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Bayesian filter for pricing grid

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23/01/2023

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Risk management

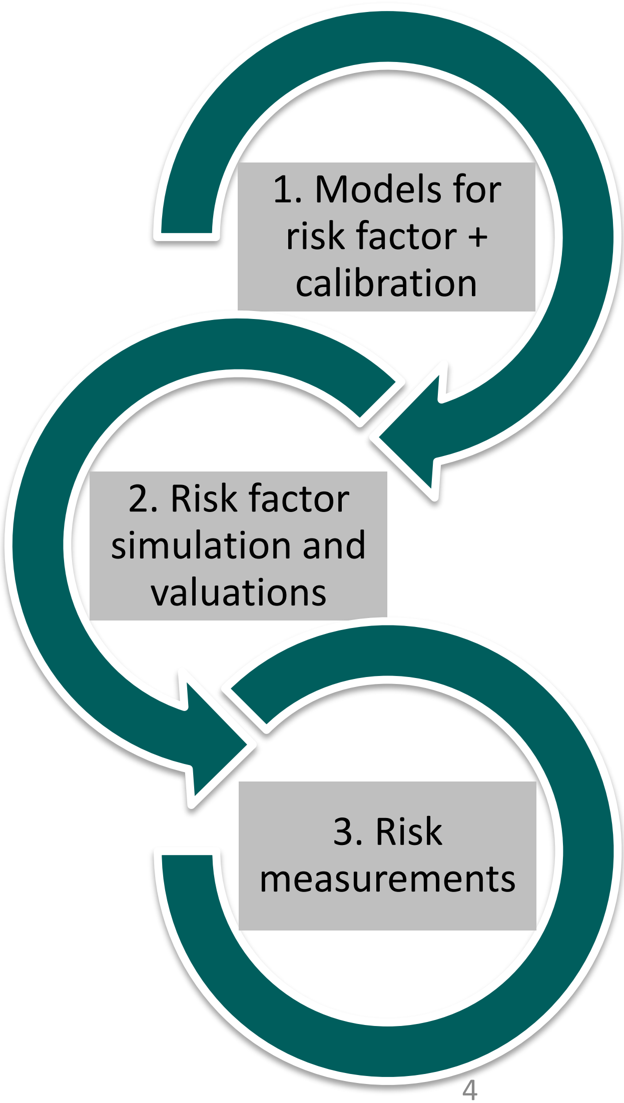
- Financial risk management is the practice which uses financial instruments to manage exposure to risks, incl. market risk, liquidity risk, credit risk, operational risk etc.

- To achieve this, a abstract mathematical representation (a model) is required to analyse the portfolio and make forecast of the likely losses that would be incurred for a variety of risks

- The risk metrics can be:
  - Expected losses (provision)
  - Value at risk (VaR)
  - Economic capital: the unexpected losses, i.e. the difference between VaR and EL

- Usages of the risk metrics:
  - Fulfil the regulatory requirements
  - Hedging
  - Pricing: i.e. XVA

How risk models works in risk measuring



Credit risk and loss valuation

- Credit risk: the possibility of a loss resulting from a borrower's failure to repay a loan or meet contractual obligations.
  - Probability of default (PD)
  - Loss given default (LGD)
  - Exposure at default EAD
  - Loss at time t: $\mathrm{L_t} = 1_{D_t} * \mathrm{LGD}_t * \mathrm{EAD}_t$

- Transition model: $\mathrm{P}_{i\to j,t} = \mathrm{P}\left(\mathrm{R_{i,t}} \to \mathrm{R_{j,t+1}} \middle| X_s, s \leq t\right), \forall\, t$, e.g. the logit model

