(t,r)dt + \sigma(t,r)dW, \\
dB &= r(t)Bdt.
\end{aligned}
$$

Question: Are bond prices uniquely determined by the P-dynamics of r, and the requirement of an arbitrage free bond market?

NO!!

WHY?

Stock Models $\sim$ Interest Rates

Black-Scholes:

$$\begin{aligned} dS &= \alpha S dt + \sigma S dw, \\ dB &= rB dt. \end{aligned}$$

Interest Rates:

$$\begin{aligned} dr &= \mu(t,r)dt + \sigma(t,r)dW, \\ dB &= r(t)B dt. \end{aligned}$$

Question: What is the difference?

Answer: The short rate r is **not the price of a traded asset!**

1. **Meta Theorem:**

$$\begin{aligned} N &= 0, \text{ (No risky asset)} \\ R &= 1, \text{ (One source of randomness, } W) \end{aligned}$$

   Thus $M < R$. The market is incomplete.

2. **Martingale Measures:**
   If the money-account B is the only exogenously given asset, then **every** $Q \sim P$ is a martingale measure.
   The martingale measure is not unique, so the market is not complete.

3. **Hedging portfolios:**
   You are only allowed to invest your money in the bank, and then sit back and wait.

We have not enough underlying assets in order to price bonds.

- There is **not** a unique price for a **particular** $T-$bond.

- In order to avoid arbitrage, bonds of **different maturities** have to satisfy internal **consistency** relations.

- If we take **one** "benchmark" T_0-bond as given, then all other bonds can be priced **in terms of** the market price of the benchmark bond.

Assumption:

$$\begin{aligned} p(t,T) &= F(t,r(t);T) \\ p(t,T) &= F^T(t,r(t)), \\ F^T(T;T) &= 1. \end{aligned}$$

Self Financing Portfolios

Definition:
A portfolio is **self-financing** if the value process satisfies

$$dV_h(t) = \sum_{i=0}^{N} h_i(t) dS_i(t)$$

Introduce **portfolio weights**

$$u^i = \frac{h_i(t)S_i(t)}{V_t}$$

Portfolio dynamics:

$$dV_t = V_t \cdot \sum_i u_i(t) \frac{dS_i(t)}{S_i(t)}$$

(Compare with CAPM)

Program:

- Form portfolio based on $T-$ and $S-$bonds. Use Itô on $F^T(t, r(t))$ to get bond- and portfolio dynamics.

$$dV = V \left\{ u_T \frac{dF^T}{F^T} + u_S \frac{dF^S}{F^S} \right\}$$

- Choose portfolio weights such that the $dW-$ term vanishes. Then we have

$$dV = V \cdot k dt,$$

  (“synthetic bank” with k as the short rate)

- Absence of arbitrage $\Rightarrow k = r$.

- Read off the relation $k = r$!

From Itô:

$$dF^T = F^T \alpha_T dt + F^T \sigma_T d\tilde{W},$$

where

$$\begin{cases} \alpha_T &= \frac{F^T_t + \mu F^T_r + \frac{1}{2}\sigma^2 F^T_{rr}}{F^T}, \\ \sigma_T &= \frac{\sigma F^T_r}{F^T}. \end{cases}$$

Portfolio dynamics

$$dV = V \left\{ u^T \frac{dF^T}{F^T} + u^S \frac{dF^S}{F^S} \right\}.$$

Reshuffling terms gives us

$$dV = V \cdot \left\{ u^T \alpha_T + u^S \alpha_S \right\} dt + V \cdot \left\{ u^T \sigma_T + u^S \sigma_S \right\} dW.$$

Let the portfolio weights solve the system

$$\begin{cases} u^T + u^S &= 1, \\ u^T \sigma_T + u^S \sigma_S &= 0. \end{cases}$$

$$\begin{cases} u^T & = & -\frac{\sigma_S}{\sigma_T - \sigma_S}, \\ u^S & = & \frac{\sigma_T}{\sigma_T - \sigma_S}, \end{cases}$$

Portfolio dynamics

$$dV = V \cdot \left\{ u^T \alpha_T + u^S \alpha_S \right\} dt.$$

i.e.

$$dV = V \cdot \left\{ \frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S} \right\} dt.$$

Absence of arbitrage requires

$$\frac{\alpha_S \sigma_T - \alpha_T \sigma_S}{\sigma_T - \sigma_S} = r$$

which can be written as

$$\frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}.$$

$$\frac{\alpha_S(t) - r(t)}{\sigma_S(t)} = \frac{\alpha_T(t) - r(t)}{\sigma_T(t)}.$$

Note!

The quotient does **not** depend upon the particular choice of maturity date.

Result:
Assume that the bond market is free of arbitrage. Then there exists a universal process λ, such that

$$\frac{\alpha_T(t) - r(t)}{\sigma_T(t)} = \lambda(t),$$

holds for all t and for every choice of maturity T.

NB: The same λ for all choices of T.

$$\begin{aligned} \lambda &= \text{Risk premium per unit of volatility} \\ &= \text{“Market Price of Risk” (cf. CAPM).} \end{aligned}$$

Slogan:
“On an arbitrage free market all bonds have the same market price of risk.”

The relation

$$\frac{\alpha_T - r}{\sigma_T} = \lambda$$

is actually a PDE!

