A Geometric View of Interest Rate Theory

Tomas Björk
Department of Finance
Stockholm School of Economics

Camilla Landén
Department of Mathematics
KTH, Stockholm

Lars Svensson
Department of Mathematics
KTH, Stockholm

Definitions:

$\mathbf{p}(t,x)$: Price, at t of zero coupon bond maturing at $t+x$,

$\mathbf{r}(t,x)$: Forward rate, contracted at t, maturing at $t+x$

$R(t)$: Short rate.

$$
\begin{aligned}
r(t,x) &= -\frac{\partial \log p(t,x)}{\partial x} \\
p(t,x) &= e^{-\int_0^x r(t,s)ds} \\
R(t) &= r(t,0).
\end{aligned}
$$

Heath-Jarrow-Morton-Musiela

Idea: Model the dynamics for the **entire forward rate curve**.

The yield curve itself (rather than the short rate R) is the explanatory variable.

Model forward rates. Use observed forward rate curve as initial condition.

Q-dynamics:

$$\begin{aligned} dr(t,x) &= \alpha(t,x)dt + \sigma(t,x)dW(t), \\ r(0,x) &= r_0^\star(x), \quad \forall x \end{aligned}$$

W: d-dimensional Wiener process

One SDE for every fixed x.

Theorem: (HJMM drift Condition) The following relations must hold, under a martingale measure Q.

$$\alpha(t,x) = \frac{\partial}{\partial x} r(t,x) + \sigma(t,x) \int_0^x \sigma(t,s)ds.$$

Moral: Volatility can be specified freely. The forward rate drift term is then uniquely determined.

The Interest Rate Model

$$r_t = r_t(\cdot), \quad \sigma(t,x) = \sigma(r_t, x)$$

Heath-Jarrow-Morton-Musiela equation:

$$dr_t = \mu_0(r_t)dt + \sigma(r_t)dW_t$$

$$\begin{aligned} \mu_0 &= \frac{\partial}{\partial x} r_t(x) + \sigma(r_t, x) \int_0^x \sigma(r_t, s) ds \\ \sigma &= \sigma(r_t, x) \end{aligned}$$

The HJMM equation is an **infinite dimensional SDE** evolving in the space $\mathcal{H}$ of forward rate curves.

A Hilbert Space

Definition:
For each $(\alpha, \beta) \in R^2$, the space $\mathcal{H}_{\alpha,\beta}$ is defined by

$$\mathcal{H}_{\alpha,\beta} = \{f \in C^{\infty}[0, \infty);\ \|f\| < \infty\}$$

where

$$\|f\|^2 = \sum_{n=0}^{\infty} \beta^{-n} \int_0^{\infty} \left[f^{(n)}(x)\right]^2 e^{-\alpha x} dx$$

where

$$f^{(n)}(x) = \frac{d^n f}{dt^n}(x).$$

We equip $\mathcal{H}$ with the inner product

$$(f, g) = \sum_{n=0}^{\infty} \beta^{-n} \int_0^{\infty} f^{(n)}(x) g^{(n)}(x) e^{-\alpha x} dx$$

Properties of $\mathcal{H}$

Proposition:
The following hold.

- The linear operator

  $$\mathbf{F} = \frac{\partial}{\partial x}$$

  is bounded on $\mathcal{H}$

- $\mathcal{H}$ is complete, i.e. it is a Hilbert space.

- The elements in $\mathcal{H}$ are real analytic functions on R (not only on R_+).

NB: Filipovic and Teichmann!

Stratonovich Form of HJMM

$$dr_t = \mu(r_t)dt + \sigma(r_t) \circ dW_t$$

where

$$\mu(r_t) = \mu_0(r_t) - \frac{1}{2}\frac{d\langle \sigma, W \rangle}{dt}$$

Main Point:
Using the Stratonovich differential we have no Itô second order term. Thus we can treat the SDE above as the ODE

$$\frac{dr_t}{dt} = \mu(r_t) + \sigma(r_t) \cdot v_t$$

where $v_t =$ "white noise".

Natural Questions

- What do the forward rate curves look like?
- What is the support set of the HJMM equation?
- When is a given model (e.g. Hull-White) consistent with a given family (e.g. Nelson-Siegel) of forward rate curves?
- When is the short rate Markov?
- When is a finite set of benchmark forward rates Markov?
- When does the interest rate model admit a realization in terms of a finite dimensional factor model?
- If there exists an FDR how can you construct a concrete realization?

