Signature SDEs as affine and polynomial processes

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(based on ongoing joint works with Guido Gazzani, Francesca Primavera, Sara Svaluto-Ferro and Josef Teichmann)

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Data driven models in finance

- Highly parametric and overparametrized models gain in importance: instead of a few parameters, learn the model's characteristics as a whole from data.
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⇒ Signature based models: the model itself or its characteristics are parameterized as linear functions of the signature of a primary underlying process, e.g.
  - a classical driving signal, e.g. Brownian motions or Lévy processes;
  - more general tractable processes describing well observable quantities.

- Compare e.g. with
  - I. Perez Arribas, C. Salvi, L. Szpruch "Sig-SDEs for quantitative finance"
  - T. Lyons, S. Nejad and I. Perez Arribas "Nonparametric pricing and hedging of exotic derivatives"

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- Other recent applications of signature methods in finance: Bayer et al. ('21) Bühler et al. ('20), Ni et al. ('20)), etc.

Part I

Signature based models for finance - theory and calibration

mainly based on

- "Signature based models for finance - theory and calibration" (joint work in progress with Guido Gazzani and Sara Svaluto-Ferro)

Key facts about signature - Universal approximation

Signature (see K. Chen ('57)) plays a prominent role in rough path theory (T. Lyons ('98), P. Friz & N. Victoir ('10), P. Friz & M. Hairer ('14)). It serves as linear regression basis for continuous path functionals.

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  - The signature of a d-dimensional geometric rough path, including continuous semimartingales, uniquely determines the path up to tree-like equivalences (see H. Boedihardjo et al.('16)).
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- Linear functions on the signature form an algebra that contains 1:
  - Every polynomial on signature may be realized as a linear function via the so-called shuffle product $\sqcup\!\sqcup$.

$\Rightarrow$ By the Stone-Weierstrass theorem continuous (with respect of to a variation distance of the lifted path) path functionals on compact sets can be uniformly approximated by a linear function of the time extended signature.

$\Rightarrow$ Universal approximation theorem (UAT).

Signature in a nutshell

- The signature takes values in the extended tensor algebra $T((\mathbb{R}^d))$ given by

$$T((\mathbb{R}^d)) := \{(a_0, a_1, \dots, a_n, \dots) \mid \text{ for all } n \geq 0,\ a_n \in (\mathbb{R}^d)^{\otimes n}\}.$$

  Elements of $T((\mathbb{R}^d))$ are denoted in bold face, e.g. $\mathbf{a} = (a_0, a_1, \dots, a_n, \dots)$.

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- Let $I = (i_1, \ldots, i_n)$ be a multi-index with entries in $\{1, \ldots, d\}$ and denote by $e_I = e_{i_1} \otimes \cdots \otimes e_{i_n}$ the basis elements of $(\mathbb{R}^d)^{\otimes n}$.
- We write $\langle e_I, \boldsymbol{a} \rangle$ to extract the I^{th} component from a_n. More generally we often write $\boldsymbol{u}(\boldsymbol{x}) = \langle \boldsymbol{u}, \boldsymbol{x} \rangle$ if $\sum_I |u_I x_I| < \infty$ and call this linear maps in $\boldsymbol{x}$.

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Definition

The coordinate signature indexed by $I = (i_1, \dots, i_n)$ of an $\mathbb{R}^d$-valued semimartingale $\widehat{X}$ is defined via iterated Stratonovic/Marcus integrals (denoted by $\circ$)

$$\langle e_I, \widehat{\mathbb{X}}_T \rangle := \int_{0 < t_1 < \dots < t_n < T} \circ d\widehat{X}^{i_1}_{t_1} \cdots \circ d\widehat{X}^{i_n}_{t_n},$$

Hence, $\widehat{\mathbb{X}}_T = 1 + \sum_{n=1}^{\infty} \sum_{|I|=n} \langle e_I, \widehat{\mathbb{X}}_T \rangle e_I \in T((\mathbb{R}^d))$.