$$\mathrm{P}_{i\to j,t} \propto g_{ij} * \exp\left(Z_{ij}^{\top} X_t\right)$$

$$X_t = AX_{t-1} + \epsilon_t, \epsilon_t \sim N(0, \Omega)$$

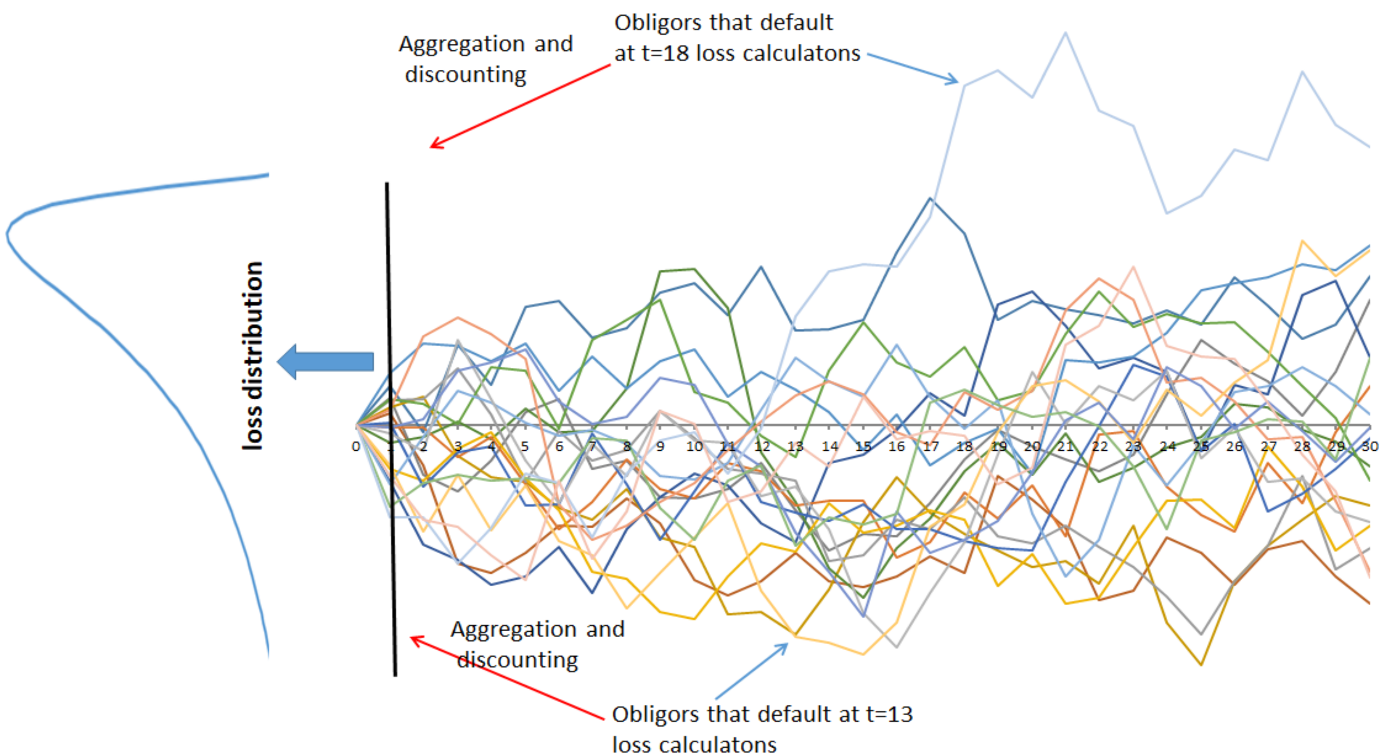
- LGD model: collateral model $\mathrm{LGD}(c_t)$ or structural model $\mathrm{LGD(X_t)}$

- EAD model: on- or off- balance

- Valuation model: $\mathrm{L_t} = V_t - V_0$, (NPV: V_t, coupon: s_t, principle at maturity T: PC)

$$V_t^{(r)} = \sum_{i}^{t} \delta_t s_i + \sum_{k=t+1}^{T} \delta_k E\left[1_{\tau(r)\in(k-1,k]}(1 - \mathrm{LGD}_k)\mathrm{EAD}_k \mid F_t\right] + \sum_{k=t+1}^{T} \delta_k s_k E\left[1_{\tau(r)>k} \mid t\right] + \delta_T PCE\left[1_{\tau(r)>k} \mid F_t\right]$$

Monte Carlo simulation



- The complexity of Monte Carlo simulation is $O(\mathrm{Nsim}_x \times \mathrm{Nclient})$
- If pooling: $\mathrm{L_t} = \mathrm{PD}_t * \mathrm{LGD}_t * \mathrm{EAD}_t$, still simulate the PD per scenario is very time consuming since it requires huge amout of matrix multiplication

Pricing grids

- Instead of simulating and aggregating the realized losses over the whole time periods, the aggregated losses after the horizon (i.e. 1 year), are approximated by the expected losses

