"No-Good-Deal" Bounds

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A Short History of Bounds

Finding bounds on the values of derivatives is an old "art form":

- Merton (1973),
  - no arbitrage bounds,
- Perrakis and Ryan (1984), Levy (1985), Ritchken and Kuo (1989), Basso and Pianca (1994),
  - bounds based on stochastic dominance (or similar).

Interest in this topic has intensified, with more interest in:

- Levy processes, and other work related to
- incomplete markets

"No-Good-Deal" Bounds

"No-Good-Deal" Bounds were:

- introduced by Cochrane and Saá-Requejo in 1996,
- modified by Hodges (Generalized Sharpe Ratio) in 1997,
- generalized to a more abstract setting
  by Cerny and Hodges, 1998 (presented at Bachalier 2000),
- related to Artzner *et al* "coherent risk measures"
  by Jaschke and Kuchler (2001)
  (also anticipated in Mejía-Pérez, 1998, and Hodges 1998).

We examine these four themes in more detail

Cochrane and Saá-Requejo

Pricing bounds are constructed relative to a Sharpe Ratio (expected excess return / standard deviation).

Two cases are provided:

1. Unconstrained: the dual pricing vector is linear in wealth (and must go negative somewhere, unless it is a constant)
2. Constrained: the dual pricing vector is piece-wise linear in wealth, and is set equal to zero where it would otherwise go negative.

The Analysis

The first case is pure mean-variance analysis.
[See Cochrane (2001) for a clear exposition].
The formulation is:

$$\underline{C} = \min_{m} E(mx^c) \text{ s.t. } \mathbf{p} = E(m\mathbf{x}), E(m^2) \leq A^2.$$

The payoff x^c is decomposed into its projection in the space of traded assets (the approximate hedge) and an orthogonal residual, w.

$$x^c = \hat{x}^c + w, \text{ where } \hat{x}^c = E(x^c\mathbf{x}')E(\mathbf{x}\mathbf{x}')^{-1}\mathbf{x}.$$

We can get further insights using the Treynor-Black (1973) analysis:

Treynor-Black (1973) analysis:

The square of the Sharpe Ratio is the sum of the squares of the Sharpe Ratios of each separate orthogonal bet.
If we let h_0 denote the Sharpe Ratio attainable from the basis assets, then in the notation of the paper it follows immediately that

$$SR^2 = h_0^2 + \frac{FV^2(\underline{c} - E[m\hat{x}^c])^2}{\sigma^2(w)} = h^2,$$

which enables us to solve for the bounds as:

$$\left(\underline{c} - E[m\hat{x}^c]\right)^2 = \frac{\sigma^2(w)}{FV^2}\left(h^2 - h_0^2\right) = \frac{\sigma^2(w)}{FV^2}\left(A^2 - E\left[x^{*2}\right]\right) \text{ as in Proposition 4.}$$

Extensions

Optimization subject to the pricing vector *m* being non-negative is similar but slightly more complicated.

Essentially, it now becomes necessary to search numerically for the shadow prices of the two constraints.

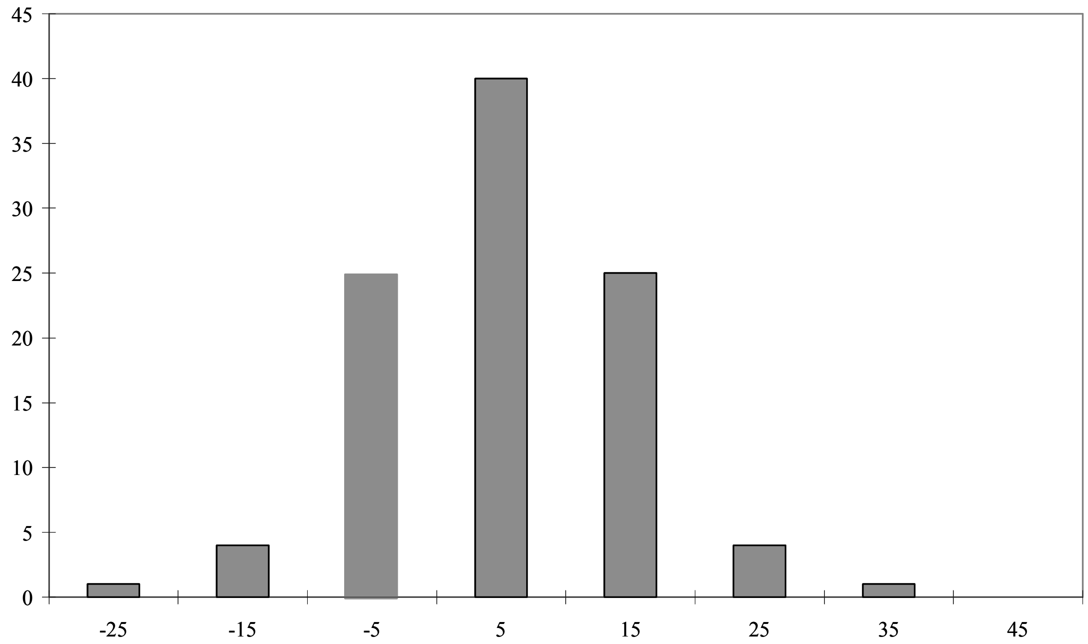
In a multiperiod context, these bounds can be calculated recursively, (but the numerical implementation is non-trivial).