The Term Structure Equation

$$\begin{aligned} F_t^T + \{\mu - \lambda\sigma\} F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T &= 0, \\ F^T(T,r) &= 1. \end{aligned}$$

P-**dynamics:**

$$dr = \mu(t,r)dt + \sigma(t,r)dW.$$

$$\lambda = \frac{\alpha_T - r}{\sigma_T}, \quad \text{for all } T$$

In order to solve the TSE we need to know λ.

General Term Structure Equation

Contingent claim:

$$X = \Phi(r(T))$$

Result:

The price is given by

$$\Pi[t; X] = F(t, r(t))$$

where F solves

$$\begin{aligned} F_t + \{\mu - \lambda\sigma\} F_r + \frac{1}{2}\sigma^2 F_{rr} - rF &= 0, \\ F(T, r) &= \Phi(r). \end{aligned}$$

In order to solve the TSE we need to know λ.

Question:

Who determines λ?

Answer:

THE MARKET!

Moral

- Since the market is incomplete the requirement of an arbitrage free bond market will not lead to unique bond prices.

- Prices on bonds and other interest rate derivatives are determined by two main factors.

  1. **Partly** by the requirement of an arbitrage free bond market (the pricing functions satisfies the TSE).

  2. **Partly** by supply and demand on the market. These are in turn determined by attitude towards risk, liquidity consideration and other factors. All these are aggregated into the particular λ used (implicitly) by the market.

Risk Neutral Valuation

Using Feynmac–Kač we obtain

$$\Pi\left[t;X\right] = E^{Q}_{t,r}\left[e^{-\int_t^T r(s)ds} \times X\right].$$

Q-dynamics:

$$dr = \{\mu - \lambda\sigma\}dt + \sigma dW$$

III

Short Rate Models

Risk Neutral Valuation

$$\Pi\left[t; X\right] = E^{Q}_{t,r}\left[e^{-\int_t^T r(s)ds} \times X\right]$$

Q**-dynamics:**

$$dr = \{\mu - \lambda\sigma\}dt + \sigma dW$$

- Price = expected value of future payments
- The expectation should **not** be taken under the "objective" probabilities P, but under the "risk adjusted" probabilities Q.

Martingale Modelling

- All prices are determined by the Q-dynamics of r.

- Model dr directly under Q!

Problem: Parameter estimation!

Martingale pricing

Q-dynamics:

$$dr = \mu(t,r)dt + \sigma(t,r)dW$$

$$\begin{aligned} \Pi\left[t;X\right] &= E^Q\left[e^{-\int_t^T r(s)ds} \times X \middle| \mathcal{F}_t \right] \\ p(t,T) &= E^Q\left[e^{-\int_t^T r(s)ds} \times 1 \middle| \mathcal{F}_t \right] \end{aligned}$$

The Case $\mathbf{X} = \Phi(r(T))$:

The price is given by

$$\Pi\left[t;X\right] = F\left(t, r(t)\right)$$

$$\begin{cases} F_t + \mu F_r + \frac{1}{2}\sigma^2 F_{rr} - rF &= 0, \\ F(T,r) &= \Phi(r(T)). \end{cases}$$

(Term Structure Equation)

1. VasičeK

$$dr = (b - ar)\,dt + \sigma dV,$$

2. Cox-Ingersoll-Ross

$$dr = (b - ar)\,dt + \sigma\sqrt{r}dV,$$

3. Dothan

$$dr = ardt + \sigma r dV,$$

4. Black-Derman-Toy

$$dr = a(t)rdt + \sigma(t)rdV,$$

5. Ho-Lee

$$dr = a(t)dt + \sigma dV,$$

6. Hull-White (extended Vasiček)

$$dr = \{\Phi(t) - ar\}\,dt + \sigma dV,$$

Bond Options

European call on a T-bond with strike price K and delivery date S.

$$\begin{aligned} X &= \max\left[p(S,T) - K,\ 0\right] \\ X &= \max\left[F^T(S, r(S)) - K,\ 0\right] \end{aligned}$$

$$\begin{aligned} F_t^T + \mu F_r^T + \frac{1}{2}\sigma^2 F_{rr}^T - rF^T &= 0, \\ F^T(T, r) &= 1. \end{aligned}$$

$$\Phi(r) = \max\left[F^T(S, r) - K,\ 0\right]$$

$$\begin{aligned} F_t + \mu F_r + \frac{1}{2}\sigma^2 F_{rr} - rF &= 0, \\ F(S, r) &= \Phi(r(S)). \end{aligned}$$

$$\Pi\left[t; X\right] = F(t, r(t))$$

Affine Term Structures

Lots of equations!

Need analytic solutions.

We have an **Affine Term Structure** if

$$F(t, r; T) = e^{A(t,T) - B(t,T)r},$$

where A and B are deterministic functions.

Problem: How do we specify μ and σ in order to have an ATS?

