Stochastic Price Formation in Call Auctions

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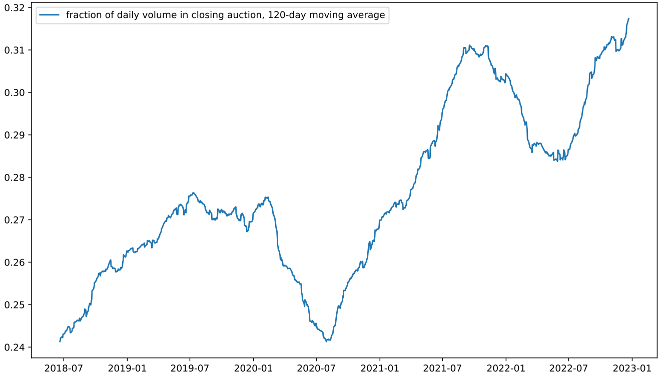
Outline

The Call Auction

- During the day, stocks are traded continuously using so called limit order books.
- To open and close the market, a call auction is conducted.
- In the call auction, orders are aggregated for a while, without executing trades.
- Then a price X is set that maximizes transactable volume and all possible transactions are against this price.

Importance of the closing auction

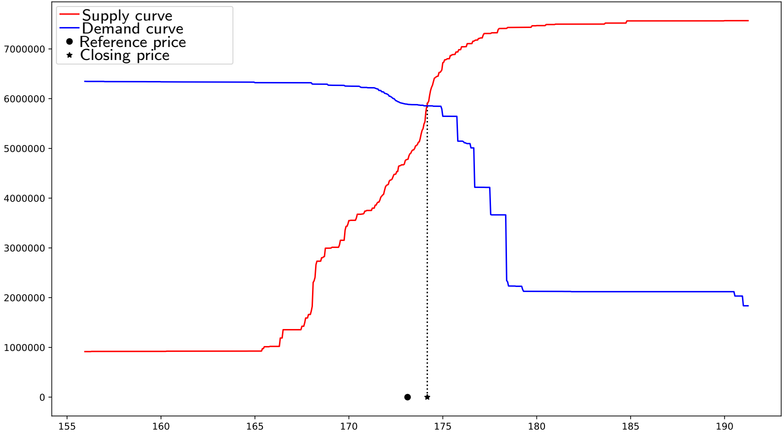


- Nowadays around 30%(!) of the daily volume on the European main exchanges is transacted in the closing auction.

Order types

- Limit order: an order to buy (or sell) a specified quantity of the stock for a price not higher (lower) than a specified limit price.
- Market order: an order to buy (or sell) a specified quantity of the stock against any price.
- A market buy (sell) order can be viewed as a limit buy order with limit price $+\infty$ ($-\infty$).

Closing Auction ASML 2018-03-16



Call auction model[1]

- Order placement distributions F_A, F_B on $(-\infty, \infty)$
- N_A sell limit orders $A_1, \ldots, A_{N_A} \sim F_A$ *i.i.d.*, N_B buy limit orders $B_1, \ldots, B_{N_B} \sim F_B$ *i.i.d.*
- Buy and sell side are independent.
- Gives empirical dfs:

$$\mathbb{F}_A(x) = \frac{1}{N_A} \sum_{i=1}^{N_A} \mathbf{1}_{\{A_i \leq x\}}, \ \mathbb{F}_B(x) = \frac{1}{N_B} \sum_{i=1}^{N_B} \mathbf{1}_{\{B_i \leq x\}}$$

- Supply and demand curves:

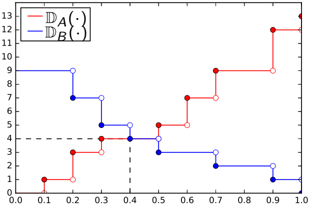
$$\mathbb{D}_A(x) = N_A \mathbb{F}_A(x), \ \mathbb{D}_B(x) = N_B(1 - \mathbb{F}_B(x))$$

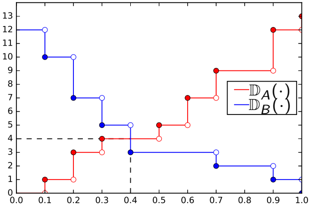
[1] Derksen, M., Kleijn, B. and De Vilder, R., Clearing price distributions in call auctions. Quantitative Finance 20(9):1475-1493, 2020.

Call auction model

- Clearing price X defined by *the clearing equation*

$$\mathbb{D}_A(X) = \mathbb{D}_B(X)$$





Definition

For given supply curve $\mathbb{D}_A$ and demand curve $\mathbb{D}_B$, the *clearing price* is defined by

$$X = \inf\{x \in \mathbb{R} : \mathbb{D}_A(x) \geq \mathbb{D}_B(x))\} \tag{1}$$

Clearing price distribution

- $X \leq x \Leftrightarrow \mathbb{D}_A(x) \geq \mathbb{D}_B(x)$.
- $(\mathbb{D}_A(x), \mathbb{D}_B(x)) \overset{\mathcal{D}}{=} \text{Bin}(N_A, F_A(x)) \times \text{Bin}(N_B, 1 - F_B(x))$.
- Combining gives:

$$\mathbb{P}(X \leq x | N_A, N_B) \\ = \sum_{k=0}^{N_A} \sum_{l=0}^{N_B \wedge k} \binom{N_A}{k} F_A(x)^k (1 - F_A(x))^{N_A - k} \binom{N_B}{l} (1 - F_B(x))^l F_B(x)^{N_B - l}.$$