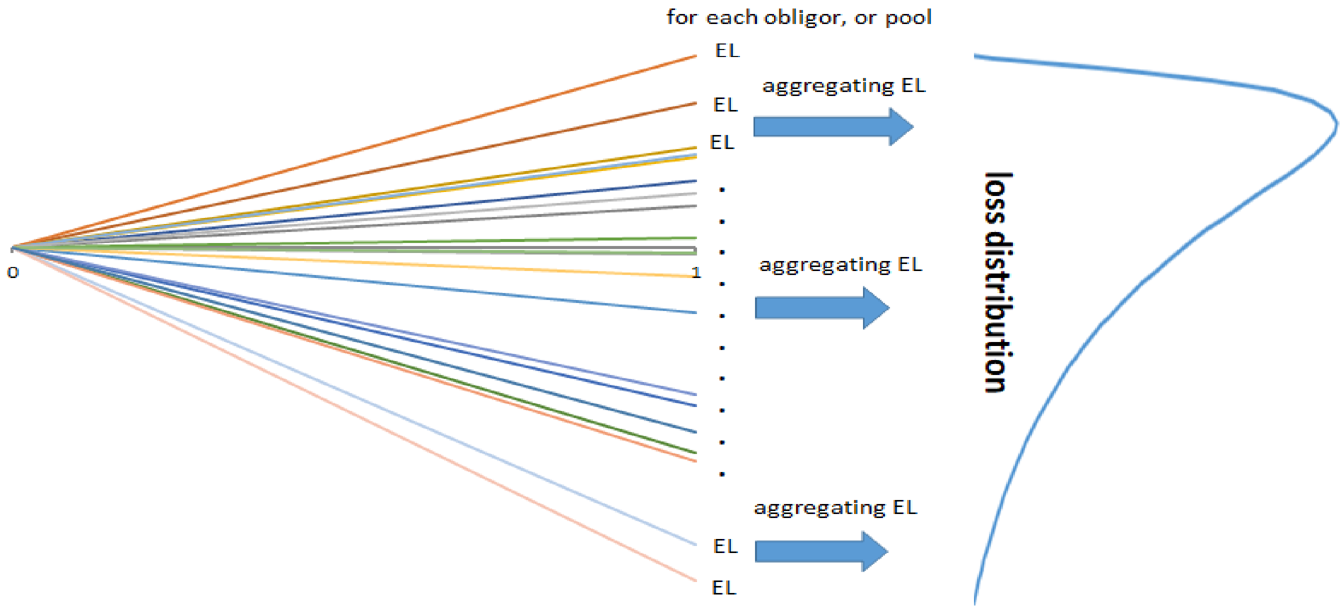
- The expected loss depends on the rating at one year R_1, the value of the state X_1 and the value of collateral c_1. Specifically

$$E[\mathrm{L}_1|\mathrm{R}_1, X_1, c_1] = \sum_{t>1} E[\,\mathrm{PD}(\mathrm{R}_1, X_1, \ldots, X_t) * \mathrm{LGD}(c_t) * \mathrm{EAD}_t|\mathrm{R}_1, X_1, c_1]$$

- Therefore, these expected losses forms a pricing grid for the expected losses with grid points R_1, X_1, c_1(actually loan to value at time 1)

- This means the simulation stops at the 1 year.
  - For the losses within the first year: full MC applies
  - For the losses after one year: R_1, X_1, c_1, the expected losses $E[\mathrm{L}_1|\mathrm{R}_1, X_1, c_1]$ approximated by using the pricing grid

MC with pricing grid



- Issues remain: curse of dimensionality
  - Complexity of training the pricing grid increases rapidly with dimension increasing
  - While a higher dimensional model for the transition is desired in order to capture the transitions of the whole matrix
  - A dimension reduction approach is needed