Model ingredients

- Goal: provide a good model for a set of traded assets $S = (S^1, \dots, S^m)$

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- Examples for X:
  - Market inferred Brownian motion (under $\mathbb{Q}$, i.e. $\mathbb{P}$-BM + market price of risk) or Lévy process
  - assets whose dynamics are well understood and which can be described by so-called Sig-SDEs of the form

$$dX_t = \boldsymbol{b}(\widehat{\mathbb{X}}_t)dt + \sqrt{\mathbf{a}(\widehat{\mathbb{X}}_t)}dB_t, \qquad \text{(SigSDE)}$$

    where B is a Brownian motion, $(\widehat{\mathbb{X}}_t)_{t\geq 0}$ denotes the signature of $t \mapsto (X_t, t)$ and $\boldsymbol{b}$ and $\boldsymbol{a}$ are linear maps.

$\Rightarrow$ Truly general class of diffusions whose coefficients can depend on the whole path.

Model framework

The model

The traded assets $(S^1, \ldots, S^m)$ *are modeled via*

$$S^j_t(\ell^j) = \ell^j(\widehat{\mathbb{X}}_t) = \ell^j_0 + \sum_{0<|I|\leq n} \ell^j_I \langle e_I, \widehat{\mathbb{X}}_t \rangle, \quad j = 1, \ldots, m, \qquad \text{(Sig-model)}$$

- $\widehat{\mathbb{X}}$ *is the signature of* $\widehat{X}$,
- $n \in \mathbb{N}$ *is degree of truncation*
- ℓ^j *denotes a linear map. Here,* $\ell^j_0, \ell^j_I \in \mathbb{R}$ *the corresponding coefficients with respect to the basis elements* 1 *and* e_I *of the tensor algebra.*

The parameters ℓ can also be taken time-dependent.

Properties of signature based models

Universality Any classical model driven by Brownian motion can be arbitrarly well approximated; extensions to Lévy/Poisson random measure driven models are possible.

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Calibration to options via precomputed Monte-Carlo samples of $\widehat{\mathbb{X}}$ exploiting linearity of the model

Tractable option pricing formulas relying on approximations via so-called signature payoffs of the form $\langle e_J, \widehat{\mathbb{S}}_T(\ell)\rangle$.

Generic primary processes $\widehat{X}$ of form (SigSDE) are projections of a extended tensor algebra valued affine and polynomial process.

$\Rightarrow \mathbb{E}_{\mathbb{Q}}[\widehat{\mathbb{S}}_T(\ell)]$ can thus be computed via polynomial technology, i.e. by solving a (usually infinite dimensional) linear ODE.

Calibration to time-series data

- Goal: match N market prices $(S^M_{t_1}, \dots, S^M_{t_N})$
- Assumption: Time series data $\widehat{X}_{t_1}, \dots, \widehat{X}_{t_N}$ is available.
- Procedure:
  - Compute from $\widehat{X}_{t_1}, \dots, \widehat{X}_{t_N}$ the path of the signature $\widehat{\mathbb{X}}$.
  - Use the path of $\widehat{\mathbb{X}}$ as linear regression basis to find $\boldsymbol{\ell}$ by matching the prices, i.e.

$$\operatorname*{argmin}_{\boldsymbol{\ell}} \sum_{i=1}^{N} \left(S_0 + \sum_{k=1}^{d} \sum_{0 \leq |I| \leq n} \ell_{I,k} \langle e_I, \widehat{\mathbb{X}}_{t_i} \rangle - S^M_{t_i} \right)^2$$

  - Important: since the dimension of $\boldsymbol{\ell}$ is typically high, introducing a regularization (Lasso, Ridge) is necessary.

Results for a Heston and a SABR market model

- Learn a Heston and a SABR market using the signature of estimated $\mathbb{Q}$-Brownian motions (i.e. $\mathbb{P}$-BM with market price of risk)
- Compare the learned (Sig-model) with new Heston/SABR trajectories.