Finite Dimensional Realizations

Main Problem:
When does a given interest rate model possess a finite dimensional realisation, i.e. when can we write r as

$$\begin{aligned} z_t &= \eta(z_t)dt + \delta(z_t) \circ dW(t), \\ r(t,x) &= G(z_t, x), \end{aligned}$$

where z is a **finite-dimensional** diffusion, and

$$G : R^d \times R_+ \to R$$

or alternatively

$$G : R^d \to \mathcal{H}$$

$\mathcal{H} =$ the space of forward rate curves

Examples:

$$\sigma(r,x) = e^{-ax},$$

$$\sigma(r,x) = xe^{-ax},$$

$$\sigma(r,x) = e^{-x^2},$$

$$\sigma(r,x) = \log\left(\frac{1}{1+x^2}\right),$$

$$\sigma(r,x) = \int_0^\infty e^{-s} r(s) ds \cdot x^2 e^{-ax}.$$

Which of these admit a finite dimensional realisation?

Invariant Manifolds

Def:
Consider an interest rate model

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t) \circ dW_t$$

on the space $\mathcal{H}$ of forward rate curves. A manifold (surface) $\mathcal{G} \subseteq \mathcal{H}$ is an **invariant manifold** if

$$r_0 \in \mathcal{G} \Rightarrow r_t \in \mathcal{G}$$

P-a.s. for all $t > 0$

Main Insight

There exists a finite dimensional realization.

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There exists a finite dimensional invariant manifold.

Characterizing Invariant Manifolds

Proposition: (Björk-Christensen)
Consider an interest rate model on Stratonovich form

$$dr_t = \mu(r_t)dt + \sigma(r_t) \circ dW_t$$

A manifold $\mathcal{G}$ is invariant under r if and only if

$$\begin{aligned} \mu(r) &\in T_{\mathcal{G}}(r), \\ \sigma(r) &\in T_{\mathcal{G}}(r), \end{aligned}$$

at all points of $\mathcal{G}$. Here $T_{\mathcal{G}}(r)$ is the tangent space of $\mathcal{G}$ at the point $r \in \mathcal{G}$.

Main Problem

Given:

- An interest rate model on Stratonovich form

$$dr_t = \mu(t, r_t)dt + \sigma(t, r_t) \circ dW_t$$

- An inital forward rate curve r_0:

$$x \longmapsto r_0(x)$$

Question:

When does there exist a finite dimensional manifold $\mathcal{G}$, such that

$$r_0 \in \mathcal{G}$$

and

$$\begin{aligned} \mu(r) &\in T_{\mathcal{G}}(r), \\ \sigma(r) &\in T_{\mathcal{G}}(r), \end{aligned}$$

A manifold satisfying these conditions is called a **tangential manifold**.

Lie Brackets

Definition:
Given two vector fields $f_1(r)$ and $f_2(r)$, their **Lie bracket** $[f_1, f_2]$ is a vector field defined by

$$[f_1, f_2] = (Df_2)f_1 - (Df_1)f_2$$

where D is the Frechet derivative.

Fact:
If $\mathcal{G}$ is tangential to f_1 and f_2, then it is also tangential to $[f_1, f_2]$.

Definition:
Given vector fields $f_1(r), \ldots, f_n(r)$, the **Lie algebra**

$$\{f_1(r), \ldots, f_n(r)\}_{LA}$$

is the smallest linear space of vector fields, containing $f_1(r), \ldots, f_n(r)$, which is closed under the Lie bracket.

Main result

- Given any fixed initial forward rate curve r_0, there exists a finite dimensional invariant manifold $\mathcal{G}$ with $r_0 \in \mathcal{G}$ if and only if the Lie-algebra

$$\mathcal{L} = \{\mu, \sigma\}_{LA}$$

is finite dimensional.

- Given any fixed initial forward rate curve r_0, there exists a finite dimensional realization if and only if the Lie-algebra

$$\mathcal{L} = \{\mu, \sigma\}_{LA}$$

is finite dimensional. The dimension of the realization equals $dim \{\mu, \sigma\}_{LA}$.