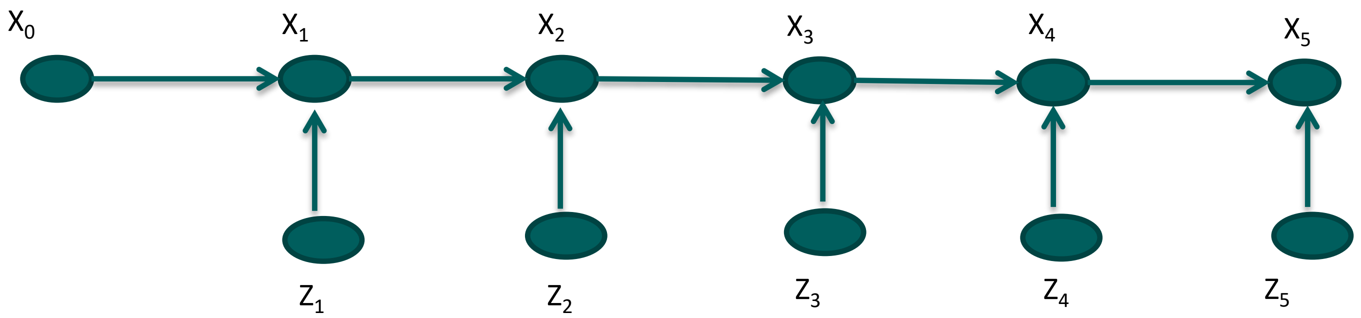
Dimension reduction approaches - Bayesian filtering (I)

- State equation

$$X_t = F_t(X_{t-1}) + \eta_t,$$

- Measurement equation:

$$Z_t = h_t(X_t) + \epsilon_t$$



1. Model update, i.e. based on estimation of the previous states, use the state equation to update the value or distribution of next states. This estimate is referred as the prior estimate

2. Measurements update, i.e. use the observation to update the prior estimates. This estimate is referred as the posterior estimate

Dimension reduction approaches - Bayesian filtering (II)