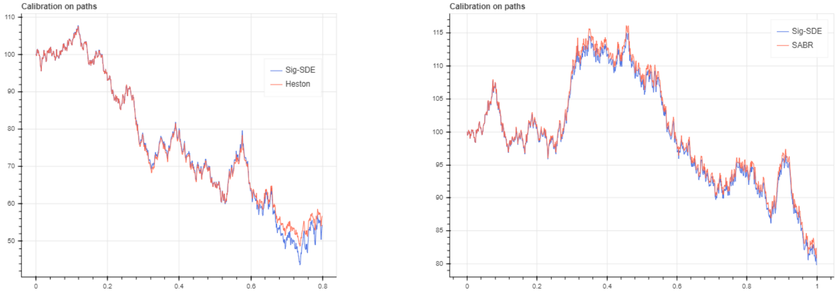
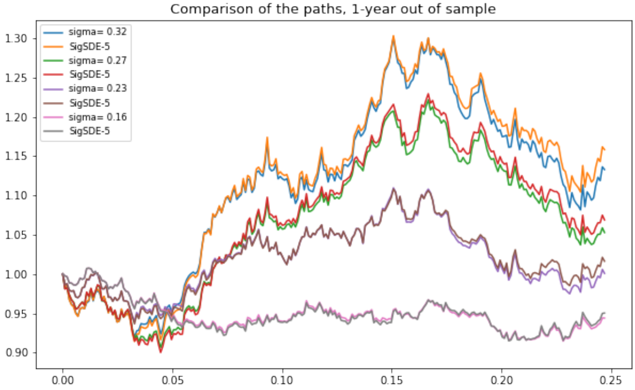


Figure: Out of sample comparison of the price trajectories

Multivariate Case - 4 dimensional correlated B&S model

- Learn a 4 dimensional Black and Scholes market using the signature of $\widehat{\mathbb{W}}$ up to order 5.
- Compare the learned (Sig-model) with new trajectories.



Calibration to options

- Goal: match N option prices $(\pi^1, \dots, \pi^N)$ with payoffs $F_i(S_{T_i}), i = 1, \dots, N$.
- Typically we calibrate to call and put options with different strikes and maturities, whose prices are expressed in terms of implied volatility.

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- Assumption: Monte-Carlo samples of $\widehat{\mathbb{X}}^j_{T_1}, \ldots, \widehat{\mathbb{X}}^j_{T_N}$ for $j = 1, \ldots M$ (under a pricing measure $\mathbb{Q}$) are available.
- Procedure: The calibration can then be formalized via

$$\operatorname*{argmin}_{\ell} \sum_{i=1}^{N} w^i \left(\frac{1}{M} \sum_{j=1}^{M} F_i(\ell(\widehat{\mathbb{X}}^j_{T_i})) - \pi_i \right)^2,$$

where w^i are weights, e.g. vega-weights to match implied volatility well.

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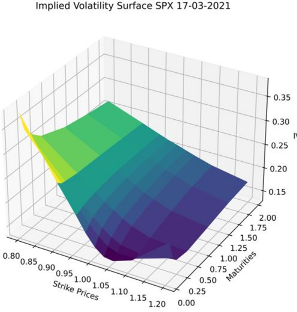
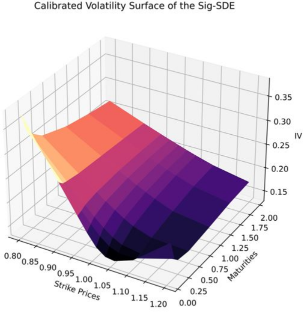
  where w^i are weights, e.g. vega-weights to match implied volatility well.
- Advantages:
  - All Monte-Carlo samples can be easily precomputed, no Monte-Carlo simulation in each optimization step!
  - For ℓ such that $\frac{1}{M} \sum_{j=1}^{M} F_i(\ell(\widehat{\mathbb{X}}^j_{T_i})) \geq \pi_i$ the optimization is convex for convex payoffs.