Proposition: Assume that μ and σ are of the form

$$\begin{array}{rcl} \mu(t,r) & = & \alpha(t)r + \beta(t), \\ \sigma^2(t,r) & = & \gamma(t)r + \delta(t). \end{array}$$

Then the model admits an affine term structure

$$F(t,r;T) = e^{A(t,T)-B(t,T)r},$$

where A and B satisfy the system

$$\left\{ \begin{array}{rcl} B_t(t,T) & = & -\alpha(t)B(t,T) + \frac{1}{2}\gamma(t)B^2(t,T) - 1, \\ B(T;T) & = & 0. \end{array} \right.$$

$$\left\{ \begin{array}{rcl} A_t(t,T) & = & \beta(t)B(t,T) - \frac{1}{2}\delta(t)B^2(t,T), \\ A(T;T) & = & 0. \end{array} \right.$$

Inverting the Yield Curve

Q-dynamics with parameter vector α:

$$dr = \mu(t, r; \alpha)dt + \sigma(t, r; \alpha)dV$$

$$\Downarrow$$

Theoretical term structure

$$\{p(0, T; \alpha);\ \ T \geq 0\}$$

Observed term structure:

$$\{p^\star(0, T);\ \ T \geq 0\}.$$

Want: A model such that **theoretical** prices fit the **observed** prices of today, i.e. choose parameter vector α such that

$$p(0, T; \alpha) \approx \{p^\star(0, T);\ \ \forall T \geq 0\}$$

Number of equations = ∞ (one for each T).
Number of unknowns =dim(α)

Need: Infinite dimensional parameter vector.

Hull-White

Q-dynamics:

$$dr = \{\Phi(t) - ar\}\, dt + \sigma dV(t),$$

$$p(t,T) = e^{A(t,T) - B(t,T) r(t)},$$

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}$$

The **instantaneous forward rate** at T, contracted at t is given by

$$f(t,T) = -\frac{\partial \log p(t,T)}{\partial T}.$$

Fit the observed forward rate curve!

Result: The Hull-White model can be fitted exactly to any observed initial term structure. The calibrated model takes the form

$$p(t,T) = \frac{p(0,T)}{p(0,t)} \times e^{C(t,r(t))}$$

where C is given by

$$B(t,T)f^{\star}(0,t) - \frac{\sigma^2}{2a^2}B^2(t,T)\left(1 - e^{-2aT}\right) - B(t,T)r(t)$$

Analytical formulas for bond-options.

Models Based on the Short Rate

Pro:

- Easy to model Markov structure for r.
- Analytical expressions for bond prices and derivatives.

Con:

- Inverting the yield curve can be hard.
- Hard to model a flexible volatility structure for forward rates.
- One factor models implies perfect correlation along the yield curve.

IV

Forward Rate Models

Riskless Interest Rates

At time t:

- Sell one S-bond.

- Buy exactly $p(t, S)/p(t, T)$ $T-$bonds.

- Zero net investment.

At time S:

- Pay out 1$

At time T:

- Receive $p(t, S)/p(t, T) \cdot 1$$.

Net Effect

- The contract is made at t.
- An investment of 1 at time S has yielded $p(t,S)/p(t,T)$ at time T.
- The equivalent constant rates, R, are given as the solutions to

  Continuous rate:

  $$e^{R\cdot(T-S)} \cdot 1 = \frac{p(t,S)}{p(t,T)}$$

  Simple rate:

  $$[1 + R \cdot (T - S)] \cdot 1 = \frac{p(t,S)}{p(t,T)}$$

Continuous Interest Rates

1. The **forward rate for the period** $[S, T]$**, contracted at** t is defined by

$$R(t; S, T) = -\frac{\log p(t, T) - \log p(t, S)}{T - S}.$$

2. The **spot rate**, $R(S, T)$, for the period $[S, T]$ is defined by

$$R(S, T) = R(S; S, T).$$

3. The **instantaneous forward rate at** T**, conracted at** t is defined by

$$f(t, T) = -\frac{\partial \log p(t, T)}{\partial T} = \lim_{S \to T} R(t; S, T).$$

4. The **instantaneous short rate at** t is defined by

$$r(t) = f(t, t).$$

Simple Rates
(LIBOR)

1. The **simple forward rate L(t;S,T)for the period** $[S,T]$**, contracted at** t is defined by

$$L(t; S, T) = \frac{1}{T - S} \cdot \frac{p(t, S) - p(t, T)}{p(t, T)}$$

2. The **simple spot rate,** $L(S,T)$, for the period $[S,T]$ is defined by

$$L(S, T) = \frac{1}{T - S} \cdot \frac{1 - p(S, T)}{p(S, T)}$$

Bond prices $\sim$ LIBOR rates

The **simple spot rate,** $L(T, T+\delta)$, for the period $[T, T+\delta]$ is given by

$$p(T, T+\delta) = \frac{1}{1 + \delta L(T, T+\delta)}$$

i.e.

$$L(T, T+\delta) = \frac{1}{\delta} \cdot \frac{1 - p(T, T+\delta)}{p(T, T+\delta)}$$

Bond Prices $\sim$ Forward Rates

$$p(t,T) = p(t,s) \cdot \exp\left\{-\int_s^T f(t,u)du\right\},$$

In particular

$$p(t,T) = \exp\left\{-\int_t^T f(t,s)ds\right\}.$$

Toolbox

Proposition:
If the forward rate dynamics under Q are given by

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW$$

Then the bond dynamics are given by

$$\begin{aligned} dp(t,T) &= p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right\}dt \\ &+ p(t,T)S(t,T)dW \end{aligned}$$

$$\begin{cases} A(t,T) &= -\int_t^T \alpha(t,s)ds, \\ S(t,T) &= -\int_t^T \sigma(t,s)ds \end{cases}$$

The Money Account

$$\begin{cases} dB(t) &= r(t)B(t)dt, \\ B(0) &= 1. \end{cases}$$

i.e.

$$B(t) = \exp\left\{\int_0^t r(s)ds\right\},$$

Model of a bank with stochastic short rate of interest r.