Note that, although the solution for $m>0$ is general, the criterion of maximizing the Sharpe Ratio was arbitrary.

Generalized Sharpe Ratio

- What's wrong with the Sharpe Ratio
- The Generalisation (GSR) and some of its properties
- Applications to:
  - Valuation bounds in Incomplete Markets
  - Value at Risk
  - Performance Measurement

A Sharpe Ratio Paradox



μ	= 5.00	→	5.10
σ	= 10.00	→	10.34
S.R. = μ / σ	= 0.50	→	0.493
GSR	= 0.498	→	0.500

Generalized Sharpe Ratio

We propose a new measure where an investor with CARA utility can choose the quantity of the prospect to hold:

- we obtain the usual value for Normal distributions
- for non-Normal distributions, we provide a

generalization based on equating expected utility.

For normal distributions we find

$$U^* = \underset{x}{\text{Maximise}}\ E[U] = -e^{-\frac{1}{2}\frac{\mu^2}{\sigma^2}T}$$

A Generalization of the Sharpe Ratio μ/σ is obtained as

$$GSR = \sqrt{\frac{-2}{T} ln(-U^*)}.$$

Computation

$Max\ E[U] = \sum p_s \exp(-y r_s)$.

First order Condition:

$\sum p_s r_s \exp(-y r_s) = 0 = f(y)$.

Solve using Newton - Raphson iteration for y with

$f'(y) = -\sum p_s r_s^2 \exp(-y r_s)$.

We can do this on a spreadsheet.

Valuation and Hedging

Even where exact replication of derivatives is impossible, the price of a contingent claim may be "cheap" or "dear".

We solve the choice problem for an investor who maximizes

$E[U(w)]$ with $U = -e^{-\lambda w}$.

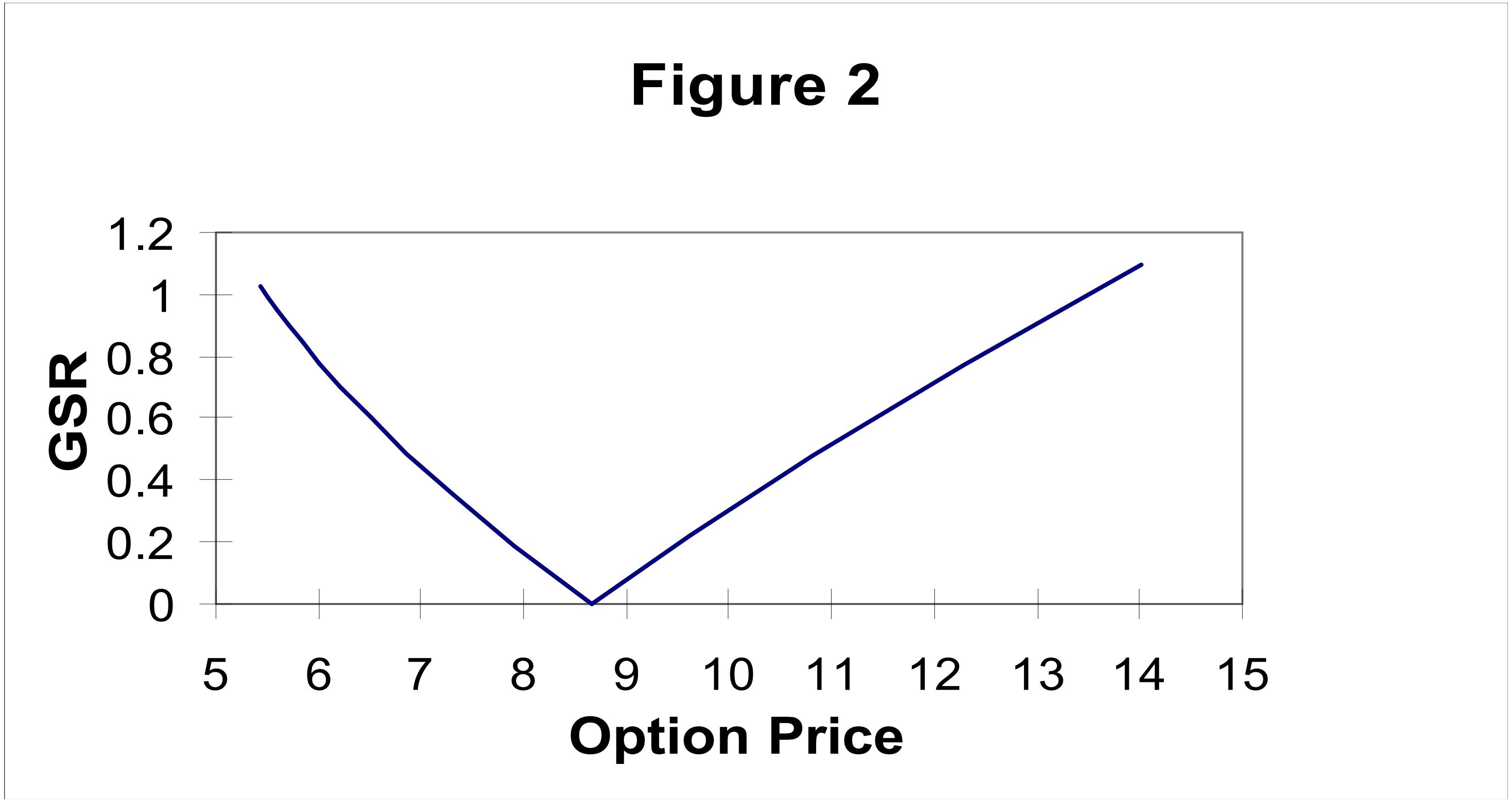
The investor buys y units of the contingent claim, and hedges with x units of the underlying:

$$\underset{x_t, y}{\text{Maximise}}\, E[U] = -Ee^{-\lambda\left[\int_0^T x_t dS_t + y(C_T - C_0)\right]}$$

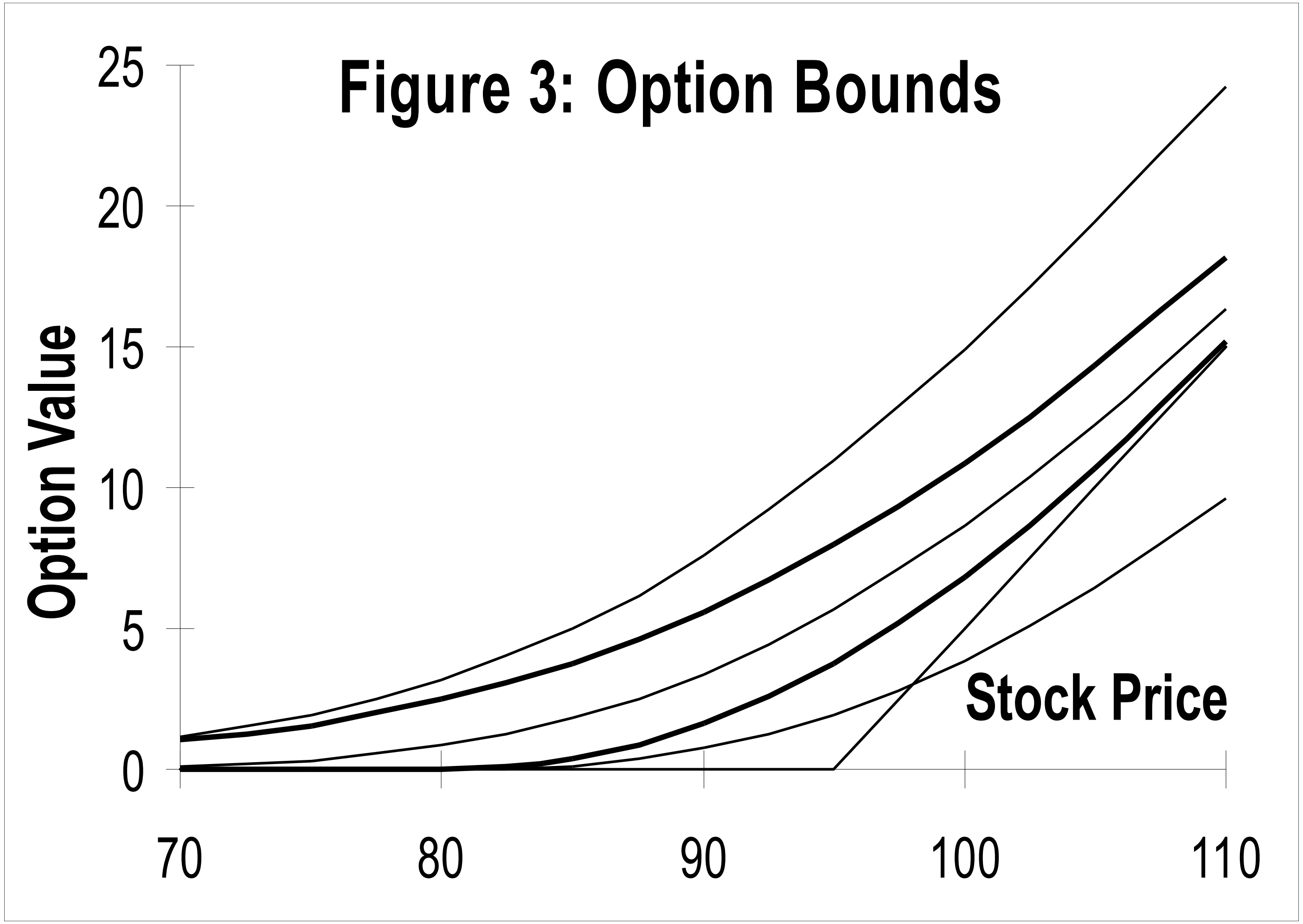
The value of the expected utility provides a GSR measure of the market opportunity provided by any particular C_0.