Calibration to real data

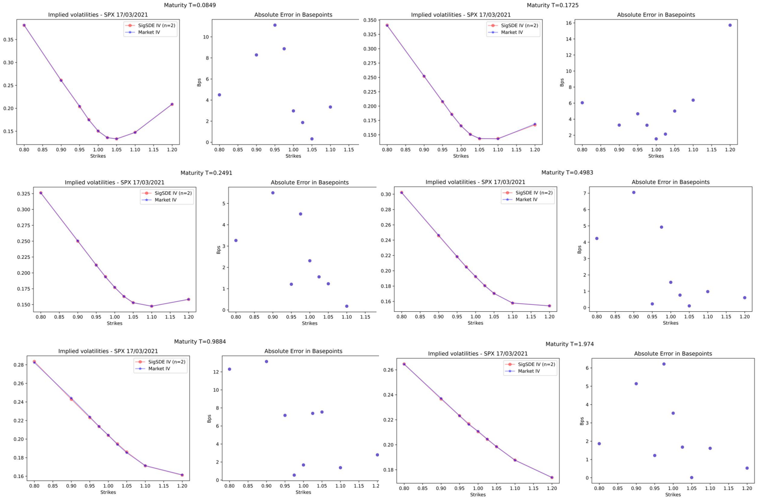
- Goal: reproduce S&P 500 volatility surface (here from 17-03-21) using as primary underlying process $\widehat{X}_t = (W_t^1, W_t^2, t)$.
- We consider here an extension of the model with time-dependent parameters and achieve a nearly perfect fit.





Calibration to real data - error analysis

- With 13 parameters per maturity the absolute error is in the range of 0 to 15 basis points.



Pricing of signature payoffs

Theorem (C.C, G. Gazzani, S.Svaluto-Ferro ('21))

The price of a signature payoff $\langle e_J, \widehat{\mathbb{S}}_{\mathbb{T}}(\boldsymbol{\ell})\rangle$ *can be expressed as*

$$\mathbb{E}_{\mathbb{Q}}[\langle e_J, \widehat{\mathbb{S}}_T(\boldsymbol{\ell})\rangle] = \langle e(J,\boldsymbol{\ell}), \mathbb{E}_{\mathbb{Q}}[\widehat{\mathbb{X}}_T]\rangle = \sum_I p_I(J,\boldsymbol{\ell})\langle e_I, \mathbb{E}_{\mathbb{Q}}[\widehat{\mathbb{X}}_T]\rangle,$$

where $p_I(J,\boldsymbol{\ell})$ *are polynomials in the coefficients of* $\boldsymbol{\ell}$*.*

- Path-dependent payoffs like Asian forwards correspond to signature payoffs.

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- Path-dependent payoffs like Asian forwards correspond to signature payoffs.
- How can we compute $\mathbb{E}_{\mathbb{Q}}[\widehat{\mathbb{X}}_T]$?
  - Case of $\widehat{\mathbb{X}} = \widehat{\mathbb{W}}$: T. Fawcett ('04)
  - Semimartingale case: P. Friz, P. Hager & N. Tapia ('21)
  - An affine and polynomial process point view works for generic primary processes $\widehat{X}$ of form (SigSDE).
    $\Rightarrow \mathbb{E}_{\mathbb{Q}}[\widehat{\mathbb{X}}_T]$ can be computed via polynomial technology, i.e. by solving (usually infinite dimensional) linear ODEs.

Part II

An affine and polynomial perspective to signature based models

based on

- Universality of affine and polynomial processes (joint work in progress with S. Svaluto-Ferro and J. Teichmann)
- Signature based affine and polynomial jump diffusions (joint work in progress with F. Primavera and S. Svaluto-Ferro)

Motivation

A plethora of stochastic models stem from the class of affine and polynomial processes, even though this is not always visible at first sight.

$\Rightarrow$ Universal model classes?

$\Rightarrow$ Mathematically precise statements for this universality?

$\Rightarrow$ Can we embed signature based models in this framework?

Definition of affine and polynomial processes

Simplest setting (for illustrative purposes): Itô diffusion in one dimension with state space S, some (bounded or unbounded) interval of $\mathbb{R}$:

$$dX_t = b(X_t)dt + \sqrt{a(X_t)}dB_t, \quad X_0 = x, \qquad (*)$$

with $a : \mathbb{R} \to \mathbb{R}_+$ and $b : \mathbb{R} \to \mathbb{R}$ continuous functions and B a Brownian motion.

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Definition

A weak solution X of $(*)$ is called polynomial process if

- b is an affine function, i.e. $b(x) = b + \beta x$ for some constants b and β and
- a is a quadratic function, i.e. $a(x) = a + \alpha x + Ax^2$ for some constants a, α and A.

If additionally $A = 0$, then the process is called affine.[a]

[a] In this diffusion setting all affine processes are polynomial (in general this only holds true under moment conditions).