Money account as roll-over:

"Put all your money in just maturing bonds"

$$dV_t^x = h_t^x dp(t, t+x)$$

We have

$$h_t^x = \frac{V_t^x}{p(t, t+x)}$$

so

$$dV_t^x = V_t^x \frac{dp(t, t+x)}{p(t, t+x)}$$

From toolbox

$$dV_t^x = V_t^x \frac{r_t}{p(t, t+x)} dt + V_t^x \frac{1}{p(t, t+x)} S(t, t+x) dW_t$$

As $x \to 0$, $p(t, t+x) \to 1$ and $S(t, t+x) \to 0$ so we obtain:

Roll-over dynamics:

$$dV = r(t)V dt.$$

We need measure valued portfolios!

Heath-Jarrow-Morton

Idea: Model the dynamics for the **entire yield curve**.

The yield curve itself (rather than the short rate r) is the explanatory variable.

Model forward rates. Use observed yield curve as boundary value.

Dynamics:

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t), \\ f(0,T) &= f^{\star}(0,T). \end{aligned}$$

One SDE for every fixed maturity time T.

Existence of martingale measure

$$
\begin{aligned}
f(t,T) &= \frac{\partial \log p(t,T)}{\partial T} \\
p(t,T) &= \exp\left\{-\int_t^T f(t,s)ds\right\}
\end{aligned}
$$

Thus:

Specifying forward rates.

$\Longleftrightarrow$

Specifying bond prices.

Thus:

No arbitrage

$\Downarrow$

restrictions on α and σ.

$\mathbb{P}$-dynamics:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\tilde{W}(t)$$

Look for Girsanov transformation $P \to Q$, s.t.

$\mathbb{Q}$-dynamics:

$$dp(t,T) = r(t)p(t,T)dt + p(t,T)v(t,T)d\tilde{W}(t)$$

Toolbox:

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)d\tilde{W}$$

$$\Downarrow$$

$$\begin{aligned} dp(t,T) &= p(t,T)\left\{r(t) + A(t,T) + \frac{1}{2}\|S(t,T)\|^2\right\}dt \\ &+ p(t,T)S(t,T)dW \end{aligned}$$

$$\begin{cases} A(t,T) &= -\int_t^T \alpha(t,s)ds, \\ S(t,T) &= -\int_t^T \sigma(t,s)ds \end{cases}$$

Girsanov:

$$\begin{aligned} dL(t) &= L(t)G(t)d\tilde{W}(t), \\ L(0) &= 1. \end{aligned}$$

Q-dynamics:

$$\begin{aligned} dp(t,T) &= p(t,T)r(t)dt \\ &+ \left\{A(t,T) + \frac{1}{2}||S(t,T)||^2 + S(t,T)g(t)\right\} dt \\ &+ p(t,T)S(t,T)dW(t), \end{aligned}$$

Proposition:
$\exists$ a martingale measure

$$\Updownarrow$$

$\exists$ process $g(t) = [g_1(t), \cdots g_d(t)]$ s.t.

$$A(t,T) + \frac{1}{2}||S(t,T)||^2 + S(t,T)g(t) = 0, \quad \forall t, T$$

alternatively

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds - \sigma(t,T)g(t), \quad \forall t, T$$

- Specify arbitrary volatilities $\sigma(t,T)$.

- Fix d "benchmark" maturities $T_1, \cdots, T_d$. For these maturities, specify drift terms $\alpha(t,T_1), \cdots \alpha(t,T_1)$.

- The Girsanov kernel is uniquely determined (for each fixed t) by

$$\begin{aligned}\sum_{i=1}^{d} \sigma_i(t,T_j) g_i(t) &= \sum_{i=1}^{d} \sigma_i(t,T_j) \int_0^T \sigma_i(t,s)ds \\ &- \alpha(t,T_j), \quad j = 1, \cdots d.\end{aligned}$$

- Thus Q is uniquely determined.

- All other drift terms will be uniquely defined by

$$\alpha(t,T) = \sigma(t,T) \int_t^T \sigma(t,s)ds - \sigma(t,T)g(t), \;\; \forall t, T$$

Martingale Modelling

Q-**dynamics:**

$$df(t,T) = \alpha(t,T)dt + \sigma(t,T)dW(t)$$

Specifying forward rates.

$$\Longleftrightarrow$$

Specifying bond prices.

Thus:

The process $P(t,T)/B_t$ is a Q martingale for every T

$$\Downarrow$$

restrictions on α and σ.

Which?

$$\text{Martingale modelling}$$
$$\Updownarrow$$
$$P = Q$$
$$\Updownarrow$$
$$g \equiv 0$$

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds - \sigma(t,T)g(t), \quad \forall t, T$$

Theorem: (HJM drift Condition) The bond market is arbitrage free if and only if

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds.$$

Moral: Volatility can be specified freely. The forward rate drift term is then uniquely determined.

