4. MOT & SEP

Discrete time

Ex Revisit the proof of Duality to show that $\forall \mu, \nu \in \mathcal{P}_1(\mathbb{R})$, $\mu \leq_{cx} \nu$, $\forall c$ lsc & ...

$$\underline{P}^{M}(\mu,\nu) = \inf_{\pi \in \mathcal{M}(\mu,\nu)} \int c\, d\pi = \sup_{\substack{\varphi, \psi, h \in C_b \\ \underbrace{\varphi(x)+\psi(y)+h(x)(y-x)}_{\varphi \oplus \psi + h^{\otimes}} \leq c(x,y)}} \int \varphi\, d\mu + \int \psi\, d\nu$$

More generally: for $\mu_1 \leq_{cx} \dots \leq \mu_n$ & $c : \mathbb{R}^n \to \mathbb{R}$

$$\inf_{\pi \in \mathcal{M}(\mu_1, \dots, \mu_n)} \int c\, d\pi = \sup \left\{ \sum_i \int \varphi_i\, d\mu_i : \sum_i \varphi_i(x_i) + \sum_{i=1}^{n-1} h_i(x_1, \dots, x_i)(x_{i+1} - x_i) \leq c(x_1, \dots, x_n) \right\}$$

Financial interpretation: Stock market prices are modelled with a stochastic process $(S_t)_{1 \leq t \leq n}$.

At time t, we can use the available info $\mathcal{F}_t \sim h(S_1 \dots S_t)$ to buy $h(S_1, \dots S_t)$ shares at price $h(S_1, \dots, S_t) S_t$. We sell these at time $t+1$ for $h(S_1, \dots S_t) S_{t+1}$. We repay then the loan used to buy shares & end up with:

$$h(S_1, \dots, S_t)(S_{t+1} - S_t).$$

$\Rightarrow$ A "self-financing" trading strategy is of the form $\sum_{t=1}^{n-1} h_t(S_1, \dots, S_t)(S_{t+1} - S_t) = (h \circ S)_n$

Then if S is defined on $(\Omega, \mathcal{F}, \mathbb{P})$ & we want to use $\mathbb{E}_{\mathbb{P}}$ to price we need $\mathbb{E}_{\mathbb{P}}[(h \circ S)_n] = 0$ $\forall h$ above $\Leftrightarrow$ (moments ass.) S is a $\mathbb{P}$-martingale.

Suppose then that in the market we can observe the prices of call & put options, say

$$C(K, t) = \mathbb{E}_{\mathbb{P}}[(S_t - K)^+], \quad K \in \mathbb{R}, \ t = 1, \dots, n.$$

$$\Rightarrow \frac{\partial C}{\partial K}(K,t) = \frac{\partial}{\partial K} \int_K^{\infty} (x - K)\, \mathcal{L}(S_t)(dx) = -\mathcal{L}(S_t)([K, +\infty)) = -\mathbb{P}(S_t \geq K)$$

$$\Rightarrow \frac{\partial^2 C}{\partial K^2}(dK) = \frac{\partial}{\partial K} \measuredangle = \mathcal{L}(S_t)(dK) \text{ gives the distribution of } S_t. \quad (\text{B-L})$$

Let $\mu_t := \frac{\partial^2}{\partial K^2} C(t, dK) \Rightarrow$ compatible $\mathbb{P}$s are $\mathcal{M}(\delta_{S_0}, \mu_1, \dots, \mu_n)$.

If we now want to understand the range on admissible prices for an exotic / non-traded option ξ, this will be

$$\left[\inf_{\pi \in \mathcal{M}(\mu_0, \dots, \mu_n)} \mathbb{E}_{\pi}\{\xi\}, \ \sup_{\pi \in \mathcal{M}(\mu_0, \dots, \mu_n)} \mathbb{E}_{\pi}\{\xi\} \right].$$

So the bounds are given by MOT values!