Key properties of affine and polynomial processes

From this definition, ...

- ... they appear as a narrow class.
- ... follow some remarkable implications.
  - All marginal moments of a polynomial process, i.e. $\mathbb{E}[X_t^n]$ can be computed by solving a system of linear ODEs, i.e. the Feynman-Kac PDE reduces to a linear ODE.
  - Additionally, exponential moments of affine processes, i.e. $\mathbb{E}[\exp(uX_t)]$ for $u \in \mathbb{C}$ can be expressed in terms of solutions of Riccati ODEs whenever $\mathbb{E}[|\exp(uX_t)|] < \infty$, i.e. the Cole-Hopf transform of the Feynman-Kac PDE reduces to a Riccati ODE.

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For much more generality and details about these processes classes we refer to

- D. Duffie, D. Filipović & W. Schachermayer ('03); D. Filipović & E. Mayerhofer ('09);
- C., M. Keller-Ressel & J. Teichmann ('12); D. Filipovic & M. Larsson ('16).

Linear processes

- We consider here an even simpler subset of affine (hence polynomial) processes where $b(x) = \beta x$ and $a(x) = \alpha x$ are just linear functions and call the corresponding stochastic processes linear processes.
- Their infinitesimal generator if given by $\mathcal{A}f(x) = f'(x)\beta x + \frac{1}{2}f''(x)\alpha x$ and acts on $f(x) = \exp(ux)$ and polynomials $f(x) = \sum_{i=0}^{k} c_i x^i$ as follows

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  - Affine property: $\mathcal{A}\exp(ux) = \exp(ux)R(u), \quad R(u) = \frac{1}{2}\alpha u^2 + \beta u$

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  - Affine property: $\mathcal{A}\exp(ux) = \exp(ux)R(u), \quad R(u) = \frac{1}{2}\alpha u^2 + \beta u$
  - Polynomial property: $\mathcal{A}\left(\sum_{i=0}^k c_i x^i\right) = \sum_{i=1}^k (ic_i\beta + \frac{1}{2}(i+1)ic_{i+1}\alpha 1_{\{i \leq k+1\}})x^i = \sum_{i=1}^k L(c)_i x^i$ with matrix L applied to the vector $c = (c_0, c_1, \ldots, c_k)^\top$

$$L = \begin{pmatrix} 0 & \cdots & & & & & 0 \\ 0 & \beta & \alpha & 0 & \cdots & & 0 \\ 0 & 0 & 2\beta & 3\alpha & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & & i\beta & \frac{(i+1)i}{2}\alpha & \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Affine transform formula and moment formula

Theorem (Duffie, Filipovic, Schachermayer ('03), C.C., Keller-Ressel, Teichmann ('12))

Let $T > 0$ be fixed and let X be a linear process.

- *Let $u \in \mathbb{C}$ such that $\mathbb{E}[|\exp(uX_T)|] < \infty$ and denote by $\psi(t)$ the solution of the following Riccati ODE*

$$\partial_t \psi(t) = R(\psi(t)), \quad \psi(0) = u.$$

  Then $\mathbb{E}\left[\exp(uX_T)\right] = \exp(\psi(t)X_0)$

Affine transform formula and moment formula

Theorem (Duffie, Filipovic, Schachermayer ('03), C.C., Keller-Ressel, Teichmann ('12))

Let $T > 0$ be fixed and let X be a linear process.

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$$\partial_t \psi(t) = R(\psi(t)), \quad \psi(0) = u.$$

  Then $\mathbb{E}\left[\exp(uX_T)\right] = \exp(\psi(t) X_0)$

- *Denote by $c(t) = (c_0(t), \ldots, c_k(t))^\top$ the solution of the following linear ODE*

$$\partial_t c(t) = Lc(t), \quad c(0) = c \in \mathbb{R}^{k+1}.$$

  Then its moments are given by
  $\mathbb{E}\left[\sum_{i=0}^k c_i X_T^i\right] = \sum_{i=0}^k c_i(T) x^i = \sum_{i=0}^k (\exp(LT)c)_i X_0^i.$

One dimensional diffusions with analytic characteristics

- Consider a one-dimensional diffusion process X on S given by

$$dX_t = b(X_t)dt + \sqrt{a(X_t)}dB_t, \quad X_0 = x_0,$$

where the functions b and a are real analytic of the form

$$b(x) = \sum_{n=0}^{\infty} b_n x^n \qquad \text{and} \qquad a(x) = \sum_{n=0}^{\infty} a_n x^n, \qquad x \in S,$$

converging on an open neighborhood of S.