Uniqueness of Q

Proposition: The following conditions are equivalent

1. The martingale measure Q is unique.

2. For each fixed t, there exist maturities $T_1, \cdots, T_d$ (which may depend on t) such that the matrix

$$D(t; T_1, \cdots, T_d)_{i,j} = \sigma_i(t, T_j)$$

is nonsingular.

3. For each fixed t, there exist maturities $T_1, \cdots, T_d$ (which may depend on t) such that the matrix

$$H(t; T_1, \cdots, T_d)_{i,j} = S_i(t, T_j)$$

is nonsingular.

Proposition Assume that

1. For each t, ω the functions

$$\sigma_1(t,T), \cdots, \sigma_d(t,T)$$

   are real analytic in the T-variable.

2. For each t, ω the functions

$$\sigma_1(t,T), \cdots, \sigma_d(t,T)$$

   are linearly independent (as functions of T).

Then, for each fixed t, it is possible to choose volatilities $T_1, \cdots, T_d$ such that the matrix

$$\left\{ S_i(t,T_j) \right\}_{i,j}$$

is nonsingular. Apart from a finite set of forbidden points these volatilities can be chosen freely as long as they are distinct.

Musiela parametrization

Parameterize forward rates by the time **to** maturity (x), rather than time **of** maturity (T).

Def:

$$r(t, x) = f(t, t + x).$$

Q-dynamics:

$$df(t, T) = \alpha(t, T)dt + \sigma(t, T)dV.$$

$$dr(t, x) = \alpha_m(t, x)dt + \sigma_m(t, x)dV.$$

What are the relations between drifts and volatilities under Q?

$$\begin{aligned} dr(t,x) &= d\left[f(t,t+x)\right] \\ &= df(t,t+x) + f_T(t,t+x)dt \\ &= \left\{\alpha(t,t+x) + r_x(t,x)\right\} dt + \sigma(t,t+x)dV \end{aligned}$$

$$\begin{aligned} \alpha_m(t,x) &= \alpha(t,t+x) + r_x(t,x) \\ \sigma_m(t,x) &= \sigma(t,t+x). \end{aligned}$$

HJM-condition:

$$\alpha(t,T) = \sigma(t,T)\int_t^T \sigma(t,s)ds.$$

HJMM forward rate equation:

$$\begin{aligned} dr(t,x) &= \left\{\frac{\partial}{\partial x}r(t,x) + \sigma_m(t,x)\int_0^x \sigma_m(t,y)dy\right\} dt \\ &+ \sigma_m(t,x)dV \end{aligned}$$

This is an SDE in infinite dimensional space. Connections to control theory.

Forward Rate Models

Pro:

- Easy to model flexible volatility structure for forward rates.
- Easy to include multiple factors.

Con:

- The short rate will typically not be a Markov process.
- Computational problems.

V

Change of Numeraire

Change of Numeraire

(Geman, Jamshidian, El Karoui)

Valuation formula:

$$\Pi\left[t;X\right]=E^{Q}\left[e^{-\int_t^T r(s)ds}\times X\middle|\mathcal{F}_t\right]$$

Hard to compute. Double integral.

Note: If X and r are **independent** then

$$\begin{aligned}\Pi\left[t;X\right] &= E^{Q}\left[e^{-\int_t^T r(s)ds}\middle|\mathcal{F}_t\right]\cdot E^{Q}\left[X\left|\mathcal{F}_t\right.\right]s \\ &= p(t,T)\cdot E^{Q}\left[X\middle|\mathcal{F}_t\right].\end{aligned}$$

Nice! We do not have to compute $p(t,T)$. It can be observed directly on the market! Single integral!

Sad Fact: X and r are (almost) never independent!

Idea: Use T-bond (for a fixed T) as numeraire. Define the **T-forward measure** Q^T by the requirement that

$$\frac{\Pi(t)}{p(t,T)}$$

is a Q^T-martingale for every price process $\Pi(t)$.

Then

$$\frac{\Pi[t;X]}{p(t,T)} = E^T\left[\left.\frac{\Pi[T;X]}{p(T,T)}\right|\mathcal{F}_t\right]$$

$$\Pi[T;X] = X, \qquad p(T,T) = 1.$$

$$\Pi[t;X] = p(t,T)E^T[X\,|\mathcal{F}_t]\,s$$

Do such measures exist?.

“The forward measure takes care of the stochastics over the interval $[t, T]$.”

Enormous computational advantages.

Useful for interest rate derivatives, currency derivatives and derivatives defined by several underlying assets.

General change of numeraire.

Idea: Use a fixed asset price process $S(t)$ as numeraire. Define the measure Q^S by the requirement that

$$\frac{\Pi(t)}{S(T)}$$

is a Q^S-martingale for every arbitrage free price process $\Pi(t)$.