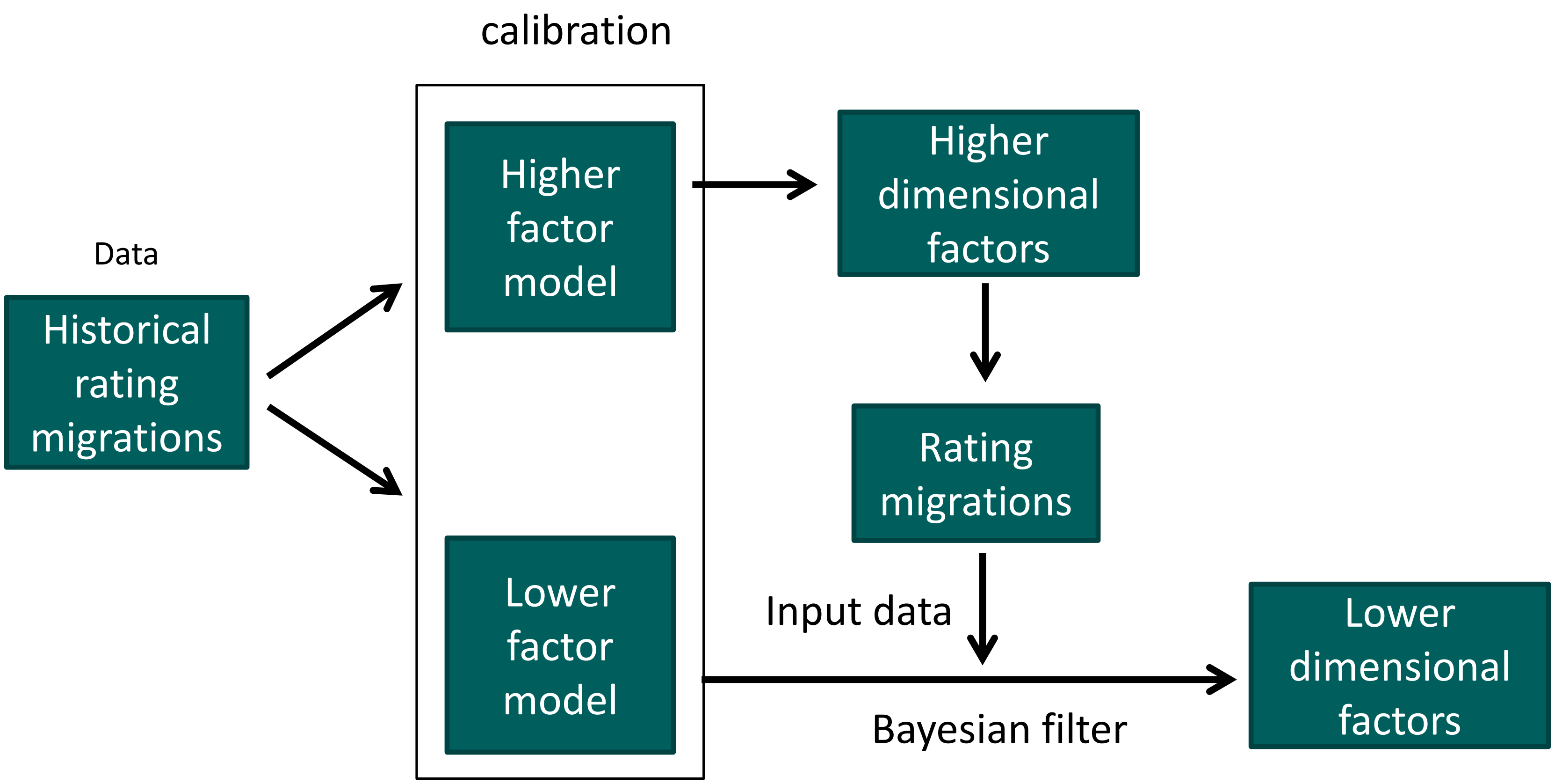
- **Kalman filter and its extensions**

- **Particle filter**

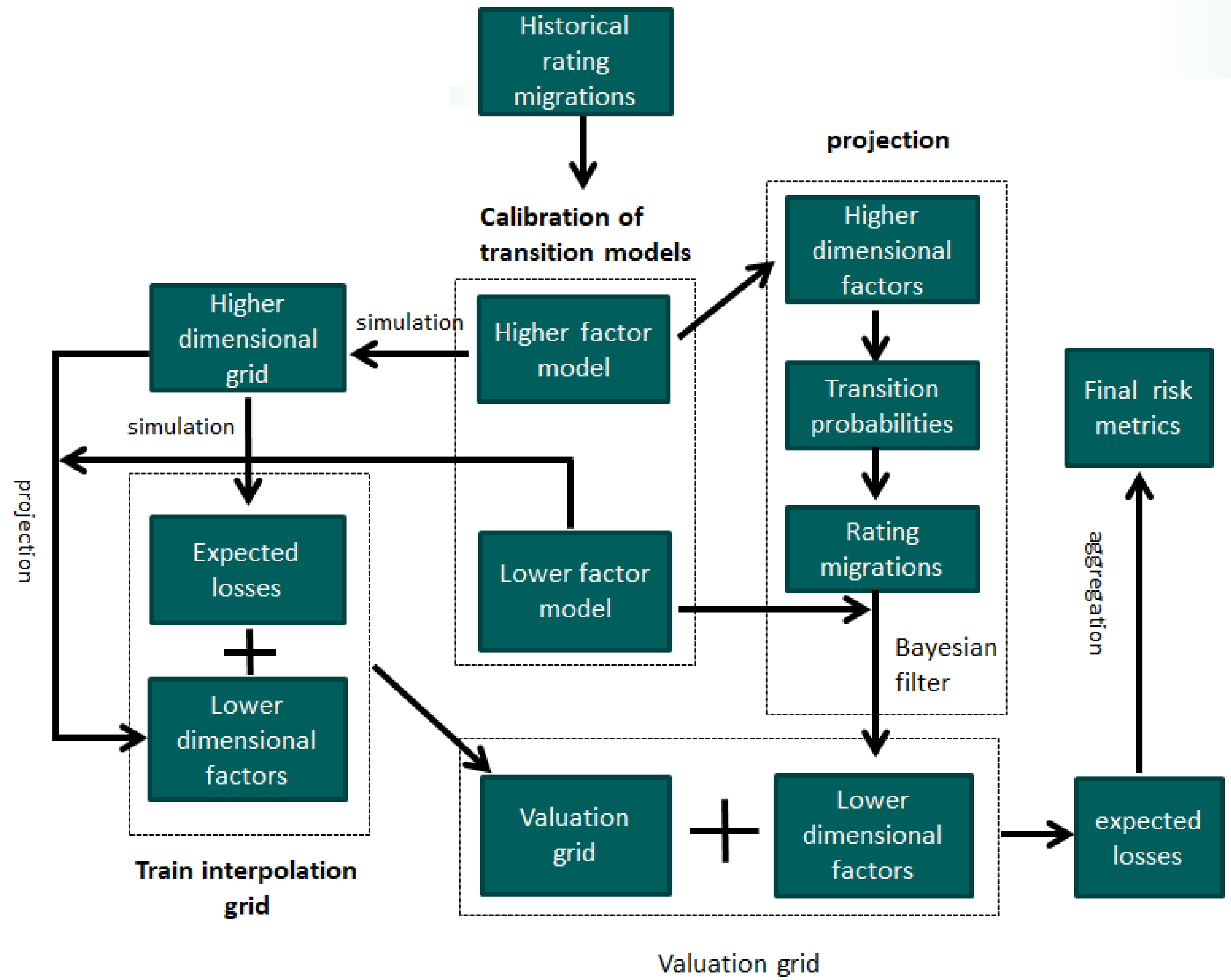
- **A Kalman particle filter for online parameter estimation with applications to affine models**. He, J., Khedher, A. & Spreij, P. *Stat Inference Stoch Process 24, 353–403 (2021).* https://doi.org/10.1007/s11203-021-09239-3