- Note that up to a slight reparametrization analytic functions are linear functions in the signature of X as $\mathbb{X}_t = (1, X_t - x_0, \frac{(X_t-x_0)^2}{2}, \frac{(X_t-x_0)^3}{3!}, \ldots)$.
- Let

$$\mathcal{U} := \{\mathbf{u} \in T((\mathbb{R})) \colon |\sum_{n=0}^{\infty} |u_n x^n| < \infty \text{ for all } x \in S + B_\varepsilon(0),\ \varepsilon > 0\}$$

and denote by $\star$ the discrete convolution, i.e.

$$(\mathbf{u} \star \mathbf{c})_i := \sum_{j_1+j_2=i} u_{j_1} c_{j_2}, \qquad \mathbf{u}, \mathbf{c} \in T((\mathbb{R})).$$

Affine case

Theorem (C.C, S. Svaluto-Ferro, J. Teichmann ('22))

The process $\mathbb{X} := (1, X, X^2, \dots, X^n, \dots)$ *is affine with respect to the operator* $R : \mathcal{U} \to T((\mathbb{R}))$ *given by*

$$R(\mathbf{u}) = \mathbf{b} \star \mathbf{u}^{(1)} + \frac{1}{2}\mathbf{a} \star \left(\mathbf{u}^{(2)} + \mathbf{u}^{(1)} \star \mathbf{u}^{(1)}\right), \quad u_k^{(\ell)} := u_{k+\ell}\frac{(k+\ell)!}{k!},$$

meaning that $\mathcal{A}\exp(\sum_{n=0}^\infty u_n x^n) = \exp(\sum_{n=0}^\infty u_n x^n)\sum_{n=0}^\infty R_n(\mathbf{u})x^n$.

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$$\partial_t \psi(t) = R(\psi(t)), \quad \psi(0) = \mathbf{u}$$

admits an $\mathcal{U}$*-valued (weak) solution and that certain exponential moment conditions hold true. Then*

$$\mathbb{E}\left[\exp\left(\sum_{n=0}^{\infty} u_n X_T^n\right)\right] = \exp\left(\sum_{n=0}^{\infty} \psi_n(T) X_0^n\right).$$

Examples: Brownian motion & geometric Brownian motion

- Let $\mathcal{B}$ be the set of entire functions f such that $|f| \leq \exp(a(|\cdot|+1))$ on $\mathbb{R}$ for some $a \in \mathbb{R}_+$ and $\mathcal{D} := \{f \in \mathcal{B} \colon f', f'' \in \mathcal{B}\}$.
- Let $f \in \mathcal{D}$ such that $f(x) = \exp(\sum_{n=0}^{\infty} u_n x^n)$. Then

$$\mathbb{E}[\exp(\sum_{n=0}^{\infty} u_n(x+B_t)^n)] = \exp(\sum_{n=0}^{\infty} \psi_n(T)x^n).$$

  with $R(\mathbf{u}) = \frac{1}{2}(1,0,\ldots) \star \left(\mathbf{u}^{(2)} + \mathbf{u}^{(1)} \star \mathbf{u}^{(1)}\right)$.

- Choose $f(x) = \exp((i\lambda - \kappa)\exp(x))$ for $\lambda \in \mathbb{R}$ and $\kappa \in \mathbb{R}_+$, to obtain an expression for the Fourier-Laplace transform of geometric Brownian motion $S_t = \exp(x + B_t)$

$$\mathbb{E}[\exp((i\lambda - \kappa)S_t)] = \mathbb{E}[\exp((i\lambda - \kappa)\sum_{n=0}^{\infty} \frac{(x+B_t)^n}{n!}] = \exp(\sum_{n=0}^{\infty} \psi_n(T)\log(S_0)^n)$$

  with $\psi_n(0) = \frac{i\lambda - \kappa}{n!}$.