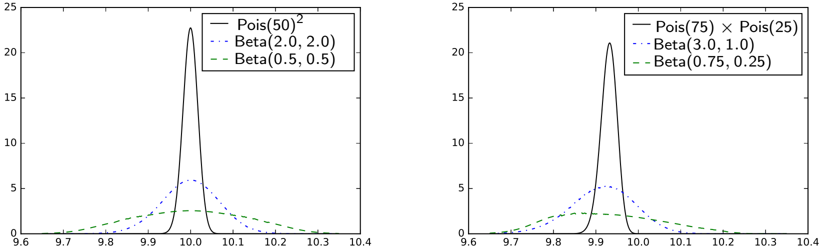


Figure: Clearing price density for $F_A = F_B = \Phi_{10,0.1}$ and various balanced (left panel) and unbalanced (right panel) choices for the order flow distribution of (N_A, N_B), for the beta-distributions set $N_A = \alpha N$, $N_A + N_B = N$, $\alpha \sim$ Beta.

Market orders in the model

- $M_A = \#$sell market orders, $M_B = \#$buy market orders.
- Only play a role through *market order imbalance*

$$\Delta = M_B - M_A$$

.

- Market order imbalance alters the market clearing equation:

$$\mathbb{D}_A(X) = \mathbb{D}_B(X) + \Delta$$

Clearing price distribution, given N_A, N_B, Δ

$$\mathbb{P}(X \leq x | N_A, N_B, \Delta)$$

$$= \sum_{k=0}^{N_A} \sum_{l=0}^{U(k)} \binom{N_A}{k} F_A(x)^k (1 - F_A(x))^{N_A - k} \binom{N_B}{l} (1 - F_B(x))^l F_B(x)^{N_B - l},$$

where $U(k) = (k - \Delta) \wedge N_B$.

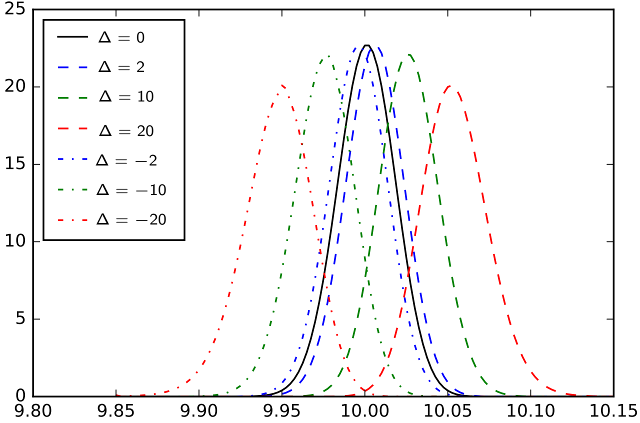


Figure: Clearing price densities, for market order imbalance Δ, with $F_A = F_B = \Phi_{10,0.1}$ and $(N_A, N_B) \sim \text{Pois}(50)^2$.

Large number of orders

- Set $N := N_A + N_B$, $N_A = \alpha N$, for some $\alpha \in (0, 1)$.
- Denote by x_E the *real equilibrium price* defined by

$$\alpha F_A(x_E) = (1 - \alpha)(1 - F_B(x_E)). \tag{2}$$

- Consider the high liquidity limit, what happens as $N \to \infty$?

High liquidity limit

Theorem

Assume that F_A and F_B are strictly increasing with densities f_A and f_B. Additionally, assume that $\Delta = \sqrt{N}D$, for some $D > 0$. Then, as $N \to \infty$,

$$\sqrt{N}(X - x_E) \xrightarrow{w.} N(\mu(x_E), \sigma^2(x_E)), \tag{3}$$

where the asymptotic mean and standard deviation are given by,

$$\mu(x_E) = \frac{D}{\alpha f_A(x_E) + (1-\alpha) f_B(x_E)},$$

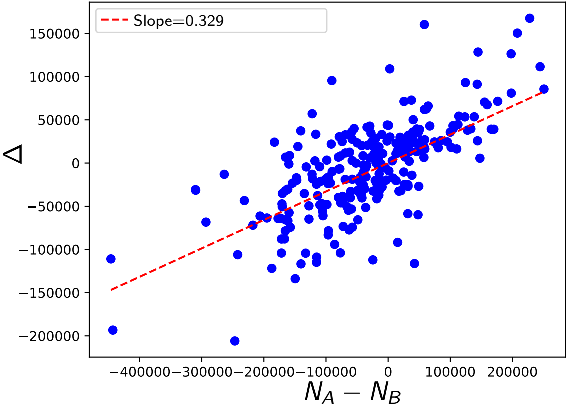
$$\sigma(x_E) = \frac{\sqrt{\alpha F_A(x_E)(1 - F_A(x_E)) + (1-\alpha) F_B(x_E)(1 - F_B(x_E))}}{\alpha f_A(x_E) + (1-\alpha) f_B(x_E)}.$$

Towards a better model

- Modern closing auctions are not sealed.
- Market participants can react on price and imbalance information.
- Notion of time.

Strategic behavior

$\Delta = M_B - M_A \propto (N_A - N_B)$: limit order flow goes against market order imbalance



Concluding remarks

- Around 30% of the daily volume is transacted in the closing auction.
- Stochastic call auction model, leading to closed form solutions for the distribution of price.
- Order distributions are free parameters, can be fitted empirically.
- Limit order submitters behave strategically.

Derksen, M., Kleijn, B. and De Vilder, R., Clearing price distributions in call auctions. *Quantitative Finance*, 2020, 20(9): 1475-1493.

Derksen, M., Kleijn, B. and De Vilder, R., Heavy tailed distributions in closing auctions, *Physica A: Statistical Mechanics and its Applications*, 2022, 593: 126959.

Thank you