We have, by duality,

$$\inf_{\pi \in \mathcal{M}(\mu_0,\dots,\mu_n)} \int \xi \, d\pi = \sup \Big\{ \underbrace{x + \overbrace{\sum_{i=1}^{t} \sum_{j=1}^{m_t} \alpha_j^t C(K_j^t, t)}^{\sum_t \int \psi_t \, d\mu_t}}_{\text{price to setup}} : x + (h \circ S)_n + \sum \sum \alpha_j^t (S_t \dots \le \xi(S_1, \dots, S_n) \Big\}$$

$\mathbb{R}$

Each such element on the right can be written as ψ_t^j $\int \psi_t^j \, d\mu_t$

$$x + \sum_i \sum_j \alpha_j^t C(K_{j}^t, t) + \underbrace{(h \circ S)_n + \sum_i \sum_j \alpha_j^t ((S_t - K_j^t)^+ - C(K_j^t, t))}_{H_n := \text{ self financing}} \le \xi($$

so the lowest admissible price = the cost of the most expensive strategy sub-replicating ξ.

At any lower price I can make riskless profit buying ξ at $p <$ above & setup hedge

$$-p + \xi - H_n \overset{\text{(above)}}{\geq} -p + x + \sum\sum \alpha_j^t C(K_j^t, t) > 0.$$

Likewise, at price higher than sup, I sell & hedge.

Continuous time

Consider now a continuous time setting $(S_t : t \le T)$. As before (take limits)

$(H \circ S)_T = \int_0^T h_t \, dS_t$ models outcomes of a self-financing strategy.

We also need some admissibility constraint to avoid ∞-credit lines (e.g. $(H \cdot S)_t \ge -K, t$

$\mathbb{E}_{\mathbb{P}}[(H \cdot S)_T] = 0 \implies S$ is a $\mathbb{P}$-mg.

$C(K) = \mathbb{E}_{\mathbb{P}}[(S_T - K)^+]$ given $\implies S_T \sim_{\mathbb{P}} \nu$ given. $S_0 = s$ also given.

We are interested in $\inf/\sup \, \mathbb{E}[\xi(S_t : t \le T)]$ over all cont. mg $S_0 = s$, $S_T \sim \nu$.

Suppose $\xi(\cdot)$ is invariant under time changes (cont.), e.g., $\xi((S_t : t \le T)) = \mathbb{1}_{\max_{t \le T} S_t \ge b}$

Then using D-D-Sch. $\inf/\sup \, \mathbb{E}[\xi(B_t : t \le \tau)]$ over all st. times $B_\tau \sim \nu$ & $(B_{t \wedge \tau})$ is a UI m

Such a st. time is called an embedding, or a solution to:

(SEP) Given a centred $\nu \in \mathcal{P}_1(\mathbb{R})$, find a st. time τ s.t. $B_\tau \sim \nu$ & $(B_{\tau \wedge t})$

A simple solution using randomised stopping times (Hall '68)

Let $\rho_\nu(dr, ds) = \frac{s-r}{\int_0^\infty x\,\nu(dx)}\, \mathbb{1}_{r\le 0\le s}\, \nu(dr)\,\nu(ds)$

Let $(R,S)\sim\rho_\nu$ indep of (B_t) & $\tau := \inf\{t\ge 0 : B_t \notin (R,S)\}$

indeed $\mathbb{E}[f(B_\tau)] = \mathbb{E}\left[f(S)\frac{-R}{S-R} + f(R)\frac{S}{S-R} \right]$

$$= \int_{-\infty}^0\int_0^\infty -r f(s)\, \frac{\nu(dr)\,\nu(ds)}{\int_0^\infty x\,\nu(dx)} + \int_{-\infty}^0\int_0^\infty s f(r)\, \frac{\nu(dr)\,\nu(ds)}{\int_0^\infty x\,\nu(dx)}$$