- **Time Series Analysis by State Space Methods.** Durbin, James & Koopman, Siem Jan. (2001).

Algorithm- Bayesian filter projection



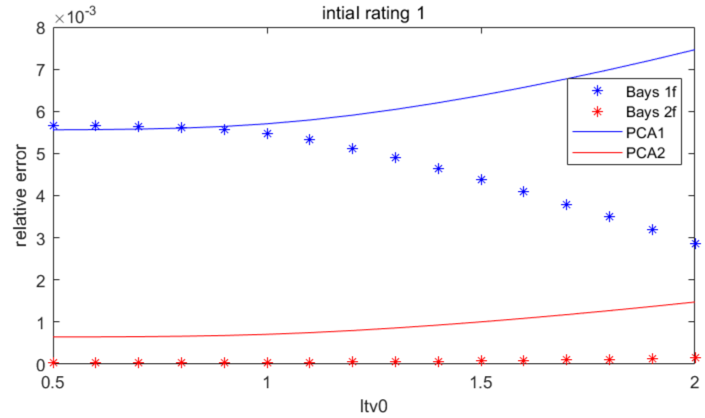
Algorithm- risk metrics calculation



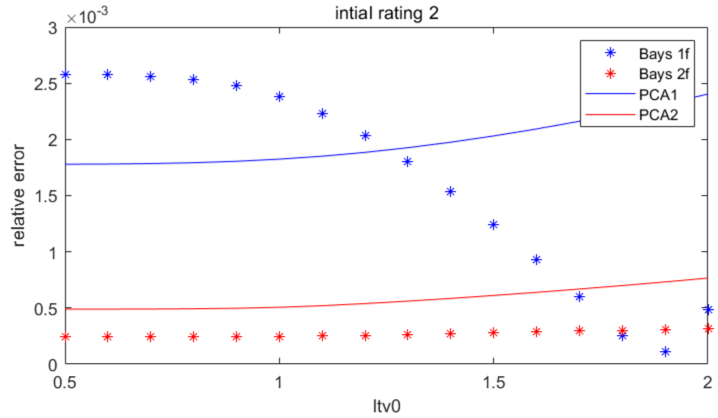
Numerical experiments

- Four rating system: three performing ratings and one default rating
- Number of clients of each rating: 100,000, 1,000, 300
- PD: logit model; LGD: Collateral model; EAD: unit amortizing loan, on-balance
- PD and LGD are assumed to be independent to simplify the calculation → PD-LGD grid is not necessary → only need PD grid, ELGD has analytical BS formula
- Simulated observed migrations → Calibration four, two and one factor model
- The four factor model is the higher dimensional model for the transitions and the two or one factor models are for the valuation grid
- The time span is 30 periods, i.e. T=30.
- Pools only, i.e. the idiosyncratic migrations are ignored
- Interest rates ignored: no discounting, no interest gains.
- Grid: linear interpolation
- One million scenarios, i.e. X_t are simulated, the EL and VaR, per rating and per initial LTV, are computed based on the Bayesian and PCA projection approach. The PCA approach is used for comparison.
- Benchmark: EL and VaR computed based on the four factor model.