Constructing $\mathbf{Q}^S$: Fix a T-claim X. From general theory:

$$\Pi[0; X] = E^Q\left[\frac{X}{B(T)}\right]$$

Assume that Q^S exists and denote

$$L(t) = \frac{dQ^S}{dQ}, \quad \text{on } \mathcal{F}_t$$

Then

$$\begin{aligned}
\frac{\Pi[0; X]}{S(0)} &= E^S\left[\frac{\Pi[T; X]}{S(T)}\right] = E^S\left[\frac{X}{S(T)}\right] \\
&= E^Q\left[L(T)\frac{X}{S(T)}\right]
\end{aligned}$$

Thus we have

$$\Pi[0; X] = E^Q\left[L(T)\frac{X \cdot S(0)}{S(T)}\right],$$

Natural candidate:

$$L(t) = \frac{dQ_t^S}{dQ_t} = \frac{S(t)}{S(0)B(t)}$$

Proposition:

$$\Pi\left(t\right)/B(t) \text{ is a } Q\text{-martingale.}$$
$$\Downarrow$$
$$\Pi\left(t\right)/S(t) \text{ is a } Q^{\star}\text{-martingale.}$$

Proof.

$$E^{\star}\left[\left.\frac{\Pi(t)}{S(t)}\right|\mathcal{F}_s\right]=\frac{E^Q\left[\left.L(t)\frac{\Pi(t)}{S(t)}\right|\mathcal{F}_s\right]}{L(s)}$$

$$=\quad\frac{E^Q\left[\left.\frac{\Pi(t)}{B(t)S(0)}\right|\mathcal{F}_s\right]}{L(s)}=\frac{\Pi(s)}{B(s)S(0)L(s)}$$

$$=\quad\frac{\Pi(s)}{S(s)}.\blacksquare$$

Result:

$$\Pi\left[t;X\right]=S(t)E^{S}\left[\left.\frac{X}{S(T)}\right|\mathcal{F}_t\right]$$

We can observe $S(t)$ directly on the market.

Example: $X=S(T)\cdot Y$

$$\Pi\left[t;X\right]=S(t)E^{S}\left[\left.Y\right|\mathcal{F}_t\right]$$

Several underlying:

$$X = \Phi\left[S_0(T), S_1(T)\right]$$

Assume Φ is linearly homogeous. Transform to Q^0.

$$\begin{aligned} \Pi\left[t; X\right] &= S_0(t) E^0\left[\left.\frac{\Phi\left[S_0(T), S_1(T)\right]}{S_0(T)}\right| \mathcal{F}_t\right] \\ &= S_0(t) E^0\left[\left.\varphi\left[Z(T)\right]\right| \mathcal{F}_t\right] \end{aligned}$$

$$\varphi\left[z\right] = \Phi\left[1, z\right], \qquad Z(t) = \frac{S_1(t)}{S_0(t)}$$

Exchange option:

$$X = \max\left[S_1(T) - S_0(T),\ 0\right]$$

$$\Pi\left[t; X\right] = S_0(t)E^0\left[\max\left[Z(T) - 1,\ 0\right] \middle| \mathcal{F}_t\right]$$

European Call on Z with strike price K. Zero interest rate.

Piece of cake!

Identifying the Girsanov Transformation

Assume Q-dynamics of S known as

$$dS(t) = r(t)S(t)dt + S(t)v(t)dW(t)$$

$$L(t) = \frac{S(t)}{S(0)B(t)}$$

Thus

$$dL(t) = L(t)v(t)dW(t).$$

The Girsanov kernel is given by the numeraire volatility $v(t)$.

Forward Measures

Use price of T-bond as numeraire.

$$L^T(t) = \frac{p(t,T)}{p(0,T)B(t)}$$

$$\begin{aligned} dp(t,T) &= r(t)p(t,T)dt + p(t,T)v(t,T)dW(t), \\ dL^T(t) &= L^T(t)v(t,T)dW(t) \end{aligned}$$

Result:

$$\Pi\left[t; X\right] = p(t,T)E^T\left[X \middle| \mathcal{F}_t\right]$$

Common Conjecture: "The forward rate is an unbiased estimator of the future spot rate:"

Lemma:

$$f(t,T) = E^T\left[r(T) \middle| \mathcal{F}_t\right]$$

A new look on option pricing

(Geman, El Karoui, Rochet)

European call on asset S with strike price K and maturity T.

$$X = \max\left[S(T) - K,\ 0\right]$$

$$\begin{aligned} \Pi\left[0; X\right] &= S(0) \cdot Q^{S}\left[S(T) \geq K\right] \\ &- K \cdot p(0,T) \cdot Q^{T}\left[S(T) \geq K\right] \end{aligned}$$

Analytical Results

Assumption: Assume that $Z_{S,T}$, defined by

$$Z_{S,T}(t) = \frac{S(t)}{p(t,T)},$$

has dynamics

$$dZ_{S,T}(t) = Z_{S,T}(t) m_T^S(t) dt + Z_{S,T}(t) \sigma_{S,T}(t) dW,$$

where $\sigma_{S,T}(t)$ is **deterministic**.