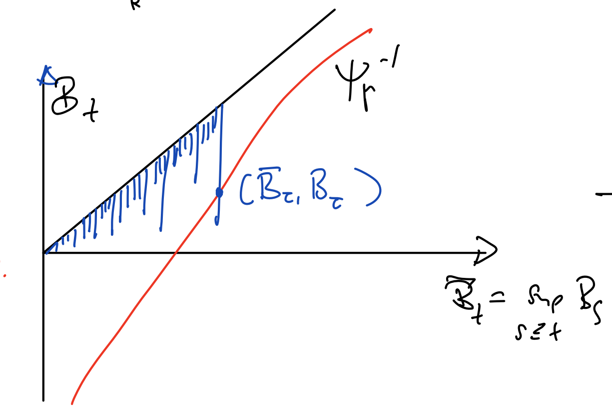
$\int_{-\infty}^\infty x\,\nu(dx)=0$

$$= \int_{-\infty}^\infty f(x)\,\nu(dx).$$

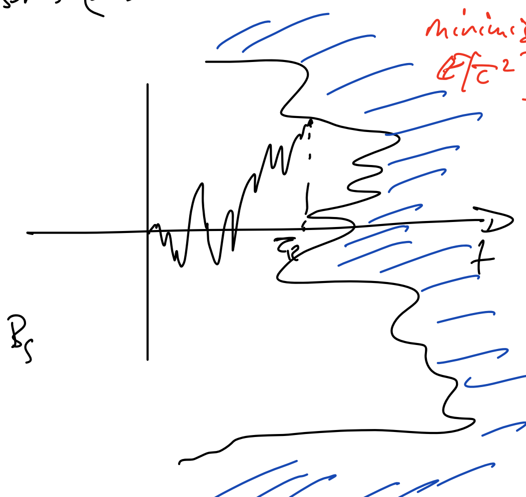
Many solutions exist in the natural filtration $\mathcal{F}^B$. Two are important to us:

- AY: $\exists\ \psi_\nu \uparrow$ s.t. $\tau_{AY} = \inf\{t\ge 0 : \psi_\nu(B_t) \le \bar B_t\}$ solves (SEP)
  $= \inf\{t\ge 0 : B_t \le \psi_\nu^{-1}(\bar B_t)\}$ for $\bar B_t := \sup_{s\le t} B_s$.
- Root: $\exists$ barrier $\mathcal{R} \subseteq \mathbb{R}_+\times\mathbb{R}$, $(x,t)\in\mathcal{R} \Rightarrow (x,s)\in\mathcal{R}\ \forall_{s\ge t}$.
  s.t. $\tau_{\mathcal{R}} = \inf\{t\ge 0 : (t, B_t)\in\mathcal{R}\}$ solves (SEP)

maximises $\mathbb{E}[f(\bar B_\tau)]$ for increasing $f\ge 0$.



minimises $\mathbb{E}[\tau^2]$



OT & SEP (Beiglböck, Cox & Huesmann '17)

$u(y,t) := U\mathcal{L}(B_{t\wedge\tau})(y) = -\mathbb{E}|B_{t\wedge\tau} - y| \ge U_\mu$!

Randomised stopping times

A measure $\xi\in\mathcal{P}(C(\mathbb{R}_+)\times\mathbb{R}_+)$ s.t. $\xi(d\omega, dt) = \mathbb{W}(d\omega)\otimes\xi_\omega(dt)$ is a randomised stopping time if

$A^\xi_\omega(t) := \xi_\omega([0,t])$ is an optional process (here adapted w.r.t. completed nat. filtr.)

(eq) on $C(\mathbb{R}_+)\times[0,1]$ with $\mathbb{W}\otimes Leb$: $\rho(\omega,u) = \inf\{t\ge 0 : \xi_\omega([0,t])\ge u\}$ is an $\bar{\mathcal{F}}_t = \sigma(\mathcal{F}^X_t \cup \mathcal{B}([0,1]))^a$ - stopping time

For an optional Y, $\mathcal{L}(Y_\xi) := Y_\# \xi$, where $Y : C(\mathbb{R}_+)\times\mathbb{R} \to \mathbb{R}$, $(\omega,t)\mapsto Y_t(\omega)$

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ be a p-ty space with a BM (B_t) & $\mathcal{F}_0$ supports an indep $U[0,1]$ r.v..