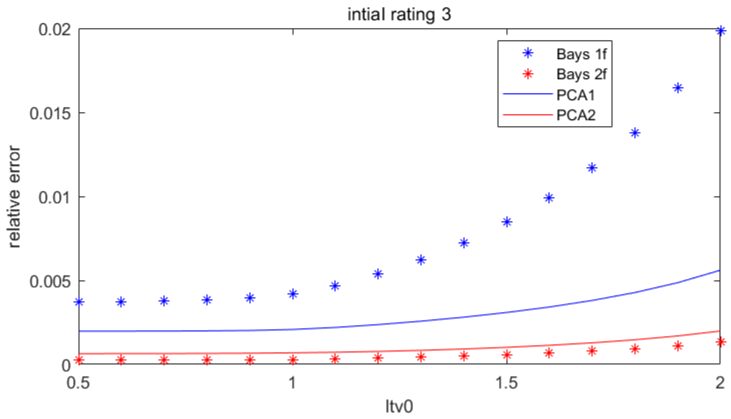
Results: EL



intial rating 1
×10^-3
8
7
6
5
4
3
2
1
0
0.5
1
1.5
2
relative error
ltv0
Bays 1f
Bays 2f
PCA1
PCA2

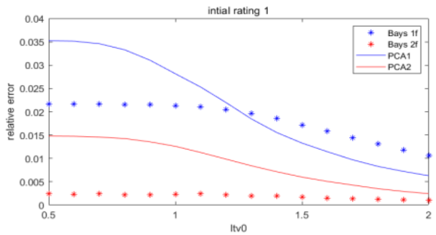


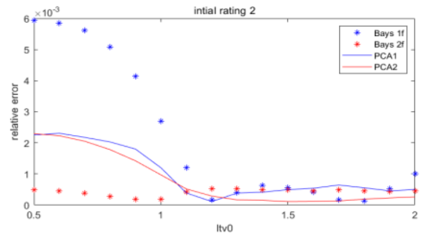
intial rating 2
×10^-3
3
2.5
2
1.5
1
0.5
0
0.5
1
1.5
2
relative error
ltv0
Bays 1f
Bays 2f
PCA1
PCA2

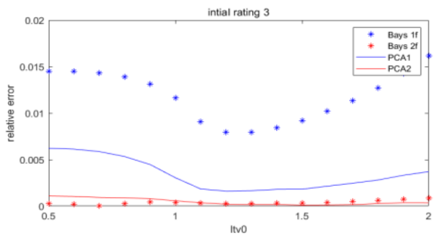


intial rating 3
0.02
0.015
0.01
0.005
0
0.5
1
1.5
2
relative error
ltv0
Bays 1f
Bays 2f
PCA1
PCA2

Results: VaR







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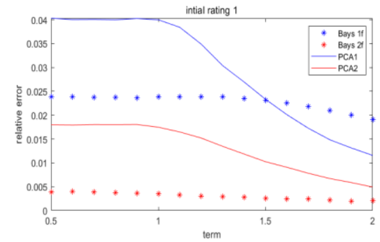

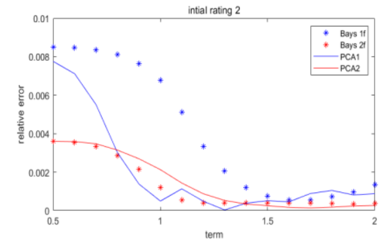

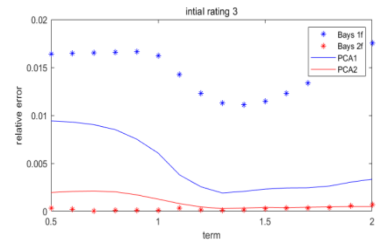

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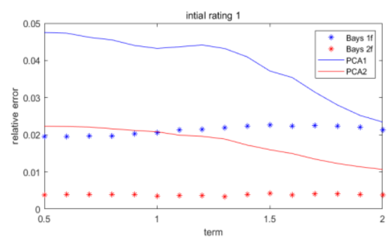

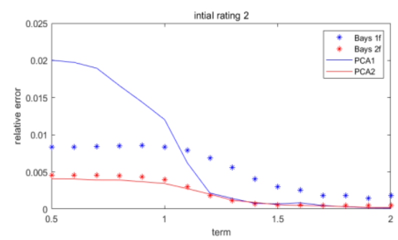

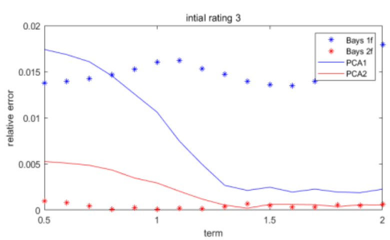

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