We have to compute

$$Q^T\left[S(T) \geq K\right]$$

and

$$Q^S\left[S(T) \geq K\right]$$

$$
\begin{aligned}
Q^T\left(S(T) \geq K\right) &= Q^T\left(\frac{S(T)}{p(T,T)} \geq K\right) \\
&= Q^T\left(Z_{S,T}(T) \geq K\right)
\end{aligned}
$$

By definition $Z_{S,T}$ is a Q^T-martingale, so Q^T-dynamics are given by

$$
dZ_{S,T}(t) = Z_{S,T}(t)\sigma_{S,T}(t)dW^T,
$$

with the solution

$$
Z_{S,T}(T) =
$$

$$
\frac{S(0)}{p(0,T)} \times \exp\left\{-\frac{1}{2}\int_0^T \sigma_{S,T}^2(t)dt + \int_0^T \sigma_{S,T}(t)dW^T\right\}
$$

Lognormal distribution!

The integral

$$\int_0^T \sigma_{S,T}(t)dW^T$$

is Gaussian, with zero mean and variance

$$\Sigma^2_{S,T}(T) = \int_0^T \|\sigma_{S,T}(t)\|^2 dt$$

Thus

$$Q^T\left(S(T) \geq K\right) = N[d_2],$$

$$d_2 = \frac{\ln\left(\frac{S(0)}{Kp(0,T)}\right) - \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}$$

$$
\begin{aligned}
Q^S\left(S(T) \geq K\right) &= Q^S\left(\frac{p(T,T)}{S(T)} \leq \frac{1}{K}\right) \\
&= Q^S\left(Y_{S,T}(T) \leq \frac{1}{K}\right),
\end{aligned}
$$

$$
Y_{S,T}(t) = \frac{p(t,T)}{S(t)} = \frac{1}{Z_{S,T}(t)}.
$$

$Y_{S,T}$ is a Q^S-martingale, so Q^S-dynamics are

$$
dY_{S,T}(t) = Y_{S,T}(t)\delta_{S,T}(t)dW^S.
$$

$$
Y_{S,T} = Z_{S,T}^{-1}
$$

$$
\Downarrow
$$

$$
\delta_{S,T}(t) = -\sigma_{S,T}(t)
$$

$$Y_{S,T}(T) =$$

$$\frac{p(0,T)}{S(0)} \exp\left\{-\frac{1}{2}\int_0^T \sigma_{S,T}^2(t)dt - \int_0^T \sigma_{S,T}(t)dW^S\right\},$$

$$Q^S\left(S(T) \geq K\right) = N[d_1],$$

$$d_1 = d_2 + \sqrt{\Sigma_{S,T}^2(T)}$$

Proposition: Price of call is given by

$$\Pi\,[0;X] = S(0)N[d_2] - K\cdot p(0,T)N[d_1]$$

$$d_2 \;=\; \frac{\ln\left(\frac{S(0)}{Kp(0,T)}\right) - \frac{1}{2}\Sigma^2_{S,T}(T)}{\sqrt{\Sigma^2_{S,T}(T)}}$$

$$d_1 \;=\; d_2 + \sqrt{\Sigma^2_{S,T}(T)}$$

$$\Sigma^2_{S,T}(T) \;=\; \int_0^T \|\sigma_{S,T}(t)\|^2 dt$$

Hull-White

Q-dynamics:

$$dr = \{\Phi(t) - ar\}\, dt + \sigma dW.$$

Affine term structure:

$$p(t,T) = e^{A(t,T)-B(t,T)r(t)},$$

$$B(t,T) = \frac{1}{a}\left\{1 - e^{-a(T-t)}\right\}.$$

Ckeck if Z has deterministic volatility

$$Z = \frac{S(t)}{p(t,T_1)}, \qquad S(t) = p(t,T_2)$$

$$Z(t) = \frac{p(t,T_2}{p(t,T_1)},$$

$$Z(t) = \exp\left\{\Delta A(t) - \Delta B(t) r(t)\right\},$$

$$Z(t) = \exp\left\{\Delta A(t) - \Delta B(t) r(t)\right\},$$

$$\begin{aligned} \Delta A(t) &= A(t, T_2) - A(t, T_1), \\ \Delta B(t) &= B(t, T_2) - B(t, T_1), \end{aligned}$$

$$dZ(t) = Z(t)\left\{\cdots\right\} dt + Z(t) \cdot \sigma_z(t) dW,$$

$$\sigma_z(t) = -\sigma \Delta B(t) = \frac{\sigma}{a} e^{at}\left[e^{-aT_1} - e^{-aT_2}\right]$$

Deterministic volatility!

VI

Some New Directions

1. LIBOR Models

(Miltersen-Sandman-Sondermann,
Brace-Gatarek-Musiela)

Problem: Many popular interest rate models (Vasicec, Hull-White) lead to negative interest rates with positive probability.

The Dothan model

$$dr = ardt + \sigma rdV,$$

gives a **lognormal** short rate of interest.

This implies $r > 0$. (Good!)

Lognormality also implies $E\left[B(t)\right] = +\infty$ (Bad!)

This also implies infinite values for Eurodollar futures (Ouch!)

A dilemma

- Lognormal modelling of **instantaneuos** forward rates implies exploding forward rates, infinite interest rates, zero bond prices and arbitrage possibilities.

- Despite this, the market continues happily to use Black type formulas, which assumes lognormality.

- If you cannot beat them, join them.

- Construct a model which leads to theoretically sound pricing formulas of the Black type!

Main Idea:

Focus on (non-infinitesimal) **market** forward rates (LIBOR rates).

$$L(t, T; \delta) = \frac{p(t, T) - p(t, T + \delta)}{\delta \cdot p(t, T + \delta)}$$

Typically $\delta = 1/4$ i.e. three months.