Rk RST $\equiv$ $(\mathcal{F}_t)$ st. times.

Let $\mu \in \mathcal{P}_2(\mathbb{R})$, $\int x\,d\mu = 0$ & $\gamma : S \to \mathbb{R}$
$(\omega, t) \to \gamma(\omega, t)$

$S = \{(\omega, s) : \omega : [0,s] \to \mathbb{R}$ is continuous with $\omega(0) = 0\}$ are stopped paths.

(OptSEP) $\quad P_\gamma(\mu) = \inf\{\mathbb{E}[\gamma((B_t)_{t \le \tau}, \tau)] : \tau$ solves SEP$\}$

Rk $\mu \in \mathcal{P}_2(\mathbb{R})$, (B^τ) UI iff $\mathbb{E}[\tau] = \int x^2 d\mu$.

Thm Suppose γ is lsc & bdd from below. Then (OptSEP) admits a solution.

Proof Show the set of RST ξ with $B_\xi = \mu$ is compact.

Thm Suppose γ is lsc & bdd from below. Let

$D_\gamma(\mu) = \sup\{\int \psi\, d\mu : \psi \in C(\mathbb{R}),\ |\psi(y)| \le a + by^2,\ \exists M$ a cont. $(\mathcal{F}_t)$-martingale $M_0 = 0$, $|M_t| \le a + bt + cB_t^2$, $M_t + \psi(B_t) \le \gamma((B_s)_{s \le t}, t)\ \forall t \ge 0\}$

Then $P_\gamma(\mu) = D_\gamma(\mu)$.

Now we combine p-tic tools with the geometric intuition from OT.

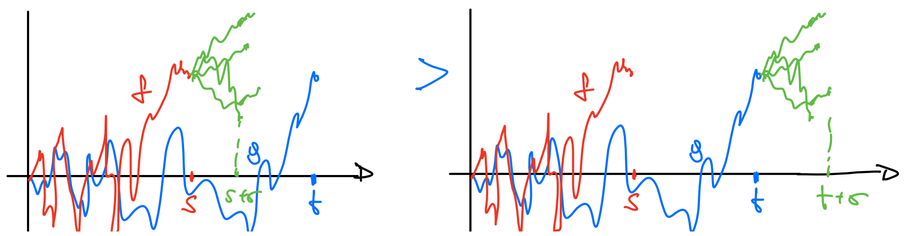
For $(f, s), (h, u) \in S$ let $(f \oplus h)(r) := \begin{cases} f(r) & : r \le s \\ f(s) + h(r - s), & s \le r \le s + u \end{cases}$

$\gamma^{(f,s)\oplus}(h, u) := \gamma(f \oplus h, s + u)$

Def The pair $((f, s), (g, t)) \in S \times S$ is a stop-go pair if $f(s) = g(t)$ and

$\mathbb{E}[\gamma^{(f,s)\oplus}((B_u)_{u \le \sigma}, \sigma)] + \gamma(g, t) > \gamma(f, s) + \mathbb{E}[\gamma^{(g,t)\oplus}((B_u)_{u \le \sigma}, \sigma)] \quad \forall (\mathcal{F}^B)$ s.t.

(provided all defined & finite).



Akin to c-cm but with just two pair of points...

Def A set $\Gamma \subseteq S$ is called γ-monotone if
$$SG \cap (\Gamma^< \times \Gamma) = \emptyset$$
where $SG \subseteq S \times S$ are stop-go pairs & $\Gamma^< = \{(f,s) : \exists (\tilde{f},\tilde{s}) \in \Gamma : s < \tilde{s}, f = \tilde{f} \text{ on } [0,s]\}$

Thm $\gamma : S \to \mathbb{R}$ is Borel measurable s.t. (OptSEP) is well posed & has an optimiser τ. Then $\exists\, \Gamma \subset S$ γ-monotone s.t. $((B_t)_{t \le \tau}, \tau) \in \Gamma$ a.s.