Deterministic Volatility

$$\sigma(t, r, x) = \sigma(x)$$

Consider a **deterministic** volatility function $\sigma(x)$. Then the Ito and Stratonovich formulations are the same:

$$dr = \{\mathbf{F}r + S\}\, dt + \sigma dW$$

where

$$\mathbf{F} = \frac{\partial}{\partial x}, \quad S(x) = \sigma(x) \int_0^x \sigma(s) ds.$$

The Lie algebra $\mathcal{L}$ is generated by the two vector fields

$$\mu(r) = \mathbf{F}r + S, \quad \sigma(r) = \sigma$$

Proposition:
For an FDR to exist σ has to be "quasi exponential", i.e. of the form

$$\sigma(x) = \sum_{i=1}^{n} p_i(x) e^{\alpha_i x}$$

where p_i is a polynomial.

Constant Direction Volatility

$$\sigma(t, r, x) = \varphi(r)\lambda(x)$$

Theorem
The model admits a finite dimensional realization if and only if λ is quasi-exponential. The scalar field $\varphi(r)$ can be arbitrary.

Short Rate Realizations

Question:
When is a given forward rate model realized by a short rate model?

$$\begin{aligned} r(t,x) &= G(t,R_t,x) \\ dR_t &= a(t,R_t)dt + b(t,R_t) \circ dW \end{aligned}$$

Answer:
There must exist a 2-dimensional realization. (With the short rate R and running time t as states).

Proposition: The model is a short rate model only if

$$dim\, \{\mu, \sigma\}_{LA} \leq 2$$

Theorem: The model is a generic short rate model if and only if

$$[\mu, \sigma]\, //\sigma$$

“All short rate models are affine“

Theorem: (Jeffrey) Assume that the forward rate volatitliy is of the form

$$\sigma(R_t, x)$$

Then the model is a generic short rate model if and only if σ is of the form

$$\begin{array}{rcll} \sigma(R,x) & = & c & \text{(Ho-Lee)} \\ \sigma(R,x) & = & ce^{-ax} & \text{(Hull-White)} \\ \sigma(R,x) & = & \lambda(x)\sqrt{aR+b} & \text{(CIR)} \end{array}$$

(λ solves a certain Ricatti equation)

Slogan:
Ho-Lee, Hull-White and CIR are the **only generic** short rate models.

Constructing an FDR

Problem:
Suppose that there actually **exists** an FDR, i.e. that

$$dim\,\{\mu, \sigma\}_{LA} < \infty.$$

How do you **construct** a realization?

Constructing the invariant manifold

Proposition:

Suppoose that the Lie algebra $\{\mu, \sigma\}_{LA}$ is spanned by the vector fields $f_1, \ldots, f_n$. Fix a point $r_0 \in X$. Then the induced invariant manifold is parametrized by

$$G : R^n \to \mathcal{G}$$

where

$$G(t_1, \ldots, t_n) = e^{f_n t_n} \ldots e^{f_2 t_2} e^{f_1 t_1} r_0$$

Here the operator $e^{f_i t}$ is defined as the flow mapping of the ODE

$$\frac{dr_t}{dt} = f_i(r_t)$$

Constructing a Realization:

- Choose a finite number of vector fields $f_1, \ldots, f_d$ which span $\{\mu, \sigma\}_{LA}$.

- Compute the invariant manifold $G(z_1, \ldots, z_d)$ using

$$G(t_1, \ldots, t_n) = e^{f_n t_n} \ldots e^{f_2 t_2} e^{f_1 t_1} r_0$$

- Make the *Ansatz*

$$dZ_t = a(Z_t)dt + b(Z_t) \circ dW_t.$$

- From $r_t = G(Z_t)$ it follows that

$$G_\star a = \mu, \qquad G_\star b = \sigma.$$

- Use these (linear!) equations to solve for the vector fields a and b.

Example: Deterministic Direction Volatility

Model:

$$\sigma_i(r, x) = \varphi(r)\lambda(x).$$

Minimal Realization:

$$\begin{cases} dZ_0 &= dt, \\ dZ_0^1 &= [c_0 Z_n^1 + \gamma\varphi^2(G(Z))]dt + \varphi(G(Z))dW_t, \\ dZ_i^1 &= (c_i Z_n^1 + Z_{i-1}^1)dt, \qquad i = 1, \dots, n, \\ dZ_0^2 &= [d_0 Z_q^2 + \varphi^2(G(Z))]dt, \\ dZ_j^2 &= (d_j Z_q^2 + Z_{j-1}^2)dt, \qquad j = 1, \dots, q. \end{cases}$$

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