Model, for a fixed compounding period δ, the LIBOR rates "lognormally" as

$$dL(t, T) = \mu(t, T)dt + \gamma(t, T)L(t, T)dW,$$

where γ is **deterministic**.

Note:
Under the forward measure $Q^{T+\delta}$ the LIBOR rate $L(t, T) = L(t, T, T + \delta)$ is a **martingale**

Caps

Basic idea: Buy an insurance against high interest rates in the future.

1. The contract is written at $t = 0$. At that time also the **principal**, K, and the fixed **cap rate**, R are determined.

2. A cap is a sum of elementary contracts, so called **caplets**.

3. A caplet is active over the period $[T, T+\delta]$,

4. At time $T + \delta$ the holder of the caplet receives

   $$X = K\delta \max\left[L_T - R, 0\right] = K\delta\left(L_T - R\right)^+$$

   where L_t is the simple forward rate (LIBOR) for the period $[T, T + \delta]$, i.e.

   $$L_t = L(t, T, T + \delta)$$

Pricing a caplet

At time $T+\delta$ the holder of the caplet receives

$$X = K\delta\,\text{max}\,[L-R, 0] = K\delta\,(L-R)^{+}$$

$$\Pi\,[t; X] = p(t, T+\delta)K\delta E^{T+\delta}\left[(L-R)^{+}\Big|\,\mathcal{F}_t\right]$$

$Q^{T+\delta}$ dynamics of $L_t = L(t, T, T+\delta)$:

$$dL_t = \gamma(t, T)L_t dW_t$$

where W is $Q^{T+\delta}$ Wiener.

- If γ is deterministic we have lognormal forward rates under $Q^{t+\delta}$.
- We can use Black-Scholes type formulas.
- Calibrate γ from the cap curve.

2. Interest Rates with Jumps

(Shirakawa, Björk-Kabanov-Runggaldier, Jarrow-Madan, Duffie-Singelton)

BKR model:

$$\begin{aligned} df(t,T) &= \alpha(t,T)dt + \sigma(t,T)dW(t) \\ &+ \int_E \delta(t,T,x)\mu(dt,dx) \end{aligned}$$

$$dr(t) = a(t)r(t)dt + b(t)dW(t) + \int_E q(t,x)\mu(dt,dx),$$

Results.

- Need to consider measure valued portfolios.
- HJM type drift condition.
- Uniqueness of MG-measure only implies hedgeability on dense subspace.
- The hedging equation is generically ill posed.
- Existence of Affine Term Structures.
- Hard computational problems.

3. Risky Bonds

$$\begin{aligned} \tau &= \text{time of default} \\ X &= \text{nominal claim at time } T \\ X \cdot I\{\tau > T\} &= \text{actual pay-out at time } T \end{aligned}$$

How to model default?

- Default is triggered when the underlying value of the firm hits a barrier. (τ is predictable). Longstaff-Schwartz, Merton.

- Default is triggered by a Poisson type point event. (τ is totally inaccessible). Jarrow, Lando, Turnbull.

The simplest intensity model

Assumption:
The default time τ is the first jump time of a Cox process N with intensity process λ. The process λ is independent of r and X.

Result:
Denote risky bond price by $q(t,T)$. Then we have

$$q(0,T) = E^Q\left[e^{-\int_0^T R(s)ds} \cdot X\right]$$

where

$$R(s) = r(s) + \lambda(s).$$

"Risk adjusted discount factor"

$$Q\left(N(t)=k\,\middle|\,\mathcal{F}_t^{\lambda}\right)=e^{-\int_0^T\lambda(s)ds}\cdot\frac{\left(\int_0^T\lambda(s)ds\right)^k}{k!}$$

$$\begin{aligned}
q(0,T) &= E^Q\left[e^{-\int_0^T r(s)ds}\cdot X\cdot I\left\{N_T=0\right\}\right]\\
&= E^Q\left[E^Q\left[e^{-\int_0^T r(s)ds}\cdot X\cdot I\left\{N_T=0\right\}\middle|\,\mathcal{F}_T^{\lambda,r,X}\right]\right]\\
&= E^Q\left[e^{-\int_0^T r(s)ds}\cdot X\cdot E^Q\left[I\left\{N_T=0\right\}\middle|\,\mathcal{F}_T^{\lambda}\right]\right]\\
&= E^Q\left[e^{-\int_0^T r(s)ds}\cdot X\cdot e^{-\int_0^T\lambda(s)ds}\right]\\
&= E^Q\left[e^{-\int_0^T R(s)ds}\cdot X\right]
\end{aligned}$$

Markov chain models

- The company is described by a finite state Markov Chain (state = credit rating).

- The default state is absorbing.

- Easy to obtain good P-statistics for the intensity matrix. Hard to get Q-statisatics.

- Jarrow-Lando-Turnbull, Duffie-Singleton.

4. Further topics

- The potential approach to interest rates. (Rogers, Jin-Glasserman)

- Functional models. (Hunt-Kennedy-Pelsser)

- Positive interest rates. (Brody-Hughston)

- The geometric approach to the HJMM equation. (Björk, Christenssen, Filipovič, Landen, Svensson, Teichmann).

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