Prop Let $\gamma(f,t) = h(t)$ for a strictly convex $h : \mathbb{R} \to \mathbb{R}$ for which (OptSEP) is well posed. Then a minimiser exists & is a Root stopping time $\tau = \tau_R$ for a barrier R.

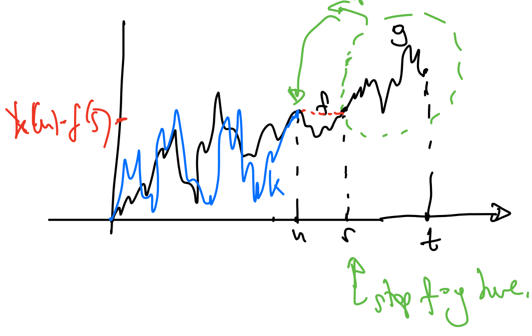
Proof · By above $\exists\, \tilde{\tau}$ & $\exists\, \Gamma$ $((B_t)_{t \le \tilde{\tau}}, \tilde{\tau}) \in \Gamma$ a.s. & $(\Gamma^< \times \Gamma) \cap SG = \emptyset$.

· We have $((f,s),(g,t)) \in SG$ if $f(s) = g(t)$ and
$$\mathbb{E}[h(s+\sigma)] + h(t) > h(s) + \mathbb{E}[h(t+\sigma)] \quad \text{i.e. } h(t) - h(s) > \mathbb{E}[h(t+\sigma) - h(s+\sigma)]$$
& strict convexity of $h \Rightarrow$ iff $t < s$.

Let $R_{cl} = \{(s,x) : \exists\, (g,t) \in \Gamma,\ g(t) = x,\ t \le s\}$
$\cup$
$R_{op} = \{ \text{—} \ \text{''} \ \text{—},\ \text{—},\ t < s\}$

Take $(g,t) \in \Gamma \Rightarrow (t, g(t)) \in R_{cl}$ by definition. Also if $\inf\{s \le t : (s, g(s)) \in R_{op}\} < t$ then $(f,s) = (g_{|[0,s]}, s) \in \Gamma^<$ & $(s, f(s)) \in R_{op}$ for some $s < t$. By def of R_{op}, but $\exists\, (k,u) \in \Gamma : k(u) = f(s)$ & $u < s$. Then $((f,s),(k,u)) \in SG \cap (\Gamma^< \times \Gamma)$ a contradiction.

$\Rightarrow$ $(g,t) \in \Gamma \Rightarrow \inf\{s \le t : (s, g(s)) \in R_{cl}\} \le t \le \inf\{s \le t : (s, g(s)) \in R_{op}\}$



$h(u) + h(t) > h(u+t-s) + h(s)$
$\Updownarrow$ $h(t) - h(s) > h(u+t-s) - h(u)$ & $u < s$. ✓

$$\Rightarrow \tau_{R_{cl}} \le \tilde{\tau} \le \tau_{R_{op}}$$ but the two Roots are $=$ by strong Markov & $\inf\{t > 0 : B_t = 0\} = 0$.

∎

$\Rightarrow$ minimizer $\tilde{\tau}$ is unique. If two τ_1, τ_2 then also $\tau_1 \mathbb{1}_{u \leq 1/2} + \tau_2 \mathbb{1}_{u > 1/2} =: \hat{\tau}$ & by

Thm above $\hat{\tau} = \tau_{Rst}$ $\Rightarrow$ $Rst_1 = Rst_2$.

<u>Rk</u> Here we only showed $\underbrace{\text{optimal}}$ $\Rightarrow$ γ-monotone. The reverse is not conjectured. Probably requires n-tuples of paths?

<u>Rk</u>

$$\mathcal{M}^{cont}(\mu, \nu) = \{ \mathbb{P} \in C([0,1], \mathbb{R}) : \text{continuous martingales with } X_0 \sim \mu, \; X_1 \sim \nu \}$$

is not compact. $\Rightarrow$ things <u>break down big time.</u>