Conditional Bounds

We obtain valuation bounds which are much tighter than could be obtained by riskless arbitrage arguments.



Bounds at Different Asset Price Levels (GSR = ½)



Other Properties

These GSR bounds defined by the class of negative exponential utility functions have a number of advantages and disadvantages:

- The bounds do not explicitly depend
  - on risk aversion, or
  - on wealth levels.
- Losses (negative wealth) is not ruled out
  - as it would be for power or log utility.
- Some claims have very weak (and in some cases infinite) bounds.
  - in particular, any finite certain loss is preferred to a short position in a log-normal distribution, which makes the expected utility infinitely negative .

Performance Measurement

Under a continuous diffusion process with a constant price of risk μ/σ , a CARA investor will have constant risk exposure.

The terminal distribution is Normal.

Hence, odd shaped distributions are **not** preferred.

The Generalised Sharpe Ratio is robust in the sense that the maximum *ex ante* GSR **equals** the conventional Sharpe Ratio.

General Theory of Good-Deal Pricing

Cerny and Hodges (2001) have proposed a more general framework of "no-good-deal" pricing which places

- no-arbitrage, and
- representative agent equilibrium

at the two ends of a spectrum of possibilities.

A **desirable claim** is one which provides a specific level of von Neumann-Morgenstern expected utility.
A **good deal** is a desirable claim with zero or negative price.

Extension Theorem

In an incomplete market it is often convenient to suppose that the market is augmented in such a way that the resulting complete market contains no arbitrages.

We can more powerfully augment the market so that the complete market contains no arbitrages.

We obtain a set of pricing functionals which form a subset of those which simply preclude arbitrage.

Pricing Theorem

The link between no arbitrage and strictly positive pricing rules carries over to good deals, and enables price restrictions to be placed on non-marketed claims.

Under suitable technical assumptions (see C&H):

- The no-good-deal price region *P* for a set of claims is a convex set,
- Redundant assets have unique good-deal prices

Coherent Bounds

GSR and *G-NGD* bounds satisfy the properties advocated by Artzener et al, 1997 for coherent risk measures (*SR* ones don't):

Linearity:

$$B[\alpha \tilde{C}] = \alpha B[\tilde{C}], \text{and}$$
$$B[\beta + \tilde{C}] = \beta + B[\tilde{C}]$$

Subadditivity:

$$LB[\tilde{C}] + LB[\tilde{D}] \leq LB[\tilde{C} + \tilde{D}]$$
$$UB[\tilde{C} + \tilde{D}] \leq UB[\tilde{C}] + UB[\tilde{D}]$$

Monotonicity

$$\tilde{C} \leq \tilde{D} \Rightarrow B[\tilde{C}] \leq B[\tilde{D}]$$

(where *B* denotes any bound, *LB* lower bound, *UB* upper bound).

Jaschke and Küchler

There is a one-to-one correspondence between:

1. "coherent risk measures"
2. Cones of "desirable claims"
3. Partial orderings
4. Valuation bounds
5. Sets of "admissible" price systems.

Tail Areas

The GSR tail area is always strictly less than -U*.

This makes it suitable as an alternative coherent substitute for VaR to the "downside" risk measure which has also been suggested.

Conclusions

The no-good-deal bound framework has been considerably extended from its original Sharpe Ratio definition.

It provides a powerful method for obtaining:

- Valuation bounds in incomplete markets
- Coherent risk measures for Value at Risk

It is computationally attractive, for example:

- Values can be characterized in terms of the attractiveness of different prices (Generalized Sharpe Ratio).
- We can solve under suitable Markov processes or add as a heuristic to Monte Carlo simulations.

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