- We approximate the solution of the sequence-valued Riccati ODE by neural networks and deep learning methods for ODEs.

Polynomial case

Theorem (C.C, S. Svaluto-Ferro, J. Teichmann ('22))

The process $\mathbb{X} := (1, X, X^2, \dots, X^n, \dots)$ *is polynomial with respect to the operator* $L : \mathcal{U} \to T((\mathbb{R}))$ *given by*

$$L(\mathbf{u}) = \mathbf{b} \star \mathbf{u}^{(1)} + \frac{1}{2}\mathbf{a} \star \mathbf{u}^{(2)}, \quad u_k^{(\ell)} := u_{k+\ell}\frac{(k+\ell)!}{k!},$$

meaning that $\mathcal{A}(\sum_{n=0}^{\infty} u_n x^n) = \sum_{n=0}^{\infty} L_n(\mathbf{u}) x^n$.

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$$\mathbb{E}\left[\sum_{n=0}^\infty u_n X_T^n\right] = \sum_{n=0}^\infty \boldsymbol{c}_n(T) X_0^n.$$

Examples

For the following examples we can e.g. compute the moment generating function

$$\mathbb{E}[\exp(uX_T)] = \sum_{n=0}^{\infty} c_n(T) X_0^n$$

for appropriate u by solving the above infinite dimensional linear ODE with inital value $\mathbf{u} = (1, u, \frac{u}{2}, \ldots, \frac{u^k}{k!}, \ldots)$.

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- Polynomial processes on compact state spaces
- Classically non-affine and non-polynomial examples:
  - $dX_t = \sqrt{X_t}(1 - X_t)dB_t$ on $[0, 1]$
  - $dX_t = \kappa \sum_{i=1}^{\infty} \pi(X_t^i - X_t)dt + \sqrt{X_t(1 - X_t)}dB_t$ on $[0, 1]$
- Affine Feller diffusion: $dX_t = \sqrt{a_1 X_t} dB_t$ on $\mathbb{R}_+$. For $u < 0$, the solution of the linear ODE leads to the well known expression for the Laplace transform

$$\mathbb{E}[\exp(uX_T)] = \sum_{n=0}^{\infty} \underbrace{\frac{u^n}{(1 - \frac{a_1}{2} uT)^n n!}}_{c_n(T)} X_0^n = \exp\left(\frac{uX_0}{1 - \frac{a_1}{2} uT}\right).$$

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Question: can the signature process of generic SDEs be treated as an infinite dimensional affine and/or polynomial process?

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Indeed, if X is of the truly generic form

$$dX_t = \boldsymbol{b}(\widehat{\mathbb{X}}_t)dt + \sqrt{\mathbf{a}(\widehat{\mathbb{X}}_t)}dB_t, \quad X_0 \in S \subseteq \mathbb{R}^d \qquad \text{(SigSDE)}$$

where $(\widehat{\mathbb{X}}_t)_{t\geq 0}$ is the signature of $t \mapsto (X_t, t)$ and $\boldsymbol{b}$ and $\boldsymbol{a}$ are linear maps, then

- $\Rightarrow$ Ito's formula yields that the characteristics of $(\widehat{\mathbb{X}}_t)_{t\geq 0}$ are linear
- $\Rightarrow$ $\mathcal{A}\exp(\langle \boldsymbol{u}, \boldsymbol{x}\rangle) = \exp(\langle \boldsymbol{u}, \boldsymbol{x}\rangle)\langle R(\boldsymbol{u}), \boldsymbol{x}\rangle$ and $\mathcal{A}(\langle \boldsymbol{u}, \boldsymbol{x}\rangle) = \langle L(\boldsymbol{u}), \boldsymbol{x}\rangle$, $\boldsymbol{x} \in \mathcal{S}(S)$

$$R(\mathbf{u}) = \mathbf{b} \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup \mathbf{u}^{(1)} + \frac{1}{2}\mathrm{tr}(\mathbf{a} \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup (\mathbf{u}^{(2)} + \mathbf{u}^{(1)} \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup \mathbf{u}^{(1)})),$$

$$L(\mathbf{u}) = \mathbf{b} \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup \mathbf{u}^{(1)} + \frac{1}{2}\mathrm{tr}(\mathbf{a} \sqcup\mathchoice{\mkern-7mu}{\mkern-7mu}{\mkern-3.2mu}{\mkern-3.8mu}\sqcup \mathbf{u}^{(2)})$$

- $\Rightarrow$ $(\widehat{\mathbb{X}}_t)_{t\geq 0}$ is a $T((\mathbb{R}^d))$ valued linear, hence affine and polynomial process.

Sig-SDEs as affine and polynomial processes

- This means that under appropriate conditions...
  - ... $\mathbb{E}[\widehat{\mathbb{X}}_T]$ can be computed via polynomial technology, i.e. by solving an infinite dimensional linear ODE.
  - ... $\log \mathbb{E}[\exp(\langle \boldsymbol{u}, \widehat{\mathbb{X}}_T \rangle)]$ can be computed via affine technology, i.e. by solving an infinite dimensional Riccati ODE.

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- The SigSDE setting goes beyond Markovian settings due to possibly path-dependent coefficients. The signature itself remains Markovian with linear characteristics, which is the essential feature.
- Special cases of (SigSDE) are Makovian SDEs with $\boldsymbol{b}$ and $\boldsymbol{a}$ analytic in X. The infinite dimensional linear PDE for the expected signature (valid for any Markovian diffusion) reduces to an infinite dimensional linear ODE.
- If $\boldsymbol{b}$ and $\boldsymbol{a}$ only depend on the signature up to order 1 and 2 respectively, then $(\widehat{\mathbb{X}}_t^{\leq N})_{t\geq 0}$ is a finite dimensional polynomial process. This holds true in particular for X being a classical polynomial processes.

Jump Sig-SDEs as affine and polynomial processes

- In the case of jumps we use the Marcus signature, given as solution of

$$d\widehat{\mathbb{X}} = \sum_{i=1}^{d} \widehat{\mathbb{X}} \otimes \diamond d\widehat{X}^i, \quad \widehat{\mathbb{X}}_0 = (1, 0, 0, \dots) \in T((\mathbb{R}^d)),$$

where $\diamond$ denotes the Marcus integral (giving rise to a first order calculus)
$\int_0^t f(Z_s) \diamond dZ_s :=$
$\int_0^t f(Z_{s-}) dZ_s + \frac{1}{2} \int_0^t f'(Z_{s-}) d[Z^c, Z^c]_s + \sum_{0<s\leq t} \Delta Z_s \left(\int_0^1 f(Z_{s-} + \theta \Delta Z_s) - f(Z_{s-}) d\theta \right).$

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- Then analogous statements hold true for Sig-SDEs with jumps of the form

$$dX_t = \boldsymbol{b}(\widehat{\mathbb{X}}_t)dt + \sqrt{\mathbf{a}(\widehat{\mathbb{X}}_t)}dB_t + \int \xi(\mu^X(d\xi, dt) - \boldsymbol{K}(\widehat{\mathbb{X}}_t, d\xi)dt),$$

where the compensator $\boldsymbol{K}$ is such that $\mathbf{x} \mapsto \boldsymbol{K}(\mathbf{x}, d\xi)$ is a linear map.

$\Rightarrow$ $(\widehat{\mathbb{X}}_t)_{t\geq 0}$ is a $T((\mathbb{R}^d))$ valued affine and polynomial process.

$\Rightarrow$ If X is a classical polynomial process, then $(\widehat{\mathbb{X}}_t^{\leq N})_{t\geq 0}$ is a finite dimensional polynomial process. The infinite dimensional linear PIDE for the expected signature becomes a finite dimensional ODE.

Conclusion

- Signature based models distinguish themselves in
  - universality, as the dynamics of all classical models can be approximated
  - criteria for no-arbitrage
  - efficient pricing, hedging and calibration (also extension to VIX options).
- Extension to Lévy type signature models is possible (joint work with F.Primavera and S.Svaluto-Ferro).
- Generic classes of SDEs can be proved to be affine and polynomial, in particular SDEs with analytic coefficients $\Rightarrow$ one step in the direction of universality of affine processes
- For (jump) SigSDEs
  - its expected signature can be computed via polynomial technology
  - the Fourier-Laplace transform of its signature can be computed via affine technology

Thank you for your attention!